A Lagrangian approach for the density-dependent incompressible Navier-Stokes equations

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The inhomogeneous incompressible Navier-Stokes equations

\[
\begin{aligned}
\partial_t \rho + u \cdot \nabla \rho &= 0 \quad \text{(Mass equation)} \\
\rho(\partial_t u + u \cdot \nabla u) - \text{div} (\mu(\rho)(D u + T D u)) + \nabla P &= \rho f \quad \text{(Momentum equation)} \\
\text{div} u &= 0 \quad \text{(Incompressibility)}
\end{aligned}
\]
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\partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0 \\
\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \text{div} (\mu(\rho)(\mathbf{D} \mathbf{u} + \mathbf{T} \mathbf{u})) + \nabla P &= \rho \mathbf{f} \\
\text{div} \mathbf{u} &= 0
\end{cases}
\end{aligned}
\]

\begin{align*}
\text{(Mass equation)} & \quad \text{(Momentum equation)} \\
\text{(Incompressibility)}
\end{align*}

- Density: $\rho = \rho(t, x) \in \mathbb{R}^+$
- Velocity: $\mathbf{u} = \mathbf{u}(t, x) \in \mathbb{R}^n$
- Pressure: $P = P(t, x) \in \mathbb{R}$
- Viscosity: $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ given
- Space variable: $x \in \mathbb{R}^n$
- Time variable: $t \in \mathbb{R}^+$
- Data: $\rho_0 = \rho(x) \in \mathbb{R}^+, \ f = f(t, x) \in \mathbb{R}^n$, $\rho_0$ goes to $\bar{\rho}$ at infinity.
- Boundary conditions: $\mathbf{u}_0$ goes to 0 and $\rho_0$ goes to $\bar{\rho}$ at infinity.

We shall take $\bar{\rho} = 1$, $\ f \equiv 0$ and $\mu = \text{cste}$ to simplify the presentation.
Historical background

With the above assumptions, the considered equations read

\[
\begin{align*}
\partial_t \rho + u \cdot \nabla \rho &= 0 \\
\rho (\partial_t u + u \cdot \nabla u) - \mu \Delta u + \nabla P &= 0 \\
\text{div } u &= 0.
\end{align*}
\]

- The strong solution theory (in bounded domains with different types of boundary conditions) has been carried out by Ladyzhenskaya and Solonnikov (see also the book by Antontsev, Kazhikhov and Monakhov).
- Global weak solutions with finite energy: see Lions et al.
- ...
The homogeneous case $\rho \equiv 1$

This corresponds to the classical incompressible Navier-Stokes equations:

\[
\begin{aligned}
(NS): \quad & \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla P = 0 \\
& \text{div} \, u = 0.
\end{aligned}
\]
The homogeneous case $\rho \equiv 1$

This corresponds to the classical incompressible Navier-Stokes equations:

\[
\begin{align*}
(NS) : \quad \frac{\partial u}{\partial t} + u \cdot \nabla u - \mu \Delta u + \nabla P &= 0, \\
\text{div } u &= 0.
\end{align*}
\]

The system is invariant by the rescaling:

\[
(u(t, x), P(t, x)) \rightarrow (\lambda u(\lambda^2 t, \lambda x)), \lambda^2 P(\lambda^2 t, \lambda x))
\]

which corresponds to the following transformation for the initial data:

\[
u_0(x) \rightarrow \lambda u_0(\lambda x).
\]

Functional spaces with the above invariance are suitable for solving $(NS)$ globally by means of the Banach fixed point theorem, if the data $u_0$ is small in a space $E_0$ with norm invariant by $(2)$. 
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Functional spaces with the above invariance are suitable for solving $(NS)$ globally by means of the Banach fixed point theorem, if the data $u_0$ is small in a space $E_0$ with norm invariant by (2).

Examples of critical spaces for the initial data:

- $E_0 = \dot{H}^{n-1}_{\frac{n}{2}}(\mathbb{R}^n)$ (Fujita and Kato, 1964);
- $E_0 = L^n(\mathbb{R}^n)$ (Giga-Miyakawa, Kato, 84 Furioli-Lemarié-Terraneo, 98);
- $E_0 = \dot{B}^{n-1}_{p,1}(\mathbb{R}^n)$ and more general Besov spaces (Cannone-Meyer-Planchon, Kozono-Yamazaki, 94).
The homogeneous case $\rho \equiv 1$

This corresponds to the classical incompressible Navier-Stokes equations:

$$(NS) : \begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla P = 0 \\
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**Examples of critical spaces for the initial data:**

- $E_0 = \dot{H}^{n/2-1}(\mathbb{R}^n)$ (Fujita and Kato, 1964);
- $E_0 = L^n(\mathbb{R}^n)$ (Giga-Miyakawa, Kato, 84 Furioli-Lemarié-Terraneo, 98);
- $E_0 = \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ and more general Besov spaces (Cannone-Meyer-Planchon, Kozono-Yamazaki, 94).

If one chooses for $X$ a space with a negative index of regularity then $\|u_0\|_X$ may be small if it has fast oscillations even though it has large modulus.

**Example:** Let $p$ satisfy $-1 + n/p < 0$ and $u_0^\varepsilon : x \mapsto \phi(x) \sin(\varepsilon^{-1} x \cdot \omega) \vec{n}$ with $\omega$ and $\vec{n}$ in $S^{n-1}$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\|u_0^\varepsilon\|_{\dot{B}^{\frac{n}{p}-1}_{p,r}} \leq C\varepsilon^{1 - \frac{n}{p}}.$$
The homogeneous case $\rho \equiv 1$

This corresponds to the classical incompressible Navier-Stokes equations:

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$$\|u_0^\varepsilon\|_{\dot{B}^{p,-1}_{p,r}} \leq C\varepsilon^{1-\frac{n}{p}}.$$  

**Aim of the talk:** Solving $(NSI)$ in critical spaces with negative index by means of the Banach fixed point theorem.
A critical functional framework for (NSI)

\[
\begin{aligned}
\rho(t, x) &\rightarrow \rho(\lambda^2 t, \lambda x), \\
u(t, x) &\rightarrow \lambda u(\lambda^2 t, \lambda x), \\
P(t, x) &\rightarrow \lambda^2 P(\lambda^2 t, \lambda x).
\end{aligned}
\]

Scaling invariance of the system:

Loosely speaking, this means that one more derivative is required for \(\rho_0\) : it should be taken in a space with the same scaling invariance as \(L^\infty\).
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Loosely speaking, this means that one more derivative is required for \( \rho_0 \) : it should be taken in a space with the same scaling invariance as \( L^\infty \).

System (NSI) is a nonlinear coupling between:

- **A transport equation**: Lipschitz regularity needed to preserve the initial regularity.
- **The Stokes system**: parabolic-like gain of regularity over the velocity (maximal regularity estimates).
The model
The Eulerian approach
The Lagrangian approach

Linear estimates

Stokes system:

\[(S): \begin{cases} 
\partial_t u - \mu \Delta u + \nabla P = f \\
\text{div } u = 0.
\end{cases}\]

Given \(f \in L^1(\mathbb{R}^+; \dot{B}^s_p,1)\) and \(u_0 \in \dot{B}^s_p\), System \((S)\) has a unique solution \((u, \nabla P)\) in

\[E^s_p := \left\{ (u, \nabla P) : u \in C_b(\mathbb{R}^+; \dot{B}^s_p) \text{ with } (\partial_t u, \mu \nabla^2 u, \nabla P) \in L^1(\mathbb{R}^+; \dot{B}^s_p,1) \right\}\]

and we have

\[
\|u\|_{L^\infty(\dot{B}^s_p,1)} + \|\partial_t u, \mu \nabla^2 u, \nabla P\|_{L^1(\dot{B}^s_p,1)} \lesssim \|u_0\|_{\dot{B}^s_p} + \|f\|_{L^1(\dot{B}^s_p,1)}.
\] (1)
Linear estimates

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\[ \|u\|_{L^\infty(\dot{B}^s_{p,1})} + \|\partial_t u, \mu \nabla^2 u, \nabla P\|_{L^1(\dot{B}^s_{p,1})} \lesssim \|u_0\|_{\dot{B}^s_{p,1}} + \|f\|_{L^1(\dot{B}^s_{p,1})}. \quad (1) \]

Transport equation: \((T) : \partial_t a + u \cdot \nabla a = 0.\)

Given \( a_0 \in \dot{B}^s_{p,1} \) with \(-n/p' - 1 < s \leq n/p + 1\), Equation (T) has a unique solution \( a \) such that.

\[ \|a(t)\|_{\dot{B}^s_{p,1}} \leq \|a_0\|_{\dot{B}^s_{p,1}} \exp\left( \int_0^t \|\nabla u\|_{\dot{B}^{n/p}_{p,1}} \, d\tau \right). \quad (2) \]
Linear estimates

Stokes system:

\[ (S) : \begin{cases} \partial_t u - \mu \Delta u + \nabla P = f \\ \text{div} u = 0. \end{cases} \]

Given \( f \in L^1(\mathbb{R}^+; \dot{B}^s_{p,1}) \) and \( u_0 \in \dot{B}^s_{p,1} \), System \((S)\) has a unique solution \((u, \nabla P)\) in

\[ E^s_p := \left\{(u, \nabla P) : u \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^s_{p,1}) \text{ with } (\partial_t u, \mu \nabla^2 u, \nabla P) \in L^1(\mathbb{R}^+; \dot{B}^s_{p,1}) \right\} \]

and we have

\[ \|u\|_{L^\infty(\dot{B}^s_{p,1})} + \|\partial_t u, \mu \nabla^2 u, \nabla P\|_{L^1(\dot{B}^s_{p,1})} \lesssim \|u_0\|_{\dot{B}^s_{p,1}} + \|f\|_{L^1(\dot{B}^s_{p,1})}. \] (1)

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\[ \|a(t)\|_{\dot{B}^s_{p,1}} \leq \|a_0\|_{\dot{B}^s_{p,1}} \exp \left( \int_0^t \|\nabla u\|_{\dot{B}^n_{p,1}} \, d\tau \right). \] (2)

A good space for \( u_0 \) is the homogeneous Besov space \( \dot{B}^{n/p-1}_{p,1} \) because \((1)\) implies that \( \nabla u \in L^1(\mathbb{R}^+; \dot{B}^{n/p}_{p,1}) \). So, from \((2)\) we gather that the \( \dot{B}^{n/p}_{p,1} \) regularity for the density is going to be conserved, too.
About the proof of existence

We assume that the following smallness condition is satisfied:

$$\|\rho_0 - 1\|_{\dot{B}^{n/p-1}_{p,1}} + \mu^{-1}\|u_0\|_{\dot{B}^{n/p-1}_{p,1}} \leq c.$$ 

Step 1 : A priori estimates in large norm. We have to estimate \((\rho - 1)\) in \(L^\infty(\mathbb{R}; \dot{B}^{n/p}_{p,1})\) and \((u, \nabla P)\) in \(E^{n/p-1}_p\).

The main ingredients are the above estimates for the transport and Stokes equations and product estimates : we use that \(\dot{B}^{n/p}_{p,1}\) is a Banach algebra and that the product maps

$$\dot{B}^{n/p}_{p,1} \times \dot{B}^{n/p-1}_{p,1} \rightarrow \dot{B}^{n/p-1}_{p,1} \quad \text{if} \quad 1 \leq p < 2n.$$
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\(\dot{B}^{n/p}_{p,1} \times \dot{B}^{n/p-1}_{p,1} \rightarrow \dot{B}^{n/p-1}_{p,1}\) if \(1 \leq p < 2n\).

Step 2 : Stability estimates in small norm. The difference \(\delta \rho := \rho_2 - \rho_1,\) 
\(\delta u := u_2 - u_1\) and \(\nabla \delta P := \nabla P_2 - \nabla P_1\) between two solutions satisfies

\[
\begin{cases}
\partial_t \delta \rho + u_2 \cdot \nabla \delta \rho = -\delta u \cdot \nabla \rho_1 \\
\rho_2 (\partial_t \delta u + u_2 \cdot \nabla \delta u) - \mu \Delta \delta u + \nabla \delta P = -\delta \rho (\partial_t u_1 + (\rho_2 u_2 - \rho_1 u_1) \cdot \nabla u_1)
\end{cases}
\]

\(\implies\) loss of one derivative in the stability estimates.
We need to use that the product maps 
\(\dot{B}^{n/p-1}_{p,1} \times \dot{B}^{n/p-1}_{p,1} \rightarrow \dot{B}^{n/p-2}_{p,1}\).
But this is true if and only if \(1 \leq p < n\) and \(n \geq 2\).
Theorem (D. 03, Abidi 06)

Let $1 \leq p < 2n$. Let $u_0 \in \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)$ with $\text{div} u_0 = 0$. Assume that $\rho_0 - 1$ belongs to $\dot{B}^{n/p}_{p,1}$. There exists a constant $c$ depending only on $p$ and on $n$ such that if

$$\|\rho_0 - 1\|_{\dot{B}^{n/p}_{p,1}} + \mu^{-1} \|u_0\|_{\dot{B}^{n/p-1}_{p,1}} \leq c$$

(3)

then System $(NSI)$ has:

- a global solution $(\rho, u, \nabla P)$ with $(\rho - 1) \in C(\mathbb{R}^+; \dot{B}^{n/p}_{p,1})$ and $(u, \nabla P) \in E^{n/p}_p$
- this solution is unique if $1 \leq p \leq n$. 
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Good news:

- Local-in-time statement if $u_0$ is large and $\rho_0$ is bounded away from 0.
- This statement extends to the case $\mu = \mu(\rho)$.
- One may take different Lebesgue exponents for $\rho$ and $u$ (Abidi-Paicu 07).
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- This statement extends to the case $\mu = \mu(\rho)$.
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Bad news:

- Loss of one derivative in the stability estimates.
- The condition $p \leq n$ precludes us from considering $u_0$ in Besov spaces with negative indices.
- The regularity assumption for $\rho_0$ implies uniform continuity.
Goal of the talk

- **Improve stability estimates**: Proving existence by means of Banach fixed point theorem. *Will improve the uniqueness condition and yield the continuity of the flow map.*

- **Weaken the conditions over the density**: *Is it possible to consider discontinuities across an interface?*
Lagrangian change of coordinates

Flow of $u = u(t, x)$:

$$X_u(t, y) = y + \int_0^t u(\tau, X_u(\tau, y)) \, d\tau$$
Lagrangian change of coordinates

Flow of $u = u(t, x)$:

$$X_u(t, y) = y + \int_0^t u(\tau, X_u(\tau, y)) \, d\tau = y + \int_0^t \bar{u}(\tau, y) \, d\tau.$$ 

Change of coordinates: $(t, x) \mapsto (t, y)$ with $x = X_u(t, y)$.

$$\bar{u}(t, y) = u(t, x),$$
$$\bar{P}(t, y) = P(t, x).$$
Flow of $u = u(t, x)$:

$$X_u(t, y) = y + \int_0^t u(\tau, X_u(\tau, y)) \, d\tau = y + \int_0^t \tilde{u}(\tau, y) \, d\tau.$$ 

Change of coordinates: $(t, x) \longrightarrow (t, y)$ with $x = X_u(t, y)$.

$$\tilde{u}(t, y) = u(t, x),$$

$$\tilde{P}(t, y) = P(t, x).$$

Chain rule:

$$\nabla_y \tilde{P} = \nabla_y X_u \cdot \nabla_x P.$$ 

Hence the divergence-free condition recasts in

$$\text{div}_y \tilde{u} = g := D_y \tilde{u} : (\text{Id} - A) \quad \text{with} \quad A := (D_y X_u)^{-1}.$$ 

In general, $\text{div} \tilde{u}$ need not be 0 for $t > 0$. 

---

Lagrangian change of coordinates
The generalized Stokes equations

The linearized momentum equation now reads:

\[
(S): \begin{cases}
\partial_t u - \mu \Delta u + \nabla P = f \\
\text{div} u = g.
\end{cases}
\]

Set \( u = v + w \) with \( w \) s.t. \( \text{div} w = g \). One can take (formally) \( w = -\nabla (-\Delta)^{-1} g \).

Then \( v \) has to satisfy

\[
\partial_t v - \mu \Delta v + \nabla P = f - \nabla (-\Delta)^{-1} \partial_t g + \mu \nabla g.
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The generalized Stokes equations

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Then \( v \) has to satisfy

\[ \partial_t v - \mu \Delta v + \nabla P = f - \nabla (-\Delta)^{-1} \partial_t g + \mu \nabla g. \]

Needed conditions for \( g \):

- \( \nabla g \in L^1(\mathbb{R}^+; \dot{B}^{s}_{p,1}) \)
- \( \partial_t g = \text{div} R \) with \( R \in L^1(\mathbb{R}^+; \dot{B}^{s}_{p,1}) \).

If so, then we get

\[ \| (u, \nabla P) \|_{E^s_p} \lesssim \| u_0 \|_{\dot{B}^{s}_{p,1}} + \| f \|_{L^1(\dot{B}^{s}_{p,1})} + \mu \| \nabla g \|_{L^1(\dot{B}^{s}_{p,1})} + \| R \|_{L^1(\dot{B}^{s}_{p,1})}. \]

We are interested in the case \( s = n/p - 1 \).
Recall that $g = D_y \tilde{u} : (\text{Id} - A)$ with $A = (D_y X_u)^{-1}$ and that

$$D_y X_u(t) - \text{Id} = \int_0^t D\tilde{u}(\tau) \, d\tau \in \dot{B}^{n/p}_{p,1}.$$
Recall that \( g = D_y \bar{u} : (\text{Id} - A) \) with \( A = (D_y X_u)^{-1} \) and that

\[
D_y X_u(t) - \text{Id} = \int_0^t D\bar{u}(\tau) \, d\tau \in \dot{B}^{n/p}_{p,1}.
\]

As \( \dot{B}^{n/p}_{p,1} \) is a Banach algebra, if the red term is small enough then one may write

\[
A = (\text{Id} + (D_y X_u - \text{Id})) = \sum_{k=0}^{+\infty} (-1)^k \left( \int_0^t D\bar{u} \, d\tau \right)^k.
\]

Hence

\[
\|\text{Id} - A(t)\|_{\dot{B}^{n/p}_{p,1}} \lesssim \|D\bar{u}\|_{L^1(0,t;\dot{B}^{n/p}_{p,1})},
\]

whence

\[
\|g\|_{L^1(0,t;\dot{B}^{n/p}_{p,1})} \lesssim \|D\bar{u}\|_{L^1(0,t;\dot{B}^{n/p}_{p,1})}^2.
\]
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Hence

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\| \text{Id} - A(t) \|_{\dot{B}^{n/p}_{p,1}} \lesssim \| D\bar{u} \|_{L^1(0,t;\dot{B}^{n/p}_{p,1})},
\]

whence

\[
\| g \|_{L^1(0,t;\dot{B}^{n/p}_{p,1})} \lesssim \| D\bar{u} \|^2_{L^1(0,t;\dot{B}^{n/p}_{p,1})}.
\]

Do we have \( \partial_t g = \text{div} \, R \) with \( R \in L^1(\mathbb{R}^+; \dot{B}^{n/p-1}_{p,1}) \)?
Estimates for $g$ (continued)

**Key observation:** The flow $X_u$ is measure preserving hence

$$\text{div}_x u = D_y \bar{u} : A = \text{div}_y (A\bar{u}).$$

Hence

$$\partial_t g = \text{div} R \quad \text{with} \quad R = -\partial_t A \bar{u} + (\text{Id} - A) \partial_t \bar{u}.$$
Estimates for $g$ (continued)

**Key observation**: The flow $X_u$ is measure preserving hence

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Hence

$$\partial_t g = \text{div } R \quad \text{with} \quad R = -\partial_t A \bar{u} + (\text{Id} - A) \partial_t \bar{u}.$$

Under the same smallness condition as in the previous slide, one can write

$$\partial_t A = D \bar{u} \sum_{k \geq 1} (-1)^k \left( \int_0^t D \bar{u} \, d\tau \right)^{k-1}.$$

So finally, if $1 \leq p < 2n$ then we easily gather from product laws that $R \in L^1(\mathbb{R}^+; \dot{B}_{p,1}^{n/p-1})$ and

$$\|R\|_{L^1(\dot{B}_{p,1}^{n/p-1})} \lesssim \|D \bar{u}\|_{L^1(\dot{B}_{p,1}^{n/p})} (\|\bar{u}\|_{L^\infty(\dot{B}_{p,1}^{n/p-1})} + \|\partial_t \bar{u}\|_{L^1(\dot{B}_{p,1}^{n/p-1})}).$$
Important fact: In Lagrangian coordinates $\rho_0$ is time-independent, hence no loss of derivatives in the stability estimates!

As for the velocity, we have

$$\begin{cases}
\rho_0 \partial_t \bar{u} - \mu \text{div}_y (A_u T A_u \nabla_y \bar{u}) + T A_u \nabla_y \bar{P} = 0 \\
\text{div}_y (A_u \bar{u}) = 0.
\end{cases}$$

with $A_u = (D_y X_u)^{-1}$
A priori estimates

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This equation rewrites

$$\begin{cases}
\partial_t \bar{u} - \mu \Delta \bar{u} + \nabla_y \bar{P} = (1 - \rho_0) \partial_t \bar{u} + \mu \text{div}_y ((A_u^T A_u - \text{Id}) \nabla_y \bar{u}) + (\text{Id} - T A_u) \nabla_y \bar{P} \\
\text{div}_y \bar{u} = g := \text{div}_y ((\text{Id} - A_u) \bar{u}) = D\bar{u} : (\text{Id} - A_u).
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From the above estimates for \( g \), \( A_u \) and for the Stokes equations, we thus get

\[
U(t) \lesssim \|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + U(t)^2 + \int_0^t \| (1 - \rho_0) \partial_t \bar{u} \|_{\dot{B}^{n/p-1}_{p,1}} d\tau
\]

with \( U(t) := \| \bar{u} \|_{L^\infty(0,t;\dot{B}^{n/p-1}_{p,1})} + \| \partial_t \bar{u}, \mu D^2 \bar{u}, \nabla \bar{P} \|_{L^1(0,t;\dot{B}^{n/p-1}_{p,1})} \).
A priori estimates

This equation rewrites

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From the above estimates for \(g, A_u\) and for the Stokes equations, we thus get

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Let \(\mathcal{M}(\dot{B}^{n/p-1}_{p,1})\) be the multiplier space for \(\dot{B}^{n/p-1}_{p,1}\). By definition,

\[
\|(1 - \rho_0) \partial_t \bar{u}\|_{\dot{B}^{n/p-1}_{p,1}} \leq (1 - \rho_0) \|\mathcal{M}(\dot{B}^{n/p-1}_{p,1}) \| \|\partial_t \bar{u}\|_{\dot{B}^{n/p-1}_{p,1}}.
\]

So we just need \(\|(1 - \rho_0)\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1})} \ll 1\) and \(\|u_0\|_{\dot{B}^{n/p-1}_{p,1}} \ll \mu\) to close the estimates.
Implementing the fixed point argument

Let $\Phi : (\bar{v}, \nabla \bar{Q}) \mapsto (\bar{u}, \nabla \bar{P})$ where $(\bar{u}, \nabla \bar{P})$ stands for the solution to the linear system

$$\begin{cases} 
\rho_0 \partial_t \bar{u} - \mu \text{div} (A_v T A_v \nabla \bar{u}) + T A_v \nabla \bar{P} = 0 \\
\text{div} (A_v \bar{u}) = 0,
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with $A_v := (DX_v)^{-1}$ and $X_v(y) := y + \int_0^t \bar{v}(\tau, y) \, d\tau$. 
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**Step 1.** Existence of $\Phi$.

If $(\bar{v}, \nabla \bar{Q})$ belongs to a small ball $B_R$ of $E_{n/p}$ and $X_v$ is measure preserving in the “original” Eulerian coordinates then the previous slide implies that the same holds for $(\bar{u}, \nabla \bar{P})$.

**Important:** the corresponding set $\mathcal{E}_R$ is a closed subset of $E_{n/p}$.
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**Step 1. Existence of $\Phi$.**

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**Step 2. Contraction estimates for $\Phi$.**

One just has to write $\Phi(\vec{v}_2, \nabla \vec{Q}_2) - \Phi(\vec{v}_1, \nabla \vec{Q}_1)$ as a solution to the Stokes estimate and slightly generalize the previous estimates. No loss of derivative here!

Applying the Banach fixed point theorem allows to conclude to the existence of a solution in $\mathcal{E}_R$. 
Implementing the fixed point argument

Let $\Phi : (\bar{v}, \nabla \bar{Q}) \mapsto (\bar{u}, \nabla \bar{P})$ where $(\bar{u}, \nabla \bar{P})$ stands for the solution to the linear system

$$\begin{cases}
    \rho_0 \partial_t \bar{u} - \mu \text{div} \left( A_v T A_v \nabla \bar{u} \right) + T A_v \nabla \bar{P} = 0 \\
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**Step 1.** Existence of $\Phi$.

If $(\bar{v}, \nabla \bar{Q})$ belongs to a small ball $B_R$ of $E_p^{n/p}$ and $X_v$ is measure preserving in the “original” Eulerian coordinates then the previous slide implies that the same holds for $(\bar{u}, \nabla \bar{P})$.

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One just has to write $\Phi(\bar{v}_2, \nabla \bar{Q}_2) - \Phi(\bar{v}_1, \nabla \bar{Q}_1)$ as a solution to the Stokes estimate and slightly generalize the previous estimates. No loss of derivative here!

**Applying the Banach fixed point theorem allows to conclude to the existence of a solution in $\mathcal{E}_R$.**

**Step 3.** Uniqueness. This is a straightforward modification of Step 2.
Theorem

Let \( p \in [1, 2n) \) and \( u_0 \) be a divergence-free vector field in \( \dot{B}_{p,1}^{n/p-1}(\mathbb{R}^n) \). Assume that the initial density \( \rho_0 \) belongs to the multiplier space \( \mathcal{M}(\dot{B}_{p,1}^{n/p-1}) \). There exists a constant \( c \) depending only on \( p \) and on \( n \) such that if

\[
\|\rho_0 - 1\|_{\mathcal{M}(\dot{B}_{p,1}^{n/p-1})} + \mu^{-1}\|u_0\|_{\dot{B}_{p,1}^{n/p-1}} \leq c
\]  

then System (LNSI) has a unique global solution \((\bar{u}, \nabla \bar{P})\) in \( E_p \). Moreover,

\[
\|\bar{u}\|_{L^{\infty}(\dot{B}_{p,1}^{n/p-1})} + \|\mu \nabla^2 \bar{u}, \partial_t \bar{u}, \nabla \bar{P}\|_{L^1(\dot{B}_{p,1}^{n/p-1})} \leq C\|u_0\|_{\dot{B}_{p,1}^{n/p-1}}
\]

for some constant \( C \) depending only on \( n \) and on \( p \), and the flow map \((\rho_0, u_0) \mapsto (\bar{u}, \nabla \bar{P})\) is Lipschitz continuous from \( \mathcal{M}(\dot{B}_{p,1}^{n/p-1}) \times \dot{B}_{p,1}^{n/p-1} \) to \( E_p \).
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then System (LNSI) has a unique global solution \((\bar{u}, \nabla \bar{P})\) in \( E_p \). Moreover,

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for some constant \( C \) depending only on \( n \) and on \( p \), and the flow map \((\rho_0, u_0) \mapsto (\bar{u}, \nabla \bar{P})\) is Lipschitz continuous from \( \mathcal{M}(\dot{B}^{n/p-1}_{p,1}) \times \dot{B}^{n/p-1}_{p,1} \) to \( E_p \).

Remark

- If only \( \rho_0 - 1 \) is small then one gets a local-in-time statement.
- If \( n/p - 1 < 1/p \) then one can take \( \rho_0 = 1 + c_1 \Omega \) with \( c \) small enough, whenever \( \Omega \) is a bounded or exterior \( C^1 \) domain, or the half-space.
A few open problems

- Is the Lagrangian approach useful to handle the case where $\rho_0$ is not close to a positive constant?

- In real life, $\mu$ depends on $\rho$. Is Lagrangian approach still efficient?

- Use of Lagrangian approach for other models (e.g. compressible Navier-Stokes equations).