Periodic solutions of Hamiltonian systems coupling twist with an isochronous center

Alessandro Fonda and Wahid Ullah

Abstract

We extend the Poincaré–Birkhoff Theorem to a Hamiltonian system which couples two systems with fairly different behaviours; the first one involves a twist assumption, while the second one is generated from a nonresonant isochronous center. By a suitable change of variables we modify the second system into a perturbation of a nonresonant linear one, and then prove that there exist multiple periodic solutions.

1 Introduction and statement of the main result

The celebrated Poincaré–Birkhoff Theorem [25] has been recently extended in the framework of Hamiltonian systems to any even dimension [20], with possible coupling with some nonresonant asymptotically linear system [11]. It is the aim of this paper to show that, in this coupling, linearity can be replaced by positive homogeneity, still preserving the already established multiplicity results.

Consider for instance the periodic problem associated with a four-dimensional system of the type

$$\begin{cases} \dot{q} = \partial_p \mathcal{H}(t,q,p) + \partial_p P(t,q,p,w)\,, \\ \dot{p} = -\partial_q \mathcal{H}(t,q,p) - \partial_q P(t,q,p,w)\,, \\ J\dot{w} = \nabla \mathscr{H}(w) + \nabla_w P(t,q,p,w)\,, \end{cases} \tag{1.1}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the standard symplectic 2×2 matrix. All functions involved in (1.1) are continuous and T-periodic in t. Here are our structural assumptions.

$A1$. The function $\mathcal{H}(t,q,p)$ is 2π-periodic in q.

*A*2. There are $a < b$ such that, for every C^1-function $\mathcal{W} : [0,T] \to \mathbb{R}^2$, all the solutions (q,p) of the system

$$\begin{cases} \dot q = \partial_p \mathcal{H}(t,q,p) + \partial_p P(t,q,p,\mathcal{W}(t))\,, \\ \dot p = -\partial_q \mathcal{H}(t,q,p) - \partial_q P(t,q,p,\mathcal{W}(t))\,, \end{cases} \tag{1.2}$$

starting with $p(0) \in [a,b]$, are defined on $[0,T]$ and satisfy

$$\begin{cases} p(0) = a \quad \Rightarrow \quad q(T) - q(0) < 0\,, \\ p(0) = b \quad \Rightarrow \quad q(T) - q(0) > 0\,. \end{cases}$$

The above is the well known *twist condition.*

*A*3. The Hamiltonian function $\mathscr{H}$ is positively homogeneous of degree 2 and positive, i.e.,

$$\mathscr{H}(\lambda w) = \lambda^2 \mathscr{H}(w) > 0\,, \quad \text{for every } w \in \mathbb{R}^2 \setminus \{0\} \text{ and } \lambda > 0\,.$$

In this setting, the origin is an isochronous center for the autonomous system

$$J\dot w = \nabla \mathscr{H}(w)\,; \tag{1.3}$$

aside from the origin, which is an equilibrium, all solutions of system (1.3) are periodic and have the same minimal period, which will be denoted by τ.

*A*4. The function $P(t,q,p,w)$ is 2π-periodic in q and has a bounded gradient with respect to (q,p,w).

We are now ready to state our first main result.

Theorem 1.1. *Assume that* $A1$ – $A4$ *hold true, and*

$$\frac{T}{\tau} \notin \mathbb{N}\,. \tag{1.4}$$

Then there are at least two geometrically distinct T*-periodic solutions of system* (1.1)*, with* $p(0) \in \,]a,b[\,$.

Clearly enough, when a T-periodic solution (q,p,w) of system (1.1) has been found, we may obtain infinitely many others by just adding an integer multiple of 2π to the q-th component. We say that two solutions are *geometrically distinct* if they cannot be obtained from each other in this way. Notice that the period 2π in assumption $A1$ is not necessary; any period would be possible.

The above theorem extends the two-dimensional Poincaré–Birkhoff Theorem as stated in [20]. Its proof will be given is Section 2. By a suitable change of variables, similarly as in [1, 6, 15], we first modify the second system into a perturbation of a nonresonant linear one, and then use the results in [11] to prove the multiplicity of periodic solutions. Some examples of applications will be provided in Section 3. In Section 4, we will show how to generalize Theorem 1.1 in some higher dimensional settings. Finally, in Section 5, we propose several possible applications in higher dimensions.

Different extensions of the results in [11, 20] have been proposed, coupling the twist assumption with the existence of well-ordered lower/upper solutions [10, 19], still obtaining multiplicity of periodic solutions. Our result represents a step forward in this field of investigation.

2 Proof of Theorem 1.1

The proof is divided into two parts: first we deal with the autonomous system (1.3) and show how to construct an appropriate symplectic change of variables to linearize it. Then, in the second part, we use this change of variables and modify the original system (1.1) so to be able to apply a result from [11].

2.1 The autonomous system

By $A3$, we have that $\mathscr{H}(0) = 0$, and the Euler Identity holds true, i.e.,

$$\langle \nabla \mathscr{H}(w), w \rangle = 2\mathscr{H}(w)\,, \quad \text{for every } w \in \mathbb{R}^2. \tag{2.1}$$

Choose the positive constant

$$\gamma = \tfrac{1}{2} \min\{\mathscr{H}(w) : |w| = 1\}\,, \tag{2.2}$$

and let $\eta : \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that $\eta'(s) \le 0$ for all $s \in \mathbb{R}$ and

$$\eta(s) = \begin{cases} 1\,, & \text{if } s \le 1\,, \\ 0\,, & \text{if } s \ge 2\,. \end{cases}$$

Set

$$\widehat{\mathscr{H}}(w) = \eta(|w|)\gamma|w|^2 + (1 - \eta(|w|))\mathscr{H}(w)\,, \tag{2.3}$$

and consider the new system

$$J\dot{w} = \nabla\widehat{\mathscr{H}}(w)\,. \tag{2.4}$$

Notice that $\widehat{\mathscr{H}}(0) = 0$. We first show that the origin is still a global center for system (2.4). For any $w \neq 0$, we have

$$\nabla\widehat{\mathscr{H}}(w) = \Big(\gamma\eta'(|w|)|w| + 2\gamma\eta(|w|) - \frac{\eta'(|w|)}{|w|}\mathscr{H}(w)\Big)w + (1-\eta(|w|))\nabla\mathscr{H}(w)\,.$$

Then, using (2.1) and (2.2), if $w \neq 0$ we have

$$\begin{aligned}\langle\nabla\widehat{\mathscr{H}}(w), w\rangle &= \eta'(|w|)|w|^3\left(\gamma - \mathscr{H}\left(\frac{w}{|w|}\right)\right)\\ &\quad + 2|w|^2\left(\eta(|w|)\,\gamma + (1-\eta(|w|))\mathscr{H}\left(\frac{w}{|w|}\right)\right) > 0\,.\end{aligned}$$

This shows that the origin is a global center for system (2.4) (cf. [1, Lemma 2.1]). For any $w_0 \in \mathbb{R}^2 \setminus \{0\}$, we denote by $\hat{\tau}(w_0)$ the minimal period of the solution of (2.4) passing through w_0. (Notice that this solution is unique, even if we are not assuming $\nabla\mathscr{H}$ to be locally Lipschitz continuous, cf. [26].) The function $\hat{\tau} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ thus defined is continuously differentiable (see [1]).

Define

$$\delta^\star = [0, +\infty[\times\{0\}\,,$$

and a function $\xi :]0, +\infty[\to]0, +\infty[$ as follows: for every $c > 0$, the level line $\{w \in \mathbb{R}^2 : \widehat{\mathscr{H}}(w) = c\}$ intersects $\delta^\star$ at the point $(\xi(c), 0)$. By the above arguments, such a point is unique.

Now define $H : \mathbb{R}^2 \to \mathbb{R}$ as

$$H(w) = \frac{1}{\tau}\int_0^{\widehat{\mathscr{H}}(w)} \hat{\tau}(\xi(c), 0)\,dc\,. \tag{2.5}$$

This function is continuously differentiable, and

$$\nabla H(w) = \frac{\hat{\tau}(w)}{\tau}\nabla\widehat{\mathscr{H}}(w)\,.$$

Hence, the origin is an *isochronous* center for the system

$$J\dot{w} = \nabla H(w)\,, \tag{2.6}$$

since all solutions except the equilibrium 0 have minimal period τ. Moreover,

$$H(w) = \frac{\pi}{\tau}|w|^2\,, \qquad \text{when } |w| \le 1\,.$$

Now, for every $w_0 \in \mathbb{R}^2 \setminus \{0\}$, let $\zeta(t\,;w_0)$ be the solution of system (2.6) satisfying $\zeta(0\,;w_0) = w_0$, and define $\theta(w_0) \in [0, 2\pi[$ as the minimum time for which

$$\zeta\Big(-\frac{\tau}{2\pi}\theta(w_0)\,;w_0\Big) \in \delta^\star.$$

As shown in [1], the restricted function $\theta : \mathbb{R}^2 \setminus \delta^\star \to]0, 2\pi[$ is continuously differentiable, and its gradient $\nabla\theta$ can be continuously extended to $\mathbb{R}^2 \setminus \{0\}$. We will still denote this extension by $\nabla\theta : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$.

Notice that, if $w(t) \neq 0$ is a solution of system (2.6), then $w(t_0) \in \delta^\star$ for some $t_0 \in]t - \tau, t]$. Thus $\theta(w(t)) = \frac{2\pi}{\tau}(t - t_0)$ for all $t \in]t_0, t_0 + \tau[$, hence

$$\frac{d}{dt}\theta(w(t)) = \frac{2\pi}{\tau}\,, \qquad \text{for every } t \in]t_0, t_0 + \tau[\,.$$

Now define $\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Lambda(w) = \begin{cases} \sqrt{\dfrac{\tau}{\pi}H(w)}\big(\cos\theta(w),\, -\sin\theta(w)\big), & \text{if } w \neq 0\,,\\ 0\,, & \text{if } w = 0\,. \end{cases}$$

As a consequence of [1, Proposition 2.2], we have that the map Λ is a symplectic diffeomorphism and satisfies

$$\det \Lambda'(w) = 1\,, \qquad \text{for every } \; w \in \mathbb{R}^2.$$

By the change of variables $z = \Lambda(w)$, system (2.6) becomes

$$J\dot z = \frac{2\pi}{\tau} z\,.$$

Hence, this change of variables considerably simplifies the problem, leading to a linear system. The sequel of the proof of Theorem 1.1 will be based on the idea of applying this change of variables to the general system (1.1), so to obtain the coupling of a twisting system with a linear one.

2.2 Back to the original system

By the global existence assumption in $A2$, there exists a constant $C > 0$ such that, for any solution (q, p) of (1.2) starting with $p(0) \in [a, b]$, one has that

$$|p(t)| \le C\,, \qquad \text{for every } t \in [0, T]\,.$$

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that

$$\sigma(s) = \begin{cases} 1\,, & \text{if } \ s \le C\,, \\ 0\,, & \text{if } \ s > C+1\,, \end{cases}$$

and set

$$\widehat{\mathcal{H}}(t,q,p) = \sigma(|p|)\mathcal{H}(t,q,p)\,. \tag{2.7}$$

Then $\widehat{\mathcal{H}}$ has a bounded gradient with respect to (q,p).

In addition, we define

$$\widetilde{P}(t,q,p,z) = P(t,q,p,\Lambda^{-1}(z))\,.$$

Lemma 2.1. *The function $\widetilde{P}$ has a bounded gradient with respect to (q,p,z).*

Proof. By *A*4, both

$$\partial_q \widetilde{P}(t,q,p,z) = \partial_q P(t,q,p,\Lambda^{-1}(z))\,, \quad \partial_p \widetilde{P}(t,q,p,z) = \partial_p P(t,q,p,\Lambda^{-1}(z))$$

are bounded, and

$$\begin{aligned}\nabla_z \widetilde{P}(t,q,p,z) &= \left[(\Lambda^{-1}(z))'\right]^{tr} \nabla_w P(t,q,p,\Lambda^{-1}(z)) \\ &= \left[(\Lambda'(\Lambda^{-1}(z)))^{tr}\right]^{-1} \nabla_w P(t,q,p,\Lambda^{-1}(z)))\,.\end{aligned}$$

By *A*4 again, $\nabla_w P(t,q,p,w)$ is bounded, so it is sufficient to show that $(\Lambda'(w))^{-1}$ is bounded. For $|w|$ large enough, we have that $H(w) = \mathscr{H}(w)$. By denoting $c(w) = \cos\theta(w)$ and $s(w) = \sin\theta(w)$, with $w = (u,v) \in \mathbb{R}^2$, we have

$$\Lambda'(w) = \begin{bmatrix} a_{11}(w) & a_{12}(w) \\ a_{21}(w) & a_{22}(w) \end{bmatrix}\,,$$

where

$$a_{11}(w) = \sqrt{\frac{\tau}{\pi}}\left(\frac{\partial_u \mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}}\,c(w) - \sqrt{\mathscr{H}(w)}\,\partial_u\theta(w)\,s(w)\right),$$

$$a_{12}(w) = \sqrt{\frac{\tau}{\pi}}\left(\frac{\partial_v \mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}}\,c(w) - \sqrt{\mathscr{H}(w)}\,\partial_v\theta(w)\,s(w)\right),$$

$$a_{21}(w) = \sqrt{\frac{\tau}{\pi}}\left(-\frac{\partial_u \mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}}\,s(w) - \sqrt{\mathscr{H}(w)}\,\partial_u\theta(w)\,c(w)\right),$$

$$a_{22}(w) = \sqrt{\frac{\tau}{\pi}}\left(-\frac{\partial_v \mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}}\,s(w) - \sqrt{\mathscr{H}(w)}\,\partial_v\theta(w)\,c(w)\right).$$

Recalling that $\det \Lambda'(w) = 1$, the inverse matrix is

$$(\Lambda'(w))^{-1} = \begin{bmatrix} a_{22}(w) & -a_{12}(w) \\ -a_{21}(w) & a_{11}(w) \end{bmatrix}.$$

From the definition of θ, for $w \neq 0$ and $\lambda > 0$ we see that $\theta(\lambda w) = \theta(w)$, hence $\nabla\theta(\lambda w) = \lambda \nabla\theta(w)$, and thus

$$\nabla\theta(w) = \frac{1}{|w|}\nabla\theta\left(\frac{w}{|w|}\right), \qquad \text{when } w \neq 0 .$$

Therefore, for $|w|$ large, since $\mathscr{H}$ is positively homogeneous of degree 2, we have

$$\begin{aligned} |a_{22}(w)| &\leq \sqrt{\frac{\tau}{\pi}}\left(\frac{|\partial_v \mathscr{H}(w)|}{2\sqrt{\mathscr{H}(w)}} + \sqrt{\mathscr{H}(w)}\,|\partial_v \theta(w)|\right) \\ &\leq \sqrt{\frac{\tau}{\pi}}\left(\frac{\left|\nabla\mathscr{H}\left(\frac{w}{|w|}\right)\right|}{2\sqrt{\mathscr{H}\left(\frac{w}{|w|}\right)}} + \sqrt{\mathscr{H}\left(\frac{w}{|w|}\right)}\left|\nabla\theta\left(\frac{w}{|w|}\right)\right|\right). \end{aligned}$$

This shows that $|a_{22}(w)|$ is bounded. Similarly we can show that all the other elements of matrix $(\Lambda'(w))^{-1}$ are bounded, thus proving that the map $\widetilde{P}$ has a bounded gradient with respect to z. □

Now we consider the modified system

$$\begin{cases} \dot{q} = \partial_p \widehat{\mathcal{H}}(t,q,p) + \partial_p \widetilde{P}(t,q,p,z)\,, \\ \dot{p} = -\partial_q \widehat{\mathcal{H}}(t,q,p) - \partial_q \widetilde{P}(t,q,p,z)\,, \\ J\dot{z} = \dfrac{2\pi}{\tau} z + \nabla_z \widetilde{P}(t,q,p,z)\,, \end{cases} \tag{2.8}$$

where we have applied the change of variables $z = \Lambda(w)$. The new Hamiltonian function is defined as

$$\widetilde{H}(t,q,p,z) = \widehat{\mathcal{H}}(t,q,p) + \frac{2\pi}{\tau}|z|^2 + \widetilde{P}(t,q,p,z)\,.$$

Using $A2$ and (1.4), we conclude by [11, Corollary 2.4] that the modified system (2.8) has at least two geometrically distinct T-periodic solutions such that $p(0) \in \,]a,b[\,$.

Recalling that Λ is a diffeomorphism, we can apply the inverse change of variables $w = \Lambda^{-1}(z)$ and obtain the solutions of system (1.1) we were looking for. □

3 Examples of applications

There are many possible applications of our result. The twist condition is encountered in the framework of scalar second order differential equations, with many different possible behaviours of the retraction forces (see, e.g., [8] for an updated list of references). Concerning the positively homogeneous case, various particular cases have been treated, starting with the pioneering papers by Fucik [21] and Dancer [3] (see also, e.g., [7] and the references therein).

As a first example, consider the coupling of a pendulum-like equation with an asymmetric oscillator, i.e.,

$$\begin{cases} \ddot{q} + A\sin q = e(t) + \partial_q P(t,q,u)\,, \\ \ddot{u} + \mu u^+ - \nu u^- = \partial_u P(t,q,u)\,, \end{cases} \tag{3.1}$$

where the constants A, μ, ν are positive. In the above, we have used the notation $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$. Assume that $P(t,q,u)$ is T-periodic in t and 2π-periodic in q, and that it has a bounded gradient with respect to (q,u). Setting $E(t) = \int_0^t e(s)\,ds$, system (3.1) is equivalent to

$$\begin{cases} \dot{q} = p + E(t)\,, \quad \dot{p} = -A\sin q + \partial_q P(t,q,u)\,, \\ \dot{u} = v\,, \quad \dot{v} = -\mu u^+ + \nu u^- + \partial_u P(t,q,u)\,. \end{cases}$$

Assuming $e(t)$ to be T-periodic with

$$\int_0^T e(t)\,dt = 0\,,$$

the function $E(t)$ is T-periodic, as well.

Let us verify the twist condition $A2$. Notice that there exists $K > 0$ such that, for every C^1-function $\mathcal{U} : [0,T] \to \mathbb{R}$, all the solutions (q,p) of the system

$$\dot{q} = p + E(t)\,, \quad \dot{p} = -A\sin q + \partial_q P(t,q,\mathcal{U}(t))$$

are defined on $[0,T]$ and satisfy

$$|\dot{p}(t)| \le K\,, \qquad \text{for every } t \in [0,T]\,.$$

Define $b = KT + \|E\|_\infty + 1$ and $a = -(KT + \|E\|_\infty + 1)$. Then, if $p(0) = b$, we have

$$\dot{q}(t) = p(t) + E(t) = p(0) + \int_0^t \dot{p}(s)\,ds + E(t) \ge b - KT - \|E\|_\infty > 0\,,$$

for every $t \in [0,T]$, and so $q(T) - q(0) > 0$. Similarly, if $p(0) = a$, then $q(T) - q(0) < 0$. Assumption $A2$ is thus satisfied.

On the other hand, if we define $\mathscr{H}$ by

$$\mathscr{H}(u,v) = \frac{1}{2}v^2 + \frac{\mu}{2}(u^+)^2 + \frac{\nu}{2}(u^-)^2\,,$$

then $\mathscr{H}$ is positive, positively homogeneous of degree 2, and all the solutions of system $J\dot{w} = \mathscr{H}(w)$ with $w = (u,v)$ are periodic with a fixed period

$$\tau = \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}\,.$$

All the assumptions of Theorem 1.1 are satisfied, and we can thus state the following.

Corollary 3.1. *In the above setting, assume moreover that*

$$\frac{\sqrt{\mu\nu}}{\sqrt{\mu}+\sqrt{\nu}} \neq \frac{n\pi}{T}\,, \qquad \textit{for every } n \in \mathbb{N}\,.$$

Then system (3.1) *has at least two geometrically distinct T-periodic solutions.*

A variant of the previous example is obtained replacing the second equation in (3.1) as follows:

$$\begin{cases} \ddot{q} + A\sin q = e(t) + \partial_q P(t,q,u)\,, \\ J\dot{w} + \mu w^+ - \nu w^- = \partial_u P(t,q,u)\,, \end{cases} \tag{3.2}$$

where, being $w = (u,v)$, one has $w^+ = (u^+, v^+)$ and $w^- = (u^-, v^-)$. Assuming $\mu\nu > 0$, as shown in [7], the minimal period of the isochronous system is

$$\tau = \frac{\pi}{2}\left(\frac{1}{\sqrt{|\mu|}} + \frac{1}{\sqrt{|\nu|}}\right)^2.$$

We thus get the following.

Corollary 3.2. *In the above setting, assume moreover that*

$$\frac{\mu\nu}{(\sqrt{|\mu|}+\sqrt{|\nu|})^2} \neq \frac{n\pi}{2T}\,, \qquad \textit{for every } n \in \mathbb{N}\,.$$

Then system (3.2) *has at least two geometrically distinct T-periodic solutions.*

The above results generalize a classical theorem in [24] by Mawhin and Willem on the multiplicity of periodic solutions for the pendulum equation.

4 Higher dimensional systems

We now consider a Hamiltonian system in $\mathbb{R}^{2N}$,

$$\begin{cases} \dot q = \nabla_p \mathcal{H}(t,q,p) + \nabla_p P(t,q,p,w)\,, \\ \dot p = -\nabla_q \mathcal{H}(t,q,p) - \nabla_q P(t,q,p,w)\,, \\ J\dot w = \nabla \mathscr{H}(w) + \nabla_w P(t,q,p,w)\,, \end{cases} \tag{4.1}$$

where we assume that all the involved functions are continuous, and T-periodic in their first variable t. Here $N = M + L$, and we write $w = (u, v)$, with

$$q = (q_1, \dots, q_M) \in \mathbb{R}^M, \quad p = (p_1, \dots, p_M) \in \mathbb{R}^M,$$
$$u = (u_1, \dots, u_L) \in \mathbb{R}^L, \quad v = (v_1, \dots, v_L) \in \mathbb{R}^L\,.$$

Moreover, we assume that $\mathscr{H} : \mathbb{R}^{2L} \to \mathbb{R}$ is of the type

$$\mathscr{H}(u,v) = \sum_{j=1}^{L} \mathscr{H}_j(u_j, v_j)\,,$$

for some functions $\mathscr{H}_j : \mathbb{R}^2 \to \mathbb{R}$.

In order to introduce the twist condition, let us recall some definitions. A closed convex bounded subset $\mathcal{D}$ of $\mathbb{R}^M$ having nonempty interior is said to be a convex body of $\mathbb{R}^M$. If we assume that $\mathcal{D}$ has a smooth boundary, then we denote the unit outward normal at $\zeta \in \partial\mathcal{D}$ by $\nu_{\mathcal{D}}(\zeta)$. Moreover, we say that $\mathcal{D}$ is strongly convex if for any $p \in \partial\mathcal{D}$, the map $\mathcal{F} : \mathcal{D} \to \mathbb{R}$ defined by $\mathcal{F}(\xi) = \langle \xi - p, \nu_{\mathcal{D}}(p)\rangle$ has a unique maximum point at $\xi = p$.

Here are our hypotheses.

A1′. The function $\mathcal{H}(t,q,p)$ is 2π-periodic in each variable $q_1, \dots, q_M$.

A2′. There are a strongly convex body $\mathcal{D}$ of $\mathbb{R}^M$ having a smooth boundary and a symmetric regular $M \times M$ matrix $\mathbb{A}$ such that for every C^1-function $\mathcal{W} : [0,T] \to \mathbb{R}^{2L}$, all the solutions (q,p) of system

$$\begin{cases} \dot q = \nabla_p \mathcal{H}(t,q,p) + \nabla_p P(t,q,p,\mathcal{W}(t))\,, \\ \dot p = -\nabla_q \mathcal{H}(t,q,p) - \nabla_q P(t,q,p,\mathcal{W}(t))\,, \end{cases} \tag{4.2}$$

starting with $p(0) \in \mathcal{D}$ are defined on $[0,T]$, and

$$p(0) \in \partial\mathcal{D} \quad \Rightarrow \quad \langle q(T) - q(0)\,,\, \mathbb{A}\nu_{\mathcal{D}}(p(0))\rangle > 0\,.$$

A3′. For every $j \in \{1, \dots, L\}$, the Hamiltonian function $\mathscr{H}_j : \mathbb{R}^2 \to \mathbb{R}$ is positively homogeneous of degree 2 and positive, i.e.,

$$\mathscr{H}_j(\lambda x, \lambda y)) = \lambda^2 \mathscr{H}_j(x, y) > 0 \,, \quad \text{for every } (x, y) \in \mathbb{R}^2 \setminus \{0\} \text{ and } \lambda > 0 \,.$$

In this setting, the origin $(0, 0)$ is an isochronous center for the planar autonomous system

$$J\dot{\zeta} = \nabla \mathscr{H}_j(\zeta) \,. \tag{4.3}$$

Similar to the case of low dimension, for every $j \in \{1, \dots, L\}$, besides the origin all solutions of system (4.3) are periodic and have the same minimal period, which will be denoted by τ_j.

A4. The function $P(t, q, p, w)$ is 2π-periodic in $q_1, \dots, q_M$ and has a bounded gradient with respect to (q, p, w).

Here is our first generalization of Theorem 1.1.

Theorem 4.1. *Assume that $A1'$ – $A4'$ hold true and*

$$\frac{T}{\tau_j} \notin \mathbb{N} \,, \qquad \text{for every } j \in \{1, \dots, L\} \,. \tag{4.4}$$

Then there are at least $M + 1$ geometrically distinct T-periodic solutions of system (4.1)*, with $p(0) \in \mathring{\mathcal{D}}$.*

Proof. Most of the arguments will be similar to the ones provided in Section 2, so we will be very brief. We can define $\widehat{\mathscr{H}_j} : \mathbb{R}^2 \to \mathbb{R}$ as in (2.3), and $H_j : \mathbb{R}^2 \to \mathbb{R}$ as in (2.5), so that the origin is an isochronous center for the system

$$J\dot{\zeta} = \nabla H_j(\zeta) \,, \tag{4.5}$$

i.e., for every $j \in \{1, \dots, L\}$, all solutions of system (4.5) except the origin are periodic and have the same minimal period τ_j. Now similar to the case of Section 2, for every $j \in \{1, \dots, L\}$, there exists a symplectic diffeomorphism $\Lambda_j : \mathbb{R}^2 \to \mathbb{R}^2$ such that by the change of variables $\rho = \Lambda_j(\zeta)$, system (4.5) becomes

$$J\dot{\rho} = \frac{2\pi}{\tau_j} \rho \,.$$

Define $\Lambda : \mathbb{R}^{2L} \to \mathbb{R}^{2L}$ by

$$\Lambda(u, v) = (\Lambda_1(u_1, v_1), \dots, \Lambda_L(u_L, v_L)) \,.$$

Then Λ is a symplectic diffeomorphism. Let us define

$$\widetilde{P}(t,q,p,z) = P(t,q,p,\Lambda^{-1}(z)) .$$

As in Lemma 2.1, we can show that the function $\widetilde{P}$ has a bounded gradient with respect to (q,p,z).

By a cut-off function σ, we modify the Hamiltonian $\mathcal{H}$ like in (2.7), setting

$$\widehat{\mathcal{H}}(t,q,p) = \sigma(|p|)\mathcal{H}(t,q,p) ,$$

so that the new Hamiltonian $\widehat{\mathcal{H}}$ has a bounded gradient with respect to (q,p).

We now consider the modified system

$$\begin{cases} \dot{q} = \nabla_p \widehat{\mathcal{H}}(t,q,p) + \nabla_p \widetilde{P}(t,q,p,z) , \\ \dot{p} = -\nabla_q \widehat{\mathcal{H}}(t,q,p) - \nabla_q \widetilde{P}(t,q,p,z) , \\ J\dot{z}_j = \dfrac{2\pi}{\tau_j} z_j + \nabla_{z_j} \widetilde{P}(t,q,p,z) , \quad j = 1,\dots,L , \end{cases} \tag{4.6}$$

where we have applied the change of variables $z = \Lambda(w)$. Using $A2'$ and (4.4), we conclude by [11, Corollary 2.4] that the modified system (4.6) has at least $M+1$ geometrically distinct T-periodic solutions, such that $p(0) \in \mathring{\mathcal{D}}$.

Recalling that Λ is a diffeomorphism, we can apply the inverse change of variables $w = \Lambda^{-1}(z)$ and obtain the solutions of system (4.1) we are looking for. □

We now consider some variants of Theorem 4.1. Let us first state the following "avoiding rays" assumption.

A2''. There exists a convex body $\mathcal{D}$ of $\mathbb{R}^M$, having a smooth boundary, such that for $\sigma \in \{-1,1\}$ and for every C^1-function $\mathcal{W} : [0,T] \to \mathbb{R}^{2L}$, all the solutions (q,p) of system (4.2) starting with $p(0) \in \mathcal{D}$ are defined on $[0,T]$, and

$$p(0) \in \partial\mathcal{D} \quad \Rightarrow \quad q(T) - q(0) \notin \{\sigma\lambda\,\nu_{\mathcal{D}}(p(0)) : \lambda \geq 0\} .$$

Theorem 4.2. *If in the statement of Theorem 4.1 we replace assumption $A2'$ by $A2''$, the same conclusion holds.*

Proof. The argument is the same as the one in the proof of Theorem 4.1, with the only difference that instead of applying [11, Corollary 2.3], we apply [11, Corollary 2.1]. □

Now we consider the case when $\mathcal{D}$ is a rectangle in $\mathbb{R}^M$, i.e.

$$\mathcal{D} = [a_1, b_1] \times \cdots \times [a_M, b_M] .$$

We state the following assumption.

A2‴. There exists an M-tuple $\sigma = (\sigma_1, \ldots, \sigma_M) \in \{-1, 1\}^M$ such that for every C^1-function $\mathcal{W} : [0,T] \to \mathbb{R}^{2L}$, all the solutions (q,p) of system (4.2) starting with $p(0) \in \mathcal{D}$ are defined on $[0,T]$, and for every $i = 1, \ldots, M$ we have

$$\begin{cases} p_i(0) = a_i \quad \Rightarrow \quad \sigma_i(q_i(T) - q_i(0)) < 0 \,, \\ p_i(0) = b_i \quad \Rightarrow \quad \sigma_i(q_i(T) - q_i(0)) > 0 \,. \end{cases}$$

Theorem 4.3. *If in the statement of Theorem 4.1 we replace assumption A2′ by A2‴, the same conclusion holds.*

Proof. Apply [11, Corollary 2.4] instead of [11, Corollary 2.3]. □

5 Further examples and remarks

As an example of application, consider the system

$$\begin{cases} \ddot{q}_i + A_i \sin q_i = e_i(t) + \partial_{q_i} P(t,q,u) \,, \quad i = 1, \ldots, M \,, \\ \ddot{u}_j + \mu_j u_j^+ - \nu_j u_j^- = \partial_{u_j} P(t,q,u) \,, \quad j = 1, \ldots, L \,, \end{cases} \tag{5.1}$$

where all constants A_i, μ_j, ν_j are positive. Assume that $P(t,q,u)$ is T-periodic in t and 2π-periodic in each q_i, and has a bounded gradient with respect to (q,u). Following the lines of the example given in Section 3, we assume the functions $e_i(t)$ to be T-periodic with

$$\int_0^T e_i(t)\, dt = 0 \,, \qquad \text{for every } i = 1, \ldots, M \,,$$

so that the functions $E_i(t) = \int_0^t e_i(s)\, ds$ are also T-periodic. We can find a constant $K > 0$ such that, for every C^1-function $\mathcal{U} : [0,T] \to \mathbb{R}^L$, the solutions of the system

$$\dot{q}_i = p_i + E_i(t) \,, \quad \dot{p}_i = -A \sin q_i + \partial_{q_i} P(t,q,\mathcal{U}(t)) \,, \quad i = 1, \ldots, M$$

are defined on $[0,T]$ and satisfy

$$|\dot{p}(t)| \leq K \,, \qquad \text{for every } t \in [0,T] \,.$$

Define $b_i = KT + \|E_i\|_\infty + 1$ and $a_i = -(KT + \|E_i\|_\infty + 1)$ and

$$\mathcal{D} = [a_1, b_1] \times \cdots \times [a_M, b_M] \,.$$

Now, for every $i \in \{1, \ldots, M\}$, if $p_(0) = b_i$, then for every $t \in [0, T]$ we have

$$\dot{q}_(t) = p_i(t) + E_i(t) = p_i(0) + \int_0^t \dot{p}_i + E_i(t) \geq b - KT - \|E_i\|_\infty > 0 \,,$$

and so $q_i(T) - q_i(0) > 0$. Similarly, if $p_i(0) = a_i$, then $q_i(T) - q_i(0) < 0$ for every $i \in \{1, \ldots, M\}$. Then, the twist condition $A2'''$ is verified.

On the other hand, if we define $\mathscr{H}_j : \mathbb{R}^2 \to \mathbb{R}$ by

$$\mathscr{H}_j(x, y) = \frac{1}{2} y^2 + \frac{\mu_j}{2} (x^+)^2 + \frac{\nu_j}{2} (x^-)^2 \,,$$

we see that $A3'$ is satisfied. We can thus state the following higher dimensional version of Corollary 3.1.

Corollary 5.1. *In the above setting, assume moreover that*

$$\frac{\sqrt{\mu_j \nu_j}}{\sqrt{\mu_j} + \sqrt{\nu_j}} \neq \frac{n\pi}{T} \,, \qquad \text{for every } n \in \mathbb{N} \text{ and } j = 1, \ldots, L \,.$$

Then system (5.1) *has at least* $M + 1$ *geometrically distinct* T*-periodic solutions.*

One can similarly provide a generalization of Corollary 3.2, as well. We avoid the details, for briefness.

Several other situations can be tackled using our results. Here we sketch some possible examples. Consider, e.g., a system of the type

$$\begin{cases} \ddot{q}_i + g_i(t, q_i) = \partial_{q_i} P(t, q, w) \,, & i = 1, \ldots, M \,, \\ J \dot{w}_j = \nabla \mathscr{H}_j(w_j) + \nabla_{w_j} P(t, q, w) \,, & j = 1, \ldots, L \,, \end{cases}$$

where the functions $\mathscr{H}_j : \mathbb{R}^2 \to \mathbb{R}$ are as above, and the functions $g_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and T-periodic in their first variable.

When the retraction functions g_i have a superlinear growth at infinity, one can follow the approach in [4, 16] to prove the existence of infinitely many T-periodic solutions. Indeed, the large amplitude solutions of the first system rotate around the origin faster and faster, with a time of rotation going to zero with the amplitude. Passing to suitably rotating polar coordinates, we recover the necessary twist condition (see [16] for the details).

A similar argument applies when the functions g_i have different behaviours near the origin and near infinity, thus generating a twisting behaviour in the associated phase planes, like in [23], or for systems involving a parameter (see, e.g., [2] and the references therein), where the same situation is recovered after a change of coordinates. Perturbation results can also be obtained, as shown, e.g., in [9]. Let us finally mention the possibility of treating, with the same techniques, problems with one or more singularities (see, e.g., [13, 17] and the references therein). In all these mentioned situations, multiplicity of T-periodic solutions is obtained.

On the other hand, if the functions g_i have a sublinear growth at infinity, we can use the approach developed in [5, 18] to get an infinite number of *subharmonic* solutions, i.e., periodic solutions having as minimal period an integer multiple of T. In this case, the large amplitude solutions of the first system rotate around the origin very slowly, with a time of rotation going to infinity.

Similar results for two-point boundary value problems have been recently obtained in [12, 14, 22] (see also [8]). The main novelty there is that no twist condition is needed. The extension of Theorem 1.1 to this setting will appear elsewhere.

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Authors' addresses:

Alessandro Fonda and Wahid Ullah
Dipartimento di Matematica e Geoscienze
Università degli Studi di Trieste
P.le Europa 1, 34127 Trieste, Italy
e-mail: a.fonda@units.it, wahid.ullah@phd.units.it

Mathematics Subject Classification: 34C25.

Keywords: Hamiltonian systems; periodic solutions; multiplicity; Poincaré–Birkhoff Theorem; positively homogeneous systems.