Optimal profiles in a phase-transition model with a saturating flux

Denis Bonheure *
Département de Mathématique
Université Libre de Bruxelles
CP 214, Boulevard du Triomphe, 1050 Bruxelles, Belgique
E-mail: denis.bonheure@ulb.ac.be

Franco Obersnel †
Dipartimento di Matematica e Geoscienze
Università degli Studi di Trieste
Via A. Valerio 12/1, 34127 Trieste, Italia
E-mail: obersnel@units.it

Abstract

It is well known that for the Allen-Cahn equation, the minimizing transition in an infinite cylinder $\mathbb{R} \times \omega$ is one-dimensional and unique up to a translation in the first variable. We analyze in this paper the existence and symmetry of optimal profiles for transitions in a similar phase-separation model with a saturating flux. This amounts to consider transitions in the space of BV functions as we consider the area integral instead of the Dirichlet energy to penalize the creation of wild interfaces.

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1 Introduction

A classical model in phase-transition modeling is given by the so-called Allen-Cahn energy functional

$$\frac{\varepsilon}{2}\int |\nabla u|^2\,dx + \frac{1}{4\varepsilon}\int (|u|^2-1)^2\,dx \tag{1}$$

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where u is a scalar function taking values between -1 and $+1$. This energy functional is used to describe the pattern and the separation of the (stable) phases ± 1 of a substance or a material within the van der Waals-Cahn-Hilliard gradient theory of phase transitions [16]. For instance, it has important physical applications in the study of interfaces in both gasses and solids, e.g. for binary metallic alloys [3] or bi-phase separation in fluids [41]. In this model the function u describes the pointwise state of the material or the fluid. The constant equilibria corresponding to the global minimum points ± 1 of the potential $\frac{1}{4}(|u|^2-1)^2$ are called the pure phases, whereas other configurations u represent mixed states and, if asymptotic to ± 1, they describe phase transitions.

Let us mention that the Allen-Cahn energy functional is relevant too in the theory of superconductors and superfluids where it appears as a Ginzburg-Landau free energy functional, u being then a complex-valued function, see e.g. [11], as well as in cosmology [28] where the motivation is the detection of the shape of the interfaces which "separate" the different regions of the universe which possibly arose from the big-bang.

The classical van der Waals-Cahn-Hilliard theory postulates that interface formation is driven by a variational principle, namely the pattern is the outcome of the minimization of the potential energy. This is clearly not satisfactory since any pattern that takes the values ± 1 only minimizes the potential energy $\frac{1}{4\varepsilon}\int(|u|^2-1)^2\,dx$ so that the separation between the two phases could be as complicated as we want and dramatically non-smooth patterns occur. Since such wild patterns are not observed in experiments, one has to modify the model and a classical way consists in penalizing the creation of unnecessary interfaces by adding a gradient term such as $\frac{\varepsilon}{2}\int|\nabla u|^2\,dx$. The parameter ε accounts somehow for the thickness of the interface. A function u which minimizes the full energy functional now tries to minimize the potential energy without creating too many interfaces since this would increase the gradient term. So basically, the presence of the gradient term has a smoothing effect on the phase separation. To recover the van der Waals-Cahn-Hilliard theory, one then let ε go to zero. It has been shown that the level sets of the minimizers then approach (in a suitable way) hypersurfaces of least possible area [34, 35, 36, 14], meaning that the optimal profiles tend to minimize the potential energy and the area of the interfaces.

In a series of papers (see, e.g., [38, 12, 32]) it was pointed out that, in some realistic diffusion processes, characterized for small gradients by linear gradient-flux relations, the flux response to an increase of gradients is expected to slow down and ultimately to approach saturation at large gradients. Accordingly, it was proposed in these contexts to penalize interfaces by a gradient term which is still quadratic for small values of the norm of the gradient but asymptotically linear. A simple model is given by the area integral

$$\int\sqrt{1+|\nabla u|^2}\,dx.$$

When the saturation of the diffusion flux is incorporated into these processes, it may cause a fundamental change in the morphology of the ensuing response. It may happen in particular that transitions connecting the equlibrium states may, when the potential exceeds a critical threshold, exhibit one or more discontinuities. In [32] a detailed numerical analysis of the morphology of the responses was performed for the Euler-Lagrange equation of the one-dimensional model

$$\left(u'/\sqrt{1+u'^2}\right)'=F'(u), \tag{2}$$

where F is a potential with several states at the lower energy level. At this stage, it is worth mentioning closely related models of flux limited diffusion equations studied in [1, 15] and the references therein.

Apart from its physical relevance, the Allen-Cahn energy functional presents interesting mathematical features. The stationary Allen-Cahn equation in $\mathbb{R}^N$, namely

$$\Delta u = F'(u), \tag{3}$$

where F is the double-well potential

$$F(s) = \tfrac{1}{4}(1-s^2)^2, \tag{4}$$

has attracted much attention in the last thirty years and in particular regarding the qualitative properties of bounded entire solutions. The most challenging question is known as De Giorgi's Conjecture which has stimulated many developments in the area of the calculus of variations, nonlinear analysis and semilinear elliptic PDEs.

De Giorgi's Conjecture (1978). *Assume $N > 1$. Let $u \in C^2(\mathbb{R}^N)$ be a solution of (3), with F given by (4), such that, for all $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, $|u(x)| \le 1$ and $\partial_{x_1} u(x) > 0$. Then, at least up to dimension $N = 8$, the level sets of u are all parallel hyperplanes; or, equivalently, there exist $v \in C^2(\mathbb{R})$ and $a = (a_2, \dots, a_N) \in \mathbb{R}^{N-1}$ such that $u(x) = v(x_1 + a_2x_2 + \cdots + a_Nx_N)$.*

De Giorgi's Conjecture has been proved in dimension $N = 2$ by N. Ghoussoub and C. Gui [30] and in dimension $N = 3$ by L. Ambrosio and X. Cabré [4]. The proofs for $N = 2$ and $N = 3$ use some techniques developed by H. Berestycki, L. Caffarelli and L. Nirenberg in [9] for the study of symmetry properties of solutions of semilinear equations in half spaces. The conjecture is still open when $4 \le N \le 8$, though a positive answer was given by O. Savin [37] under the additional assumption

$$\lim_{x_1 \to \pm\infty} u(x_1, x_2, \dots, x_N) = \pm 1 \quad \text{for every } (x_2, \dots, x_N) \in \mathbb{R}^{N-1}, \tag{5}$$

while in dimension $N \ge 9$ a counterexample has been established by M. del Pino, M. Kowalczyk and J. Wei in [19]. We refer to the survey [27] for more details.

Motivated by an application in cosmology, G.W. Gibbons (see [17, 28]) proposed a variant of De Giorgi's Conjecture, where condition (5) is strengthened by assuming uniformity in the limits.

Gibbons' Conjecture. *Assume $N > 1$. Let $u \in C^2(\mathbb{R}^N)$ be a solution of (3), with F given by (4), such that*

$$\lim_{x_1 \to \pm\infty} u(x_1, x_2, \dots, x_N) = \pm 1 \quad \textit{uniformly in } (x_2, \dots, x_N) \in \mathbb{R}^{N-1}.$$

Then the level sets of u are all parallel hyperplanes.

This conjecture was settled in any dimension, by using different approaches, by A. Farina in [20, 21], by M.T. Barlow, R.F. Bass and C. Gui in [8] and by H. Berestycki, F. Hamel and R. Monneau in [10]. It was also studied by A. Farina and E. Valdinoci [25] in an abstract setting which covers non-uniformly elliptic operators and, in particular, the case of smooth solutions of the bistable curvature equation

$$\operatorname{div}\Big(\nabla u/\sqrt{1+|\nabla u|^2}\Big) = F'(u), \tag{6}$$

or of the bistable p-Laplace equation

$$\Delta_p u = \operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = F'(u). \tag{7}$$

In the preceding paper [22], assuming $p \geq 2$, A. Farina also proved that if $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ solves (7) and there exists an open bounded set $\omega \subset \mathbb{R}^{N-1}$ such that u minimizes the energy functional

$$\frac{1}{2}\iint_{\mathbb{R}\times\omega} |\nabla u|^p \, dx + \frac{1}{4}\iint_{\mathbb{R}\times\omega} (|u|^2-1)^2 \, dx \tag{8}$$

in the set of all functions $v \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}\times\omega)$ satisfying

$$\lim_{x_1\to\pm\infty} u(x_1, x_2, \ldots, x_N) = \pm 1 \quad \text{uniformly a.e. with respect to } (x_2,\ldots,x_N) \in \omega, \tag{9}$$

then u is one-dimensional. The rigidity result for minimizers in cylinders was previously considered by G. Carbou in [17] for $p = 2$. The method was also extended by G. Alberti in [2] for a phase-separation model with non-local interaction energy.

One-dimensional symmetry of solutions of non-uniformly elliptic operators has also been dealt with in [13, 18, 23, 40, 26]. In particular, A. Farina, B. Sciunzi and E. Valdinoci [24] studied several alternative conditions to the uniformity of the limits which lead to rigidity results. Among these they proved that a solution u of equation (6), in dimension $N = 2$, or respectively $N = 3$, is one-dimensional provided that u is of class C^1, has a uniformly bounded gradient and is stable (i.e. the second variation of the associated functional is non-negative), or respectively $\partial_{x_3} u > 0$.

Our aim in this paper is to face similar questions for equation (6), giving in particular partial positive answers to Gibbons' conjecture and other related rigidity results, when it is referred to this equation. As we already noticed, solutions of (6) may exhibit discontinuities, even in dimension $N = 1$. Hence a natural setting where to settle this problem, as usual when the prescribed mean curvature operator is involved, is the space of functions having local bounded variation. Namely, in analogy with the above mentioned papers [17, 22, 2], we prove the one-dimensional character of any minimizer of the relaxation $\mathcal{I}_\omega$ of the functional

$$\iint_{\mathbb{R}\times\omega} (\sqrt{1+|\nabla u|^2} - 1)\, dx + \iint_{\mathbb{R}\times\omega} F(u)\, dx \tag{10}$$

to the set $\mathcal{E}_\omega$ of all functions $v : \mathbb{R}\times\omega \to \mathbb{R}$ having bounded variation in any cylinder $]-T, T[\times\omega$ and satisfying (9) (with lim replaced by $\operatorname{ess\,lim}$), where ω is an open bounded set in $\mathbb{R}^{N-1}$ and

(h_0) $F : \mathbb{R} \to [0, +\infty[$ *is continuous and satisfies* $F(s) = 0$ *if and only if* $s = \pm 1$.

A typical example is given by (4).

As for (8), the first term of the functional (10) represents the interfacial energy and it penalizes sharp transitions, while the second one is associated with the volume energy density and penalizes the states far away from the equilibria. Our main result is as follows.

Theorem 1.1. *Let ω be a non-empty open bounded set in $\mathbb{R}^{N-1}$. Then the functional $\mathcal{I}_\omega : \mathcal{E}_\omega \to [0,+\infty]$ defined by*

$$\mathcal{I}_\omega(v) = \mathcal{J}_\omega(v) + \iint_{\mathbb{R}\times\omega} F(v)\, dtdx$$

achieves its minimum in $\mathcal{E}_\omega$ and if u is a minimizer then u coincides with its increasing one-dimensional rearrangement $u^\star$.

We refer to Section 2 for the precise definition of $\mathcal{J}_\omega$ and $u^\star$. Motivated by the above conclusion, we study in detail the regularity properties of the one-dimensional minimizers. In particular, we deduce the following consequence with respect to Gibbon's conjecture in $\mathbb{R}^N$. Assuming (h_0) and

$$\max_{[-1,1]} F(s) < 1,$$

our result combined with [13, Theorem 5.1] leads to a counterpart to Farina's result [22] for smooth solutions of (6) which minimize $\mathcal{I}_\omega$ for some bounded $\omega \subset \mathbb{R}^{N-1}$. On the other hand, it is worth emphasizing that when

$$\max_{[-1,1]} F(s) \geq 1,$$

any rigidity result for smooth phase transitions, say $C^{1,\alpha}$, of (6) is in fact a nonexistence result.

The structure of our paper is as follows. In Section 2, we prove some preliminary results related to the study of the functional $\mathcal{J}_\omega$, as well as an approximation property for functions in the set $\mathcal{E}_\omega$. Section 3 contains the analysis of one-dimensional minimizers and a detailed study of their qualitative properties. Section 4 is devoted to the proof of the rigidity result concerning the minimizers of $\mathcal{I}_\omega$ in $\mathcal{E}_\omega$. We stress that performing all this program requires to extend to the BV-setting some delicate results that have been established in [2] within the frame of Sobolev spaces.

2 Preliminaries

We dedicate this section to the required preliminaries. After introducing our notation, we work out the functional setting and we recall a notion of monotone rearrangement which is the key in our analysis.

2.1 Notations

We start with a list of notation that we use throughout the paper.

We set $\mathbb{R}^+ = [0,+\infty[$ and $\mathbb{R}^+_0 =]0,+\infty[$. Moreover, we write $\mathbb{N}_0 = \{n \in \mathbb{N} : n \geq 1\}$. The convention $\pm\infty + r = \pm\infty$, for every $r \in \mathbb{R}$, and $\pm\infty \cdot r = \pm\infty$, for every $r \in \mathbb{R}^+_0$, are made. For any set E, the symbol χ_E is used to denote the characteristic function of E. The k-dimensional Hausdorff measure is denoted by $\mathcal{H}^k$. The Lebesgue measure is denoted by meas or by $\mathcal{L}$.

For functions $u, v \in E \to \mathbb{R}$ we write $(u \wedge v)(x) = \min\{u(x), v(x)\}$ and $(u \vee v)(x) = \max\{u(x), v(x)\}$ for all $x \in E$.

Throughout the paper, ω is a non-empty bounded open set in $\mathbb{R}^{N-1}$, and $(t,x) = (t, x_1, \ldots, x_{N-1})$ denotes a generic point of $\mathbb{R} \times \omega$.

If $u \in BV_{\text{loc}}(\mathbb{R} \times \omega)$ and for each $t \in \mathbb{R}$, we denote by $u(t_-, \cdot)$ and $u(t_+, \cdot)$, with $u(t_-, \cdot), u(t_+, \cdot) \in L^1_{\text{loc}}(\omega)$, the left and the right trace of u at t, as defined in [29, Section 2].

If $N = 1$ and $\omega = \mathbb{R}^0$ is a singleton, we identify $\mathbb{R} \times \omega$ with $\mathbb{R}$. In this case $u(t_-, \cdot) = u(t_-)$ and $u(t_+, \cdot) = u(t_+)$ are the left and the right essential limits of u at t. We will write $u_+(t)$ for $u(t_+)$ and $u_-(t)$ for $u(t_-)$.

For any $v \in BV(]t_1, t_2[\times \omega)$, with $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2$, $Dv = (Dv)^a dtdx + (Dv)^s$ is the Lebesgue decomposition of the measure Dv in its absolutely continuous part $(Dv)^a dtdx$, with density function $(Dv)^a$, and its singular part $(Dv)^s$ with respect to the Lebesgue measure in $\mathbb{R}$; $|Dv|$ denotes the total variation of the measure Dv, $|Dv| = |Dv|^a dtdx + |Dv|^s$ is the Lebesgue decomposition of $|Dv|$, and $\frac{Dv}{|Dv|}$ is the density function of Dv with respect to its total variation $|Dv|$.

2.2 Functional framework

For each $T > 0$ and every $v \in BV(] - T, T[\times \omega)$, we set

$$\mathcal{J}^T_\omega(v) = \iint_{]-T,T[\times\omega} \sqrt{1 + |Dv|^2} \; - 2T \, \text{meas}(\omega),$$

where meas(ω) denotes the $N-1$-dimensional Lebesgue measure of ω and

$$\iint_{]-T,T[\times\omega} \sqrt{1 + |Dv|^2} = \iint_{]-T,T[\times\omega} \sqrt{1 + |(Dv)^a|^2} dtdx + \iint_{]-T,T[\times\omega} |Dv|^s, \tag{11}$$

or, equivalently,

$$\iint_{]-T,T[\times\omega} \sqrt{1 + |Dv|^2} = \sup \Big\{ \iint_{]-T,T[\times\omega} \big(v \, \text{div} w_1 + w_2\big) \, dtdx : \\ w_1 \in C^1_0(] - T, T[\times \omega)^N, w_2 \in C^1_0(] - T, T[\times \omega), \text{ and } \| |w_1|^2 + w_2^2 \|_{L^\infty} \le 1 \Big\}.$$

Let us denote by $\mathcal{E}_\omega$ the set of all functions $u : \mathbb{R} \times \omega \to \mathbb{R}$ such that

(i) $u_{|]-T,T[\times\omega} \in BV(] - T, T[\times \omega)$, for every $T > 0$,

(ii) $\operatorname*{ess\,lim}_{t \to \pm\infty} u(t, x) = \pm 1$, uniformly a.e. with respect to $x \in \omega$.

From (11) it follows that, for any fixed $v \in \mathcal{E}_\omega$, the function $T \mapsto \mathcal{J}^T_\omega(v)$ is increasing with respect to $T > 0$. This allows to define a functional $\mathcal{J}_\omega : \mathcal{E}_\omega \to [0, +\infty]$ by setting

$$\mathcal{J}_\omega(v) = \lim_{T \to +\infty} \mathcal{J}^T_\omega(v) = \sup_{T > 0} \mathcal{J}^T_\omega(v). \tag{12}$$

It is suggestive to write

$$\mathcal{J}_\omega(v) = \iint_{\mathbb{R}\times\omega} \Big(\sqrt{1 + |Dv|^2} - 1\Big).$$

Remark 2.1 If $N = 1$, $\omega = \mathbb{R}^0$ is meant to be a singleton and we identify $\mathbb{R} \times \omega$ with $\mathbb{R}$. In this case we simply write $\mathcal{E}$ instead of $\mathcal{E}_\omega$, $\mathcal{J}^T$ instead of $\mathcal{J}^T_\omega$, and $\mathcal{J}$ instead of $\mathcal{J}_\omega$.

Remark 2.2 Let ω be a non-empty bounded open set in $\mathbb{R}^{N-1}$, with $N \geq 2$. By Fubini Theorem and (11), we immediately see that, for each $T > 0$ and every $v \in BV(-T,T)$,

$$\mathcal{J}_\omega^T(v) = \text{meas}(\omega)\, \mathcal{J}^T(v).$$

Hence, for every $v \in \mathcal{E}$, we have, by (12),

$$\mathcal{J}_\omega(v) = \text{meas}(\omega)\, \mathcal{J}(v).$$

The functional $\mathcal{J}_\omega$ enjoys the following semicontinuity property.

Proposition 2.1. *Let ω be a non-empty bounded open set in $\mathbb{R}^{N-1}$ and let $v \in \mathcal{E}_\omega$ be given. If $(v_n)_n$ is a sequence in $\mathcal{E}_\omega$ such that, for each $T > 0$, the sequence $(v_{n|]-T,T[\times\omega})_n$ converges in $L^1(]-T,T[\times\omega)$ to $v_{|]-T,T[\times\omega}$, then*

$$\mathcal{J}_\omega(v) \leq \liminf_{n\to+\infty} \mathcal{J}_\omega(v_n).$$

Proof. Take a function $v \in \mathcal{E}_\omega$ and a sequence $(v_n)_n$ in $\mathcal{E}_\omega$ such that, for each $T > 0$, the sequence $(v_{n|]-T,T[\times\omega})_n$ converges in $L^1(]-T,T[\times\omega)$ to $v_{|]-T,T[\times\omega}$. Suppose, by contradiction, that

$$\mathcal{J}_\omega(v) > \liminf_{n\to+\infty} \mathcal{J}_\omega(v_n).$$

Then there exist $\varepsilon > 0$, $T_0 > 0$ and a subsequence of $(v_n)_n$, we still denote by $(v_n)_n$, such that, for all n,

$$\mathcal{J}_\omega^{T_0}(v) > \sup_{T>0} \mathcal{J}_\omega^T(v_n) + \varepsilon \geq \mathcal{J}_\omega^{T_0}(v_n) + \varepsilon.$$

On the other hand, the lower semicontinuity of $\mathcal{J}_\omega^{T_0}$ with respect to the L^1-convergence in $BV(]-T_0,T_0[\times\omega)$ (see [29, Theorem 14.2]) implies that

$$\mathcal{J}_\omega^{T_0}(v) \leq \liminf_{n\to+\infty} \mathcal{J}_\omega^{T_0}(v_n),$$

thus yielding a contradiction. $\square$

Let us denote by $\mathcal{F}_\omega$ the set of all functions $u : \mathbb{R}\times\omega \to \mathbb{R}$ such that

(i) $u_{|]-T,T[\times\omega} \in W^{1,1}(]-T,T[\times\omega)$ for every $T > 0$,

(ii) $\operatorname*{ess\,lim}_{t\to\pm\infty} u(t,x) = \pm 1$ uniformly a.e. with respect to $x \in \omega$.

The following approximation property holds in $\mathcal{E}_\omega$.

Proposition 2.2. *Let ω be a non-empty bounded open set in $\mathbb{R}^{N-1}$ and let $v \in \mathcal{E}_\omega$ be such that $\|v\|_\infty \leq 1$. Then there exist a convex function $g : [0,+\infty[\to [0,+\infty[$, such that $g(s) = 0$ if and only if $s = 0$, and a sequence $(v_n)_n$ in $\mathcal{F}_\omega$ such that $(v_n)_n$ converges to v a.e. in $\mathbb{R}\times\omega$,*

$$\lim_{n\to+\infty} \iint_{\mathbb{R}\times\omega} g(|v_n - v|)dtdx = 0 \tag{13}$$

and

$$\lim_{n\to+\infty} \mathcal{J}_\omega(v_n) = \mathcal{J}_\omega(v). \tag{14}$$

Proof. Take $v \in \mathcal{E}_\omega$ with $\|v\|_\infty \leq 1$. An elementary, but tedious, argument shows that we can construct a strictly decreasing function $h : [0,+\infty[\to]0,+\infty[$, with $h(0) > 2$,

$$\lim_{t\to+\infty} h(t) = 0$$

and

$$\operatorname*{ess\,sup}_{x\in\omega} |v(t,x) - \operatorname{sgn}(t)| \leq h(|t|)$$

for a.e. $t \in \mathbb{R}$, in such a way that the function $g : [0,+\infty[\to [0,+\infty[$, defined by $g(0) = 0$, $g(s) = \left(\frac{1}{h^{-1}(s)}\right)^2$ for $s \in]0,2]$ and linear otherwise, is convex. Accordingly, g is continuous, increasing, and satisfies

$$\iint_{\mathbb{R}\times\omega} g(|v - \operatorname{sgn}(\cdot)|)dtdx < +\infty. \tag{15}$$

Using the approximation property in $BV(]-T,T[\times\omega)$ (see [29, Theorem 1.17]) and the Lipschitz character of the sets $\{\pm T\}\times\omega$, we can prove, arguing as in [6, p. 498], that, for every $T > 0$, there exists a sequence $(w_k^T)_k$ in $W^{1,1}(]-T,T[\times\omega)$, with $w_k^T((-T)^+,\cdot) = -1$ and $w_k^T(T^-,\cdot) = 1$ in ω, for all k, which converges in $L^1(]-T,T[\times\omega)$ and a.e. in $]-T,T[\times\omega$ to $v_{|]-T,T[\times\omega}$, and satisfies

$$\begin{aligned}\lim_{k\to+\infty} \iint_{]-T,T[\times\omega} \sqrt{1+|\nabla w_k^T|^2}\,dtdx = &\iint_{]-T,T[\times\omega} \sqrt{1+|Dv|^2} \\ &+ \int_\omega \big(|v((-T)^+,x)+1| + |v(T^-,x)-1|\big)\,dx.\end{aligned}$$

For each n, take k_n such that both

$$\iint_{]-n,n[\times\omega} \left|w_{k_n}^n - v\right| dtdx < \frac{1}{n} \tag{16}$$

and

$$\begin{aligned}\Big|\iint_{]-n,n[\times\omega} \sqrt{1+|Dv|^2} + \int_\omega \big(|v((-n)^+,x)+1| + |v(n^-,x)-1|\big)\,dx \\ - \iint_{]-n,n[\times\omega} \sqrt{1+|\nabla w_{k_n}^n|^2}\,dtdx\Big| < \frac{1}{n}\end{aligned} \tag{17}$$

hold. Then define $v_n \in \mathcal{F}_\omega$ by setting, for $(t,x) \in \mathbb{R}\times\omega$,

$$v_n(t,x) = \begin{cases} -1 & \text{if } t < -n, \\ w_{k_n}^n(t,x) & \text{if } t \in [-n,n], \\ 1 & \text{if } t > n. \end{cases}$$

The definition of g implies the existence of a constant $c > 0$ such that

$$\begin{aligned}\iint_{\mathbb{R}\times\omega} g(|v_n - v|)dtdx \leq c &\iint_{]-n,n[\times\omega} \left|v_n - v\right| dtdx \\ &+ \iint_{]-\infty,-n]\times\omega} g(|v+1|)dtdx + \iint_{[n,+\infty[\times\omega} g(|v-1|)dtdx.\end{aligned}$$

Hence, using (15) and (16), we deduce the validity of (13). Finally, we have

$$
\begin{aligned}
|\mathcal{J}_\omega(v) - \mathcal{J}_\omega(v_n)| \leq & \Big| \mathcal{J}_\omega(v) - \iint_{]-n,n[\times\omega} \big(\sqrt{1+|Dv|^2} - 1\big) \Big| \\
& + \Big| \iint_{]-n,n[\times\omega} \big(\sqrt{1+|Dv|^2} - 1\big) + \int_\omega \big(|v((-n)^+,x)+1| + |v(n^-,x)-1|\big)\,dx \\
& \qquad - \iint_{]-n,n[\times\omega} \big(\sqrt{1+|\nabla v_n|^2} - 1\big)\,dtdx \Big| \\
& + \Big| \int_\omega \big(|v((-n)^+,x)+1| + |v(n^-,x)-1|\big)\,dx \Big|.
\end{aligned}
$$

Thus we conclude, using (17) and the definitions of $\mathcal{E}_\omega$ and $\mathcal{J}_\omega$, that (14) holds. $\square$

Remark 2.3 If ω has a Lipschitz boundary, then Remark 3.22 in [5] implies that each w_k^T can be chosen in $H^1(]-T,T[\times\omega)$ and hence $\nabla v_n \in L^2(\mathbb{R}\times\omega)$.

The following lattice-type property is useful.

Proposition 2.3. *Let ω be a non-empty bounded open set in $\mathbb{R}^{N-1}$. For every $u \in \mathcal{E}_\omega$, we have*

$$
\mathcal{J}_\omega((u \vee -1)\wedge 1) \leq \mathcal{J}_\omega(u).
$$

Proof. Let us take $u \in \mathcal{E}_\omega$. We first observe that $(u\vee -1)\wedge 1 \in \mathcal{E}_\omega$: this easily follows from [5, Exercise 3.12, p. 209]. Next, we fix $T>0$. Using [39, Theorem 1.56], we see that, for every $v, w, z \in W^{1,1}(]-T,T[\times\omega)$,

$$
\iint_{]-T,T[\times\omega} (\sqrt{1+|\nabla(w\vee -1)|^2} - 1)\,dtdx \leq \iint_{]-T,T[\times\omega} (\sqrt{1+|\nabla w|^2} - 1)\,dtdx,
$$

$$
\iint_{]-T,T[\times\omega} (\sqrt{1+|\nabla(z\wedge 1)|^2} - 1)\,dtdx \leq \iint_{]-T,T[\times\omega} (\sqrt{1+|\nabla z|^2} - 1)\,dtdx,
$$

and hence

$$
\iint_{]-T,T[\times\omega} (\sqrt{1+|\nabla((v\vee -1)\wedge 1)|^2} - 1)\,dtdx \leq \iint_{]-T,T[\times\omega} (\sqrt{1+|\nabla v|^2} - 1)\,dtdx.
$$

The approximation property in $BV(]-T,T[\times\omega)$ (see [29, Theorem 1.17]) and the semicontinuity property of $\mathcal{J}_\omega^T$ easily yield

$$
\mathcal{J}_\omega^T((u\vee -1)\wedge 1) \leq \mathcal{J}_\omega^T(u).
$$

Finally, letting $T \to +\infty$, we get the conclusion. $\square$

We finally notice that the following additivity property holds.

Proposition 2.4. *Let $a, b \in \mathbb{R}$ be such that $a<b$. Then, for every $v \in BV_{\mathrm{loc}}(\mathbb{R})$, we have*

$$
\begin{aligned}
\mathcal{J}(v) = & \lim_{T\to+\infty} \Big(\int_{-T}^a \sqrt{1+|Dv|^2} - (T+a) \Big) + |v(a^+) - v(a^-)| + |v(b^+) - v(b^-)| \\
& + \int_a^b \sqrt{1+|Dv|^2} - (b-a) + \lim_{T\to+\infty} \Big(\int_b^T \sqrt{1+|Dv|^2} - (T-b) \Big),
\end{aligned}
$$

$$\lim_{h\to 0^+}\int_{a+h}^{b}\sqrt{1+|Dv|^2}=\lim_{h\to 0^+}\int_{a}^{b-h}\sqrt{1+|Dv|^2}=\int_a^b\sqrt{1+|Dv|^2},$$

and

$$\lim_{h\to 0^+}\int_{a-h}^{a+h}\sqrt{1+|Dv|^2}=|v(a^+)-v(a^-)|.$$

2.3 A notion of monotone rearrangement

We are going to use a notion of monotone rearrangement that we recall here for the reader's convenience. This notion of rearrangement, which was first introduced in [17], see also [22] and [2] for a deeper study, is a generalization of the classical monotone rearrangement for functions of a single variable. For the one-dimensional setting, we refer to [31].

Let ω be a non-empty open bounded set in $\mathbb{R}^{N-1}$. Given a function $v\in\mathcal{E}_\omega$, with $\|v\|_\infty\le 1$, we denote, for $s\in\,]-1,1[$, the s-superlevel of v by

$$E_s(v)=\{(t,x)\in\mathbb{R}\times\omega : v(t,x)\ge s\}.$$

The increasing rearrangement of v is the function $v^\star\in\mathcal{E}_\omega$ whose s-superlevel, for $s\in\,]-1,1[$, is the half-cylinder

$$E_s(v^\star)=[c_v(s),+\infty[\,\times\,\omega,$$

where

$$c_v(s)=\frac{\operatorname{meas}((\mathbb{R}^+\times\omega)\setminus E_s(v))-\operatorname{meas}(E_s(v)\setminus(\mathbb{R}^+\times\omega))}{\operatorname{meas}(\omega)}$$

Of course, if $N=1$, then $c_v(s)=\operatorname{meas}(\mathbb{R}^+\setminus E_s(v))-\operatorname{meas}(E_s(v)\setminus\mathbb{R}^+)$ and $E_s(v^\star)=[c_v(s),+\infty[$. As $v^\star$ is independent of $x\in\omega$, the indication of this variable will be sometimes omitted, keeping only the indication of the variable $t\in\mathbb{R}$ and thus identifying $v^\star$ with a single variable function.

The following Pólya-Szegö type theorem proved by G. Alberti [2] makes this rearrangement useful when looking for minimizers of the functional (10). We do not need it for BV functions to get existence of the minimizer but later on we somehow extend it to prove 1D-symmetry of the optimal profile.

Theorem 2.5. *[2, Theorem 2.10] Assume $g:[0,+\infty)\to[0,+\infty)$ is convex, null at 0 and strictly increasing. Then, for every $u\in W^{1,1}_{loc}(\mathbb{R}\times\omega)$ we have*

$$\iint_{\mathbb{R}\times\omega} g(|\nabla u|)\ge\iint_{\mathbb{R}\times\omega} g(|\nabla u^\star|).$$

Moreover, when the right-hand side is finite, equality holds if and only if $u=u^\star$ a.e.

3 Analysis of the one-dimensional problem

We will discuss in this section the existence and the regularity of optimal transitions for (2). Assume (h_0) and define a functional $\mathcal{I}:\mathcal{E}\to[0,+\infty]$ by setting

$$\mathcal{I}(v)=\mathcal{J}(v)+\int_{-\infty}^{+\infty}F(v)\,dt.$$

Proposition 3.1. *Assume* (h_0). *Suppose that* $u \in \mathcal{E}$ *is a minimizer of* $\mathcal{I}$ *in* $\mathcal{E}$. *Then* u *satisfies the following properties:*

(i_1) *the left-trace function* u_- *and the right-trace function* u_+ *are increasing on* $\mathbb{R}$ *and strictly increasing on the sets* $\{t \in \mathbb{R} : |u_-(t)| < 1\}$ *and* $\{t \in \mathbb{R} : |u_+(t)| < 1\}$ *respectively; moreover, for every* $t_0 \in \mathbb{R}$, $u_-(t_0) = u(t_0^-) \leq u(t_0^+) = u_+(t_0)$;

(i_2) *for every* $t_0 \in \mathbb{R}$, *if either* $F(u(t_0^-)) < 1$ *or* $F(u(t_0^+)) < 1$, *then* u *is continuous at* t_0;

(i_3) *for every* $t_0 \in \mathbb{R}$, *if there exists* $\varepsilon > 0$ *such that* $F(s) \geq 1$ *either for all* $s \in [u(t_0^-), u(t_0^-) + \varepsilon]$ *or for all* $s \in [u(t_0^+) - \varepsilon, u(t_0^+)]$, *then* $u(t_0^+) \geq u(t_0^-) + \varepsilon$;

(i_4) *for every* $t_0 \in \mathbb{R}$, *if* $u(t_0^-) < u(t_0^+)$, *then* $F(s) \geq 1$ *for all* $s \in [u(t_0^-), u(t_0^+)]$.

Proof. Let us prove (i_1). We first show that the left-trace function u_- is strictly increasing on the set $U_- = \{t \in \mathbb{R} : |u_-(t)| < 1\}$. Suppose, by contradiction, that there exist $t_1, t_2 \in U_-$, with $t_1 < t_2$, such that $u(t_1^-) \geq u(t_2^-)$. Set

$$t_3 = \inf\{t \geq t_2 : u(t^-) \geq u(t_1^-)\}.$$

We have $t_3 > t_1$, $u(t_3^+) \geq u(t_1^-)$ and $u(t_3^-) \leq u(t_1^-)$. Define $v \in \mathcal{E}$ by

$$v(t) = \begin{cases} u(t) & \text{if } t \leq t_1, \\ u(t + (t_3 - t_1)) & \text{if } t > t_1. \end{cases}$$

We observe that

$$\begin{aligned} \mathcal{I}(u) - \mathcal{I}(v) = |u(t_1^+) - u(t_1^-)| + \int_{t_1}^{t_3} (\sqrt{1 + |Du|^2} - 1) + |u(t_3^+) - u(t_3^-)| \\ + \int_{t_1}^{t_3} F(u)\, dt - |v(t_1^+) - v(t_1^-)| > 0, \end{aligned}$$

as $F(u(t)) > 0$ on a subset of $[t_1, t_3]$ of positive measure and

$$|v(t_1^+) - v(t_1^-)| = |u(t_3^+) - u(t_1^-)| \leq |u(t_3^+) - u(t_3^-)|.$$

This implies that $\mathcal{I}(v) < \min_{\mathcal{E}} \mathcal{I}$ which is a contradiction.

Now we can prove that u_- is increasing on $\mathbb{R}$. Assume, by contradiction, that there exist t_2, t_3, with $t_2 < t_3$, such that, e.g., $u(t_2^-) = 1$ and $u(t_3^-) < 1$. We can also suppose that $u(t_3^-) > -1$ because, otherwise, $J(u) \geq 2$, which is impossible as u is a minimizer. If there exists $t_1 < t_3$ such that $|u(t_1^-)| < 1$ and $u(t_1^-) \geq u(t_3^-)$, we get a contradiction. Otherwise for all $t < t_3$ either $u(t^-) = 1$ or $u(t^-) < u(t_3^-)$. In this case there is $t_1 < t_3$ such that $-1 < u(t_1^-) < u(t_3^-)$ and $u(t_1^+) = 1$. Pick $t_4 \in]t_2, t_3[$ such that $u(t_1^-) < u(t_4^-) < 1$ and define a function $v \in \mathcal{E}$ by

$$v(t) = \begin{cases} u(t) & \text{if } t \leq t_1, \\ u(t + (t_4 - t_1)) & \text{if } t > t_1. \end{cases}$$

We have

$$\begin{aligned} \mathcal{I}(u) - \mathcal{I}(v) = |u(t_1^+) - u(t_1^-)| + \int_{t_1}^{t_4} (\sqrt{1 + |Du|^2} - 1) + |u(t_4^+) - u(t_4^-)| \\ + \int_{t_1}^{t_4} F(u(t))\, dt - |v(t_1^+) - v(t_1^-)| > 0, \end{aligned}$$

as

$$|v(t_1^+) - v(t_1^-)| = |u(t_4^+) - u(t_1^-)| < |1 - u(t_1^-)| = |u(t_1^+) - u(t_1^-)|.$$

This implies that $\mathcal{I}(v) < \min_{\mathcal{E}} \mathcal{I}$ which is again a contradiction. A similar argument allows to get the corresponding conclusions for u_+.

Finally let us prove that $u(t_0^-) \leq u(t_0^+)$ for every $t_0 \in \mathbb{R}$. Suppose, by contradiction, that there exists $t_0 \in \mathbb{R}$ such that $u(t_0^-) > u(t_0^+)$. Set

$$t_1 = \inf\{t \geq t_0 : u(t^+) \geq u(t_0^-)\}.$$

We have $t_1 > t_0$, $u(t_1^+) \geq u(t_0^-)$ and $u(t_1^-) \leq u(t_0^-)$. Define $v \in \mathcal{E}$ by

$$v(t) = \begin{cases} u(t) & \text{if } t \leq t_0, \\ u(t + (t_1 - t_0)) & \text{if } t > t_0. \end{cases}$$

Therefore we observe again that

$$\begin{aligned} \mathcal{I}(u) - \mathcal{I}(v) = |u(t_0^+) - u(t_0^-)| + \int_{t_0}^{t_1} (\sqrt{1 + |Du|^2} - 1) + |u(t_1^+) - u(t_1^-)| \\ + \int_{t_0}^{t_1} F(u(t))\, dt - |v(t_0^+) - v(t_0^-)| > 0, \end{aligned}$$

as

$$|v(t_0^+) - v(t_0^-)| = u(t_1^+) - u(t_0^-), \quad u(t_0^-) - u(t_1^-) \geq 0 \text{ and } u(t_0^-) - u(t_0^+) > 0.$$

This leads one more time to the contradiction that $\mathcal{I}(v) < \min_{\mathcal{E}} \mathcal{I}$.

Let us prove (i_2). Assume that $F(u(t_0^-)) < 1$ and suppose, still by contradiction, that $u(t_0^-) < u(t_0^+)$. Pick $\sigma \in \,]0, 1[$ and $\varepsilon > 0$ such that $\varepsilon \leq u(t_0^+) - u(t_0^-)$ and $F(s) \leq 1 - \sigma$ for all $s \in [u(t_0^-), u(t_0^-) + \varepsilon]$. Take $\delta > 0$ so small that

$$\sqrt{\delta^2 + \varepsilon^2} - \delta\sigma < \varepsilon$$

and define $v \in \mathcal{E}$ by

$$v(t) = \begin{cases} u(t) & \text{if } t \leq t_0, \\ u(t_0^-) + \frac{\varepsilon}{\delta}(t - t_0) & \text{if } t \in \,]t_0, t_0 + \delta], \\ u(t - \delta) & \text{if } t > t_0 + \delta. \end{cases}$$

Then we have

$$\begin{aligned} \mathcal{I}(v) - \mathcal{I}(u) &= \int_{t_0}^{t_0+\delta} \left(\sqrt{1 + |Dv|^2} - 1\right) + |v((t_0 + \delta)^+) - v((t_0 + \delta)^-)| \\ &\quad + \int_{t_0}^{t_0+\delta} F(v(t))\, dt - |u(t_0^+) - u(t_0^-)| \\ &= \int_{t_0}^{t_0+\delta} \left(\sqrt{1 + |Dv|^2} - 1\right) + |u(t_0^+) - u(t_0^-) - \varepsilon| \\ &\quad + \int_{t_0}^{t_0+\delta} F(v(t))\, dt - |u(t_0^+) - u(t_0^-)| \\ &\leq \sqrt{\delta^2 + \varepsilon^2} - \delta\sigma - \varepsilon < 0, \end{aligned}$$

leading to the usual contradiction. The proof in case $F(u(t_0^+)) < 1$ is similar.

Let us prove (i_3). Assume that $F(s) \geq 1$ for all $s \in [u(t_0^-), u(t_0^-) + \varepsilon]$ and suppose, by contradiction, that $u(t_0^+) < u(t_0^-) + \varepsilon$. Take $t_1 > t_0$ such that $u(t_1^+) \leq u(t_0^-) + \varepsilon$. Define a function $v \in \mathcal{E}$ by

$$v(t) = \begin{cases} u(t) & \text{if } t \leq t_0, \\ u(t + (t_1 - t_0)) & \text{if } t > t_0. \end{cases}$$

We have

$$\begin{aligned} \mathcal{I}(v) - \mathcal{I}(u) &= u(t_1^-) - u(t_0^+) - \int_{t_0}^{t_1} \left(\sqrt{1 + |Du|^2} - 1\right) - \int_{t_0}^{t_1} F(u(t))\,dt \\ &\leq u(t_1^+) - u(t_0^+) - \int_{t_0}^{t_1} \sqrt{1 + |Du|^2} < 0, \end{aligned}$$

leading again to the same contradiction. Assuming $F(s) \geq 1$ for all $s \in [u(t_0^+) - \varepsilon, u(t_0^+)]$, we argue in a similar way.

Finally, let us prove (i_4). Assume that $u(t_0^-) < u(t_0^+)$ and suppose, by contradiction, that there exist $\sigma > 0$ and s_1, s_2, with $u(t_0^-) \leq s_1 < s_2 \leq u(t_0^+)$, such that $F(s) \leq 1 - \sigma$ for all $s \in [s_1, s_2]$. Arguing as when proving (i_2), with $\varepsilon = s_2 - s_1$, we easily contradict the minimality of u. □

Remark 3.1 Assume (h_0). Let $u \in \mathcal{E}$ be a minimizer of $\mathcal{I}$ in $\mathcal{E}$ and set $\mathcal{O} = \{t \in \mathbb{R} : F(u_-(t)) < 1\}$. By Proposition 3.1, $\mathcal{O} = \{t \in \mathbb{R} : F(u_+(t)) < 1\}$, $\mathcal{O}$ is open and u is continuous in $\mathcal{O}$. Note that, as $u \in \mathcal{E}$ and (h_0) holds, there exist $\tau_0 \in \,]-\infty, +\infty]$ and $\tau_1 \in [-\infty, +\infty[$ such that $F(u(t)) < 1$ for all $t \in \,]-\infty, \tau_0[\, \cup \,]\tau_1, +\infty[$. Therefore, either $\mathcal{O} = \mathbb{R}$, or we can represent $\mathcal{O}$ as an at most countable union of connected components, i.e., $\mathcal{O} = \dot{\bigcup}_{i \in N} K_i$, where $N \subseteq \mathbb{N}$ and, say, $K_0 = \,]-\infty, \tau_0[$ and $K_1 = \,]\tau_1, +\infty[$.

Proposition 3.2. *Assume (h_0). Suppose that $u \in \mathcal{E}$ is a minimizer of $\mathcal{I}$ in $\mathcal{E}$. Let $I \subset \mathcal{O}$ be a bounded interval, $\mathcal{O}$ being defined in Remark 3.1. Then $u_{|I} \in W^{1,1}(I)$.*

Proof. Assume that $t_0 \in \mathbb{R}$ and either $F(u(t_0^-)) < 1$ or $F(u(t_0^+)) < 1$. Then, by Proposition 3.1, u is continuous at t_0. By the continuity of the function $F(u(\cdot))$, there is a neighbourhood U of t_0 such that $F(u(t)) < 1$ for all $t \in U$. In particular u is continuous in U. Pick $\sigma \in \,]0, 1[$ and $\eta > 0$ such that

$$F(s) < 1 - \sigma,$$

for all $s \in [u(t_0) - \eta, u(t_0) + \eta]$, and

$$|F(s_1) - F(s_2)| < \tfrac{\sigma}{4},$$

for all $s_1, s_2 \in [u(t_0) - \eta, u(t_0) + \eta]$. By the standard decomposition of BV functions (see e.g. [5, Corollary 3.33]) we can write

$$u_{|U} = u_A + u_C$$

where $u_A \in W^{1,1}(U)$ is the absolutely continuous part of $u_{|U}$ and u_C is the Cantor part of $u_{|U}$. Take an interval $I \subset U$, neighbourhood of t_0, such that, for all $t_1, t_2 \in I$,

$$\begin{aligned} |u_A(t_1) - u_A(t_0)| &< \tfrac{1}{2}\eta, \\ |u_C(t_1) - u_C(t_0)| &< \tfrac{1}{2}\eta, \\ |u_C(t_2) - u_C(t_1)| &< \tfrac{1}{2}\eta. \end{aligned}$$

Observe that u also satisfies

$$|u_A(t) + u_C(t_1) - u(t_0)| < \eta$$

for all $t, t_1 \in I$,

$$|u(t) - u(t_0)| < \eta$$

for all $t \in I$, and

$$\left|u_A(t_2) + u_C(t_1) + \frac{u_C(t_2) - u_C(t_1)}{t_2 - t_1}(t - t_1) - u(t_0)\right| < 2\eta$$

for all $t, t_1, t_2 \in I$, with $t_1 < t < t_2$.

We aim to prove that $u_C(t) = 0$ for all $t \in I$. If this does not hold, then $\int_I |Du_C| > 0$. We set $\delta = \mathcal{L}(I)$ and $V = \int_I |Du_C|$. Recall that the measure $|Du_C|$ is singular with respect to the Lebesgue measure $\mathcal{L}$, therefore there exists a set $N_0 \subset I$ such that $\mathcal{L}(N_0) = 0$ and

$$V = \int_I |Du_C| = \int_{N_0} |Du_C|.$$

Claim. There exists a sequence of pairwise disjoint intervals $(I_n)_n$, with $I_n \subset I$ for all n and $N_0 \subset \bigcup_{n\in\mathbb{N}} I_n$, such that, for some $n \in \mathbb{N}$,

$$\sqrt{\delta_n^2 + V_n^2} - \tfrac{3}{4}\sigma\delta_n < V_n,$$

where $\delta_n = \mathcal{L}(I_n)$ and $V_n = \int_{I_n} |Du_C|$. Arguing by contradiction, we suppose that, for all $\varepsilon > 0$, we can take a sequence $(I_n)_n$ with $\mathcal{L}(I_n) < \varepsilon\, 2^{-n}$, $N_0 \subset \bigcup_{n\in\mathbb{N}} I_n$ and, for all $n \in \mathbb{N}$,

$$\sqrt{\delta_n^2 + V_n^2} - \tfrac{3}{4}\sigma\delta_n \geq V_n.$$

Therefore we have

$$\tfrac{3}{2}\sigma\delta_n V_n \leq \delta_n^2\big(1 - \tfrac{9\sigma^2}{16}\big)$$

and, summing over $\mathbb{N}$,

$$\tfrac{3}{2}\sigma V \leq 2\varepsilon\big(1 - \tfrac{9\sigma^2}{16}\big),$$

thus contradicting the assumption $V > 0$. This concludes the proof of the claim.

Let $I_0 = [t_1, t_2] \subset I$ be an interval, whose existence is proved in the claim, satisfying

$$\sqrt{\delta_0^2 + V_0^2} - \tfrac{3}{4}\sigma\delta_0 < V_0,$$

where $\delta_0 = \mathcal{L}(I_0)$ and $V_0 = \int_{I_0} |Du_C|$. We define $v \in \mathcal{E}$ by

$$v(t) = \begin{cases} u(t) & \text{if } t \leq t_1, \\ u_C(t_1) + u_A(t) & \text{if } t \in \,]t_1, t_2], \\ u_A(t_2) + u_C(t_1) + \frac{u_C(t_2) - u_C(t_1)}{\delta_0}(t - t_2) & \text{if } t \in \,]t_2, t_2 + \delta_0], \\ u(t - \delta_0) & \text{if } t > t_2 + \delta_0. \end{cases}$$

We claim that $\mathcal{I}(v) < \mathcal{I}(u)$. Indeed, we compute

$$\begin{aligned}\mathcal{I}(v) - \mathcal{I}(u) = &\int_{t_1}^{t_2} \sqrt{1 + (u_A')^2}\, dt - \delta_0 + \int_{t_1}^{t_2} F\big(u_C(t_1) + u_A(t)\big)\, dt \\ &+ \Big(\sqrt{\delta_0^2 + \big(u_C(t_2) - u_C(t_1)\big)^2} - \delta_0 \\ &+ \int_{t_1}^{t_2} F\Big(u_A(t_2) + u_C(t_1) + \frac{u_C(t_2) - u_C(t_1)}{\delta_0}(t - t_1)\Big)\, dt \\ &- \int_{t_1}^{t_2} \sqrt{1 + (u_A')^2}\, dt + \delta_0 - \int_{t_1}^{t_2} |Du_C| - \int_{t_1}^{t_2} F\big(u(t)\big)\, dt.\end{aligned}$$

Since $u_C(t_2) - u_C(t_1) \le V_0$, we observe that

$$\begin{aligned}\mathcal{I}(v) - \mathcal{I}(u) \le &\int_{t_1}^{t_2} \Big(F\big(u_C(t_1) + u_A(t)\big) - F\big(u(t)\big)\Big)\, dt \\ &+ \sqrt{{\delta_0}^2 + {V_0}^2} - \delta_0 - V_0 \\ &+ \int_{t_1}^{t_2} F\Big(u_A(t_2) + u_C(t_1) + \frac{u_C(t_2) - u_C(t_1)}{\delta_0}(t - t_1)\Big)\, dt \\ &\le \delta_0\big(\tfrac{\sigma}{4} - \sigma\big) - V_0 + \sqrt{{\delta_0}^2 + {V_0}^2} < 0.\end{aligned}$$

Since u was assumed to be a minimizer of $\mathcal{I}$, we reached a contradiction. We proved that, for all $t \in \mathbb{R}$, if either $F(u(t^-)) < 1$ or $F(u(t^+)) < 1$, then there is a neighbourhood I of t where $u_C = 0$ and therefore $u \in W^{1,1}(I)$. The statement of the proposition immediately follows. $\square$

Proposition 3.3. *Assume*

(h_1) $F : \mathbb{R} \to [0, +\infty[$ *is of class* C^1 *and satisfies* $F(s) = 0$ *if and only if* $s = \pm 1$.

Let $u \in \mathcal{E}$ *be a minimizer of* $\mathcal{I}$ *in* $\mathcal{E}$*. Then the following conclusions hold:*

(i_1) $u \in C^2(\mathcal{O})$ *and satisfies in* $\mathcal{O}$ *the equation*

$$\Big(u' / \sqrt{1 + u'^2}\Big)' = F'(u), \tag{18}$$

$\mathcal{O}$ *being defined in Remark 3.1;*

(i_2) $\lim_{t \to -\infty} u'(t) = \lim_{t \to +\infty} u'(t) = 0$*;*

(i_3) *for each* $i \in N$,

$$\lim_{t \to \inf K_i^+} u'(t) = +\infty,$$

if K_i *is bounded from below, and*

$$\lim_{t \to \sup K_i^-} u'(t) = +\infty,$$

if K_i *is bounded from above,* N *and* K_i *being defined in Remark 3.1;*

(i_4) *u is uniquely determined up to translations.*

Proof. Let $]a,b[\subset \mathcal{O}$ be any bounded interval. Since $u_{|]a,b[}$ is a minimizer in $BV(a,b)$ for

$$\int_a^b \sqrt{1+|Dv|^2} + |v(a^+)-u(a^+)| + |v(b^-)-u(b^-)| + \int_a^b F(v)\,dt,$$

by [6] u satisfies

$$\int_a^b \frac{(Du)^a\,(D\phi)^a}{\sqrt{1+|(Du)^a|^2}}\,dt + \int_a^b \operatorname{sgn}\left(\frac{Du}{|Du|}\right)(D\phi)^s + \int_a^b F'(u)\phi\,dt = 0,$$

for every $\phi \in BV(a,b)$ such that $|D\phi|^s$ is absolutely continuous with respect to $|Du|^s$, $\phi(a^+) = u(a^+)$ and $\phi(b^-) = u(b^-)$. By Proposition 3.2 $u_{|]a,b[} \in W^{1,1}(a,b)$, hence $u_{|]a,b[}$ is a weak solution of the Dirichlet problem

$$\begin{cases} \left(v'/\sqrt{1+(v')^2}\right)' = F'(v), \\ v(a) = u(a^+),\ v(b) = u(b^-). \end{cases} \tag{19}$$

Since $u'/\sqrt{1+(u')^2} \in W^{1,1}(a,b)$ and $F(u(t)) < 1$ for all $t \in]a,b[$, we easily obtain that $u_{|]a,b[} \in C^2(a,b)$ and $u_{|]a,b[}$ satisfies the equation in (19). Therefore (i_1) holds.

Let us prove (i_2). Let $K_0 =]-\infty,\tau_0[$ as in Remark 3.1. For each $t \in]-\infty,\tau_0[$ let us consider the energy function

$$E(t) = 1 - 1/\sqrt{1+u'(t)^2} - F(u(t)). \tag{20}$$

It follows from the equation in (19) that E is constant on $]-\infty,\tau_0[$. Since $\lim_{t\to-\infty} F(u(t)) = 0$ we deduce the the limit $\lim_{t\to-\infty} u'(t)$ does exist. Since $\lim_{t\to-\infty} u(t) = -1$, we conclude that $\lim_{t\to-\infty} u'(t) = 0$. Similarly we prove that $\lim_{t\to+\infty} u'(t) = 0$. Observe that, in particular, we have $E = 0$ on $]-\infty,\tau_0[$.

Let us prove (i_3). Suppose $K = K_i =]a,b[\subset \mathcal{O}$, as defined in Remark 3.1, is bounded from below. As in the previous paragraph we observe that the energy function E defined in (20) is constant on K, hence the limit $\lim_{t\to a^+} u'(t)$ does exist. We aim to show that $\lim_{t\to a^+} u'(t) = +\infty$. Arguing by contradiction, suppose that $\lim_{t\to a^+} u'(t) < +\infty$. By the continuity of the functions $F(u(\cdot))$ and u', we can take numbers $0 \le k_1 < k_2$, $\varepsilon > 0$ and $\delta > 0$ such that $a+\delta < b$,

$$\varepsilon^2 + 2\varepsilon k_2 < 1,$$

$$F(u(t)) > 1-\varepsilon$$

for all $t \in]a, a+\delta[$,

$$k_2{}^2 - k_1{}^2 < 1 - \varepsilon^2 - 2\varepsilon k_2,$$

and

$$0 \le k_1 \le u'(t) \le k_2$$

for all $t \in]a, a+\delta[$.

Let us define $v \in \mathcal{E}$ by setting

$$v(t) = \begin{cases} u(t) & \text{if } t \le a, \\ u(t+\delta) & \text{if } t > a. \end{cases}$$

We claim that $\mathcal{I}(v) < \mathcal{I}(u)$. By the intermediate value theorem we have, for some $\sigma \in]0,1[$,

$$u(a+\delta) - u(a) = u'(a+\sigma\delta)\delta,$$

hence, observing that $k_2 < \sqrt{1+k_1^2} - \varepsilon$, we compute

$$\begin{aligned} \mathcal{I}(v) - \mathcal{I}(u) &= \big(u(a+\delta) - u(a)\big) - \int_a^{a+\delta} \sqrt{1+(u')^2}\,dt + \delta - \int_a^{a+\delta} F(u(t))\,dt \\ &\le (k_2 - \sqrt{1+k_1^2} + \varepsilon)\delta < 0. \end{aligned}$$

Since u was assumed to be a minimizer of $\mathcal{I}$ we have a contradiction. Therefore $\lim\limits_{t\to a^+} u'(t) = +\infty$. Similarly we prove that, if $K_i =]a,b[\subset \mathcal{O}$, as defined in Remark 3.1, is bounded from above, we have $\lim\limits_{t\to b^-} u'(t) = +\infty$. Observe that, in particular, we have $E = 0$ on $\mathcal{O}$.

Let us prove (i_4). Let u_1 and u_2 be minimizers of $\mathcal{I}$ on $\mathcal{E}$. Following the notation introduced in Remark 3.1, we denote the intervals K_i associated to u_j, for $j = 1,2$, by K_i^j, and write $\mathcal{O}^j = \bigcup_{i\in N^j} K_i^j$, $K_0^j =]-\infty, \tau_0^j[$. Possibly translating u_2 we may assume that $K_0 := K_0^1 = K_0^2$ and $\tau_0 := \tau_0^1 = \tau_0^2$. By energy considerations we deduce that $u_1 = u_2$ on K_0. If $K_0 = \mathbb{R}$ we are done. Otherwise we note that $u_1(\tau_0^-) = u_2(\tau_0^-)$. By Proposition 3.1 we must have $u_1(\tau_0^+) = u_2(\tau_0^+)$. Let $i_1 \in N^1$ be such that $K_{i_1}^1 =]\tau_0, \tau_{i_1}^1[$ and $K_{i_2}^2 =]\tau_0, \tau_{i_2}^2[$ and set $\tau = \min\{\tau_{i_1}^1, \tau_{i_2}^2\}$. Again using the fact that the energy is zero for both u_1 and u_2 on $]\tau_0, \tau[$ we conclude that $K_i := K_{i_1}^1 = K_{i_2}^2$ and $u_1 = u_2$ on K_i. A recursive argument finally shows that actually $N := N^1 = N^2$, $K_i := K_i^1 = K_i^2$ and $u_1 = u_2$ on K_i for all $i \in N$. □

Taking advantage of the properties of the possible minimizers of $\mathcal{I}$ in $\mathcal{E}$, highlighted in Proposition 3.1, we can easily prove the existence of such minimizers.

Theorem 3.4. *Assume (h_0). Then there exists $u \in \mathcal{E}$ such that $\mathcal{I}(u) = \min\limits_{\mathcal{E}} \mathcal{I}$.*

Proof. Claim 1. There exists a minimizer of $\mathcal{I}$ in the set

$$\hat{\mathcal{E}} = \{u \in \mathcal{E} : u \textit{ is increasing a.e. in } \mathbb{R}\}.$$

Let $(u_n)_n$ be a sequence in $\hat{\mathcal{E}}$ such that

$$\lim_{n\to+\infty} \mathcal{I}(u_n) = \inf_{\hat{\mathcal{E}}} \mathcal{I}.$$

For each n, we have

$$|u_n(t)| \le 1$$

for a.e. $t \in \mathbb{R}$. Since $\mathcal{I}$ is invariant under translations, i.e., for any given $\tau \in \mathbb{R}$ and every $v \in \mathcal{E}$, $\mathcal{I}(v(\cdot - \tau)) = \mathcal{I}(v)$, we can also suppose that

$$u_n(t) \le 0 \quad \text{for a.e. } t \le 0 \quad \text{ and } \quad u_n(t) \ge 0 \quad \text{for a.e. } t \ge 0.$$

As $\sup_n \mathcal{I}(u_n) \leq 2$, we have

$$\sup_n \mathcal{J}(u_n) \leq 2.$$

Fix $N \in \mathbb{N}^+$ and denote by u_n^N the restriction to $]-N,N[$ of the function u_n. Since the sequence $(u_n^N)_n$ is bounded in $BV(-N,N)$, there exists a subsequence which converges in $L^1(-N,N)$ and a.e. in $]-N,N[$ to a function $u^N \in BV(-N,N)$. By a diagonal argument, we can extract a subsequence of $(u_n)_n$, still denoted by $(u_n)_n$, which converges to a function $u \in BV_{\text{loc}}(\mathbb{R})$ a.e. in $\mathbb{R}$ and in $L^1_{\text{loc}}(\mathbb{R})$.

Since each u_n is increasing a.e. in $\mathbb{R}$, the same holds true for u. Moreover, for every $\varepsilon \in]0,1[$, there exist $s_n, t_n \in \mathbb{R}$, with $s_n \leq 0 \leq t_n$, such that

$$u_n(t) \leq -1+\varepsilon \quad \text{for a.e. } t \leq s_n,$$

$$u_n(t) \geq 1-\varepsilon \quad \text{for a.e. } t \geq t_n,$$

and, if $s_n < t_n$,

$$-1+\varepsilon \leq u_n(t) \leq 1-\varepsilon \quad \text{for a.e. } t \in [s_n, t_n].$$

From the inequality

$$2 \geq \mathcal{I}(u_n) \geq \int_{s_n}^{t_n} F(u_n)\,dt \geq \big(\min_{[-1+\varepsilon,1-\varepsilon]} F(s)\big)\ (t_n - s_n), \tag{21}$$

we infer that the sequence $(t_n - s_n)_n$ is bounded and therefore both $(t_n)_n$ and $(s_n)_n$ are bounded. Hence, we may suppose that $(s_n)_n$ converges to some $\bar{s} \in \mathbb{R}$ and $(t_n)_n$ converges to some $\bar{t} \in \mathbb{R}$, with $\bar{s} \leq 0 \leq \bar{t}$. Accordingly, we have

$$-1 \leq u(t) \leq -1+\varepsilon \quad \text{for a.e. } t \leq \bar{s} \quad \text{and} \quad 1-\varepsilon \leq u(t) \leq 1 \quad \text{for a.e. } t \geq \bar{t}.$$

Thus we conclude that $u \in \hat{\mathcal{E}}$.

Combining the lower semicontinuity of $\mathcal{J}$, proved in Proposition 2.1, and Fatou's lemma, we deduce that

$$\begin{aligned}\mathcal{I}(u) = \mathcal{J}(u) + \int_{-\infty}^{+\infty} F(u)\,dt &\leq \liminf_{n\to+\infty} \mathcal{J}(u_n) + \liminf_{n\to+\infty} \int_{-\infty}^{+\infty} F(u_n)\,dt \\ &\leq \liminf_{n\to+\infty} \mathcal{I}(u_n) = \inf_{\hat{\mathcal{E}}} \mathcal{I},\end{aligned}$$

i.e. $\mathcal{I}(u) = \min_{\hat{\mathcal{E}}} \mathcal{I}$, thus concluding the proof of our first claim.

Claim 2. We have $\min_{\hat{\mathcal{E}}} \mathcal{I} = \inf_{\mathcal{E}} \mathcal{I}$. If this is not true, there exists $u \in \mathcal{E}$ such that $\mathcal{I}(u) \leq \min_{\hat{\mathcal{E}}} \mathcal{I} - \varepsilon$ for some $\varepsilon > 0$. Then, using the approximation property stated in Proposition 2.2, we can assume $u \in \mathcal{F}_\omega$ and $\mathcal{I}(u) \leq \min_{\hat{\mathcal{E}}} \mathcal{I} - \varepsilon/2$. It then follows from Theorem 2.5 that the rearranged function $u^\star$ satisfies

$$\mathcal{I}(u^\star) \leq \mathcal{I}(u) < \min_{\hat{\mathcal{E}}} \mathcal{I}$$

which is obviously a contradiction.

Conclusion. Combining Claim 1 and Claim 2, we deduce that

$$\inf_{\mathcal{E}} \mathcal{I} = \min_{\mathcal{E}} \mathcal{I},$$

since this level is achieved by a function in $\hat{\mathcal{E}}$. $\square$

4 Analysis of the N-dimensional problem

In this section we study the N-dimensional equation (6), in connection with the version of the Gibbons' conjecture for (6) mentioned in the introduction. Namely, we discuss the one-dimensional character of the minimizers of the functional $\mathcal{I}_\omega : \mathcal{E}_\omega \to [0, +\infty]$ defined by

$$\mathcal{I}_\omega(v) = \mathcal{J}_\omega(v) + \iint_{\mathbb{R}\times\omega} F(v)\, dtdx,$$

having assumed condition (h_0).

Theorem 4.1. *Let ω be a non-empty open bounded set in $\mathbb{R}^{N-1}$. Assume (h_0). Let $u \in \mathcal{E}_\omega$ be a minimizer of $\mathcal{I}_\omega$. Then u coincides with its increasing rearrangement $u^\star$.*

Proof. Let $u \in \mathcal{E}_\omega$ be a minimizer of $\mathcal{I}_\omega$ and denote by $u^\star \in \mathcal{E}$ its increasing rearrangement. Note that, by Proposition 2.3, both $\|u\|_\infty \le 1$ and $\|u^\star\|_\infty \le 1$ hold.

Claim 1. There is a sequence $(u_n)_n$ in $\mathcal{F}_\omega$, with $\|u_n\|_\infty \le 1$ for all n, such that, for each $T > 0$, $(u_{n|]-T,T[\times\omega})_n$ converges in $L^1(]-T,T[\times\omega)$ to $u_{|]-T,T[\times\omega}$ and

$$\lim_{n\to+\infty} \mathcal{J}_\omega(u_n) = \mathcal{J}_\omega(u).$$

In addition this sequence can be chosen so that the sequence $(u_n^\star)_n$ of the corresponding increasing rearrangements is such that, for each $T > 0$, $(u^\star_{n|]-T,T[\times\omega})_n$ converges in $L^1(]-T,T[\times\omega)$ to ${u^\star}_{|]-T,T[\times\omega}$, and

$$\lim_{n\to+\infty} \mathcal{J}_\omega(u_n^\star) = \mathcal{J}_\omega(u^\star),$$

with

$$\mathcal{J}_\omega(u^\star) = \mathcal{J}_\omega(u). \tag{22}$$

Let $(v_n)_n$ be a sequence in $\mathcal{F}_\omega$, whose existence is guaranteed by Proposition 2.2, satisfying (13), (14), with $v = u$, and

$$\lim_{n\to+\infty} \mathcal{I}_\omega(v_n) = \mathcal{I}_\omega(u).$$

Set, for each n,

$$u_n = (v_n \vee -1) \wedge 1.$$

By Proposition 2.3 we have

$$\iint_{\mathbb{R}\times\omega} \big(\sqrt{1+|\nabla u_n|^2} - 1\big)\, dtdx \le \iint_{\mathbb{R}\times\omega} \big(\sqrt{1+|\nabla v_n|^2} - 1\big)\, dtdx;$$

moreover, it is easy to see that

$$\iint_{\mathbb{R}\times\omega} g(|u_n - u|)\, dtdx \le \iint_{\mathbb{R}\times\omega} g(|v_n - u|)\, dtdx \tag{23}$$

and

$$\iint_{\mathbb{R}\times\omega} F(u_n)\, dtdx \le \iint_{\mathbb{R}\times\omega} F(v_n)\, dtdx.$$

Therefore, from

$$\limsup_{n\to+\infty} \mathcal{I}_\omega(u_n) \le \lim_{n\to+\infty} \mathcal{I}_\omega(v_n) = \mathcal{I}_\omega(u),$$

we infer

$$\lim_{n\to+\infty} \mathcal{J}_\omega(u_n) = \mathcal{J}_\omega(u). \tag{24}$$

By Theorem 2.5, we have, for any $v \in \mathcal{E}_\omega \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}\times\omega)$,

$$\iint_{\mathbb{R}\times\omega} \big(\sqrt{1+|\nabla v^\star|^2}-1\big)dtdx \le \iint_{\mathbb{R}\times\omega} \big(\sqrt{1+|\nabla v|^2}-1\big)dtdx.$$

Hence, (24) implies

$$\limsup_{n\to+\infty} \mathcal{J}_\omega(u_n^\star) \le \lim_{n\to+\infty} \mathcal{J}_\omega(u_n) = \mathcal{J}_\omega(u).$$

On the other hand, [2, Proposition 2.7] yields

$$\iint_{\mathbb{R}\times\omega} g(|u_n^\star - u^\star|)dtdx \le \iint_{\mathbb{R}\times\omega} g(|u_n - u|)dtdx.$$

Hence, combining (23) and (13), we obtain

$$\lim_{n\to+\infty} \iint_{\mathbb{R}\times\omega} g(|u_n^\star - u^\star|)\, dtdx = 0. \tag{25}$$

This implies, as g is increasing, that the sequence $(u_n^\star)_n$ converges in measure to $u^\star$ and therefore a.e. in $\mathbb{R}\times\omega$, up to the extraction of a subsequence that we still denote by $(u_n^\star)_n$. Since $\|u_n^\star\|_\infty \le 1$ for all n, $(u^\star_{n\,|]-T,T[\times\omega})_n$ converges to $u^\star{}_{|]-T,T[\times\omega}$ in $L^1(]-T,T[\times\omega)$ for each $T>0$. The semicontinuity property stated in Proposition 2.1 and the estimate (25) imply

$$\mathcal{J}_\omega(u^\star) \le \liminf_{n\to+\infty} \mathcal{J}_\omega(u_n^\star) \le \mathcal{J}_\omega(u).$$

Since, by [2, Theorem 2.6], we have

$$\iint_{\mathbb{R}\times\omega} F(u^\star)\, dtdx = \iint_{\mathbb{R}\times\omega} F(u)\, dtdx,$$

it follows that $\mathcal{I}(u^\star) = \mathcal{I}(u)$ and thus $u^\star$ is a minimizer of $\mathcal{I}$ in $\mathcal{E}$. Hence we also get

$$\mathcal{J}_\omega(u^\star) = \lim_{n\to+\infty} \mathcal{J}_\omega(u_n^\star) = \mathcal{J}_\omega(u). \tag{26}$$

This concludes the proof of Claim 1.

Without restriction we can identify $u^\star$ with the right-trace function $u^\star_+$. Proposition 3.1 states that $u^\star$ is increasing on $\mathbb{R}$ and strictly increasing on the set $\{t\in\mathbb{R} : |u^\star(t)|<1\}$. Hence, by [2, Remark 2.3] we have, for a.e. $s \in]-1,1[$,

$$\mathrm{meas}(\{(t,x)\in\mathbb{R}\times\omega : u(t,x)=s\}) = \mathrm{meas}(\{(t,x)\in\mathbb{R}\times\omega : u^\star(t,x)=s\}) = 0.$$

Let us set, for each $s\in]-1,1[$ and all n,

$$E_s(u) = \{(t,x)\in\mathbb{R}\times\omega : u(t,x)\ge s\ \}, \qquad E_s(u^\star) = \{(t,x)\in\mathbb{R}\times\omega : u^\star(t,x)\ge s\ \},$$
$$E_s(u_n) = \{(t,x)\in\mathbb{R}\times\omega : u_n(t,x)\ge s\ \}, \quad E_s(u_n^\star) = \{(t,x)\in\mathbb{R}\times\omega : u_n^\star(t,x)\ge s\ \}.$$

It is clear that (cf. [2, Lemma 1.5])

$$\mathrm{Per}(E_s(u^\star)) = \mathrm{Per}(E_s(u_n^\star)) = \mathrm{meas}(\omega).$$

Claim 2. For a.e. $s \in \,]-1,1[$, we have

$$\lim_{n\to+\infty} \chi_{E_s(u_n)}(t,x) = \chi_{E_s(u)}(t,x) \quad \text{a.e. in } \mathbb{R}\times\omega,$$

and hence the sequence $(\chi_{E_s(u_n)})_n$ converges to $\chi_{E_s(u)}$ in $L^1_{\mathrm{loc}}(\mathbb{R}\times\omega)$. We know that, for a.e. $s \in \,]-1,1[$,

$$\mathrm{meas}\{(t,x)\in\mathbb{R}\times\omega : u(t,x)=s\} = 0.$$

Fix such a number $s \in \,]-1,1[$. Since the sequence $(u_n)_n$ converges to u a.e. in $\mathbb{R}\times\omega$, we infer that, for a.e. $(t,x)\in\mathbb{R}\times\omega$, if $u(t,x) > s$, then $u_n(t,x) > s$ for all sufficiently large n. Hence the conclusions immediately follow. This concludes the proof of Claim 2.

Claim 3. For a.e. $s \in \,]-1,1[$ we have

$$\lim_{n\to+\infty} \mathrm{Per}(E_s(u_n)) = \mathrm{meas}(\omega).$$

Assume, by contradiction, that there are a constant $\delta > 0$ and a set $D \subset \,]-1,1[$, with $\mathrm{meas}(D) > 0$, such that

$$\mathrm{Per}(E_s(u_n)) > \mathrm{meas}(\omega) + \delta \tag{27}$$

for all n and every $s \in D$. We borrow now some ideas from the proof of [2, Theorem 2.10] for showing that there is a constant $\eta > 0$ such that, for infinitely many n,

$$\mathcal{J}(u_n^\star) \le \mathcal{J}(u_n) - \eta; \tag{28}$$

thus contradicting (24) and (26). Without restriction we can also suppose that D is a Borel set and there is $d \in \,]0,1[$ such that

$$D \subseteq \,]-d,d[.$$

Let us prove (28). We identify each $u_n^\star$ with a function depending only on the variable t, i.e., $u_n^\star : \mathbb{R} \to [-1,1]$. Set

$$H_n = \{s \in [-1,1] : (u_n^\star)^{-1}(\{s\}) \text{ is not a singleton}\}$$

and

$$J_n = \bigcup_{s\in H_n} (u_n^\star)^{-1}(\{s\}).$$

Since $u_n^\star$ is non-decreasing and continuous, H_n is at most countable and J_n is the countable union of closed intervals. Define $I_n = \mathbb{R}\setminus J_n$, $v_n^\star = u^\star_{n|I_n}$ and $K_n = u_n^\star(I_n) = \,]-1,1[\setminus H_n$. The function $v_n^\star$ is strictly increasing, with inverse $(v_n^\star)^{-1} : K_n \to I_n$. Clearly we have, by [39, Theorem 1.56], $u_n^{\star\prime} = 0$ a.e. in J_n and hence

$$\begin{aligned}\mathcal{J}(u_n^\star) &= \iint_{\mathbb{R}\times\omega} \big(\sqrt{1+|\nabla u_n^\star|^2}-1\big)\,dtdx \\ &= \mathrm{meas}(\omega)\int_{-\infty}^{+\infty} \big(\sqrt{1+|u_n^{\star\prime}|^2}-1\big)\,dt = \mathrm{meas}(\omega)\int_{I_n} \big(\sqrt{1+|u_n^{\star\prime}|^2}-1\big)\,dt.\end{aligned} \tag{29}$$

Moreover we can write, for each s,

$$\sqrt{1+s^2}-1 = a(s)s - b(s),$$

where

$$a(s)=\frac{s}{\sqrt{1+s^2}}\quad\text{and}\quad b(s)=s^2\Big(\frac{1}{\sqrt{1+s^2}}-\frac{1}{1+\sqrt{1+s^2}}\Big).$$

Observe that, by convexity, we also have, for every σ,

$$\sqrt{1+\sigma^2}-1\geq a(s)\sigma-b(s). \tag{30}$$

Let us set, for $s\in K_n$,

$$a_n(s)=a(|{u_n^\star}'(v_n^\star)^{-1}(s)|)\quad\text{and}\quad b_n(s)=b(|{u_n^\star}'(v_n^\star)^{-1}(s)|),$$

and, for $s\in H_n$,

$$a_n(s)=0=b_n(s).$$

Clearly $a_n,b_n\in L^\infty(-1,1)$. We have, by (29) and the definitions of a_n and b_n,

$$\begin{aligned}
\mathcal{J}(u_n^\star)&=\iint_{\mathbb{R}\times\omega}(\sqrt{1+|\nabla u_n^\star|^2}-1)\,dtdx=\operatorname{meas}(\omega)\int_{I_n}(\sqrt{1+|{u_n^\star}'|^2}-1)\,dt\\
&=\operatorname{meas}(\omega)\int_{I_n}\Big(a(|{u_n^\star}'|)|{u_n^\star}'|-b(|{u_n^\star}'|)\Big)\,dt\\
&=\operatorname{meas}(\omega)\Big(\int_{I_n}a_n(v_n^\star){u_n^\star}'\,dt-\int_{-\infty}^{+\infty}b_n(v_n^\star)\,dt\Big)\\
&=\operatorname{meas}(\omega)\int_{(v_n^\star)^{-1}(K_n)}a_n(v_n^\star){u_n^\star}'\,dt-\iint_{\mathbb{R}\times\omega}b_n(u_n^\star)\,dtdx.
\end{aligned}\tag{31}$$

Using the coarea formula (see, e.g., [33, Theorem 1.13, Theorem 16.1] and [7, Theorem 10.3.3]), it follows that

$$\begin{aligned}
\int_{(v_n^\star)^{-1}(K_n)}a_n(v_n^\star){u_n^\star}'\,dt&=\int_{-\infty}^{+\infty}\Big(\int_{(v_n^\star)^{-1}(K_n)\cap(v_n^\star)^{-1}(\{s\})}a_n(v_n^\star)(t)d\mathcal{H}^0(t)\Big)\,ds\\
&=\int_{K_n}a_n(s)\,ds.
\end{aligned}\tag{32}$$

Combining (31) and (32) and using (27) we obtain

$$\begin{aligned}
\mathcal{J}(u_n^\star)&=\int_{-1}^{1}a_n(s)\operatorname{meas}(\omega)\,ds-\iint_{\mathbb{R}\times\omega}b_n(u_n)\,dtdx\\
&\leq\int_{-1}^{1}a_n(s)\operatorname{Per}(E_s(u_n))\,ds-\delta\int_D a_n(s)\,ds-\iint_{\mathbb{R}\times\omega}b_n(u_n)\,dtdx\\
&=\int_{-1}^{1}a_n(s)\Big(\iint_{\mathbb{R}\times\omega}|D\chi_{E_s(u_n)}|\Big)\,ds-\delta\int_D a_n(s)\,ds-\iint_{\mathbb{R}\times\omega}b_n(u_n)\,dtdx\\
&=\int_{-\infty}^{+\infty}\Big(\iint_{\mathbb{R}\times\omega}a_n(u_n)|D\chi_{E_s(u_n)}|\Big)\,ds-\delta\int_D a_n(s)\,ds-\iint_{\mathbb{R}\times\omega}b_n(u_n)\,dtdx.
\end{aligned}\tag{33}$$

Applying the coarea formula one more time, we obtain, from (33),

$$\mathcal{J}(u_n^\star)\leq\iint_{\mathbb{R}\times\omega}\Big(a(|{u_n^*}'(v_n^\star)^{-1}(u_n)|)\,|\nabla u_n|-b(|{u_n^*}'(v_n^\star)^{-1}(u_n)|)\Big)\,dtdx-\delta\int_D a_n(s)\,ds$$

and hence, using (30),

$$\begin{aligned}\mathcal{J}(u_n^\star) &\le \iint_{\mathbb{R}\times\omega} (\sqrt{1+|\nabla u_n|^2}-1)\,dtdx - \delta\int_D a_n(s)\,ds\\ &= \mathcal{J}(u_n) - \delta\int_D a_n(s)\,ds.\end{aligned}$$

In order to achieve (28) it is sufficient to show that

$$\liminf_{n\to+\infty}\int_D a_n(s)\,ds > 0.$$

Suppose by contradiction that

$$\lim_{n\to+\infty}\int_D a_n(s)\,ds = 0.$$

At first, using a change of variable formula, we notice that

$$\begin{aligned}\int_D a_n(s)\,ds &= \int_{D\cap K_N} \frac{{u_n^*}'((v_n^*)^{-1}(s))}{\sqrt{1+({u_n^*}'((v_n^*)^{-1}(s)))^2}}\,ds\\ &= \int_{-\infty}^{+\infty}\Big(\int_{(v_n^\star)^{-1}(D\cap K_n)\cap(v_n^\star)^{-1}(\{s\})} \frac{{u_n^*}'(t)}{\sqrt{1+({u_n^*}'(t))^2}} d\mathcal{H}^0(t)\Big)\,ds\\ &= \int_{(v_n^*)^{-1}(D\cap K_n)} \frac{({u_n^*}'(t))^2}{\sqrt{1+({u_n^*}'(t))^2}}\,dt.\end{aligned}$$

As $D\subseteq[-d,d]$, for some $d\in\,]0,1[$, there is $a>0$ such that $u^\star(-a)<-d$, $u^\star(a)>d$, and

$$\lim_{n\to+\infty} u_n^\star(\pm a) = u^\star(\pm a).$$

Hence, by monotonicity, we get

$$(v_n^\star)^{-1}(D\cap K_n)\subseteq(u_n^*)^{-1}(D)\subseteq[-a,a],$$

for all n sufficiently large. For each $\varepsilon>0$, set

$$A_n = \{t\in[-a,a] : \chi_{(v_n^*)^{-1}(D\cap K_n)}(t)\,{u_n^\star}'(t)\ge\varepsilon\}.$$

We have

$$\begin{aligned}&\int_{(v_n^*)^{-1}(D\cap K_n)} \frac{({u_n^*}'(t))^2}{\sqrt{1+({u_n^*}'(t))^2}}\,dt\\ &= \int_{(v_n^*)^{-1}(D\cap K_n)} \frac{(\chi_{(v_n^*)^{-1}(D\cap K_n)}(t)\,{u_n^*}'(t))^2}{\sqrt{1+(\chi_{(v_n^*)^{-1}(D\cap K_n)}(t)\,{u_n^*}'(t))^2}}\,dt\\ &\ge \frac{\varepsilon^2}{\sqrt{1+\varepsilon^2}}\,\mathrm{meas}(A_n).\end{aligned}$$

Hence the sequence $(\chi_{(v_n^*)^{-1}(D\cap K_n)}\,{u_n^*}')_n$ converges to 0 in measure in $[-a,a]$ and then, maybe by taking a subsequence if necessary,

$$\lim_{n\to+\infty}\int_{(v_n^*)^{-1}(D\cap K_n)} {u_n^*}'(t)\,dt = 0.$$

As

$$\int_{(v_n^*)^{-1}(D\cap K_n)} {u_n^*}'(t)\,dt = \mathrm{meas}(D\cap K_n) = \mathrm{meas}(D),$$

we get a contradiction. This concludes the proof of Claim 3.

From Claim 2 and the lower semicontinuity of the variations with respect to the L^1_{loc}-convergence, we infer, for a.e. $s \in \,]-1,1[$,

$$\mathrm{Per}(E_s(u)) = \iint_{\mathbb{R}\times\omega} |D\chi_{E_s(u)}| \le \liminf_{n\to+\infty} \iint_{\mathbb{R}\times\omega} |D\chi_{E_s(u)}| = \liminf_{n\to+\infty} \mathrm{Per}(E_s(u_n)).$$

Claim 3 and [1, Lemma 1.5] imply that, for a.e. $s \in \,]-1,1[$, $\mathrm{Per}(E_s(u)) = \mathrm{meas}(\omega) = \mathrm{Per}(E_s(u^\star))$ and hence $E_s(u)$ coincides with $E_s(u^\star)$ up to a set having zero measure. By using the representation

$$v(x) = \int_{-\infty}^{+\infty} \kappa_s(v)(x)\,ds,$$

where

$$\kappa_s(v) = \begin{cases} \chi_{E_s(v)} & \text{if } s > 0,\\ -1 + \chi_{E_s(v)} & \text{if } s \le 0,\end{cases}$$

we finally conclude that $u = u^\star$. □

With Theorem 4.1 at hand, we obtain a complete description of the interface between the two phases. Indeed, combining Theorem 4.1 and Proposition 3.3 immediately yields the following statement.

Corollary 4.2. *Let ω be a non-empty open bounded set in $\mathbb{R}^{N-1}$ and assume (h_1). Then there exists, up to translations, a unique minimizer u of $\mathcal{I}_\omega$ in $\mathcal{E}_\omega$. Furthermore, u coincides with its increasing rearrangement $u^\star$ and $u \in C^1(\omega\times\mathbb{R})$ if and only if*

$$\max_{[-1,1]} F(s) < 1.$$

In this case $u \in C^2(\omega\times\mathbb{R})$ and it satisfies equation (6) in the classical sense.

Proof. Suppose $u \in \mathcal{E}_\omega$ is a minimizer of $\mathcal{I}_\omega$. By Theorem 4.1, we have $u = u^\star$. Since $u^\star$ is a minimizer of $\mathcal{I}$ in $\mathcal{E}$, Proposition 3.3 applies. In particular, u is uniquely determined up to translations. If $\max_{[-1,1]} F(s) < 1$, then $u^\star \in C^2(\mathbb{R})$, hence $u \in C^2(\omega\times\mathbb{R})$ and it satisfies equation (6) in the classical sense. If $\max_{[-1,1]} F(s) \ge 1$, we can take $t_0 \in \mathbb{R}$ such that $F\big(u^\star(t)\big) < 1$ for all $t < t_0$ and $F\big(u^\star(t_0)\big) = 1$. Then $(u^\star)'(t_0) = +\infty$, thus $u \notin C^1(\omega\times\mathbb{R})$. □

Observe also that even when the condition

$$\max_{[-1,1]} F(s) < 1$$

is not satisfied, the profile of the optimal transition u is smooth in the sense that the boundary of the set $\{(x,y) \in \omega\times\mathbb{R}\times\mathbb{R} \mid y < u(x)\}$ is a C^2 manifold. Indeed at the points of discontinuity, both u' and u'' diverge.

We conclude with a final remark. As a consequence of the last statement, we can derive a rigidity assertion similar to [22, Theorem 1.1]. Indeed, as a particular case of the results of [13], we infer that if $u \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ solves (6) and $\sup_{\mathbb{R}^N} |\nabla u| < \infty$, then for every $x \in \mathbb{R}^N$, we have

$$1 - 1/\sqrt{1 + |\nabla u(x)|^2} \leq F(u(x)).$$

Moreover, if the equality holds for some point $x_0 \in \mathbb{R}^N$, then there exists a function $h : \mathbb{R} \to \mathbb{R}$, $\alpha \in \mathbb{R}$ and a vector $\beta \in \mathbb{R}^N$ such that, for every $x \in \mathbb{R}^N$,

$$u(x) = h(\alpha + \beta \cdot x).$$

Therefore, if $u \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ solves (6), ∇u is bounded and there exists an open bounded set $\omega \subset \mathbb{R}^{N-1}$ such that u minimizes the energy functional $\mathcal{I}_\omega$ in $\mathcal{E}_\omega$, in particular u satisfies

$$\lim_{x_1 \to \pm\infty} u(x_1, x_2, \ldots, x_N) = \pm 1 \quad \text{uniformly a.e. with respect to } (x_2, \ldots, x_N) \in \omega,$$

then u coincides with its increasing rearrangement $u^\star$ and the equality

$$1 - 1/\sqrt{1 + |\nabla u(x)|^2} = F(u(x))$$

holds for every $x \in \mathbb{R} \times \omega$. Observe that this can happen if and only if $\max_{[-1,1]} F(s) < 1$. As a consequence of [13], u is a function of x_1 only.

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