Multiplicity of periodic solutions for nearly resonant Hamiltonian systems

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Abstract

We prove a multiplicity result for the periodic problem associated with a Hamiltonian system whose Hamiltonian function has a twisting part and a nonresonant part. The possible approach to resonance together with some kind of Landesman–Lazer conditions is also analyzed. We propose a new version of this condition, and we also treat the so-called *double resonance* situation.

1 Introduction

We consider the Hamiltonian system

$$\begin{cases} J\dot{u} = \nabla_u \mathcal{H}(t,u) + \nabla_u P(t,u,z)\,, \\ J\dot{z} = \nabla_z \mathscr{H}(t,z) + \nabla_z P(t,u,z)\,, \end{cases} \tag{HS}$$

where

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

denotes the standard symplectic matrix in any even dimension. We assume that all the involved functions $\mathcal{H}(t,u)$, $\mathscr{H}(t,z)$, and $P(t,u,z)$ are continuous, T-periodic in the variable t and continuously differentiable with respect to the variables (u,z).

System (HS) appears as the coupling of two systems, which are assumed to have a completely different behaviour. While for the first one we have a *twist* dynamics, for the second one we ask for some *nonresonance* conditions. In order to better understand this setting, let us first provide a brief historical account, which for the reader's convenience we divide in three parts.

Twist dynamics. In 1912, Poincaré [52] conjectured his *last geometrical theorem*, proved by Birkhoff in [7, 8], which is now called the Poincaré–Birkhoff Theorem (see [11] for a modern exposition). The theorem has been extended in many ways and applied to obtain multiplicity of periodic solutions for planar Hamiltonian systems of the type

$$J\dot{u} = \nabla_u \mathcal{H}(t, u) \tag{1}$$

(see [32] and the references therein). Here we state a version of it, as proposed in [38].

Theorem 1.1. *Assume $\mathcal{H}(t, u)$ to be continuous, T-periodic in t, continuously differentiable in $u = (q, p)$, and 2π-periodic with respect to q. Let $a < b$ be such that all solutions $u = (q, p)$ of (1) starting with $p(0) \in [a, b]$ are defined on $[0, T]$ and are such that*

$$\begin{cases} p(0) = a \quad \Rightarrow \quad q(T) - q(0) < 0\,, \\ p(0) = b \quad \Rightarrow \quad q(T) - q(0) > 0\,. \end{cases}$$

Then, system (1) has at least two geometrically distinct T-periodic solutions $u = (q, p)$, with $p(0) \in \,]a, b[\,$.

An analogous statement holds with the above inequalities being reversed.

In 1983, Conley and Zehnder [16] were able to obtain a multiplicity result for the periodic problem associated with a higher dimensional Hamiltonian system, opening the road towards a generalization of the Poincaré–Birkhoff Theorem in this setting. Since then, many papers have been devoted to this problem, e.g., [14, 21, 28, 39, 42, 46, 53, 55], extending the result in [16] in several directions. In [38], a generalization of the Poincaré–Birkhoff Theorem in the setting of higher dimensional Hamiltonian systems was proved, by the use of an infinite dimensional version of the Lusternik–Schnirelmann theory provided in [55]. This result was later extended in [26] to coupled systems having a twisting part and a nonresonant linear part.

Nonresonance. In 1972/73, Lazer [44] and Ahmad [1] proved an existence and uniqueness result for the T-periodic problem associated with a system in $\mathbb{R}^N$ of the type

$$\ddot{x} + \nabla \mathscr{G}(x) = p(t)\,.$$

They asked for the existence of $\mathbb{A}, \mathbb{B} \in \mathrm{Sym}(\mathbb{R}^N)$, with

$$\sigma(\mathbb{A}) = \{\alpha_1 \leq \cdots \leq \alpha_N\}\,, \qquad \sigma(\mathbb{B}) = \{\beta_1 \leq \cdots \leq \beta_N\}\,,$$

such that

$$\mathbb{A} \leq \mathscr{G}''(x) \leq \mathbb{B}\,, \qquad \text{for every } x \in \mathbb{R}^N,$$

and

$$[\alpha_k, \beta_k] \cap \left\{ \left(\frac{2\pi n}{T}\right)^2 : n \in \mathbb{N} \right\} = \varnothing\,, \quad k = 1, \ldots, N\,. \tag{2}$$

Here and in the following we denote by $\mathrm{Sym}(\mathbb{R}^L)$ the set of symmetric $L \times L$ real matrices, and by $\sigma(\mathbb{M})$ the spectrum of any $\mathbb{M} \in \mathrm{Sym}(\mathbb{R}^L)$. Moreover, given $\mathbb{A}, \mathbb{B} \in \mathrm{Sym}(\mathbb{R}^L)$ we write $\mathbb{A} \leq \mathbb{B}$ if $\langle \mathbb{A}z, z\rangle \leq \langle \mathbb{B}z, z\rangle$ for every $z \in \mathbb{R}^L$. Notice that (2) is a typical nonresonance condition involving the eigenvalues of the differential operator.

Different proofs of this theorem have been provided in [3, 10, 17, 50, 51, 57]; it has then been extended by many authors [2, 4, 5, 6, 40, 48, 56, 59] and finally found a solid abstract setting in [29]. In particular, for the system

$$J\dot{z} = \nabla_z \mathscr{H}(t, z)\,, \tag{3}$$

the following generalization of the Ahmad–Lazer result has been proved in [30].

Theorem 1.2. *Assume the function $\mathscr{H}$ to be twice continuously differentiable in z, with*

$$\mathbb{A} \leq \mathscr{H}''_z(t, z) \leq \mathbb{B}\,, \quad \textit{for every } (t, z) \in [0, T] \times \mathbb{R}^{2N},$$

for some $\mathbb{A}, \mathbb{B} \in \mathrm{Sym}(\mathbb{R}^{2N})$, and

$$\bigcup_{\lambda \in [0,1]} \sigma\big((1 - \lambda)J\mathbb{A} + \lambda J\mathbb{B}\big) \cap \frac{2\pi}{T}\, i\mathbb{Z} = \varnothing\,.$$

Then, system (3) has a unique T-periodic solution.

Approaching resonance. The so-called Landesman–Lazer condition has been indeed first introduced by Lazer and Leach in [45] for the periodic problem associated with a scalar second order ODE of the type

$$\ddot{x} + \lambda x + h(t, x) = 0\,, \tag{4}$$

assuming $h(t, x)$ to be continuous, and T-periodic in t. The following result was proved.

Theorem 1.3. *Let $\lambda = (\frac{2\pi n}{T})^2$ for some positive integer n, and $h(t,x)$ be uniformly bounded. If for every nontrivial solution of $\ddot{\xi} + \lambda\xi = 0$ one has that*

$$\int_{\{\xi<0\}} \limsup_{x\to-\infty} h(t,x)\xi(t)\,dt + \int_{\{\xi>0\}} \liminf_{x\to+\infty} h(t,x)\xi(t)\,dt > 0\,, \tag{5}$$

then equation (4) *has a T-periodic solution.*

One year later the condition has been adapted in [43] in order to deal with a Dirichlet problem associated with an elliptic PDE, and since then it is named after Landesman and Lazer. Notice that when $h(t,x)$ is increasing in x this condition happens to be *necessary and sufficient* for the existence of a solution. This remarkable fact has attracted a lot of attention, leading to a large literature on the subject (see, e.g., [12, 47, 54]). Note that, in the above theorem, condition (5) can be replaced by the symmetrical one

$$\int_{\{\xi<0\}} \liminf_{x\to-\infty} h(t,x)\xi(t)\,dt + \int_{\{\xi>0\}} \limsup_{x\to+\infty} h(t,x)\xi(t)\,dt < 0\,.$$

Even the so-called *double resonance* has been considered, assuming a Landesman–Lazer condition on both sides (see [18, 19, 20, 23, 54]).

It is the aim of this paper to couple a Poincaré–Birkhoff type system with a nonresonant one, still preserving the multiplicity of periodic solutions. We also introduce a new version of the Landesman–Lazer condition, which seems to be well fitted in order to deal with higher dimensional systems. We are then able to extend our analysis to the *double resonance* situation. Our multiplicity results thus contain in a unique setting all the above stated ones.

Let us now describe in detail how the paper is organized.

In Section 2 we introduce the general setting for system (HS), coupling a system in $\mathbb{R}^{2M}$ having a twist dynamics with a nonresonant one in $\mathbb{R}^{2N}$. Assuming the first Hamiltonian function $\mathcal{H}(t,q,p)$ to be 2π-periodic in the components of the state variable $q = (q_1,\ldots,q_M)$, we will prove the existence of at least $M+1$ periodic solutions.

In Section 3 we propose a new version of the Landesman–Lazer condition to be imposed on the second system in (HS) and prove that the above mentioned multiplicity result still holds in this case.

In Section 4 we compare our version of the Landesman–Lazer condition with the one introduced in [22] in the setting of planar systems. Then we show that, when dealing with a scalar second order equation, our condition follows from the classical one.

In Section 5 we discuss some possible extensions and applications of our results, and suggest a few open problems.

In the Appendix we provide a detailed exposition of a compactness property of the solutions of ODE's which plays an important role in the proof of our results.

2 Nonresonance

Using the notation $u = (q, p)$, $z = (x, y)$, with

$$q = (q_1, \dots, q_M) , \quad p = (p_1, \dots, p_M) ,$$

and

$$x = (x_1, \dots, x_N) , \quad y = (y_1, \dots, y_N) ,$$

system (HS) can be equivalently written as

$$\begin{cases} \dot{q} = \nabla_p \mathcal{H}(t, q, p) + \nabla_p P(t, q, p, x, y) , \\ \dot{p} = -\nabla_q \mathcal{H}(t, q, p) - \nabla_q P(t, q, p, x, y) , \\ \dot{x} = \nabla_y \mathscr{H}(t, x, y) + \nabla_y P(t, q, p, x, y) , \\ \dot{y} = -\nabla_x \mathscr{H}(t, x, y) - \nabla_x P(t, q, p, x, y) . \end{cases}$$

As already said, we assume that all functions $\mathcal{H}(t, q, p)$, $\mathscr{H}(t, x, y)$, and $P(t, q, p, x, y)$ are continuous, T-periodic in the variable t and continuously differentiable with respect to the variables (q, p, x, y).

Let us now present our further assumptions. We first ask for the *periodicity* of $\mathcal{H}$ in the state variables.

A1. The function $\mathcal{H}(t, q, p)$ is 2π-periodic in q_i for every $i \in \{1, \dots, M\}$.

We now assume that the first system in (HS) has a so-called *twist dynamics.*

A2. Given the rectangle

$$\mathcal{D} = [a_1, b_1] \times \cdots \times [a_M, b_M] ,$$

there exists an M-tuple $(s_1, \dots, s_M) \in \{-1,1\}^M$ such that, for every C^1-function $\mathcal{Z} : [0,T] \to \mathbb{R}^{2N}$, all the solutions (q,p) of the system

$$\begin{cases} \dot{q} = \nabla_p \mathcal{H}(t,q,p) + \nabla_p P(t,q,p,\mathcal{Z}(t))\,, \\ \dot{p} = -\nabla_q \mathcal{H}(t,q,p) - \nabla_q P(t,q,p,\mathcal{Z}(t))\,, \end{cases}$$

starting with $p(0) \in \mathcal{D}$, are defined on $[0,T]$ and, for every $i \in \{1,\dots,M\}$,

$$\begin{cases} p_i(0) = a_i \quad \Longrightarrow \quad s_i(q_i(T) - q_i(0)) < 0\,, \\ p_i(0) = b_i \quad \Longrightarrow \quad s_i(q_i(T) - q_i(0)) > 0\,. \end{cases}$$

Concerning the function P, we assume *periodicity* in the q-variables and that it has a *bounded gradient.*

***A*3.** The function $P(t,q,p,x,y)$ is 2π-periodic in q_i for every $i \in \{1,\dots,M\}$, and has a bounded gradient with respect to (q,p,x,y). In particular, there exists $\overline{m} > 0$ such that

$$|\nabla_z P(t,u,z)| \le \overline{m}\,, \quad \text{for every } (t,u,z) \in [0,T] \times \mathbb{R}^{2M} \times \mathbb{R}^{2N}.$$

We now introduce a *structural assumption* for the function $\mathscr{H}$.

***A*4.** There are two functions $\mathbb{M} : \mathbb{R}\times\mathbb{R}^{2N} \to \mathrm{Sym}(\mathbb{R}^{2N})$ and $Q : \mathbb{R}\times\mathbb{R}^{2N} \to \mathbb{R}$ such that

$$\nabla_z \mathscr{H}(t,z) = \mathbb{M}(t,z)z + \nabla_z Q(t,z)\,. \tag{6}$$

The above functions are continuous, T-periodic in t,

$$\mathbb{A} \le \mathbb{M}(t,z) \le \mathbb{B}\,, \quad \text{for every } (t,z) \in [0,T] \times \mathbb{R}^{2N},$$

for some $\mathbb{A}, \mathbb{B} \in \mathrm{Sym}(\mathbb{R}^{2N})$, and $Q(t,z)$ is continuously differentiable in z, with uniformly bounded gradient $\nabla_z Q(t,z)$.

Finally, a *nonresonance condition* is needed.

***A*5.** The following holds:

$$\bigcup_{\lambda\in[0,1]} \sigma\big((1-\lambda)J\mathbb{A} + \lambda J\mathbb{B}\big) \cap \frac{2\pi}{T}\, i\mathbb{Z} = \varnothing\,.$$

We are now able to state our first main result.

Theorem 2.1. *Assume $A1 - A5$. Then system (HS) has at least $M+1$ geometrically distinct T-periodic solutions satisfying $p(0) \in \mathring{\mathcal{D}}$.*

In the above setting, T-periodic solutions appear in equivalence classes made of those functions whose components $q_i(t)$ differ by an integer multiple of 2π. We say that two T-periodic solutions are *geometrically distinct* if they do not belong to the same equivalence class.

Assumption A2 is usually called a twist condition. The form given here, which involves a C^1-function $\mathcal{Z}(t)$, has already been exploited in [24, 27, 37, 36], where several examples of applications have been discussed.

Concerning assumption A4, let us point out that it is a consequence of the following condition.

***A$\widehat{4}$*.** The function $\mathscr{H}$ is twice continuously differentiable in z, with

$$\mathbb{A} \leq \mathscr{H}_z''(t,z) \leq \mathbb{B}\,, \quad \text{for every } (t,z) \in [0,T] \times \mathbb{R}^{2N}$$

for some $\mathbb{A}, \mathbb{B} \in \mathrm{Sym}(\mathbb{R}^{2N})$.

Indeed, we have

$$\nabla_z \mathscr{H}(t,z) = \left(\int_0^1 \mathscr{H}_z''(t,sz)\,ds \right) z + \nabla_z \mathscr{H}(t,0)\,,$$

hence, defining

$$\mathbb{M}(t,z) = \int_0^1 \mathscr{H}_z''(t,sz)\,ds\,,$$

condition $A4$ is readily verified. We thus immediately have the following extension of Theorems 1.1 and 1.2.

Corollary 2.2. *If $A1 - A3$, $A\widehat{4}$, and $A5$ are satisfied, the same conclusion of Theorem 2.1 holds.*

As a particular case of (HS), we have the system

$$\begin{cases} J\dot{u} = \nabla_u \mathcal{H}(t,u) + \nabla_u P(t,u,x)\,, \\ \ddot{x} + \nabla_x \mathscr{G}(t,x) = -\nabla_x P(t,u,x)\,. \end{cases} \tag{7}$$

We assume $A1 - A3$, and the following conditions corresponding to $A4$ and $A5$.

***A$\widetilde{4}$*.** There are two functions $\widetilde{\mathbb{M}} : \mathbb{R} \times \mathbb{R}^N \to \mathrm{Sym}(\mathbb{R}^N)$ and $\widetilde{Q} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that

$$\nabla_x \mathscr{G}(t,x) = \widetilde{\mathbb{M}}(t,x)x + \nabla_x \widetilde{Q}(t,x)\,.$$

The above functions above are continuous, T-periodic in t,

$$\widetilde{\mathbb{A}} \leq \widetilde{\mathbb{M}}(t,x) \leq \widetilde{\mathbb{B}}\,, \quad \text{for every } (t,x) \in [0,T] \times \mathbb{R}^N,$$

for some $\widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \in \mathrm{Sym}(\mathbb{R}^N)$, and $\widetilde{Q}(t,x)$ is continuously differentiable in x, with uniformly bounded gradient $\nabla_x \widetilde{Q}(t,x)$.

***A*$\widetilde{5}$.** Writing $\sigma(\widetilde{\mathbb{A}}) = \{\alpha_1 \leq \cdots \leq \alpha_N\}$ and $\sigma(\widetilde{\mathbb{B}}) = \{\beta_1 \leq \cdots \leq \beta_N\}$, the following nonresonance condition holds:

$$[\alpha_k, \beta_k] \cap \left\{ \left(\frac{2\pi n}{T}\right)^2 : n \in \mathbb{N} \right\} = \varnothing \,, \quad k = 1, \ldots, N \,.$$

We remark that, since $\widetilde{\mathbb{A}} \leq \widetilde{\mathbb{B}}$, the Courant–Fischer Theorem guarantees that $\alpha_k \leq \beta_k$ for every $k = 1, \ldots, N$ (see, e.g., [41, Theorem 4.2.11]).

Theorem 2.3. *Assume $A1 - A3$, $A\widetilde{4}$, and $A\widetilde{5}$. Then system* (7) *has at least $M+1$ geometrically distinct T-periodic solutions, with $p(0) \in \mathring{\mathcal{D}}$.*

Proof. Writing the second equation as

$$\dot{x} = y \qquad \dot{y} = -\nabla_x \mathscr{G}(t,x) - \nabla_x P(t,u,x) \,,$$

we can recover the setting of Theorem 2.1, with

$$\mathbb{A} = \begin{pmatrix} \widetilde{\mathbb{A}} & 0 \\ 0 & I \end{pmatrix}, \qquad \mathbb{B} = \begin{pmatrix} \widetilde{\mathbb{B}} & 0 \\ 0 & I \end{pmatrix}.$$

Indeed, it is easily seen that condition $A\widetilde{4}$ is equivalent to $A4$. □

As an immediate consequence, we have the following.

Corollary 2.4. *Assume $A1 - A3$ and that $\mathscr{G}$ is twice continuously differentiable in x, with*

$$\widetilde{\mathbb{A}} \leq \mathscr{G}''_x(t,x) \leq \widetilde{\mathbb{B}} \,, \quad \textit{for every } (t,x) \in [0,T] \times \mathbb{R}^N .$$

If also $A\widetilde{5}$ is satisfied, then the same conclusion of Theorem 2.3 holds.

2.1 Proof of Theorem 2.1

The main strategy of the proof is to modify the Hamiltonian functions in system (HS) in order to enter the setting of [26, Corollary 2.4], where the nonlinearity in the second system is a perturbation of a linear one and $\nabla_u \mathcal{H}$ is assumed to be bounded.

By (6), there is a function $\Phi : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ such that $\mathscr{H}(t,z) = \Phi(t,z) + Q(t,z)$, and

$$\nabla_z \Phi(t,z) = \mathbb{M}(t,z) z \,.$$

Writing

$$\Phi(t,z) = \Phi(t,0) + \int_0^1 \langle \nabla_z \Phi(t,sz), z \rangle \, ds \,,$$

we can assume without loss of generality that $\Phi(t,0)=0$. Hence, by $A4$ we have that

$$\tfrac{1}{2}\langle \mathbb{A}z, z\rangle \le \Phi(t,z) \le \tfrac{1}{2}\langle \mathbb{B}z, z\rangle\,. \tag{8}$$

For every $r>1$ let $\eta_r:\mathbb{R}\to[0,1]$ be a C^∞-function such that

$$\eta_r(\xi)=\begin{cases}1 & \text{if } \xi\le r\,,\\ 0 & \text{if } \xi\ge r^3\,,\end{cases}$$

and

$$-\frac{1}{\xi\ln\xi}\le \eta_r'(\xi)\le 0\,,\quad \text{for every } \xi\ge r\,. \tag{9}$$

The existence of such a function is guaranteed by the fact that

$$\int_r^{r^3}\frac{d\xi}{\xi\ln\xi}>1\,.$$

We now fix a matrix $\mathbb{D}\in\mathrm{Sym}(\mathbb{R}^{2N})$ such that $\mathbb{A}\le\mathbb{D}\le\mathbb{B}$, and define the function $\Phi_r:\mathbb{R}\times\mathbb{R}^{2N}\to\mathbb{R}$ as

$$\Phi_r(t,z)=\begin{cases}\Phi(t,z) & \text{if } |z|\le r\,,\\ \eta_r(|z|)\Phi(t,z)+(1-\eta_r(|z|))\frac{1}{2}\langle\mathbb{D}z\,,\,z\rangle & \text{if } r\le|z|\le r^3\,,\\ \frac{1}{2}\langle\mathbb{D}z\,,\,z\rangle & \text{if } |z|\ge r^3\,.\end{cases}$$

Consider the modified system

$$\begin{cases}J\dot u=\nabla_u\mathcal{H}(t,u)+\nabla_uP(t,u,z)\,,\\ J\dot z=\nabla_z\Phi_r(t,z)+\nabla_zQ(t,z)+\nabla_zP(t,u,z)\,,\end{cases}\tag{HS_r}$$

and notice that $\nabla_z\Phi_r(t,z)$ can be decomposed as

$$\nabla_z\Phi_r(t,z)=\mathbb{M}_r(t,z)z+\upsilon_r(t,z)\,,$$

where $\mathbb{M}_r:\mathbb{R}\times\mathbb{R}^{2N}\to\mathrm{Sym}(\mathbb{R}^{2N})$ is defined as

$$\mathbb{M}_r(t,z)=\begin{cases}\mathbb{M}(t,z) & \text{if } |z|\le r\,,\\ \eta_r(|z|)\mathbb{M}(t,z)+(1-\eta_r(|z|))\mathbb{D} & \text{if } r\le|z|\le r^3\,,\\ \mathbb{D} & \text{if } |z|\ge r^3\,,\end{cases}$$

and $\upsilon_r:\mathbb{R}\times\mathbb{R}^{2N}\to\mathbb{R}^{2N}$ as

$$\upsilon_r(t,z)=\begin{cases}0 & \text{if } |z|\le r\,,\\ \eta_r'(|z|)|z|^{-1}\left[\Phi(t,z)-\frac{1}{2}\langle\mathbb{D}z,z\rangle\right]z & \text{if } r\le|z|\le r^3\,,\\ 0 & \text{if } |z|\ge r^3\,.\end{cases}$$

Notice that $\mathbb{A} \le \mathbb{M}_r(t,z) \le \mathbb{B}$, for every $(t,z) \in [0,T] \times \mathbb{R}^{2N}$. Moreover, by (8) and (9), if $r \le |z| \le r^3$, then

$$|\upsilon_r(t,z)| \le \tfrac{1}{2}\|\mathbb{B} - \mathbb{A}\| \, \frac{|z|}{\ln|z|} \le \tfrac{1}{2}\|\mathbb{B} - \mathbb{A}\| \, \frac{|z|}{\ln r} \,, \tag{10}$$

while $\upsilon_r(t,z) = 0$ otherwise. We now need to prove an a priori bound for the z-component of the solutions of (HS_r).

Proposition 2.5. *There exists $\overline{r} > 1$ such that, for any $r \ge \overline{r}$, every T-periodic solution of (HS_r) satisfies $\|z\|_\infty \le \overline{r}$.*

Proof. We assume by contradiction that there is a sequence $(r_n)_n$ in $]1,+\infty[$ and a sequence of T-periodic solutions (u_n, z_n) of (HS_r), with $r = r_n$, such that $r_n \to +\infty$ and $\|z_n\|_\infty > n$. Introducing the normalized function $v_n = z_n/\|z_n\|_\infty$ we have that v_n is a solution of

$$J\dot{v}_n = \Gamma_n(t) v_n + e_n(t) \,, \qquad v_n(0) = v_n(T) \,, \tag{11}$$

with

$$\Gamma_n(t) = \mathbb{M}_{r_n}(t, z_n(t)) \,,$$

and

$$e_n(t) = \frac{1}{\|z_n\|_\infty}\Big(\upsilon_r(t, z_n(t)) + \nabla_z Q(t, z_n(t)) + \nabla_z P(t, u_n(t), z_n(t)) \Big) \,. \tag{12}$$

We notice that $\mathbb{A} \le \Gamma_n(t) \le \mathbb{B}$ and, by (10) and the boundedness of $\nabla_z Q$ and $\nabla_z P$, we have that $e_n \to 0$ uniformly in $[0,T]$. The sequence $(v_n)_n$ is thus bounded in $C^1([0,T], \mathbb{R}^{2N})$, so there exists $v \in C([0,T], \mathbb{R}^{2N})$ such that $v_n \to v$ uniformly and weakly in H^1, up to a subsequence.

Let us now define the closed convex and bounded set

$$\mathcal{C}_{\mathbb{A},\mathbb{B}} = \{ f \in L^2([0,T], \mathrm{Sym}(\mathbb{R}^{2N})) : \mathbb{A} \le f(t) \le \mathbb{B} \text{ for a.e. } t \in [0,T] \} \,.$$

Since the sequence $(\Gamma_n)_n$ is contained in $\mathcal{C}_{\mathbb{A},\mathbb{B}}$, we can find $\mathbb{G} \in \mathcal{C}_{\mathbb{A},\mathbb{B}}$ such that $\Gamma_n \rightharpoonup \mathbb{G}$, weakly in L^2, up to a subsequence. Passing to the limit in (11), we can see that v is a weak solution of

$$J\dot{v} = \mathbb{G}(t) v \,, \qquad v(0) = v(T) \,.$$

By A5, applying [30, Corollary 3] (see also [29, Corollary 1]), the previous equation admits only the trivial solution, and we get a contradiction. □

We now continue the proof of Theorem 2.1. We fix $r \geq \overline{r}$ and rewrite the Hamiltonian function for system (HS_r) as

$$\mathcal{H}_r(t,u,z) = \mathcal{H}(t,u) + \tfrac{1}{2}\langle \mathbb{D}z\,,\, z\rangle + \widetilde{P}_r(t,u,z)\,,$$

with

$$\widetilde{P}_r(t,u,z) = \Phi_r(t,z) - \tfrac{1}{2}\langle \mathbb{D}z\,,\, z\rangle + Q(t,z) + P(t,u,z)\,.$$

Proposition 2.5 provides an a priori bound in $C([0,T],\mathbb{R}^{2N})$ for the z-component of the solutions of system (HS_r). Using the second equation in that system we see that the a priori bound extends also to the derivative of z. We thus have an a priori bound in $C^1([0,T],\mathbb{R}^{2N})$. Hence, the Ascoli-Arzelà Theorem implies that z belongs to a compact set $Z \subseteq C([0,T],\mathbb{R}^{2N})$. By the global existence assumption in $A2$, using Theorem 6.1 in the Appendix with $u = (q,p)$ and $D = [0,2\pi]^M \times \mathcal{D}$, we can find a constant $C > 0$ such that all the solutions of (HS_r) starting with $p(0) \in \mathcal{D}$ satisfy $|p(t)| \leq C$ for every $t \in [0,T]$. We then introduce a C^∞-function $\delta : \mathbb{R} \to \mathbb{R}$ such that

$$\delta(\xi) = \begin{cases} 1 & \text{if } \xi \leq C\,, \\ 0 & \text{if } \xi \geq C+1\,, \end{cases}$$

and set $\widetilde{\mathcal{H}}(t,q,p) = \delta(|p|)\mathcal{H}(t,q,p)$, so that, by the periodicity in the q-components, $\widetilde{\mathcal{H}}(t,q,p)$ has a bounded gradient with respect to (q,p). In view of these considerations, in order to prove Theorem 2.1 there is no loss of generality in assuming that $\mathcal{H}(t,q,p)$ has a bounded gradient with respect to (q,p).

Since $\Phi_r(t,z) - \frac{1}{2}\langle \mathbb{D}z\,,\, z\rangle = 0$ when $|z| \geq r^3$, we have that $\nabla_{(u,z)}\widetilde{P}_r(t,u,z)$ is uniformly bounded. Hence, we can apply [26, Corollary 2.4] and obtain the existence of $M+1$ geometrically distinct T-periodic solutions of (HS_r) satisfying $p(0) \in \mathring{\mathcal{D}}$. By Proposition 2.5, the so-found solutions solve also (HS), thus concluding the proof. $\square$

3 The Landesman–Lazer conditions

We will now modify conditions $A4$ and $A5$ into

A4′. There are two functions $\mu : \mathbb{R} \times \mathbb{R}^{2N} \to [0,1]$ and $Q : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ such that

$$\nabla_z \mathscr{H}(t,z) = (1-\mu(t,z))\mathbb{A}z + \mu(t,z)\mathbb{B}z + \nabla_z Q(t,z)\,,$$

for some $\mathbb{A},\mathbb{B} \in \mathrm{Sym}(\mathbb{R}^{2N})$, with $\mathbb{A} \leq \mathbb{B}$. All functions above are continuous, T-periodic in t, and $\nabla_z Q(t,z)$ is uniformly bounded.

A5′. The following holds:

$$\bigcup_{\lambda\in\,]0,1[} \sigma\big((1-\lambda)J\mathbb{A}+\lambda J\mathbb{B}\big)\cap\frac{2\pi}{T}\,i\mathbb{Z}=\varnothing\,, \tag{13}$$

while possibly

$$\sigma(J\mathbb{A})\cap\frac{2\pi}{T}\,i\mathbb{Z}\neq\varnothing\,,\quad\text{or}\quad\sigma(J\mathbb{B})\cap\frac{2\pi}{T}\,i\mathbb{Z}\neq\varnothing\,.$$

When they are both nonempty, we also assume that $\mathbb{B}-\mathbb{A}$ is invertible.

In this setting we also need to introduce some nonresonance assumptions. The ones we propose below generalize the classical Landesman–Lazer conditions, as will be shown in Section 4. Let us first define, for every $v\in\mathbb{R}^{2N}\setminus\{0\}$ and $\theta\in\,]0,\pi/2[\,$, the cone

$$C_v(\theta)=\{z\in\mathbb{R}^{2N}:\langle z,v\rangle\geq\;|z|\,|v|\cos\theta\}\,.$$

In what follows, the constant $\overline{m}$ is the one introduced in $A3$, and $B(0,r)$ denotes the open ball of radius r centered at the origin. Here is our version of the Landesman–Lazer condition involving the matrix $\mathbb{A}$.

A6. For every nontrivial T-periodic solution $v(t)$ of $J\dot{v}=\mathbb{A}v$ there exist a null set $\mathcal{N}\subseteq[0,T]$ and three real-valued functions θ,ρ,σ, defined on $[0,T]\setminus\mathcal{N}$, such that

$$0<\theta(t)<\frac{\pi}{2}\,,\quad\rho(t)>0\,,\qquad\text{for every }t\in[0,T]\setminus\mathcal{N}\,, \tag{14}$$

and σ is integrable, with the following property: for every $t\in[0,T]\setminus\mathcal{N}$ and $z\in C_{v(t)}(\theta(t))\setminus B(0,\rho(t))$ one has

$$\langle\nabla_z\mathscr{H}(t,z)-\mathbb{A}z,v(t)\rangle\geq\sigma(t)\,, \tag{15}$$

and

$$\int_0^T\sigma(t)\,dt>\overline{m}\int_0^T|v(t)|\,dt\,.$$

The following is the analogue of our Landesman–Lazer condition involving the matrix $\mathbb{B}$.

A6′. For every nontrivial T-periodic solution $v(t)$ of $J\dot{v}=\mathbb{B}v$ there exist a null set $\mathcal{N}\subseteq[0,T]$ and three real-valued functions θ,ρ,σ, defined on $[0,T]\setminus$

$\mathcal{N}$, such that (14) holds, and σ is integrable, with the following property: for every $t \in [0,T] \setminus \mathcal{N}$ and $z \in C_{v(t)}(\theta(t)) \setminus B(0,\rho(t))$ one has

$$\langle \nabla_z \mathscr{H}(t,z) - \mathbb{B}z, v(t) \rangle \leq \sigma(t)\,,$$

and

$$\int_0^T \sigma(t)\,dt < -\overline{m} \int_0^T |v(t)|\,dt\,.$$

Here we state our result in the *double resonance* situation.

Theorem 3.1. *Assume A1 − A3, A4′, A5′, A6 and A6′. Then system* (HS) *has at least* $M+1$ *geometrically distinct* T*-periodic solutions, with* $p(0) \in \mathring{\mathcal{D}}$.

In the proof of this theorem, the following lemma will be needed.

Lemma 3.2. *Assume A4′ and A5′, and let* $\mu \in L^2(0,T)$ *be such that* $0 \leq \mu(t) \leq 1$ *for almost every* $t \in [0,T]$*. If* v *is a nontrivial solution of*

$$J\dot{v} = (1-\mu(t))\mathbb{A}v + \mu(t)\mathbb{B}v\,, \quad v(0) = v(T)\,,$$

then, either v *is a solution of* $J\dot{v} = \mathbb{A}v$*, or it is a solution of* $J\dot{v} = \mathbb{B}v$*.*

We postpone the proof of the lemma at the end of this section.

Proof of Theorem 3.1. We proceed as in the proof of Theorem 2.1. Notice that, by $A4'$, we now have

$$\mathbb{M}(t,z) = (1-\mu(t,z))\mathbb{A} + \mu(t,z)\mathbb{B}\,.$$

Choosing $\mathbb{D} = \frac{1}{2}(\mathbb{A}+\mathbb{B})$, we modify our system into (HS_r), with

$$\mathbb{M}_r(t,z) = (1-\mu_r(t,z))\mathbb{A} + \mu_r(t,z)\mathbb{B}\,,$$

where

$$\mu_r(t,z) = \begin{cases} \mu(t,z) & \text{if } |z| \leq r\,, \\ \eta_r(|z|)\mu(t,z) + \frac{1}{2}(1-\eta_r(|z|)) & \text{if } r \leq |z| \leq r^3\,, \\ \frac{1}{2} & \text{if } |z| \geq r^3\,. \end{cases}$$

Now we need to prove Proposition 2.5. We assume by contradiction that there is a sequence $(r_n)_n$ in $]1,+\infty[$ and a sequence of T-periodic solutions (u_n, z_n) of (HS_r), with $r = r_n$, such that $r_n \to +\infty$ and $\|z_n\|_\infty > n$. Introducing the normalized function $v_n = z_n/\|z_n\|_\infty$, we have that

$$J\dot{v}_n = \big[(1-\mu_n(t))\mathbb{A} + \mu_n(t)\mathbb{B}\big]v_n + e_n(t)\,, \qquad v_n(0) = v_n(T)\,,$$

where $\mu_n(t) = \mu_{r_n}(t, z_n(t)))$, and $e_n(t)$ is as in (12). Since $0 \le \mu_n(t) \le 1$ for every $t \in [0,T]$, the sequence $(\mu_n)_n$ has a subsequence which weakly converges in L^2 to some $\mu \in L^2(0,T)$, with $0 \le \mu(t) \le 1$ for almost every $t \in [0,T]$. At the same time, the sequence $(v_n)_n$ converges to some $v \in C([0,T], \mathbb{R}^{2N})$ uniformly and weakly in H^1, up to a subsequence, and v is a nontrivial solution of

$$J\dot{v} = \big[(1-\mu(t))\mathbb{A} + \mu(t)\mathbb{B}\big] v\,, \quad v(0) = v(T)\,.$$

By Lemma 3.2, either v is a solution of $J\dot{v} = \mathbb{A}v$, or it is a solution of $J\dot{v} = \mathbb{B}v$. Let us assume that $J\dot{v} = \mathbb{A}v$ and see how the contradiction is reached in this case. The other case can be treated similarly.

Since it has to be $v(t) \neq 0$ for every $t \in [0,T]$, we have that

$$\min\{|v(t)| : t \in [0,T]\} > 0\,.$$

Consequently, both

$$\lim_n |z_n(t)| = +\infty\,, \tag{16}$$

and

$$\lim_n \langle (\mathbb{B} - \mathbb{A}) z_n(t), v(t)\rangle = +\infty\,, \tag{17}$$

uniformly in $t \in [0,T]$. Recalling the second equation in (HS_r), with $r = r_n$, we have

$$J\dot{z}_n - \mathbb{A}z_n = \nabla\Phi_{r_n}(t, z_n) - \mathbb{A}z_n + \nabla_z Q(t, z_n) + \nabla_z P(t, u_n, z_n)\,.$$

Hence, multiplying by v and integrating, since

$$\int_0^T \langle J\dot{z}_n(t) - \mathbb{A}z_n(t), v(t)\rangle\, dt = \int_0^T \langle z_n(t), J\dot{v}(t) - \mathbb{A}v(t)\rangle\, dt = 0\,,$$

by $A3$ we get

$$\int_0^T \langle \nabla_z\Phi_{r_n}(t, z_n(t)) - \mathbb{A}z_n(t) + \nabla_z Q(t, z_n(t)), v(t)\rangle\, dt \le \overline{m} \int_0^T |v(t)|\, dt\,. \tag{18}$$

Let us verify that, for n sufficiently large,

$$\langle \nabla_z\Phi_{r_n}(t, z_n(t)) - \mathbb{A}z_n(t), v(t)\rangle \ge 0\,, \quad \text{for every } t \in [0,T]. \tag{19}$$

We have three cases. If $|z_n(t)| \leq r_n$, by (17), for n large enough,

$$\langle \nabla_z \Phi_{r_n}(t, z_n(t)) - \mathbb{A} z_n(t), v(t) \rangle = \mu(t, z_n(t)) \langle (\mathbb{B} - \mathbb{A}) z_n(t), v(t) \rangle \geq 0 \,.$$

If $|z_n(t)| \geq r_n^3$, then again for n large enough,

$$\langle \nabla_z \Phi_{r_n}(t, z_n(t)) - \mathbb{A} z_n(t), v(t) \rangle = \tfrac{1}{2} \langle (\mathbb{B} - \mathbb{A}) z_n(t), v(t) \rangle > 0 \,.$$

Finally, if $r_n < |z_n(t)| < r_n^3$, we just interpolate the two inequalities above, and we have that (19) holds in all cases, provided that n is large enough.

Since $\nabla_z Q$ is bounded and (19) holds, we can apply Fatou's Lemma and obtain, from (18), that

$$\int_0^T \liminf_n \langle \nabla_z \Phi_{r_n}(t, z_n(t)) - \mathbb{A} z_n(t) + \nabla_z Q(t, z_n(t)), v(t) \rangle \, dt \leq \\ \leq \overline{m} \int_0^T |v(t)| \, dt \,. \quad (20)$$

We will now use assumption $A6$ to reach the aimed contradiction. For every $t \in [0, T]$, since $z_n(t) = \|z_n\|_\infty v_n(t)$ and $v_n(t) \to v(t)$, by (16) there exists $\bar{n}_t \geq 1$ such that $z_n(t) \in C_{v(t)}(\theta(t)) \setminus B(0, \rho(t))$ for every $n \geq \bar{n}_t$. Hence, for every $t \in [0, T] \setminus \mathcal{N}$ and $n \geq \bar{n}_t$, we have that

$$\langle \nabla_z \mathscr{H}(t, z_n(t)) - \mathbb{A} z_n(t), v(t) \rangle \geq \sigma(t) \,.$$

Again we consider three cases. If $|z_n(t)| \leq r_n$, then

$$\nabla_z \Phi_{r_n}(t, z_n(t)) - \mathbb{A} z_n(t) + \nabla_z Q(t, z_n(t)) = \nabla_z \mathscr{H}(t, z_n(t)) - \mathbb{A} z_n(t) \,,$$

hence

$$\langle \nabla_z \Phi_{r_n}(t, z_n(t)) - \mathbb{A} z_n(t) + \nabla_z Q(t, z_n(t)), v(t) \rangle \geq \sigma(t) \,. \quad (21)$$

If $|z_n(t)| \geq r_n^3$, then

$$\nabla_z \Phi_{r_n}(t, z_n(t)) - \mathbb{A} z_n(t) + \nabla_z Q(t, z_n(t)) = \tfrac{1}{2} (\mathbb{B} - \mathbb{A}) z_n(t) + \nabla_z Q(t, z_n(t)) \,,$$

hence, for n large enough, (21) still holds, since the left hand side tends to $+\infty$, recalling (17). If $r_n < |z_n(t)| < r_n^3$, we just interpolate the two inequalities above, hence (21) holds in all cases, provided that n is large enough. Then, by (20), we conclude that

$$\int_0^T \sigma(t) \, dt \leq \overline{m} \int_0^T |v(t)| \, dt \,,$$

a contradiction with (15) thus ending the proof of Proposition 2.5 in this setting.

The proof of Theorem 3.1 can now be completed exactly as the one of Theorem 2.1. □

Proof of Lemma 3.2. Recalling assumption $A5'$, without loss of generality we can always assume that $\mathbb{B} - \mathbb{A}$ is invertible. It is convenient to introduce the following functional setting. Denoting by X the Hilbert space $L^2([0,T],\mathbb{R}^{2N})$ with the standard product $\langle\cdot|\cdot\rangle_2$ and the corresponding norm $\|\cdot\|_2$, we consider the linear selfadjoint operator $L : D(L) \subseteq X \to X$ defined by

$$D(L) = \{z \in H^1([0,T],\mathbb{R}^{2N}) : z(0) = z(T)\}, \qquad Lz = J\dot{z}\,.$$

Moreover, setting $\mathbb{G}(t) = (1 - \mu(t))\mathbb{A} + \mu(t)\mathbb{B}$, we introduce the bounded selfadjoint operators $A, B, N : X \to X$ defined by

$$[Az](t) = \mathbb{A}z(t)\,, \qquad [Bz](t) = \mathbb{B}z(t)\,, \qquad [Nz](t) = \mathbb{G}(t)z(t)\,.$$

We have that $A \leq N \leq B$ with the standard meaning, i.e., $\langle Az|z\rangle_2 \leq \langle Nz|z\rangle_2 \leq \langle Bz|z\rangle_2$ for every $z \in X$.

Let v be a nontrivial solution of $J\dot{v} = \mathbb{G}(t)v$ such that $v(0) = v(T)$. Then $v(t) \neq 0$ for every $t \in [0,T]$, and $Lv = Nv$. Since $\mathbb{B} - \mathbb{A}$ is a positive definite invertible matrix, the operator $B - A$ is also positive definite and invertible. We can thus define the operator $D = (B - A)^{1/2}$, which explicitly reads as

$$[Dz](t) = (\mathbb{B} - \mathbb{A})^{1/2}z(t)\,.$$

Now, following the idea in [20, 29], we set $w = Dv$. Then, w is a nontrivial solution of

$$\widetilde{L}w = \widetilde{N}w\,, \tag{22}$$

with $\widetilde{L} = D^{-1}(L - A)D^{-1}$ and $\widetilde{N} = D^{-1}(N - A)D^{-1}$. From (13) we deduce that

$$\sigma(\widetilde{L}) \cap]0,1[= \varnothing\,,$$

and $0 \leq \widetilde{N} \leq I$. Since $\widetilde{L}$ has a compact resolvent, we can write $X = X_- \oplus X_0 \oplus X_1 \oplus X_+$, where X_- is the eigenspace generated by eigenfunctions of $\widetilde{L}$ with negative eigenvalues, X_+ is the eigenspace generated by eigenfunctions of $\widetilde{L}$ with eigenvalues larger than 1, $X_0 = \ker \widetilde{L}$, and $X_1 = \ker(\widetilde{L} - I)$. Notice that, since $\widetilde{L}$ is selfadjoint, this is an orthogonal decomposition. We accordingly write $w = w_- + w_0 + w_1 + w_+$.

We want to prove that $w_- = w_+ = 0$. Recalling (22), if $w_+ \neq 0$, then $\langle \widetilde{L} w_+ \mid w_+ \rangle_2 > \|w_+\|_2^2$ and, by the symmetry of $\widetilde{N}$, we reach the contradiction

$$\begin{aligned}
0 &= \langle (\widetilde{L} - \widetilde{N})(w_- + w_0 + w_1 + w_+) \mid w_- + w_0 - w_1 - w_+ \rangle_2 \\
&= \langle \widetilde{L} w_- \mid w_- \rangle_2 - \langle \widetilde{L} w_1 \mid w_1 \rangle_2 - \langle \widetilde{L} w_+ \mid w_+ \rangle_2 \\
&\qquad - \langle \widetilde{N}(w_- + w_0) \mid w_- + w_0 \rangle_2 + \langle \widetilde{N}(w_1 + w_+) \mid w_1 + w_+ \rangle_2 \\
&\leq 0 - \|w_1\|_2^2 - \langle \widetilde{L} w_+ \mid w_+ \rangle_2 - 0 + \|w_1 + w_+\|_2^2 \\
&< -\|w_1\|_2^2 - \|w_+\|_2^2 + \|w_1 + w_+\|_2^2 = 0 \,.
\end{aligned}$$

Hence, $w_+ = 0$. If $w_- \neq 0$ then $\langle \widetilde{L} w_- \mid w_- \rangle_2 < 0$ and we get

$$\begin{aligned}
0 &= \langle (\widetilde{L} - \widetilde{N})(w_- + w_0 + w_1) \mid w_- + w_0 - w_1 \rangle_2 \\
&= \langle \widetilde{L} w_- \mid w_- \rangle_2 - \langle \widetilde{L} w_1 \mid w_1 \rangle_2 - \langle \widetilde{N}(w_- + w_0) \mid w_- + w_0 \rangle_2 + \langle \widetilde{N} w_1 \mid w_1 \rangle_2 \\
&< 0 - \|w_1\|_2^2 - 0 + \|w_1\|_2^2 = 0 \,,
\end{aligned}$$

again a contradiction. Hence, $w_- = 0$ and $w = w_0 + w_1$.

Recalling that $v = D^{-1} w$ and setting $v_0 = D^{-1} w_0$ and $v_1 = D^{-1} w_1$, it is easily seen that $J\dot{v}_0 = \mathbb{A} v_0$ and $J\dot{v}_1 = \mathbb{B} v_1$, hence, being $v = v_0 + v_1$,

$$J\dot{v} = J\dot{v}_0 + J\dot{v}_1 = \mathbb{A} v_0 + \mathbb{B} v_1 \,.$$

On the other hand, since

$$J\dot{v} = \big[(1 - \mu(t))\mathbb{A} + \mu(t)\mathbb{B}\big](v_0 + v_1) \,,$$

we deduce that

$$(1 - \mu(t))(\mathbb{B} - \mathbb{A}) v_1(t) = \mu(t)(\mathbb{B} - \mathbb{A}) v_0(t) \,, \qquad \text{for a.e. } t \in [0, T] \,.$$

Therefore, also

$$(1 - \mu(t))(\mathbb{B} - \mathbb{A})^{1/2} v_1(t) = \mu(t)(\mathbb{B} - \mathbb{A})^{1/2} v_0(t) \,, \qquad \text{for a.e. } t \in [0, T] \,,$$

i.e.,

$$(1 - \mu(t)) w_1(t) = \mu(t) w_0(t) \,, \qquad \text{for a.e. } t \in [0, T] \,.$$

In particular,

$$\langle w_0(t), w_1(t) \rangle = |w_0(t)| \, |w_1(t)| \,, \qquad \text{for a.e. } t \in [0, T] \,,$$

and since w_0 and w_1 are orthogonal in $X = L^2([0,T], \mathbb{R}^{2N})$, we have that

$$\int_0^T |w_0(t)|\,|w_1(t)|\,dt = \int_0^T \langle w_0(t), w_1(t)\rangle\,dt = 0\,.$$

So, for every $t \in [0,T]$, either $w_0(t) = 0$, or $w_1(t) = 0$, implying that either $v_0(t) = 0$, or $v_1(t) = 0$. Recalling that v_0 and v_1 solve $J\dot{v}_0 = \mathbb{A}v_0$ and $J\dot{v}_1 = \mathbb{B}v_1$, we conclude that either v_0 or v_1 are identically equal to zero, and the proof is completed. □

4 Planar systems

The aim of this section is to compare the Landesman–Lazer condition in A6 with the one introduced in [22], where the authors consider a *planar* system. For simplicity, we assume $\mu(t,z) = 0$ and $P = 0$, so that we can write the second equation in (HS) as

$$J\dot{z} = \mathbb{A}z + r(t,z)\,.$$

We assume that $\mathbb{A}$ is a 2×2 positive definite invertible symmetric matrix and $r(t,z)$ is a uniformly bounded continuous function, T-periodic in t, i.e., there is a constant $\tilde{c} \geq 0$ such that

$$|r(t,v)| \leq \tilde{c}\,, \qquad \text{for every } (t,v) \in \mathbb{R} \times \mathbb{R}^2.$$

We also assume that the planar system $J\dot{v} = \mathbb{A}v$ has nontrivial T-periodic solutions, i.e., that

$$T \in \frac{2\pi}{\sqrt{\det \mathbb{A}}}\mathbb{N}\,.$$

We denote by φ one of them, so that the others can be written as $v(t) = c\varphi(t+\alpha)$ with $c > 0$ and $\alpha \in [0,T]$. Here is the condition introduced in [22].

***A*7.** For every $\alpha \in [0,T]$,

$$\int_0^T \liminf_{\substack{\lambda \to +\infty \\ \omega \to \alpha}} \big\langle r(t, \lambda\varphi(t+\omega))\,,\, \varphi(t+\omega)\big\rangle\,dt > 0\,. \tag{23}$$

Theorem 4.1. *Assumption* A6 *is equivalent to* A7*.*

Proof. Let us prove that *A*6 implies *A*7. Let $\alpha \in [0,T]$. Then $v(t) = \varphi(t+\alpha)$ is a T-periodic solution of $J\dot{v} = \mathbb{A}v$ and we correspondingly introduce the null set $\mathcal{N}$ and the functions θ, ρ, σ as in *A*6. By the continuity of φ and the definition of the cone, for every $t \in [0,T] \setminus \mathcal{N}$ we can find $\overline{\lambda}(t) > 0$ and $\overline{\delta}(t) > 0$ such that, setting

$$F = \{\lambda\varphi(t+\omega) \,:\, \lambda > \overline{\lambda}(t)\,,\ |\omega - \alpha| < \overline{\delta}(t)\}\,,$$

we have that $F \subseteq C_{\varphi(t+\alpha)}(\theta(t)) \setminus B(0,\rho(t))$. For $z \in F$ and $t \in [0,T] \setminus \mathcal{N}$, we have

$$\begin{aligned}\big\langle r(t,\lambda\varphi(t+\omega))\,,\,\varphi(t+\omega)\big\rangle &= \big\langle r(t,\lambda\varphi(t+\omega))\,,\,\varphi(t+\alpha)\big\rangle \\ &\quad + \big\langle r(t,\lambda\varphi(t+\omega)\,,\,\varphi(t+\omega)-\varphi(t+\alpha)\big\rangle \\ &\geq \sigma(t) - \tilde{c}\,|\varphi(t+\omega)-\varphi(t+\alpha)|\,.\end{aligned}$$

Hence, we see that

$$\liminf_{\substack{\lambda\to+\infty\\ \omega\to\alpha}} \big\langle r(t,\lambda\varphi(t+\omega))\,,\,\varphi(t+\omega)\big\rangle \geq \sigma(t)\,,$$

for every $t \in [0,T] \setminus \mathcal{N}$ and we conclude.

We now prove that $A7$ implies $A6$. Let $v(t) = c\varphi(t+\alpha)$, with $c > 0$ and $\alpha \in [0,T]$, be a solution of $J\dot{v} = \mathbb{A}v$. Without loss of generality, we assume $c = 1$. Since $r(t,z)$ is bounded, we can define the function $\widetilde{\sigma} : [0,T] \to \mathbb{R}$ as

$$\widetilde{\sigma}(t) = \liminf_{\substack{\lambda\to+\infty\\ \omega\to\alpha}} \big\langle r(t,\lambda\varphi(t+\omega))\,,\,\varphi(t+\omega)\big\rangle\,.$$

Clearly enough, $\widetilde{\sigma}$ is integrable on $[0,T]$. We set $\overline{\sigma} = \int_0^T \widetilde{\sigma}(t)\,dt$ and $\mathcal{N} = \varnothing$. By (23), we have that $\overline{\sigma} > 0$. Let $\varepsilon = \overline{\sigma}/(4T)$. From the definition of $\widetilde{\sigma}$ and the uniform continuity of φ, for every $t \in [0,T]$ there are $\overline{\lambda}(t)$ and $\overline{\delta}(t)$ such that, if $\lambda > \overline{\lambda}(t)$ and $|\omega - \alpha| < \overline{\delta}(t)$, then

$$\big\langle r(t,\lambda\varphi(t+\omega))\,,\,\varphi(t+\omega)\big\rangle > \widetilde{\sigma}(t) - \varepsilon\,,$$

and

$$|\varphi(t+\alpha) - \varphi(t+\omega)| \leq \varepsilon/\tilde{c}\,.$$

We can then find two constants $\rho(t)$ and $\theta(t)$ such that

$$C_{\varphi(t+\alpha)}(\theta(t)) \setminus B(0,\rho(t)) \subseteq \{\lambda\varphi(t+\omega) \,:\, \lambda > \overline{\lambda}(t)\,,\ |\omega-\alpha| < \overline{\delta}(t)\}\,.$$

Hence, every $z \in C_{\varphi(t+\alpha)}(\theta(t)) \setminus B(0,\rho(t))$ can be written as $z = \lambda_z\varphi(t+\omega_z)$ with $\lambda_z > \overline{\lambda}(t)$ and $|\omega_z - \alpha| < \overline{\delta}(t)$. We now compute

$$\begin{aligned}\big\langle r(t,z)\,,\,v(t)\big\rangle &= \big\langle r(t,\lambda_z\varphi(t+\omega_z))\,,\,\varphi(t+\alpha)\big\rangle \\ &= \big\langle r(t,\lambda_z\varphi(t+\omega_z))\,,\,\varphi(t+\omega_z)\big\rangle \\ &\qquad + \big\langle r(t,\lambda_z\varphi(t+\omega_z))\,,\,\varphi(t+\alpha)-\varphi(t+\omega_z)\big\rangle \\ &\geq \widetilde{\sigma}(t) - \varepsilon - |r(t,\lambda_z\varphi(t+\omega_z))|\,|\varphi(t+\alpha)-\varphi(t+\omega_z)| \\ &\geq \widetilde{\sigma}(t) - 2\varepsilon\,.\end{aligned}$$

Hence, defining $\sigma(t) = \widetilde{\sigma}(t) - 2\varepsilon$ and recalling the definition of ε,

$$\int_0^T \sigma(t)\,dt = \int_0^T \widetilde{\sigma}(t)\,dt - 2\varepsilon T = \overline{\sigma} - 2\frac{\overline{\sigma}}{4T}\,T = \frac{1}{2}\overline{\sigma} > 0\,,$$

so that $A6$ holds. $\square$

Let us now focus our attention on the particular case of a scalar second order equation of the type

$$\ddot{x} + \lambda x + h(t,x) = 0\,. \tag{24}$$

For simplicity we assume $h(t,x)$ to be continuous, uniformly bounded, and T-periodic in t. If $\lambda = (\frac{2\pi n}{T})^2$ for some positive integer n, the classical Landesman–Lazer condition reads as follows.

A8. for every nontrivial solution $\xi(t)$ of $\ddot{\xi} + \lambda\xi = 0$, either

$$(a) \qquad \int_{\{\xi<0\}} \limsup_{x\to-\infty} h(t,x)\xi(t)\,dt + \int_{\{\xi>0\}} \liminf_{x\to+\infty} h(t,x)\xi(t)\,dt > 0\,,$$

or

$$(b) \qquad \int_{\{\xi<0\}} \liminf_{x\to-\infty} h(t,x)\xi(t)\,dt + \int_{\{\xi>0\}} \limsup_{x\to+\infty} h(t,x)\xi(t)\,dt < 0\,.$$

In order to have a further insight on the extent of this condition, we will now prove that $A8(a)$ implies $A6$. Similarly one can show that $A8(b)$ implies $A6'$.

By [23, Proposition 3.1], condition $A8(a)$ is equivalent to assuming the existence of some constants $\eta > 0$, $R > 0$ and two functions $\psi_\pm \in L^1(0,T)$ such that

$$\begin{cases} h(t,x) \le \psi_-(t)\,, & \text{if } x \le -R\,, \\ h(t,x) \ge \psi_+(t)\,, & \text{if } x \ge R\,, \end{cases}$$

and

$$\int_{\{\xi<0\}} \psi_-(t)\xi(t)\,dt + \int_{\{\xi>0\}} \psi_+(t)\xi(t)\,dt \ge \eta\|\xi\|_\infty\,. \tag{25}$$

Setting

$$\mathbb{A} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \qquad Q(t,z) = \int_0^x h(t,s)\,ds, \qquad z = \begin{pmatrix} x \\ y \end{pmatrix},$$

we see that (24) is equivalent to $J\dot{z} = \mathbb{A}z + \nabla_z Q(t,z)$.

Let v be a nontrivial T-periodic solution of $J\dot{v} = \mathbb{A}v$. Then

$$v(t) = \begin{pmatrix} \xi(t) \\ \dot{\xi}(t) \end{pmatrix},$$

where $\xi(t)$ is a nontrivial solution of $\ddot{\xi} + \lambda\xi = 0$. Let $\mathcal{N}$ be the set of those $t \in [0,T]$ such that $\xi(t) = 0$. It is a finite set, hence a zero-measure set. For every $t \in [0,T] \setminus \mathcal{N}$ there is a $\theta(t) \in]0, \frac{\pi}{2}[$ such that

$$C_{v(t)}(\theta(t)) \cap \{(x,y) \in \mathbb{R}^2 : x = 0\} = \varnothing\,.$$

Moreover, there is a $\rho(t) > 0$ such that

$$\big[\,\xi(t) > 0 \text{ and } (x,y) \in C_{v(t)}(\theta(t)) \setminus B(0,\rho(t))\,\big] \quad \Rightarrow \quad x \geq R\,,$$

and

$$\big[\,\xi(t) < 0 \text{ and } (x,y) \in C_{v(t)}(\theta(t)) \setminus B(0,\rho(t))\,\big] \quad \Rightarrow \quad x \leq -R\,.$$

We define

$$\sigma(t) = \psi_+(t)\xi^+(t) - \psi_-(t)\xi^-(t)\,,$$

and notice that $\int_0^T \sigma(t)\,dt > 0$, by (25). We still have to prove that

$$z \in C_{v(t)}(\theta(t)) \setminus B(0,\rho(t)) \quad \Rightarrow \quad \langle \nabla_z Q(t,z), v(t)\rangle \geq \sigma(t)\,.$$

Take $z = (x,y) \in C_{v(t)}(\theta(t))$ with $|z| \geq \rho(t)$. If $\xi(t) > 0$, then $x \geq R$, hence $h(t,x) \geq \psi_+(t)$ and

$$\langle \nabla_z Q(t,z), v(t)\rangle = h(t,x)\xi(t) \geq \psi_+(t)\xi(t) = \sigma(t)\,.$$

On the other hand, if $\xi(t) < 0$, then $x \leq -R$, hence $h(t,x) \leq \psi_-(t)$ and

$$\langle \nabla_z Q(t,z), v(t)\rangle = h(t,x)\xi(t) \geq \psi_-(t)\xi(t) = \sigma(t)\,.$$

The proof is thus completed.

5 Applications and final remarks

In this section we briefly explore some possible applications of our results assuming different behaviours of the first system in (HS), and we then suggest some open problems.

The typical situation where a twist dynamics applies is in the context of pendulum-like equations (see, e.g., [34, 49]) and their higher dimensional analogues, like those in [14, 16, 21, 28, 39, 42, 46, 53, 55]. Nonetheless, this type of dynamics also manifests in superlinear or sublinear systems when passing to action-angle variables (as seen, for instance, in [13, 33, 35]). Additionally, systems featuring singularities can also effectively be analysed using this approach (see [58] and the references therein).

The Poincaré–Birkhoff Theorem has also been employed to tackle bifurcations from nondegenerate periodic solutions (see, e.g., [32]), and situations in which the time map either exhibits some type of monotonicity in the phase plane, or has different behaviours near the origin and at infinity (or in proximity to a homoclinic or heteroclinic orbit).

Regarding assumption A2, variants of the twist condition have been proposed [38], finally leading to a general "avoiding cones" condition [25]. Our results hold also in this more general setting, but we refrain from providing the details, for the sake of brevity.

The periodicity assumption in A1 could be extended to encompass some of the components of the variable p, say $p_1, \dots, p_L$, including the case where the Hamiltonian function $\mathcal{H}$ exhibits periodicity in all variables $q_1, \dots, q_M$ and $p_1, \dots, p_M$, a situation considered in [16, Theorem 1]. In such a case, our theorem would guarantee the existence of at least $M + L + 1$ geometrically distinct T-periodic solutions.

Open Problems

1. In [15], the coupling of twist dynamics with a resonant equation involving the Ahmad–Lazer–Paul condition has been addressed. However, under such a condition, the scenario of double resonance remains unexplored, even when dealing only with a scalar second order equation. Notice that the Ahmad–Lazer–Paul condition does not guarantee an a priori bound as the one proved in Proposition 2.5, as shown in [9].

2. In [27], the asymmetric case for scalar second-order ODEs has been considered. An extension of our results to some kind of asymmetric systems would be desirable.

3. It's worth noting that, if the periodic solutions of system (HS) are known to be nondegenerate, then there are at least 2^M of them. It would be interesting to have an example where, in the opposite case, exactly $M+1$ periodic solutions appear.

4. As shown in [31], the multiplicity of solutions for the Neumann problem associated with system (1) can be proved without any twist condition. We wonder whether our results could also be rephrased in this setting.

5. The extension of our results to an infinite-dimensional setting seems to be a challenging problem.

6 Appendix - A compactness theorem

In this appendix we provide a variant of a rather classical result on the compactness of the set of solutions for a Cauchy problem, when global existence is assumed. The novelty lies in the fact that the nonlinearity may depend on a parameter which belongs to a general compact topological space Z. Let $f : [0, T] \times \mathbb{R}^d \times Z \to \mathbb{R}^d$ be a continuous function. We consider the Cauchy problem

$$\dot{u} = f(t, u, z)\,, \qquad u(0) = u_0\,. \tag{CP}$$

Theorem 6.1. *Let $D \subseteq \mathbb{R}^d$ be a compact set, and assume that all the solutions of (CP) starting with $u_0 \in D$ are defined on $[0, T]$. Then there is a constant $C > 0$ such that, for every solution of (CP) with $u_0 \in D$, one has*

$$|u(t)| \leq C\,, \quad \textit{for every } t \in [0, T]\,.$$

Proof. Let $(r_m)_m$ be a strictly increasing positive sequence, with $\lim_m r_m = +\infty$, and denote by D_m the closed ball centered at 0 with radius r_m. Assume moreover D to be contained in the interior of D_1.

Assume by contradiction that for every $k \geq 1$ there are $u_0^k \in D$, $z_k \in Z$ and a solution $u_k : [0, T] \to \mathbb{R}^d$ of the Cauchy problem

$$\dot{u} = f(t, u, z_k)\,, \qquad u(0) = u_0^k\,,$$

such that, for some $t_k \in]0, T]$, one has that $u_k(t_k) \notin D_k$.

For subsequences, $u_0^k \to u_0 \in D$ and $z_k \to z \in Z$. Let $t_k^1 \in]0, T[$ be such that

$$u_k(t) \in \mathring{D}_1\,, \quad \text{for every } t \in [0, t_k^1[\,, \quad \text{and } u_k(t_k^1) \in \partial D_1\,.$$

For subsequences, $t_k^1 \to \bar{t}_1 \in]0, T]$ and $\quad u_k(t_k^1) \to \bar{u}_1 \in \partial D_1$.

Claim. *There is a subsequence $(u_k)_k$ which converges pointwise on $[0, \bar{t}_1]$ and uniformly on every $[0, \tau]$ with $\tau \in]0, \bar{t}_1[$ to some continuous function*

$u_{(1)} : [0, \bar{t}_1] \to \mathbb{R}^d$. *Moreover,* $u_{(1)}$ *is a solution of* (CP) *on* $[0, \bar{t}_1]$ *and it satisfies*

$$u_{(1)}(t) \in D_1 \,, \quad \text{for every } t \in [0, \bar{t}_1] \,, \quad \text{and} \quad u_{(1)}(\bar{t}_1) = \bar{u}_1 \,.$$

<u>Proof of the Claim</u>. Set

$$\bar{c} = \max\{|f(t, u, z)| : t \in [0, T], u \in D_1, z \in Z\} \,.$$

Let $\delta_1 > \delta_2 > \dots > \delta_i > \dots$ be such that $\lim_i \delta_i = 0$. We can assume that $\delta_1 < \bar{t}_1$. Consider the interval $[0, \bar{t}_1 - \delta_1]$. Since $t^1_k > \bar{t}_1 - \delta_1$ for k large enough, we have that $u_k(t) \in D_1$ for every $t \in [0, \bar{t}_1 - \delta_1]$, hence $|\dot{u}_k(t)| \le \bar{c}$, for every $t \in [0, \bar{t}_1 - \delta_1]$. By the Ascoli–Arzelà theorem, there is a subsequence of $(u_k)_k$, which we denote by $(u^1_k)_k$, which converges to some $u^1 : [0, \bar{t}_1 - \delta_1] \to \mathbb{R}^d$, uniformly on $[0, \bar{t}_1 - \delta_1]$. Passing to the limit in

$$u^1_k(t) = u^k_0 + \int_0^t f(s, u^1_k(s), z_k)\, ds \,,$$

we see that u^1 is a solution of (CP) on $[0, \bar{t}_1 - \delta_1]$. Next, there is a subsequence of $(u^1_k)_k$, which we denote by $(u^2_k)_k$, which converges to some $u^2 : [0, \bar{t}_1 - \delta_2] \to \mathbb{R}^d$, uniformly on $[0, \bar{t}_1 - \delta_2]$. Again we see that u^2 is a solution of (CP) on $[0, \bar{t}_1 - \delta_2]$. It coincides with u^1 on $[0, \bar{t}_1 - \delta_1]$. Proceeding recursively, for every $i \ge 3$ we find a subsequence of $(u^{i-1}_k)_k$, which we denote by $(u^i_k)_k$, which converges to some $u^i : [0, \bar{t}_1 - \delta_i] \to \mathbb{R}^d$, uniformly on $[0, \bar{t}_1 - \delta_i]$. Again we see that u^i is a solution of (CP) on $[0, \bar{t}_1 - \delta_i]$, and it coincides with u^{i-1} on $[0, \bar{t}_1 - \delta_{i-1}]$.

Consider the diagonal subsequence $(u^k_k)_k$, which with a slight abuse of notation we denote by $(u_k)_k$. It converges to some $\bar{u} : [0, \bar{t}_1[\to \mathbb{R}^d$, uniformly on $[0, \tau]$, for every $\tau \in [0, \bar{t}_1[$. Passing to the limit in

$$u_k(t) = u^k_0 + \int_0^t f(s, u_k(s), z_k)\, ds \,,$$

we see that $\bar{u}$ is a solution of (CP) on $[0, \bar{t}_1[$. By assumption, it can be extended to a solution on $[0, T]$. Let $u_{(1)}$ be the restriction of this function to the interval $[0, \bar{t}_1]$. This will be the function we are looking for.

Indeed, $u_{(1)} : [0, \bar{t}_1] \to \mathbb{R}^d$ is a solution of (CP) on $[0, \bar{t}_1]$ and, since $u_k(t) \in D_1$ for every $t \in [0, t^1_k]$, it has to be that $u_{(1)}(t) \in D_1$ for every $t \in [0, \bar{t}_1[$. We need to prove that $u_{(1)}(\bar{t}_1) = \bar{u}_1$. Fix $\varepsilon > 0$. There exists a sufficiently small $\delta \in]0, \varepsilon/(4\bar{c})[$ such that

$$|u_{(1)}(\bar{t}_1) - u_{(1)}(\bar{t}_1 - \delta)| < \frac{\varepsilon}{4} \,,$$

and, for k sufficiently large,

$$|u_{(1)}(\bar{t}_1 - \delta) - u_k(\bar{t}_1 - \delta)| < \frac{\varepsilon}{4}, \qquad |u_k(t_k^1) - \bar{u}_1| < \frac{\varepsilon}{4}.$$

Since $u_k(t) \in D_1$ for every $t \in [0, t_k^1]$, we have that $|\dot{u}_k(t)| \leq \bar{c}$, for every $t \in [0, t_k^1]$. Taking k large enough, it will be that $t_k^1 > \bar{t}_1 - \delta$, and

$$|u_k(t_k^1) - u_k(\bar{t}_1 - \delta)| \leq \bar{c}(t_k^1 - (\bar{t}_1 - \delta)) < \bar{c}\delta < \frac{\varepsilon}{4}.$$

Hence,

$$\begin{aligned} |u_{(1)}(\bar{t}_1) - u_1| \leq |u_{(1)}(\bar{t}_1) - u_{(1)}(\bar{t}_1 - \delta)| + |u_{(1)}(\bar{t}_1 - \delta) - u_k(\bar{t}_1 - \delta)| \\ + |u_k(\bar{t}_1 - \delta) - u_k(t_k^1)| - |u_k(t_k^1) - \bar{u}_1| < 4\frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

By the arbitrariness of ε, it has to be $u_{(1)}(\bar{t}_1) = u_1$, and the proof of the Claim is completed. □

We now continue the proof of Theorem 6.1. Once we have found the subsequence $(u_k)_k$, we relabel accordingly the sequences $(u_0^k)_k$, $(z_k)_k$ and $(t_k)_k$. For $k \geq 2$, let $t_k^2 \in]0, T[$ be such that

$$u_k(t) \in \mathring{D}_2, \quad \text{for every } t \in [0, t_k^2[, \quad \text{and} \quad u_k(t_k^2) \in \partial D_2.$$

For subsequences, $t_k^2 \to \bar{t}_2 \in]0, T]$ and $u_k(t_k^2) \to \bar{u}_2 \in \partial D_2$. Adapting the Claim proved above, we find a subsequence $(u_k)_k$ which converges pointwise on $[0, \bar{t}_2]$ and uniformly on every $[0, \tau[$ with $\tau \in]0, \bar{t}_2[$ to some continuous function $u_{(2)} : [0, \bar{t}_2] \to \mathbb{R}^d$. Moreover, $u_{(2)}$ is a solution of (CP) and satisfies

$$u_{(2)}(t) \in D_2, \quad \text{for every } t \in [0, \bar{t}_2], \quad \text{and} \quad u_{(2)}(\bar{t}_2) = \bar{u}_2.$$

Notice that $u_{(2)}$ coincides with $u_{(1)}$ on $[0, \bar{t}_1]$, and $\bar{t}_2 > \bar{t}_1$.

Proceeding in this way, for every $m \geq 1$ we find $\bar{t}_1 < \bar{t}_2 < \bar{t}_3 < \cdots < \bar{t}_m$ in $[0, T]$, a point $\bar{u}_m$ in ∂D_m and a subsequence $(u_k)_k$ which converges pointwise on $[0, \bar{t}_m]$ and uniformly on every $[0, \tau[$ with $\tau \in]0, \bar{t}_m[$ to some continuous function $u_{(m)} : [0, \bar{t}_m] \to \mathbb{R}^d$. Moreover, $u_{(m)}$ is a solution of (CP) and satisfies

$$u_{(m)}(t) \in D_m, \quad \text{for every } t \in [0, \bar{t}_m], \quad \text{and} \quad u_{(m)}(\bar{t}_m) = \bar{u}_m.$$

Notice that $u_{(m)}$ coincides with $u_{(m-1)}$ on $[0, \bar{t}_{m-1}]$.

Let $\bar{t}_\infty = \lim_m \bar{t}_m$; it belongs to $]0, T]$. We define the function $u_\infty : [0, \bar{t}_\infty[\to \mathbb{R}^d$ as

$$u_\infty(t) = u_{(m)}(t) \quad \text{if } t \in [0, \bar{t}_m] \,.$$

It is a solution of (CP) on $[0, \bar{t}_\infty[$. By assumption, it can be extended to a solution of (CP) defined on $[0, T]$, which we denote by $\widetilde{u}_\infty : [0, T] \to \mathbb{R}^d$. Clearly enough, being a continuous function defined on a compact interval, the image $\widetilde{u}_\infty([0, T])$ is bounded. But we know that

$$\widetilde{u}_\infty(\bar{t}_m) = u_\infty(\bar{t}_m) = u_{(m)}(\bar{t}_m) = \bar{u}_m \,,$$

and $|\bar{u}_m| = r_m \to +\infty$, a contradiction. □

Remark 6.2. When dealing with the Cauchy problem

$$\dot{u} = F(t, u, \mathcal{Z}(t)) \,, \qquad u(0) = u_0 \,,$$

where $F : [0, T] \times \mathbb{R}^d \times \mathbb{R}^\ell \to \mathbb{R}^d$ is continuous and $\mathcal{Z} \in C^1([0, T], \mathbb{R}^\ell)$, if we know that $\mathcal{Z}$ belongs to a bounded set of $C^1([0, T], \mathbb{R}^\ell)$, then the Ascoli–Arzelà theorem tells us that $\mathcal{Z}$ also belongs to a compact set of $C([0, T], \mathbb{R}^\ell)$. Denoting by Z this compact set, we can then recover (CP) by defining $f : [0, T] \times \mathbb{R}^d \times Z \to \mathbb{R}^d$ as

$$f(t, u, z) = F(t, u, z(t)) \,.$$

It is indeed a continuous function, since on Z we have the topology of the uniform convergence. We have used this argument in the proof of Theorems 2.1 and 3.1.

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