Nonlinear Analysis: Real World Applications 74 (2023) 103948

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Periodic solutions of discontinuous second order differential equations. The porpoising effect

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A R T I C L E I N F O

Article history:
Received 18 September 2022
Received in revised form 9 June 2023
Accepted 13 June 2023
Available online xxxx

Keywords:
Discontinuous systems
Periodic boundary value problem
Poincaré–Birkhoff theorem

A B S T R A C T

We construct a mathematical model in order to study the so called porpoising effect in racing cars, and prove that, when adding a small periodic perturbation, large-amplitude subharmonic solutions may arise.

1. Introduction

Due to the fundamental role of the effect of vibrations in the stability of mechanical structures, its study is one of the main branches of Mechanical Engineering. In general, an elastic body may experience a vibration when it is displaced from the equilibrium point by the effect of an external force. A famous example is given by the nonlinear oscillations experienced by suspension bridges, which may lead to a total collapse of the structure. In fact, dynamics of suspension bridges models has become an intensively studied topic by applied mathematicians and engineers, and there is a whole body of literature devoted to it (see for instance [1] and the references therein). In comparison, other known nonlinear effects have been less studied, as for example the phenomenon of *porpoising*, which is the main motivation of this paper.

Porpoising is a bouncing effect classically observed in motorboats that above a critical speed start an oscillatory motion leaping out and striking the water alternatively. More recently, this term has become familiar among the Formula 1 racing fans, after the impressive images during the first races of the 2022 season of most of the cars subject to some rather violent periodic bounces. It is clear that not only the car mechanics is compromised, but also the driver's safety.

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https://doi.org/10.1016/j.nonrwa.2023.103948
1468-1218/

In a racing car, porpoising is closely related with the downforce produced by the *ground effect*: the small gap between the bottom of the car and the floor creates this force, which increases as long as the gap decreases until, when it reaches a certain critical level, the air flow stalls and the downforce disappears suddenly. The car then goes back to its initial height and the process starts again, thus producing a rocking motion.

As a side note, let us comment that porpoising is not new at all in Formula 1 races. In fact, it was rather common in the racing cars of the 1970's and early 1980's. (It seems that the term "porpoising" was first coined in the 1970's by the famous racer Mario Andretti.) At some moment the ground effect was considered too dangerous and banned for many years, but in 2022 a change of regulation in the cars design has brought porpoising back in the limelight.

In view of the relevance of the described problems, there is a large body of engineering-oriented work dedicated to ground effect (see the review [2] and its bibliography) and porpoising [3–6]. Our aim here is different.

We will construct a mathematical model to describe oscillations under the following basic assumptions:

- the suspension system of the car provides a repulsive force against the floor, opposing gravity;
- the ground force must be discontinuous at a certain threshold value α;
- small external periodic forces are present (whose origin will be discussed at the end of the paper).

In order to describe our model more precisely, let $x > 0$ denote the distance of the bottom of the car from the ground. We assume that the following forces are acting on the car:

- the gravitational force, $-mg$, which is a negative constant;
- the shock-absorber force, modeled by a nonnegative function $f_S(x)$, which is equal to zero when x is sufficiently large, and tends to $+\infty$ as x tends to 0;
- the ground force, modeled by a nonpositive function $f_G(x)$, which is negligible when x is large, say $x > \beta$, it becomes large and negative on the interval $]\alpha, \beta[$, and it is equal to 0 when $x \leq \alpha$;
- a small T-periodic external forcing $e(t)$.

Newton's second law of motion thus provides us the differential equation

$$mx''(t) = -mg + f_S(x(t)) + f_G(x(t)) + e(t).$$

Hence, defining

$$g(x) = \frac{1}{m}\big(mg - f_S(x) - f_G(x)\big),$$

and writing the forcing term as $e(t) = m\varepsilon\, p(t)$, we get the equation

$$x'' + g(x) = \varepsilon\, p(t).$$

We will prove that, if ε is a small parameter, there exist large-amplitude periodic solutions of the above differential equation whose minimal period is an integer multiple of T. These are called *subharmonic solutions*. The precise statement of our result will be given in Section 2. The proof, provided in Section 3, involves some careful estimates of the solutions in the phase plane, together with the use of a fruitful mathematical tool for the detection of periodic solutions of Hamiltonian systems, the so called Poincaré–Birkhoff Theorem. In Section 4 we will argue about the possible origin of the small periodic forcing and provide some other remarks.

Let us point out that, from the mathematical point of view, this seems to be the first time when the Poincaré–Birkhoff Theorem is applied to this kind of problems. Equations with discontinuities have been studied from different perspectives. We know that jump discontinuities may generate complex dynamics and bifurcations [7–9]. On the other hand, nonlinearities with an essential discontinuity (i.e. a singularity) appear profusely in many mathematical models [10] and have been studied by different methods, including the use of Poincaré-Birkhoff Theorem (see for instance [11–13]). In this paper, for the first time in the literature, we are considering a nonlinearity with both a jump discontinuity and an essential discontinuity.

2. Main result

We study the equation

$$x'' + g(x) = \varepsilon p(t)\,, \tag{1}$$

where $p : \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic, while $g : \,]a, +\infty[\, \to \mathbb{R}$ is continuous except for a discontinuity point at some $\alpha \in \,]a, +\infty[$. In the model described in the Introduction, a is a real number; here, for pure mathematical reasons, we will also treat the case when a could be equal to $-\infty$.

Here is our assumption concerning the behavior of g at α.

$A1$. (Limits condition at α) The left and right limits

$$\ell_- = \lim_{x\to\alpha^-} g(x)\,, \qquad \ell_+ = \lim_{x\to\alpha^+} g(x)$$

exist and are finite, and

$$\ell_- < 0 < \ell_+.$$

We now need to define what kind of solution we are looking for.

Definition 1. We say that $x : \mathbb{R} \to \mathbb{R}$ is a *regular solution* of (1) if it is continuously differentiable and there exists a strictly increasing double sequence $(t_j)_{j\in\mathbb{Z}}$, with no accumulation points, such that $x(t_j) = \alpha$ and the restriction of x on each interval $]t_{j-1}, t_j[$ is twice continuously differentiable. Moreover, the limits

$$\lim_{t\to t_j^-} x''(t) \quad \text{and} \quad \lim_{t\to t_j^+} x''(t)$$

exist and are finite.

Let us state our assumptions on g.

$A2$. (Sign condition) There are d_1 and d_2, with $a < d_1 < \alpha < d_2$, such that

$$x \le d_1 \;\Rightarrow\; g(x) < 0\,, \qquad x \ge d_2 \;\Rightarrow\; g(x) > 0.$$

$A3$. (Coercivity of the potential) Let $G(x) = \int_\alpha^x g(u)\,du$; then,

$$\lim_{x\to a^+} G(x) = \lim_{x\to+\infty} G(x) = +\infty.$$

$A4$. (Behavior at $+\infty$) The asymptotic growth of g is controlled by

$$\limsup_{x\to+\infty} \frac{g(x)}{x} = \ell < +\infty.$$

Here is our main result.

Theorem 2. *Under assumptions $A1$–$A4$, for any positive integers m and k such that*

$$mT\sqrt{\ell} < k\pi\,, \tag{2}$$

there exists $\bar\varepsilon > 0$ such that, if $|\varepsilon| \le \bar\varepsilon$, then equation (1) has a mT-periodic regular solution x such that $x(\cdot) - \alpha$ has exactly $2k$ simple zeros in $[0, mT[$.

Notice that, taking $k = 1$, the regular solutions thus found have *minimal* period mT. Notice moreover that condition (2) surely holds if $\ell = 0$, for every positive integers m and k.

3. Proof of Theorem 2

For any function $z = (x, y) : [0, T] \to \,]a, +\infty[\, \times \mathbb{R}$ such that $z(t) \neq (\alpha, 0)$ for every $t \in [0, T]$, passing to polar coordinates

$$x(t) = (\alpha + \rho(t)) \cos\theta(t)\,, \qquad y(t) = \rho(t) \sin\theta(t)\,, \tag{3}$$

where $\rho : [0, T] \to [0, +\infty[$ and $\theta : [0, T] \to \mathbb{R}$ are continuous functions, we can define the rotation number

$$\mathrm{Rot}(z, [0, T]) = \frac{\theta(0) - \theta(T)}{2\pi}.$$

For simplicity, we will take $m = 1$ and $k = 1$.

3.1. The autonomous equation

Let us study the autonomous equation

$$x'' + g(x) = 0\,, \tag{4}$$

which is equivalent to the system

$$x' = y\,, \qquad -y' = g(x)\,. \tag{5}$$

We have a Hamiltonian function $H : \,]a, +\infty[\, \times \mathbb{R} \to \mathbb{R}$ defined as

$$H(x, y) = \tfrac{1}{2} y^2 + G(x).$$

Notice however that H is not of class C^1, since G is continuous but not differentiable at α. The orbits of the regular solutions are symmetric with respect to the horizontal axis, and they lie on the level sets of the Hamiltonian function, i.e.,

$$L^c = \{(x, y) \in \,]a, +\infty[\, \times \mathbb{R} : H(x, y) = c\}.$$

By A3, the level sets L^c are compact. As a consequence, the regular solutions of (4) are globally defined.

Notice that all the solutions of the associated initial value problems

$$(x(0), y(0)) = (x_0, y_0) \in \,]a, +\infty[\, \times \mathbb{R}$$

are regular, with the only exception for $(x_0, y_0) = (\alpha, 0)$, in which case the solution is not defined. In this case, we agree that this point $(\alpha, 0)$ will be considered as an equilibrium.

Since, by *A3*, the function G has a minimum on $]a, +\infty[$, by adding a suitable constant we can assume without loss of generality that

$$G(x) \geq 0\,, \qquad \text{for every } x \in \,]a, +\infty[\,. \tag{6}$$

Let us concentrate on the dynamics both near the point $(\alpha, 0)$, and for large-amplitude regular solutions.

Proposition 3. *The point $(\alpha, 0)$ is a local center for system (5); if $c > G(\alpha)$ and $|c - G(\alpha)|$ is small enough, the level sets L^c are closed curves, star-shaped with respect to $(\alpha, 0)$, corresponding to periodic regular solutions of (5). The same is true is $c > 0$ is large enough.*

Proof. First of all, we can assume without loss of generality that $\alpha = 0$, just by translating the problem in the spatial variable x. Then, from now on the center is assumed to be at the origin.

By *A1*, there exists a $\delta > 0$ such that

$$x g(x) > x^2\,, \qquad \text{for every } x \in \,]-\delta, \delta[\, \setminus \{0\}.$$

If (x, y) is a regular solution of (5) with $x(t) \in \,]-\delta, \delta[\, \setminus \{0\}$ for some $t \in \mathbb{R}$, passing to polar coordinates (3) we have that

$$-\theta'(t) = \frac{x(t)y'(t) - x'(t)y(t)}{x(t)^2 + y(t)^2} = \frac{x(t)g(x(t)) + y(t)^2}{x(t)^2 + y(t)^2} \geq 1\,. \tag{7}$$

To prove that the orbit is closed, first observe that

$$x \in \,]-\delta, \delta[\, \setminus \{0\} \quad \Rightarrow \quad \nabla H(x, y) \cdot (x, y) = xg(x) + y^2 > 0.$$

Hence, near the origin, $H(x, y)$ is strictly increasing along the rays emanating from the origin, at least in a small neighborhood of it. Then, a solution starting with initial point $(x(0), y(0)) \neq (0, 0)$ sufficiently near the origin, on a given ray, will remain in the strip $\{(x, y) : |x| < \delta\}$ for every $t \in [0, 2\pi]$, and during this time it will perform an entire rotation, returning to the initial ray, by (7). Since $H(x, y)$ is constant along the orbit, it must indeed return to the same initial point. This proves that the small-amplitude regular solutions of (5) are periodic and, by (7) again, they have star-shaped orbits with respect to the origin.

Let us now consider a large-amplitude regular solution. Recall $A2$ and set $M = \max\{|xg(x)| : x \in [d_1, d_2]\}$. Then, if (x, y) is a regular solution of (5) with $(x(t), y(t)) \notin [d_1, d_2] \times [-\sqrt{M}, \sqrt{M}\,]$ and $x(t) \neq 0$ for some $t \in \mathbb{R}$, passing to polar coordinates (3) we have that

$$-\theta'(t) \begin{cases} \geq \dfrac{-M + y(t)^2}{x(t)^2 + y(t)^2} > 0 & \text{if } x(t) \in [d_1, d_2]\,, \\ > \dfrac{y(t)^2}{x(t)^2 + y(t)^2} \geq 0 & \text{if } x(t) \notin [d_1, d_2]\,. \end{cases}$$

The same argument shows that, far from the origin, $H(x, y)$ is strictly increasing along the rays emanating from the origin. Now, let

$$\overline{S} = \max\{H(x, y) : (x, y) \in [d_1, d_2] \times [-\sqrt{M}, \sqrt{M}\,]\},$$

and set $K = \sqrt{2\overline{S}}$. By $A3$, there exist $\hat{d}_1$ and $\hat{d}_2$, with $a < \hat{d}_1 \leq d_1 < 0 < d_2 \leq \hat{d}_2$, such that

$$x \notin [\hat{d}_1, \hat{d}_2] \;\Rightarrow\; G(x) > \overline{S}.$$

We need the following.

Claim. If (x, y) is a solution of (5) with $(x(0), y(0)) \notin [\hat{d}_1, \hat{d}_2] \times [-K, K]$, then $(x(t), y(t)) \notin [d_1, d_2] \times [-\sqrt{M}, \sqrt{M}\,]$ for every $t \in \mathbb{R}$.

Once the Claim is proved, the same argument used above for the small-amplitude solutions apply, thus proving that also the large-amplitude regular solutions of (5) are periodic and have star-shaped orbits with respect to the origin.

To prove the Claim, let (x, y) be a solution of (5) such that $(x(0), y(0)) \notin [\hat{d}_1, \hat{d}_2] \times [-K, K]$. We have two cases. Either $x(0) \notin [\hat{d}_1, \hat{d}_2]$, in which case $G(x(0)) > \overline{S}$, hence

$$H(x(t), y(t)) = H(x(0), y(0)) \geq G(x(0)) > \overline{S}\,, \qquad \text{for every } t \in \mathbb{R}\,;$$

or, $x(0) \in [\hat{d}_1, \hat{d}_2]$ and $y(0) \notin [-K, K]$, in which case $\frac{1}{2}y(0)^2 > \overline{S}$, hence, by (6),

$$H(x(t), y(t)) = H(x(0), y(0)) \geq \tfrac{1}{2}y(0)^2 > \overline{S}\,, \qquad \text{for every } t \in \mathbb{R}\,.$$

By the definition of $\overline{S}$, in both cases it has to be $(x(t), y(t)) \notin [d_1, d_2] \times [-\sqrt{M}, \sqrt{M}\,]$ for every $t \in \mathbb{R}$. The Claim is thus proved. $\square$

Remark 4. The above proof could have been considerably simplified if we could have used the Poincaré–Bendixson Theorem. The presence of the jump discontinuity in the function $g(x)$ prevents its use, at least in its classical version, where regularity is needed. We are not aware of a version of this theorem in this more general setting.

In order to study the periodic regular solutions of (5) whose orbits contain in their interior the point $(\alpha, 0)$, we need to introduce the so called time-map. Let (x, y) be such a regular solution, and assume that $x(0) = D > \alpha$, $y(0) = 0$. Then,

$$\tfrac{1}{2}x'(t)^2 + G(x(t)) = \tfrac{1}{2}x'(0)^2 + G(x(0)) = G(D)\,, \qquad \text{for every } t \in \mathbb{R}.$$

This regular solution will reach the line $\ell_\alpha = \{(x, y) \in \mathbb{R}^2 : x = \alpha\}$ at some time $t = \tau_D > 0$. If $G(u) < G(D)$ for every $u \in [\alpha, D[$, we can write

$$\tau_D = \frac{1}{\sqrt{2}} \int_\alpha^D \frac{du}{\sqrt{G(D) - G(u)}}.$$

The above regular solution $(x(t), y(t))$, once reached the line ℓ_α at time $t = \tau_D$, will continue its trajectory until it reaches the horizontal axis $\{(x, y) \in \,]a, +\infty[\, \times \mathbb{R} : y = 0\}$ at some point $(\widehat{D}, 0)$, with $\widehat{D} < \alpha$, and at some time $t = \tau_D + \sigma_{\widehat{D}}$. If $G(u) < G(\widehat{D})$ for every $u \in \,]\widehat{D}, \alpha]$, we can write

$$\sigma_{\widehat{D}} = \frac{1}{\sqrt{2}} \int_{\widehat{D}}^\alpha \frac{du}{\sqrt{G(\widehat{D}) - G(u)}}.$$

We have that $\widehat{D} < \alpha < D$ and $G(\widehat{D}) = G(D)$. Since $\widehat{D}$ is uniquely determined by D, we can write $\widehat{D} = \psi(D)$, thus defining a function ψ. By Proposition 3, this function is well defined on $]\alpha, \alpha + \delta[$, for some sufficiently small $\delta > 0$, and on $]\mu, +\infty[$, for some sufficiently large $\mu > \alpha$. Using the symmetry of the orbit with respect to the horizontal axis, we can thus write the period of this regular solution as $T(D) = 2(\tau_D + \sigma_{\psi(D)})$, i.e.,

$$\begin{aligned} T(D) &= 2\left(\frac{1}{\sqrt{2}} \int_\alpha^D \frac{du}{\sqrt{G(D) - G(u)}} + \frac{1}{\sqrt{2}} \int_{\psi(D)}^\alpha \frac{du}{\sqrt{G(\psi(D)) - G(u)}}\right) \\ &= \sqrt{2} \int_{\psi(D)}^D \frac{du}{\sqrt{G(D) - G(u)}}\,. \end{aligned}$$

By the above considerations, this expression is well defined for $D \in \,]\alpha, \alpha + \delta[\cup]\mu, +\infty[$.

Proposition 5. *We have that*

$$\lim_{D \to \alpha^+} T(D) = 0\,, \quad \liminf_{D \to +\infty} T(D) > T.$$

Proof. By $A1$, there exists a $\delta > 0$ such that

$$x \in \,]\alpha, \alpha + \delta[\quad \Rightarrow \quad g(x) \geq \tfrac{1}{2}\ell_+.$$

Let $(x(t), y(t))$ be a regular solution such that $(x(0), y(0)) = (D, 0)$, for some $D \in \,]\alpha, \alpha + \delta[$. For every $t \in [0, \tau_D[$, we have that $x(t) \in \,]\alpha, D]$, hence

$$x'(t) = x'(0) + \int_0^t (-g(x(s)))\, ds \leq -\tfrac{1}{2}\ell_+ t,$$

whence

$$x(t) - x(0) = \int_0^t x'(s)\, ds \leq -\tfrac{1}{4}\ell_+ t^2.$$

Taking $t = \tau_D$, we get

$$\tau_D \leq \sqrt{\frac{4D}{\ell_+}},$$

hence

$$\lim_{D\to\alpha^+} \tau_D = 0.$$

A similar estimate can be performed to the left of α, showing that

$$\lim_{D\to\alpha^+} \sigma_{\psi(D)} = 0.$$

The first limit is thus established.

The second limit follows from the results in [14,15], where it was proved that $A2$, $A3$ and $A4$ imply

$$\liminf_{D\to+\infty} T(D) \geq \frac{\pi}{\sqrt{\ell}}.$$

The conclusion follows, since we are assuming that $T\sqrt{\ell} < \pi$. □

For every positive numbers $c_1 < c_2$, we define the set

$$A(c_1,c_2) = \{(x,y) \in \mathbb{R}^2 : c_1 \leq \tfrac{1}{2}y^2 + G(x) \leq c_2\}.$$

Proposition 6. *If $c_1 > 0$ is small enough and $c_2 > c_1$ is large enough, the set $A(c_1,c_2)$ is an annulus with strictly star-shaped boundary curves and, if $z(t) = (x(t),y(t))$ is a regular solution of (5), then*

$$\begin{aligned} \tfrac{1}{2}y(0)^2 + G(x(0)) = c_1 \quad &\Rightarrow \quad \mathrm{Rot}(z,[0,T]) > 1\,, \\ \tfrac{1}{2}y(0)^2 + G(x(0)) = c_2 \quad &\Rightarrow \quad \mathrm{Rot}(z,[0,T]) < 1\,. \end{aligned}$$

Proof. By Proposition 5, the time-map is very small when the regular solution $z(t)$ is near $(\alpha,0)$. Hence, the regular solution will rotate more than once around the point $(\alpha,0)$ in the time interval $[0,T]$. This proves the first implication.

By the same Proposition 5, the time-map is greater than T when the regular solutions have a large amplitude. This implies that in the time interval $[0,T]$ the regular solutions will not have enough time to complete a whole rotation around the point $(\alpha,0)$. The second implication is thus also proved. □

From now on, we consider c_1 and c_2 satisfying the above properties as fixed.

It will be useful to define the function $\mathcal{N} : \,]a,+\infty[\, \times \mathbb{R} \to \mathbb{R}$ as follows:

$$\begin{aligned} &a = -\infty \;\Rightarrow\; \mathcal{N}(x,y) = (x-\alpha)^2 + y^2, \\ &a \in \mathbb{R} \;\Rightarrow\; \mathcal{N}(x,y) = \begin{cases} (x-\alpha)^2 + y^2 & \text{if } x \geq \alpha\,, \\ \dfrac{1}{(x-a)^2} + \dfrac{1}{(x-2\alpha+a)^2} - \dfrac{2}{(\alpha-a)^2} & \text{if } x \in \,]a,\alpha[\,. \end{cases} \end{aligned}$$

Notice that $\mathcal{N}$ is continuously differentiable, $\mathcal{N}(x,y) \geq 0$ for every $(x,y) \in \,]a,+\infty[\, \times \mathbb{R}$, and

$$\mathcal{N}(x,y) = 0 \quad \Leftrightarrow \quad (x,y) = (\alpha,0).$$

It is easy to see that we can choose a constant $\varrho > 1$ such that, for every $(x,y) \in \,]a,+\infty[\, \times \mathbb{R}$,

$$\tfrac{1}{2}y^2 + G(x) \geq c_1 \quad \Rightarrow \quad \mathcal{N}(x,y) > \varrho^{-1}\,, \tag{8}$$

and

$$\tfrac{1}{2}y^2 + G(x) \leq c_2 \quad \Rightarrow \quad \mathcal{N}(x,y) < \varrho\,. \tag{9}$$

Moreover, there is a $\bar{\rho} > 0$ such that

$$\mathcal{N}(x,y) \geq \varrho^{-1} \quad \Rightarrow \quad (x,y) \notin [\alpha-\bar{\rho},\alpha+\bar{\rho}] \times [-\bar{\rho},\bar{\rho}],$$

and there are $\bar{r} \in \,]a,\alpha[$ and $\bar{R} > \alpha$ such that

$$\mathcal{N}(x,y) \leq \varrho \quad \Rightarrow \quad (x,y) \in [\bar{r},\bar{R}] \times [-\bar{R},\bar{R}].$$

3.2. The perturbed equation

Let us write the planar system associated to Eq. (1),

$$x' = y\,, \qquad -y' = g(x) - \varepsilon p(t)\,. \tag{10}$$

We consider the approximating functions $g_n : \,]a,+\infty[\, \to \mathbb{R}$, defined for $n \geq 1$ sufficiently large as

$$g_n(x) = g(x)\,, \quad \text{when } |x-\alpha| \geq \tfrac{1}{n},$$

while, if $|x-\alpha| < \frac{1}{n}$, then

$$g_n(x) = \left(1 - \frac{n(x-\alpha)+1}{2}\right) g\big(\alpha - \tfrac{1}{n}\big) + \frac{n(x-\alpha)+1}{2} g\big(\alpha + \tfrac{1}{n}\big).$$

Precisely, if $a \in \mathbb{R}$, we need to take $n > 1/(\alpha - a)$. We can then deal with the equation

$$x'' + g_n(x) = \varepsilon p(t)\,, \tag{11}$$

with associated system

$$x' = y\,, \qquad -y' = g_n(x) - \varepsilon p(t)\,. \tag{12}$$

Note that for any n the approximating function g_n is continuous in it s domain of definition, hence any solution of (11) and (12) will be a *classical* solution.

Define $G_n(x) = \int_\alpha^x g_n(s)\,ds$, and notice that

$$\lim_n G_n(x) = G(x)\,, \qquad \text{uniformly on }]a,+\infty[\,. \tag{13}$$

Moreover, define the function $V_n : \,]a,+\infty[\, \times \mathbb{R} \to \mathbb{R}$ as

$$V_n(x,y) = \tfrac{1}{2}y^2 + G_n(x) - G_{\min} + 1,$$

where

$$G_{\min} = \min\{G(x) : x \in \,]a,+\infty[\,\}.$$

Taking n large enough, we can assume that

$$V_n(x,y) > \tfrac{1}{2}y^2 + \tfrac{1}{2} \geq \tfrac{1}{2}\,, \qquad \text{for every } (x,y) \in \mathbb{R}^2.$$

Proposition 7. *The solutions of (11) are globally defined.*

Proof. See, e.g., [12,16]. $\square$

Proposition 8. *There exist $\bar{n} \geq 1$ and $\bar{\varepsilon} > 0$ such that, if $n \geq \bar{n}$, $|\varepsilon| \leq \bar{\varepsilon}$ and $(x(t), y(t))$ is a solution of system (12), then*

$$(x(0), y(0)) \in A(c_1, c_2) \quad \Rightarrow \quad \varrho^{-1} < \mathcal{N}(x(t), y(t)) < \varrho\,, \ \textit{for every } t \in [0,T],$$

and, moreover,

$$\begin{aligned} \tfrac{1}{2}y(0)^2 + G(x(0)) = c_1 \quad &\Rightarrow \quad \mathrm{Rot}((x,y),[0,T]) > 1\,, \\ \tfrac{1}{2}y(0)^2 + G(x(0)) = c_2 \quad &\Rightarrow \quad \mathrm{Rot}((x,y),[0,T]) < 1\,. \end{aligned}$$

Proof. Let us prove the first assertion. By contradiction, assume there is a sequence of positive integers $(n_k)_k$ and a sequence $(\varepsilon_k)_k$ in $[-1,1]$ for which there exists a solution $z_k(t) = (x_k(t), y_k(t))$ of (12) with $n = n_k$ and $\varepsilon = \varepsilon_k$ such that, for some $(t_k)_k$ in $[0,T]$,

$$(x_k(0), y_k(0)) \in A(c_1, c_2) \quad \text{and} \quad \mathcal{N}(x_k(t_k), y_k(t_k)) \le \varrho^{-1}.$$

Setting

$$f_k(t) = V_{n_k}(x_k(t), y_k(t)),$$

we have that

$$f_k'(t) = \varepsilon p(t) y_k(t),$$

hence there exists a constant $c' > 0$ for which

$$|f_k'(t)| \le c'(\tfrac{1}{2} y_k(t)^2 + 1) \le c' f_k(t)\,, \qquad \text{for every } t \in [0,T] \text{ and } k \in \mathbb{N}.$$

By the Gronwall Lemma, since $f_k(0) = V_{n_k}(x_k(0), y_k(0)) \le c_2$, there exists a constant $c'' > 0$ for which

$$f_k(t) \le c''\,, \qquad \text{for every } t \in [0,T] \text{ and } k \in \mathbb{N}.$$

So, there exists a constant $c''' > 0$ such that

$$|x_k(t)| + |x_k'(t)| \le c'''\,, \qquad \text{for every } t \in [0,T] \text{ and } k \in \mathbb{N}.$$

By the Ascoli–Arzelà Theorem, there exists a subsequence $(z_{k_j})_j$, with $z_{k_j}(t) = (x_{k_j}(t), y_{k_j}(t))$, which uniformly converges in $[0,T]$ to some continuous function $\bar{z} = (\bar{x}, \bar{y}) : [0,T] \to \mathbb{R}^2$. Then,

$$x_{k_j}(t) \to \bar{x}(t) \ \text{ and } \ x_{k_j}'(t) \to \bar{y}(t)\,, \qquad \text{uniformly in } t \in [0,T].$$

As a consequence, $\bar{x} : [0,T] \to \mathbb{R}$ is differentiable, and $\bar{x}'(t) = \bar{y}(t)$, for every $t \in [0,T]$. Since $(x_{k_j}(0), y_{k_j}(0)) \in A(c_1, c_2)$ and $A(c_1, c_2)$ is closed, we have that

$$(\bar{x}(0), \bar{y}(0)) \in A(c_1, c_2).$$

Moreover, there is a subsequence of $(t_{k_j})_j$, for which we use the same notation, such that $t_{k_j} \to \bar{t}$, for some $\bar{t} \in [0,T]$. Hence, since $\mathcal{N}(x_{k_j}(t_{k_j}), y_{k_j}(t_{k_j})) \le \varrho^{-1}$, passing to the limit we have

$$\mathcal{N}(\bar{x}(\bar{t}), \bar{y}(\bar{t})) \le \varrho^{-1}.$$

On the other hand, multiplying the equation $x_{k_j}'' + g_{k_j}(x) = \varepsilon_{k_j} p(t)$ by x_{k_j}' and integrating, we get

$$\tfrac{1}{2} x_{k_j}'(t)^2 + G_{k_j}(x_{k_j}(t)) = \tfrac{1}{2} x_{k_j}'(0)^2 + G_{k_j}(x_{k_j}(0)) + \varepsilon_{k_j} \int_0^t p(s) x_{k_j}'(s)\, ds.$$

Passing to the limit, by (13), since $(x_{k_j}')_j$ is uniformly bounded, we see that

$$\tfrac{1}{2} \bar{x}'(t)^2 + G(\bar{x}(t)) = \tfrac{1}{2} \bar{x}'(0)^2 + G(\bar{x}(0))\,, \qquad \text{for every } t \in [0,T].$$

In particular, $\frac{1}{2} \bar{y}(\bar{t})^2 + G(\bar{x}(\bar{t})) \ge c_1$, and hence, by (8),

$$\mathcal{N}(\bar{x}(\bar{t}), \bar{y}(\bar{t})) > \varrho^{-1},$$

a contradiction, proving the inequality $\varrho^{-1} < \mathcal{N}(x(t), y(t))$ in the statement. A similar argument proves the inequality $\mathcal{N}(x(t), y(t)) < \varrho$.

Concerning the second part, one proceeds by contradiction, similarly as above, assuming that there is a sequence of positive integers $(n_k)_k$ and a sequence $(\varepsilon_k)_k$ in $[-1,1]$ for which there exists a solution $(x_k(t), y_k(t))$ of (12) such that

$$\tfrac{1}{2}y_k(0)^2 + G(x_k(0)) = c_1 \quad \text{and} \quad \mathrm{Rot}((x_k, y_k), [0,T]) \le 1.$$

(We now know that the rotation number is well defined, by the first part of the proof.) A subsequence of $(x_k, y_k)_k$ converges uniformly on $[0,T]$ to $(\bar{x}, \bar{y})$, a solution of (5). Since $\mathcal{N}(x_k(t), y_k(t)) > \varrho^{-1}$ for every $t \in [0,T]$, passing to the limit we have that

$$\mathcal{N}(\bar{x}(t), \bar{y}(t)) \ge \varrho^{-1}, \qquad \text{for every } t \in [0,T],$$

hence

$$(\bar{x}(t), \bar{y}(t)) \notin [\alpha - \bar{\rho}, \alpha + \bar{\rho}] \times [-\bar{\rho}, \bar{\rho}], \qquad \text{for every } t \in [0,T].$$

As a consequence, $(\bar{x}, \bar{y})$ is a regular solution of (5). Since all these solutions $(\bar{x}, \bar{y})$ and (x_k, y_k) remain at a safe distance from $(\alpha, 0)$, when k is sufficiently large, the angle variable varies continuously, hence,

$$\mathrm{Rot}((\bar{x}, \bar{y}), [0,T]) = \lim_k \mathrm{Rot}((x_k, y_k), [0,T]) \le 1,$$

contradicting Proposition 6. □

3.3. End of the proof

If $a \in \mathbb{R}$, we consider the functions $\tilde{g}_n : \mathbb{R} \to \mathbb{R}$ defined as

$$\tilde{g}_n(x) = \begin{cases} g_n(x) & \text{if } x \ge \bar{r}, \\ g_n(\bar{r}) & \text{if } x < \bar{r}, \end{cases}$$

while, if $a = -\infty$, we set $\tilde{g}_n = g_n$. We thus have the equation

$$x'' + \tilde{g}_n(x) = \varepsilon p(t), \tag{14}$$

with associated system

$$x' = y, \qquad -y' = \tilde{g}_n(x) - \varepsilon p(t). \tag{15}$$

Proposition 9. *The same statement of Proposition 8 holds for system (15), as well.*

Proof. The solutions (x, y) of (15) starting at time $t = 0$ from $\mathcal{A}(c_1, c_2)$ remain solutions of (12) as long as $x(t) \ge \bar{r}$. By Proposition 8,

$$\mathcal{N}(x(t), y(t)) < \varrho, \text{ for every } t \in [0,T],$$

hence

$$(x(t), y(t)) \in [\bar{r}, \bar{R}] \times [-\bar{R}, \bar{R}], \text{ for every } t \in [0,T].$$

Then, the solutions (x, y) of (15) starting at time $t = 0$ from $\mathcal{A}(c_1, c_2)$ remain solutions of (12) for every $t \in [0,T]$. The conclusion follows. □

By the Poincaré–Birkhoff Theorem (see, e.g., [17, Theorem 1.2]), if $n \ge \bar{n}$ and $|\varepsilon| \le \bar{\varepsilon}$, there exists a T-periodic solution $z_n^\varepsilon(t) = (x_n^\varepsilon(t), y_n^\varepsilon(t))$ of (15) such that

$$(x_n^\varepsilon(0), y_n^\varepsilon(0)) \in A(c_1, c_2) \quad \text{and} \quad \mathrm{Rot}((x_n^\varepsilon, y_n^\varepsilon), [0,T]) = 1. \tag{16}$$

Moreover, by Proposition 8,

$$\varrho^{-1} < \mathcal{N}(x_n^\varepsilon(t), y_n^\varepsilon(t)) < \varrho\,, \qquad \text{for every } t \in [0,T]\,, \tag{17}$$

hence

$$(x_n^\varepsilon(t), y_n^\varepsilon(t)) \in [\bar r, \bar R] \times [-\bar R, \bar R]\,, \qquad \text{for every } t \in [0,T].$$

As a consequence, $(x_n^\varepsilon(t), y_n^\varepsilon(t))$ is a solution of (12). Let us now fix ε such that $|\varepsilon| \le \bar\varepsilon$, and consider the sequence $(z_n^\varepsilon)_n$ in $C([0,T],\mathbb{R}^2)$. Setting

$$f_n(t) = V_n(x_n^\varepsilon(t), y_n^\varepsilon(t)),$$

we have that

$$f_n'(t) = \varepsilon p(t) y_n^\varepsilon(t)\,, \qquad \text{for every } t \in [0,T] \text{ and } n \in \mathbb{N},$$

hence there exists a constant $c' > 0$ for which

$$|f_n'(t)| \le c'\big(\tfrac12 y_n^\varepsilon(t)^2 + \tfrac12\big) \le c' f_n(t)\,, \qquad \text{for every } t \in [0,T] \text{ and } n \in \mathbb{N}.$$

By the Gronwall Lemma, since $f_n(0) = V_n(x_n^\varepsilon(0), y_n^\varepsilon(0)) \le c_2$, there exists a constant $c'' > 0$ for which

$$f_n(t) \le c''\,, \qquad \text{for every } t \in [0,T] \text{ and } n \in \mathbb{N}.$$

So, there exists a constant $c''' > 0$ such that

$$|x_n^\varepsilon(t)| + |(x_n^\varepsilon)'(t)| \le c'''\,, \qquad \text{for every } t \in [0,T] \text{ and } n \in \mathbb{N}.$$

By the Ascoli–Arzelà Theorem, there exists a subsequence $(z_{n_k}^\varepsilon)_k$, with $z_{n_k}^\varepsilon(t) = (x_{n_k}^\varepsilon(t), y_{n_k}^\varepsilon(t))$, which uniformly converges in $[0,T]$ to some function $\bar z^\varepsilon = (\bar x^\varepsilon, \bar y^\varepsilon) \in C([0,T],\mathbb{R}^2)$. Then,

$$x_{n_k}^\varepsilon(t) \to \bar x^\varepsilon(t) \;\text{ and }\; (x_{n_k}^\varepsilon)'(t) \to \bar y^\varepsilon(t)\,, \qquad \text{uniformly in } t \in [0,T].$$

As a consequence, $\bar x^\varepsilon \in C^1([0,T],\mathbb{R})$ and $(\bar x^\varepsilon)'(t) = \bar y^\varepsilon(t)$, for every $t \in [0,T]$.

We now extend $\bar x^\varepsilon$ by T-periodicity to the whole real line $\mathbb{R}$ and show that $\bar x^\varepsilon : \mathbb{R} \to \mathbb{R}$ is the regular solution we are looking for. First, notice that $\bar x^\varepsilon : \mathbb{R} \to \mathbb{R}$ is still continuously differentiable, since $\bar y^\varepsilon(0) = \bar y^\varepsilon(T)$. By (16), passing to the limit we have that

$$(\bar x^\varepsilon(0), \bar y^\varepsilon(0)) \in A(c_1, c_2) \qquad \text{and} \qquad \mathrm{Rot}((\bar x^\varepsilon, \bar y^\varepsilon), [0,T]) = 1.$$

Moreover, passing to the limit in (17) yields

$$\varrho^{-1} \le \mathcal{N}(\bar x^\varepsilon(t), \bar y^\varepsilon(t)) \le \varrho\,, \qquad \text{for every } t \in [0,T],$$

hence

$$(\bar x^\varepsilon(t), \bar y^\varepsilon(t)) \notin [\alpha - \bar\rho, \alpha + \bar\rho] \times [-\bar\rho, \bar\rho]\,, \qquad \text{for every } t \in [0,T].$$

Hence, the function $\bar x^\varepsilon(\cdot) - \alpha$ only has isolated zeros; they are precisely two in the interval $[0,T[$. Denote the zeros of $\bar x^\varepsilon(\cdot) - \alpha : \mathbb{R} \to \mathbb{R}$, in strictly increasing order, by $(t_j)_{j\in\mathbb{Z}}$, i.e.

$$\dots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \dots$$

Define $F_n : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ as

$$F_n(t,x,y) = (y, -g_n(x) + \varepsilon p(t)).$$

We know that, for every $t \in \mathbb{R}$,

$$z^{\varepsilon}_{n_k}(t) = z^{\varepsilon}_{n_k}(0) + \int_0^t F_{n_k}(s, z^{\varepsilon}_{n_k}(s))\, ds\,. \tag{18}$$

Let us fix $t > 0$ and consider the sequence of functions $\mathcal{F}_k : [0, t] \to \mathbb{R}^2$ defined as

$$\mathcal{F}_k(s) = F_{n_k}(s, z^{\varepsilon}_{n_k}(s)) = (y^{\varepsilon}_{n_k}(s), -g_{n_k}(x^{\varepsilon}_{n_k}(s)) + \varepsilon p(s)).$$

We see that, for any $s \in [0, t]$,

$$\bar{x}^{\varepsilon}(s) \neq \alpha \quad \Rightarrow \quad \lim_k \mathcal{F}_k(s) = (\bar{y}^{\varepsilon}(s), -g(\bar{x}^{\varepsilon}(s)) + \varepsilon p(s)),$$

Since the set $\{s \in [0, t] : \bar{x}^{\varepsilon}(s) = \alpha\}$ has zero measure, we have that

$$\lim_k \mathcal{F}_k(s) = (\bar{y}^{\varepsilon}(s), -g(\bar{x}^{\varepsilon}(s)) + \varepsilon p(s))\,, \qquad \text{for a.e. } s \in [0, t].$$

Moreover, by $A1$ and the uniform convergence of $(x^{\varepsilon}_{n_k})_k$ and $(y^{\varepsilon}_{n_k})_k$ there is a constant $\overline{C} > 0$ such that

$$|\mathcal{F}_k(s)| \leq \overline{C}\,, \qquad \text{for every } s \in [0, t].$$

Passing to the limit in (18), the Lebesgue's Dominated Convergence Theorem then yields

$$(\bar{x}^{\varepsilon}(t), \bar{y}^{\varepsilon}(t)) = (\bar{x}^{\varepsilon}(0), \bar{y}^{\varepsilon}(0)) + \int_0^t (\bar{y}^{\varepsilon}(s), -g(\bar{x}^{\varepsilon}(s)) + \varepsilon p(s))\, ds.$$

Now, for every $j \in \mathbb{Z}$, we can use the Fundamental Theorem on the interval $]t_{j-1}, t_j[$ and see that

$$((\bar{x}^{\varepsilon})'(t), (\bar{y}^{\varepsilon})'(t)) = (\bar{y}^{\varepsilon}(t), -g(\bar{x}^{\varepsilon}(t)) + \varepsilon p(t))\,, \qquad \text{for every } t \in\ ]t_{j-1}, t_j[,$$

i.e.,

$$(\bar{x}^{\varepsilon})''(t) + g(\bar{x}^{\varepsilon}(t)) = \varepsilon p(t)\,, \qquad \text{for every } t \in\ ]t_{j-1}, t_j[.$$

Finally, for every $j \in \mathbb{Z}$, both limits

$$\lim_{t \to t_j^-} (\bar{x}^{\varepsilon})''(t) \quad \text{and} \quad \lim_{t \to t_j^+} (\bar{x}^{\varepsilon})''(t)$$

exist and they are equal to either $\varepsilon p(t_j) - \ell_-$, or $\varepsilon p(t_j) - \ell_+$.

We have thus proved that $\bar{x}^{\varepsilon}$ is a regular T-periodic solution of (1), such that $\bar{x}^{\varepsilon}(\cdot) - \alpha$ has exactly two zeros in $[0, T[$. □

Remark 10. We have proved Theorem 2 in the case $k = m = 1$. The general case is treated in the same way, since [17, Theorem 1.2] still applies, with T replaced by mT.

4. Conclusions

A racing car is subject to the gravitational force, to the shock-absorber force which contrasts it, and to some ground effects. The so called *porpoising effect* is due to the fact that the car, when traveling with a sufficiently high speed, is attracted towards the ground by a force which increases as the distance from the ground decreases until, at a certain critical distance α, it suddenly stalls. The car then goes back to its initial height and the process starts again, thus producing a rocking motion (which may recall a porpoise diving into and out of the sea as it swims). This phenomenon results in giving an extremely unpleasant ride of the car.

We have built a mathematical model of this situation, and we have shown that adding a small periodic perturbation may produce large-amplitude *subharmonic* solutions. Denoting by T the period of the perturbation, Theorem 2 provides the existence of subharmonic solutions of any order, since the integer ℓ in assumption A4 is equal to zero in this situation.

A possible origin of this small periodic perturbation could come from some asymmetries of the wheels. A Formula 1 car's wheels have an 18-inches radius. Hence, at the speed of 200 km/h, they make about 20 rotations per second. Analyzing the car driven by Lewis Hamilton in the first races in 2022 (e.g., in https://www.youtube.com/watch?v=ubc25nMV3bE) one sees that his car oscillates approximately 5 times per second. Hence, these oscillations could be interpreted as a subharmonic response to the periodic forcing given by the wheels. In this case, $T = 0.05$ seconds, and the car's periodic oscillations are of minimal period $4T$.

Beyond the concrete model under study, mathematically speaking the presence of subharmonic solutions is a consequence of the jump discontinuity at point α. This is clear if we realize that Theorem 2 is directly applicable to the simple model

$$x'' + \mathrm{sgn}(x) = \varepsilon p(t).$$

This equation was studied in [8] under a symmetry assumption on the forcing term, which enable the use of a shooting method. Our result is new even for this simple case.

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