EXTREMAL COMPLEX POLYTOPE NORMS FOR FAMILIES OF REAL MATRICES*

N. GUGLIELMI† AND M. ZENNARO‡

Abstract. In this paper we consider finite families $\mathscr{F}$ of real $n \times n$-matrices. In particular, we focus on the computation of the *joint spectral radius* $\rho(\mathscr{F})$ via the detection of an *extremal norm* in the class of *complex polytope norms*, whose unit balls are *balanced complex polytopes* with a finite *essential system of vertices*. Such a finiteness property is very useful in view of the construction of efficient computational algorithms. More precisely, we improve the results obtained in our previous paper [GWZ05], where we gave some conditions on the family $\mathscr{F}$ which are sufficient to guarantee the existence of an extremal complex polytope norm. Unfortunately, they are such to exclude unnecessarily many interesting cases of real families. Therefore, here we relax somehow the conditions given in [GWZ05], in order to provide a more satisfactory treatment of the real case.

Keywords: Families of matrices, joint spectral radius, extremal norms, complex polytope norms, complex conjugate leading eigenpairs.

1. Introduction. In the last decade, a significant progress has been done within the theory of the *joint spectral radius* of matrix families. As is well known, given a bounded set of real or complex matrices $\mathscr{F}$, its joint spectral radius $\rho(\mathscr{F})$ determines the maximal growth of all products that can be formed by taking factors in $\mathscr{F}$. Therefore, its knowledge is important in all applications where the change of status can be described by more than one matrix. For instance, it characterizes the regularity of certain wavelets (see, e.g., [DL92, Mae95, Mae98]), the capacity of codes (see, e.g., [MAS01]), the stability of hybrid systems (see, e.g., [Bar88, Wir98]) or of the numerical solution of ordinary differential equations (see, e.g., [GZ01b]). Further references can be found in [DL01, Wir02].

In the light of this, it clearly appears that efficient methods for the computation of the joint spectral radius of a given set of matrices would be often very welcome.

Unfortunately, we are still far from the availability of such efficient methods on a general setting basis. Indeed, the theoretical forecasts in this sense are not well-disposed at all (see, e.g., [TB97]). Nevertheless, an algorithm for efficiently computing lower and upper bounds to $\rho(\mathscr{F})$ has been proposed in [Gri96] and, lately, further promising approaches for the approximation of $\rho(\mathscr{F})$ have been considered (see, e.g., [BN05b, BNT05, BN05a, Pro05]).

In [GWZ05], also the authors of the present paper have recently given a contribution in the direction of the computation of $\rho(\mathscr{F})$ by considering special classes of complex matrix families. In particular, they have determined some conditions on the family which are sufficient to guarantee the existence of an extremal *complex polytope norm*, that is a norm whose unit ball is a *balanced complex polytope* with a finite *essential system of vertices* (see [GZ07, VZss]). Such a finiteness property is very useful in view of the construction of algorithms aimed at the actual computation of $\rho(\mathscr{F})$ via the detection of an extremal norm, as is done, for example, in [GZ08].

However, the hypotheses made in [GWZ05] on the family of matrices are such to exclude unnecessarily many interesting cases of real families. In this paper, we confine ourselves to real matrix families and succeed in relaxing somehow the sufficient conditions assumed in [GWZ05], so as to provide a more satisfactory treatment of the real case. As in [GWZ05], under such relaxed conditions, we are able to construct the unit ball of an extremal norm, which is a balanced complex polytope with a finite essential system of vertices.

The organization of the paper is the following. In Section 2 we review some of the most important definitions and results available in the literature regarding the joint spectral radius of a family of matrices and the extremal norms. In Section 3 we recall the definitions of balanced complex polytope and complex polytope norm, and review some of their most important properties. In Section 4 we resume the definitions and the results given in [GWZ05]. The main results of this paper are stated and proved in Section 5. Then,

*This work was supported by MIUR and by INdAM - GNCS

†Dipartimento di Matematica Pura e Applicata, Università de L'Aquila, I-67100 L'Aquila (Italy), *e-mail:* guglielm@univaq.it

‡Dipartimento di Matematica e Informatica, Università di Trieste, I-34100 Trieste (Italy), *e-mail:* zennaro@units.it

in Section 6, we present a new computational algorithm and an illustrative example is provided in Section 7. Finally, in Section 8 we discuss the results of this paper in relation to the existing literature.

2. Preliminary results from the literature. For a bounded family of complex $n\times n$-matrices $\mathscr{F} = \{A^{(i)}\}_{i\in\mathscr{I}}$, the following definitions are given in the literature.

Let $\|\cdot\|$ be a given norm on the vector space $\mathbb{C}^n$ and let the same symbol $\|\cdot\|$ denote the corresponding induced $n\times n$-matrix norm, too. Then, for each $k\geq 1$, consider the set $\Sigma_k(\mathscr{F})$ of all possible products of length k whose factors are elements of $\mathscr{F}$, that is

$$\Sigma_k(\mathscr{F}) = \{A^{(i_1)}\cdots A^{(i_k)} \;\big|\; i_1,\ldots,i_k \in \mathscr{I}\},$$

and the number

$$\hat{\rho}_k(\mathscr{F}) = \sup_{P\in\Sigma_k(\mathscr{F})} \|P\|, \tag{2.1}$$

and define the *joint spectral radius* of $\mathscr{F}$ as

$$\hat{\rho}(\mathscr{F}) = \limsup_{k\to\infty} \hat{\rho}_k(\mathscr{F})^{1/k}$$

(see [RS60]). Note that the numbers $\hat{\rho}_k(\mathscr{F})$ depend on the particular norm $\|\cdot\|$ used in (2.1) whereas, by the equivalence of all the norms in finite dimensional spaces, $\hat{\rho}(\mathscr{F})$ is independent of it.

Analogously, let $\rho(\cdot)$ denote the spectral radius of an $n\times n$-matrix and then, for each $k\geq 1$, consider the number

$$\bar{\rho}_k(\mathscr{F}) = \sup_{P\in\Sigma_k(\mathscr{F})} \rho(P)$$

and define the *generalized spectral radius* of $\mathscr{F}$ as

$$\bar{\rho}(\mathscr{F}) = \limsup_{k\to\infty} \bar{\rho}_k(\mathscr{F})^{1/k}$$

(see [DL92]).

Recently it has been shown that

$$\hat{\rho}(\mathscr{F}) = \bar{\rho}(\mathscr{F})$$

(see [BW92, Els95, SWP97, Shi99]). This means that the joint and the generalized spectral radius of $\mathscr{F}$ are the same number, which we shall simply call the *spectral radius* of the family of matrices $\mathscr{F}$ and denote by $\rho(\mathscr{F})$. Such result generalizes the well-known Gelfand theorem for a single matrix.

Given a norm $\|\cdot\|$ on the vector space $\mathbb{C}^n$ and the corresponding induced $n\times n$-matrix norm, we shall still use the same notation to define

$$\|\mathscr{F}\| = \hat{\rho}_1(\mathscr{F}) = \sup_{i\in\mathscr{I}} \|A^{(i)}\|.$$

The following characterization of $\rho(\mathscr{F})$ can be found, for example, in [RS60] and [Els95].

THEOREM 2.1. *The spectral radius of a bounded family $\mathscr{F}$ of complex $n\times n$-matrices is characterized by the equality*

$$\rho(\mathscr{F}) = \inf_{\|\cdot\|\in\mathscr{N}} \|\mathscr{F}\|, \tag{2.2}$$

where $\mathscr{N}$ denotes the set of all possible induced $n\times n$-matrix norms.

Given a family $\mathscr{F}$, an important question to answer is whether or not the inf in (2.2) is actually attained by some induced matrix norm. To this purpose, we give the following definition.

DEFINITION 2.2. *We shall say that a norm $\|\cdot\|_*$ satisfying the condition*

$$\|\mathscr{F}\|_* = \rho(\mathscr{F})$$

is extremal *for the family $\mathscr{F}$.*

It is well known that, for a single family $\{A\}$, the existence of an extremal norm is equivalent to the fact that the matrix A is *nondefective*, i.e., all of the blocks relevant to the eigenvalues of maximum modulus are diagonal in its Jordan canonical form. Whenever $\rho(A) > 0$, another equivalent property is that, with $\hat{A} = \rho(A)^{-1}A$, the power set $\Sigma(\hat{A}) = \{\hat{A}^k \mid k \geq 1\}$ is bounded. These results generalize to a bounded family $\mathscr{F}$ as follows. Given a bounded family $\mathscr{F} = \{A^{(i)}\}_{i\in\mathscr{I}}$ of complex $n\times n$-matrices with $\rho(\mathscr{F}) > 0$, let us consider the *normalized* family

$$\hat{\mathscr{F}} = \{\rho(\mathscr{F})^{-1}A^{(i)}\}_{i\in\mathscr{I}},$$

whose spectral radius is $\rho(\hat{\mathscr{F}}) = 1$. Then consider the semigroup of matrices generated by $\hat{\mathscr{F}}$, i.e.

$$\Sigma(\hat{\mathscr{F}}) = \bigcup_{k\geq 1} \Sigma_k(\hat{\mathscr{F}}). \tag{2.3}$$

DEFINITION 2.3. *A bounded family $\mathscr{F}$ of complex $n\times n$-matrices is said to be* defective *if the corresponding normalized family $\hat{\mathscr{F}}$ is such that the semigroup $\Sigma(\hat{\mathscr{F}})$ is an unbounded set of matrices. Otherwise, if $\Sigma(\hat{\mathscr{F}})$ is bounded, then the family $\mathscr{F}$ is said to be* nondefective*.*

Note that we gave the definition of defective family without involving directly the spectral properties of its elements. The following result can be found, for example, in [Koz90] or [BW92].

PROPOSITION 2.4. *A bounded family $\mathscr{F}$ of complex $n\times n$-matrices admits an extremal norm $\|\cdot\|_*$ if and only if it is nondefective.*

In order to state another important result about defective and nondefective families (see [Bar88] or [Els95]), we give the following definition according to [RR00].

DEFINITION 2.5. *A bounded family $\mathscr{F} = \{A^{(i)}\}_{i\in\mathscr{I}}$ of complex $n\times n$-matrices is said to be* reducible *if there exist a nonsingular $n\times n$-matrix M and two integers $n_1, n_2 \geq 1$, $n_1 + n_2 = n$, such that, for all $i \in \mathscr{I}$, it holds that*

$$M^{-1}A^{(i)}M = \begin{bmatrix} A_{11}^{(i)} & A_{12}^{(i)} \\ O & A_{22}^{(i)} \end{bmatrix},$$

where the blocks $A_{11}^{(i)}$, $A_{12}^{(i)}$, $A_{22}^{(i)}$ are $n_1\times n_1$-, $n_1\times n_2$- and $n_2\times n_2$-matrices, respectively. If a family $\mathscr{F}$ is not reducible, then it is said to be irreducible*.*

THEOREM 2.6. *If a bounded family $\mathscr{F}$ of complex $n\times n$-matrices is defective, then it is reducible.*

Remark that the opposite implication is not necessarily true. For example, for $n \geq 2$ all single families $\{A\}$ are clearly reducible, but not necessarily defective.

The following corollary to Theorem 2.6 is obvious.

COROLLARY 2.7. *If a bounded family $\mathscr{F}$ of complex $n\times n$-matrices is irreducible, then it is nondefective.*

An important conjecture, the so called *Finiteness Conjecture*, arisen from work in [DL92] and stated in [LW95], whose validity would be of much help for the actual computation of the spectral radius $\rho(\mathscr{F})$ of finite families, involves the following definitions.

DEFINITION 2.8 (Finiteness Property). *A finite family of complex $n\times n$-matrices $\mathscr{F}$ is said to have the* finiteness property *if, there exist $k^* \geq 1$ and a product $\tilde{P} \in \Sigma_{k^*}(\mathscr{F})$ such that*

$$\rho(\mathscr{F}) = \bar{\rho}_{k^*}(\mathscr{F})^{1/k^*} = \rho(\tilde{P})^{1/k^*}. \tag{2.4}$$

DEFINITION 2.9. *If $\mathscr{F}$ is a bounded family of complex $n\times n$-matrices, any matrix $\tilde{P}\in\Sigma_{k^*}(\mathscr{F})$ satisfying* (2.4) *for some $k^*\geq 1$ will be called a* spectrum-maximizing product *(in short, an* s.m.p.*) for $\mathscr{F}$.*

Observe that the finiteness property means the existence of at least one s.m.p. $\tilde{P}$ for finite families. In [LW95] some sufficient conditions in terms of extremal norms, guaranteeing that the finiteness property holds, have been given. On the contrary, the Finiteness Conjecture, stating that all finite sets of matrices have the finiteness property, was instead recently disproved in [BM02] and, later, also in [BTV03].

Now we recall the following definition from [GZ01a].

DEFINITION 2.10. *Assume that $\mathscr{F}$ is a normalized bounded family of complex $n\times n$-matrices (i.e., $\rho(\mathscr{F})=1$) and that there exists a sequence of products $Q_k\in\Sigma_{d_k}(\mathscr{F})$, d_k nondecreasing integers, such that*

$$\lim_{k\to\infty} Q_k=\tilde{Q}, \tag{2.5}$$

where $\tilde{Q}\in\overline{\Sigma(\mathscr{F})}$ and $\rho(\tilde{Q})=1$. Then $\tilde{Q}$ will be called a limit spectrum-maximizing product *(in short, an* l.s.m.p.*) for $\mathscr{F}$.*

Note that, for a normalized family $\mathscr{F}$, an s.m.p. $\tilde{P}$ is an l.s.m.p., too. To see this, just put $P_k=\tilde{P}$ for all $k\geq 1$. Moreover, if the family $\mathscr{F}$ is nondefective, another possibility is to consider the power sequence $\{\tilde{P}^k\}_{k\geq 1}$ and, since $\Sigma(\mathscr{F})$ is bounded, to extract a subsequence $\{\tilde{P}^{k_s}\}_{s\geq 1}$ converging to some $\tilde{Q}\in\overline{\Sigma(\mathscr{F})}$, which obviously satisfies $\rho(\tilde{Q})=1$. For the sake of brevity, we shall say that such a limit point of the sequence $\{\tilde{P}^k\}_{k\geq 1}$ is an *infinite power* of the matrix $\tilde{P}$. For nondefective families, the following result has been proved in [GZ03a].

THEOREM 2.11. *Let $\mathscr{F}$ be a (possibly infinite) nondefective bounded family of complex $n\times n$-matrices. Then there exists an* l.s.m.p. *$\tilde{Q}$ for the normalized family $\hat{\mathscr{F}}$.*

On the contrary, for defective families, some counterexamples to the existence of l.s.m.p.'s when the dimension of the matrices is $n\geq 4$ have been given.

3. Complex polytopes and related norms. In this section we recall from [GZ07] the definition of *balanced complex polytope*, which is the generalization of symmetric real polytope (see, e.g., [Zie95]) to the complex space, along with some related results.

If $\mathscr{X}=\{x_i\}_{1\leq i\leq m}$ is a finite set of vectors, then

$$\operatorname{absco}(\mathscr{X})=\Big\{x\in\mathbb{C}^n \;\Big|\; x=\sum_{i=1}^m \lambda_i x_i \text{ with } \sum_{i=1}^m |\lambda_i|\leq 1\Big\}. \tag{3.1}$$

DEFINITION 3.1. *We shall say that a bounded set $\mathscr{P}\subset\mathbb{C}^n$ is a* balanced complex polytope *(b.c.p.) if there exists a finite set of vectors $\mathscr{X}=\{x_i\}_{1\leq i\leq m}$ such that* $\operatorname{span}(\mathscr{X})=\mathbb{C}^n$ *and*

$$\mathscr{P}=\operatorname{absco}(\mathscr{X}). \tag{3.2}$$

Moreover, if $\operatorname{absco}(\mathscr{X}')\subsetneq\operatorname{absco}(\mathscr{X})$ *for all $\mathscr{X}'\subsetneq\mathscr{X}$, then $\mathscr{X}$ will be called an* essential system of vertices *for $\mathscr{P}$, whereas any vector ux_i with $u\in\mathbb{C}$, $|u|=1$, will be called a* vertex *of $\mathscr{P}$.*

From a geometrical point of view, a b.c.p. $\mathscr{P}$ is not a classical polytope. In fact, if we identify the complex space $\mathbb{C}^n$ with the real space $\mathbb{R}^{2n}$, we see that $\mathscr{P}$ is not bounded by hyperplanes. In general, even the intersection $\mathscr{P}\bigcap\mathbb{R}^n$ is not a classical polytope. However, if the b.c.p. $\mathscr{P}$ admits an essential system of real vertices, then $\mathscr{P}\bigcap\mathbb{R}^n$ is a classical polytope.

The next proposition states the uniqueness of the essential system of vertices (modulo scalar factors of unitary modulus).

PROPOSITION 3.2. *Assume that $\mathscr{X}=\{x^{(i)}\}_{1\leq i\leq m}$ and $\hat{\mathscr{X}}=\{\hat{x}^{(i)}\}_{1\leq i\leq k}$ are two essential systems of vertices for a b.c.p. $\mathscr{P}$. Then $k=m$ and, for each $i=1,\dots,m$, there exist j_i, $1\leq j_i\leq m$, and $u_i\in\mathbb{C}$, $|u_i|=1$, such that $\hat{x}^{(i)}=u_ix^{(j_i)}$.*

Now we extend the concept of *polytope norm* to the complex case in a straightforward way.

LEMMA 3.3. *Any b.c.p. $\mathscr{P}$ is the unit ball of a norm $\|\cdot\|_{\mathscr{P}}$ on $\mathbb{C}^n$.*

DEFINITION 3.4. *We shall call* complex polytope norm *any norm $\|\cdot\|_{\mathscr{P}}$ whose unit ball is a b.c.p. $\mathscr{P}$.*

The corresponding vector norm is characterized as follows.

PROPOSITION 3.5. *Let $\mathscr{P}$ be a b.c.p. and let $\|\cdot\|_{\mathscr{P}}$ be the corresponding complex polytope norm. Then, for any $z \in \mathbb{C}^n$, it holds that*

$$\|z\|_{\mathscr{P}} = \min\Big\{ \sum_{i=1}^{m} |\lambda_i| \;\Big|\; z = \sum_{i=1}^{m} \lambda_i x_i \Big\}, \tag{3.3}$$

where $\mathscr{X} = \{x_i\}_{1\le i\le m}$ is an essential system of vertices for $\mathscr{P}$.

The next theorem shows that the set of the complex polytope norms is *dense* in the set of all norms defined on $\mathbb{C}^n$ and that, consequently, the corresponding set of induced matrix complex polytope norms is *dense* in the set of all induced $n\times n$-matrix norms (see [GWZ05]).

THEOREM 3.6. *Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. Then for any $\varepsilon > 0$ there exists a b.c.p. $\mathscr{P}_\varepsilon$ whose corresponding complex polytope norm $\|\cdot\|_\varepsilon$ satisfies the inequalities*

$$\|x\| \le \|x\|_\varepsilon \le (1+\varepsilon)\|x\| \quad \text{for all } x \in \mathbb{C}^n.$$

Moreover, denoting by $\|\cdot\|$ and $\|\cdot\|_\varepsilon$ also the corresponding induced matrix norms, it holds that

$$(1+\varepsilon)^{-1}\|A\| \le \|A\|_\varepsilon \le (1+\varepsilon)\|A\| \quad \text{for all } A \in \mathbb{C}^{n\times n}.$$

4. The small CPE theorem. In this section we resume the main definitions and results given in [GWZ05].

Complex polytope norms play a particular role. In fact, Theorem 3.6 allows us to refine Theorem 2.1.

PROPOSITION 4.1. *The spectral radius of a bounded family $\mathscr{F}$ of complex $n\times n$-matrices is characterized by the equality*

$$\rho(\mathscr{F}) = \inf_{\|\cdot\|\in\mathscr{N}_{pol}} \|\mathscr{F}\|, \tag{4.1}$$

where $\mathscr{N}_{pol}$ denotes the set of all possible induced $n\times n$-matrix complex polytope norms.

The natural question arises whether a nondefective family admits an extremal complex polytope norm. An important necessary condition for this follows.

THEOREM 4.2. *Let $\mathscr{F} = \{A^{(i)}\}_{1\le i\le m}$ be a finite nondefective family of complex $n\times n$-matrices and assume that there exists an extremal complex polytope norm $\|\cdot\|_{\mathscr{P}}$. Then $\mathscr{F}$ has at least an s.m.p. $\tilde{P}$.*

It would be nice to be able to reverse Theorem 4.2, but this has not yet been done unless assuming some additional conditions on the family $\mathscr{F}$.

For any vector $x \in \mathbb{C}^n$ and for any normalized family $\hat{\mathscr{F}}$, consider the set

$$\mathscr{T}[\hat{\mathscr{F}},x] = \{x\} \cup \{\hat{P}x \mid \hat{P} \in \Sigma(\hat{\mathscr{F}})\},$$

i.e. the *trajectory* obtained by applying all the normalized products $\hat{P}$ of matrices of $\hat{\mathscr{F}}$ to the vector x.

The following characterization holds.

PROPOSITION 4.3. *Let $\mathscr{F}$ be a bounded family of complex $n\times n$-matrices and let $x \in \mathbb{C}^n$. Then $\operatorname{span}\Big(\mathscr{T}[\hat{\mathscr{F}},x]\Big)$ is the smallest linear subspace V of $\mathbb{C}^n$ containing x such that $\mathscr{F}(V) \subseteq V$.*

COROLLARY 4.4. *Let $\mathscr{F}$ be an irreducible bounded family of complex $n\times n$-matrices and let $x \in \mathbb{C}^n$, $x \ne 0$. Then*

$$\operatorname{span}\Big(\mathscr{T}[\hat{\mathscr{F}},x]\Big) = \mathbb{C}^n. \tag{4.2}$$

For a general family of matrices $\mathscr{F}$, the sets of the type

$$\mathscr{S}[\hat{\mathscr{F}},x] = \overline{\mathrm{absco}\Big(\mathscr{T}[\hat{\mathscr{F}},x]\Big)} \tag{4.3}$$

play an important role.

PROPOSITION 4.5. *Let $\mathscr{F}$ be a nondefective bounded family of complex $n\times n$-matrices and, given a vector $x \in \mathbb{C}^n$, let (4.2) hold. Then the set $\mathscr{S}[\hat{\mathscr{F}},x]$ is the unit ball of an extremal norm for $\mathscr{F}$.*

By virtue of the foregoing result, it is interesting to find conditions under which $\mathscr{S}[\hat{\mathscr{F}},x]$ is generated by a finite number of points of the trajectory $\mathscr{T}[\hat{\mathscr{F}},x]$. So, if (4.2) holds, the set $\mathscr{S}[\hat{\mathscr{F}},x]$ is a b.c.p. and, thus, we have an extremal complex polytope norm.

Our attention is focused on families that satisfy some particular properties.

DEFINITION 4.6. *An eigenvector $x \neq 0$ of a matrix P related to an eigenvalue λ with $|\lambda| = \rho(P)$ is said to be a* leading eigenvector *of P.*

DEFINITION 4.7. *Let $\mathscr{F}$ be a nondefective bounded family of complex $n\times n$-matrices. A leading eigenvector $x \neq 0$ of either an s.m.p. $\tilde{P}$ of $\mathscr{F}$ or of an l.s.m.p. $\tilde{Q}$ of the normalized family $\hat{\mathscr{F}}$ is said to be* leading eigenvector *of $\mathscr{F}$ (and of $\hat{\mathscr{F}}$ too).*

REMARK 4.1. *Because of Theorem 2.11, any nondefective bounded family $\mathscr{F}$ has at least one leading eigenvector.*

DEFINITION 4.8. *Let $\mathscr{F}$ be a family of complex $n\times n$-matrices. A set $\mathscr{X} \subset \mathbb{C}^n$ is said to be* $\mathscr{F}$-cyclic *if for any pair $(x,y) \in \mathscr{X}\times\mathscr{X}$ there exist $\alpha,\beta \in \mathbb{C}$ with*

$$|\alpha|\cdot|\beta| = 1 \tag{4.4}$$

and two (finite) normalized products $\hat{P},\hat{Q} \in \Sigma(\hat{\mathscr{F}})$ such that

$$y = \alpha\hat{P}x \quad \text{and} \quad x = \beta\hat{Q}y.$$

REMARK 4.2. *Because of (4.4), the normalized products $\hat{P}\hat{Q}$ and $\hat{Q}\hat{P}$ determined in the above definition are s.m.p. of the normalized family $\hat{\mathscr{F}}$ and the set $\mathscr{X}$ is necessarily included in the set $\mathscr{L}$ of the leading eigenvectors of the family $\mathscr{F}$.*

DEFINITION 4.9. *A nondefective bounded family $\mathscr{F}$ of complex $n\times n$-matrices is said to be* asymptotically simple *if the set $\mathscr{L}$ of its leading eigenvectors is finite (modulo scalar nonzero factors) and $\mathscr{F}$-cyclic.*

As in Section 2, we shall say that a matrix Q is an *infinite power* of another matrix P if it is a limit point of the sequence $\{P^k\}_{k\geq 1}$. Observe that any eigenvalue λ of an infinite power Q of a matrix P satisfies either $|\lambda| = 1$ or $\lambda = 0$, since these are the only two possible limit values of the numeric power sequence $\{|\mu|^k\}_{k\geq 1}$ whenever $|\mu| \leq 1$. Moreover, given a nondefective matrix P with $\rho(P) = 1$, there exists at least an infinite power Q of P with an eigenvalue $\lambda = 1$, whose multiplicity is equal to the sum of the multiplicities of all the eigenvalues μ of P with $|\mu| = 1$. This easily follows from the fact that the power sequence $\{\mu^k\}_{k\geq 1}$ has the limit point 1 whenever $|\mu| = 1$ (see, for example, [HW79]).

REMARK 4.3. *It follows from the above observations that, for a (nondefective) asymptotically simple family $\mathscr{F}$, each s.m.p. $\tilde{P}$ and each l.s.m.p. $\tilde{Q}$ have only one leading eigenvector (modulo scalar nonzero factors). Otherwise there would exist at least one l.s.m.p. of the normalized family $\hat{\mathscr{F}}$, obtained as an infinite power, with an eigenspace of dimension ≥ 2 related to the eigenvalue $\lambda = 1$. This would contradict the finiteness (modulo scalar nonzero factors) of the set $\mathscr{L}$ of leading eigenvectors.*

Observe that all the cyclic permutations of a product P have the same eigenvalues with the same multiplicities. Thus, if $\tilde{P} = A^{(i_{k^*})}\ldots A^{(i_1)}$ is an s.m.p. for a family $\mathscr{F}$, then each of its cyclic permutations

$$A^{(i_s)}\ldots A^{(i_1)}A^{(i_{k^*})}\ldots A^{(i_{s+1})}, \qquad s = 1,\ldots,k^*-1,$$

still is an s.m.p. for $\mathscr{F}$, along with all the powers of $\tilde{P}$ and their cyclic permutations.

DEFINITION 4.10. *Let $\mathscr{F}$ be a family of complex $n \times n$-matrices. An s.m.p. $\tilde{P}$ is said to be* minimal *if it is not a power of another s.m.p. of $\mathscr{F}$.*

It is clear that, for any s.m.p. $\tilde{P}$ of a family $\mathscr{F}$, it holds that either $\tilde{P}$ is minimal or $\tilde{P}$ is a power of another s.m.p., which is minimal.

We have the following characterization of asymptotically simple families.

PROPOSITION 4.11. *A nondefective bounded family $\mathscr{F}$ of complex $n \times n$-matrices is asymptotically simple if and only if it has a minimal s.m.p. $\tilde{P}$ with only one leading eigenvector (modulo scalar nonzero factors) such that the set $\mathscr{L}$ of the leading eigenvectors of $\mathscr{F}$ is equal to the set of the leading eigenvectors of $\tilde{P}$ and of its cyclic permutations.*

The following definition selects a particular class of asymptotically simple families.

DEFINITION 4.12. *A nondefective bounded family $\mathscr{F}$ of complex $n \times n$-matrices is said to be* absolutely asymptotically simple *if it is asymptotically simple and has a unique minimal s.m.p. $\tilde{P}$ (modulo cyclic permutations).*

It is clear that, for absolutely asymptotically simple families, the unique minimal s.m.p. $\tilde{P}$ coincides with the minimal s.m.p. given by the characterizing Proposition 4.11. Moreover, the cardinality of the set $\mathscr{L}$ of its leading eigenvectors (modulo scalar nonzero factors) is equal to the number of factors of $\tilde{P}$.

LEMMA 4.13. *Let $\mathscr{F} = \{A^{(i)}\}_{1 \le i \le m}$ be a nondefective finite family of complex $n \times n$-matrices and, given a vector $x \in \mathbb{C}^n$, assume that (4.2) is satisfied and that the (bounded) set $\partial \mathscr{S}[\hat{\mathscr{F}}, x] \bigcap \mathscr{T}[\hat{\mathscr{F}}, x]$, modulo scalar factors of unitary modulus, is not finite. Then there exists a sequence of distinct vectors*

$$x^{(k)} \in \partial \mathscr{S}[\hat{\mathscr{F}}, x] \bigcap \mathscr{T}[\hat{\mathscr{F}}, x]$$

with $x^{(1)} = x$ such that, for all $k \ge 1$,

$$x^{(k+1)} = \hat{A}^{(\ell_k)} x^{(k)} \quad \text{for some } \ell_k \in \{1, \dots, m\},$$

where $\hat{A}^{(i)} = A^{(i)}/\rho(\mathscr{F}) \in \hat{\mathscr{F}}$, $1 \le i \le m$, and such that, whenever $k \ne h$,

$$x^{(k)} \ne u x^{(h)} \quad \text{for all } u \in \mathbb{C} \text{ with } |u| = 1.$$

The following theorem is the main result of [GWZ05].

THEOREM 4.14 (small CPE Theorem). *Assume that a finite family $\mathscr{F} = \{A^{(i)}\}_{1 \le i \le m}$ of complex $n \times n$-matrices is nondefective and asymptotically simple. Moreover, let $x \ne 0$ be a leading eigenvector of $\mathscr{F}$ and assume that (4.2) is satisfied. Then the set*

$$\partial \mathscr{S}[\hat{\mathscr{F}}, x] \bigcap \mathscr{T}[\hat{\mathscr{F}}, x] \tag{4.5}$$

is finite modulo scalar factors of unitary modulus. As a consequence, there exist a finite number of normalized products $\hat{P}^{(1)}, \dots, \hat{P}^{(s)} \in \Sigma(\hat{\mathscr{F}})$ such that

$$\mathscr{S}[\hat{\mathscr{F}}, x] = \operatorname{absco}\Big(\{x, \hat{P}^{(1)} x, \dots, \hat{P}^{(s)} x\}\Big), \tag{4.6}$$

so that $\mathscr{S}[\hat{\mathscr{F}}, x]$ is a b.c.p.

Remark that, if all the matrices of the family $\mathscr{F}$ are real and if also the starting leading eigenvector x is real, then Theorem 4.14 determines a classical polytope in $\mathbb{R}^n$.

The next results are useful for a deeper understanding of the structure of the b.c.p. $\mathscr{S}[\hat{\mathscr{F}}, x]$ obtained under the hypotheses of Theorem 4.14.

THEOREM 4.15. *Let the hypotheses of Theorem 4.14 hold. Then each leading eigenvector ξ of $\mathscr{F}$ in the set $\Xi = \mathscr{L} \bigcap \partial \mathscr{S}[\hat{\mathscr{F}}, x]$ satisfies one of the following two statements:*

(a) ξ *is a vertex of the b.c.p.* $\mathscr{S}[\hat{\mathscr{F}},x]$;
(b) *there exist* $s \geq 2$ *vertices* $\xi_1,\dots,\xi_s$ *of the b.c.p.* $\mathscr{S}[\hat{\mathscr{F}},x]$ *such that*

$$\xi_1,\dots,\xi_s \in \Xi \quad \text{and} \quad \xi \in \operatorname{absco}\Big(\{\xi_1,\dots,\xi_s\}\Big). \tag{4.7}$$

COROLLARY 4.16. *Let the hypotheses of Theorem 4.14 hold and, moreover, let the family* $\mathscr{F}$ *be absolutely asymptotically simple. Then all the leading eigenvectors of* $\mathscr{F}$ *(in the set* $\Xi = \mathscr{L} \bigcap \partial\mathscr{S}[\hat{\mathscr{F}},x]$*) are vertices of the b.c.p.* $\mathscr{S}[\hat{\mathscr{F}},x]$.

5. Improving the small CPE theorem for real families. Unfortunately, the hypotheses assumed by Theorem 4.14 are such to exclude some cases of real families that, on the contrary, clearly admit an extremal complex polytope norm which could be determined by a suitable modification of the procedure based on the construction of the trajectory $\mathscr{T}[\hat{\mathscr{F}},x]$. We illustrate this fact by means of the following example.

EXAMPLE 5.1. Consider the real 2×2-matrix family $\mathscr{F} = \{A,B\}$, where

$$A = \begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix} \quad \text{and} \quad B = \beta \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 \end{bmatrix} \text{ with } 0 < \beta \leq 1.$$

The eigenvalues of A are $\mathrm{e}^{-\mathrm{i}}$ and e^{i} with corresponding eigenvectors $x = [1\ \mathrm{i}]^{\mathrm{T}}$ and $\overline{x} = [1\ -\mathrm{i}]^{\mathrm{T}}$, respectively. The eigenvalues of B are $\frac{\beta\sqrt{2}}{2}$ and 0. Thus $\rho(A) = 1$ and $\rho(B) = \frac{\beta\sqrt{2}}{2}$. Now consider the b.c.p.

$$\mathscr{P} = \operatorname{absco}(\{x,\overline{x}\}), \tag{5.1}$$

whose boundary $\partial\mathscr{P}$ intersects $\mathbb{R}^2$ on the unit circle

$$\mathscr{C} = \Big\{ [x_1\ x_2]^{\mathrm{T}} \in \mathbb{R}^2 \ \Big|\ x_1^2 + x_2^2 = 1 \Big\}. \tag{5.2}$$

It is immediately seen that

$$A\mathscr{P} = \mathscr{P} \quad \text{and} \quad B\mathscr{P} = \operatorname{absco}\left([\beta\ 0]^{\mathrm{T}}\right) \subset \mathscr{P},$$

where we mean by $A\mathscr{P}$ the set $\{y = Ax : x \in \mathscr{P}\}$ and similarly by $B\mathscr{P}$. Consequently

$$\|A\|_{\mathscr{P}} = 1, \quad \|B\|_{\mathscr{P}} = \beta, \quad \rho(\mathscr{F}) = \|\mathscr{F}\|_{\mathscr{P}} = 1.$$

Consequently, whenever $\beta < 1$, the matrix A is the unique minimal s.m.p. of $\mathscr{F}$ and all the l.s.m.p.'s of $\mathscr{F}$ are (infinite) powers of A.

On the contrary, for $\beta = 1$ it is not difficult to prove that, whereas A remains the unique minimal s.m.p., there exist some l.s.m.p.'s other than the (infinite) powers of A such as the matrices $A^\infty B$ and BA^∞, where

$$A^\infty = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix},$$

which is an infinite power of A. In fact, we have

$$A^\infty B \left[\frac{\sqrt{2}}{2}\ \frac{\sqrt{2}}{2}\right]^{\mathrm{T}} = \left[\frac{\sqrt{2}}{2}\ \frac{\sqrt{2}}{2}\right]^{\mathrm{T}} \quad \text{and} \quad BA^\infty [1\ 0]^{\mathrm{T}} = [1\ 0]^{\mathrm{T}}.$$

Remark that these two l.s.m.p.'s are cyclic permutations of each other and that they have only one real leading eigenvector. In any case, for all $\beta \leq 1$, according to Remark 4.3, the identity matrix I is an l.s.m.p., so that all the vectors of $\mathbb{C}^2$ are leading eigenvectors of $\mathscr{F}$. Moreover, it is clear that the set of leading eigenvectors of $\mathscr{F}$ is not $\mathscr{F}$-cyclic. Therefore, the hypotheses of Theorem 4.14 are definitely violated.

Indeed, the set $\mathscr{S}[\mathscr{F},x]$ is not a b.c.p. In fact, it holds that

$$Bx = \left[\frac{\beta\sqrt{2}(1+\mathrm{i})}{2} \ 0\right]^{\mathrm{T}} \quad \text{and} \quad A^k Bx = \frac{\beta\sqrt{2}(1+\mathrm{i})}{2}[\cos(k) \ \sin(k)]^{\mathrm{T}}, \ k \geq 1.$$

All these vectors lie on the circle

$$\mathscr{C}_\beta = \frac{\beta\sqrt{2}(1+\mathrm{i})}{2}\mathscr{C},$$

where $\mathscr{C}$ is defined by (5.2), and form a set which is dense in $\mathscr{C}_\beta$. On the other hand,

$$BA^kBx = \left[\frac{\beta(\cos(k)+\sin(k))(1+\mathrm{i})}{2} \ 0\right]^{\mathrm{T}} \in \operatorname{absco}\left(\mathscr{C}_\beta\right),$$

and, hence, we can conclude that

$$\mathscr{S}[\mathscr{F},x] = \operatorname{absco}\{\{x\} \cup \mathscr{C}_\beta\},$$

that is not a b.c.p. Note that infinitely many vectors of the trajectory $\mathscr{T}[\mathscr{F},x]$, namely A^kBx for $k \geq 0$, which are not proportional to one another, lie on $\partial\mathscr{S}[\mathscr{F},x]$. Analogous conclusions hold for the conjugate set $\mathscr{S}[\mathscr{F},\overline{x}]$. On the other hand, it is interesting to observe that

$$\mathscr{P} = \overline{\operatorname{absco}\Big(\mathscr{T}[\mathscr{F},x] \cup \mathscr{T}[\mathscr{F},\overline{x}]\Big)}, \tag{5.3}$$

where $\mathscr{P}$ is the b.c.p. given by (5.1), and that, in the light of the previous analysis, we have

$$\partial\mathscr{P} \cap \Big(\mathscr{T}[\mathscr{F},x] \cup \mathscr{T}[\mathscr{F},\overline{x}]\Big) = \{x,\overline{x}\}$$

(modulo scalar factors of unitary modulus), whenever $\beta < 1$. On the contrary, for $\beta = 1$ the infinitely many vectors A^kBx (and $A^kB\overline{x}$), $k \geq 0$, which are not proportional to one another, lie on $\partial\mathscr{P}$. $\Diamond$

The foregoing example suggests naturally how to try to relax the hypotheses of Theorem 4.14 in order to accommodate the case of a real family such that the minimal s.m.p.'s have a pair of complex conjugate leading eigenvectors instead of a single real leading eigenvector. In order to do this, we need to modify some of the definitions given in [GWZ05] and reported in Section 2, and to give some new ones.

DEFINITION 5.1. *Let $\mathscr{F}$ be a nondefective bounded family of complex $n\times n$-matrices and let $x \neq 0$ be a leading eigenvector of $\mathscr{F}$. Then x is said to be a* standard leading eigenvector *of $\mathscr{F}$ (and of $\hat{\mathscr{F}}$ too) if it is a leading eigenvector of a minimal s.m.p. $\tilde{P}$ of $\mathscr{F}$, whereas it is said to be a* limit leading eigenvector *of $\mathscr{F}$ if it is a leading eigenvector of an l.s.m.p. (possibly, an s.m.p.) $\tilde{Q}$ of the normalized family $\hat{\mathscr{F}}$, but is not a leading eigenvector of any minimal s.m.p..*

Now, we modify Definition 4.9 for real families.

DEFINITION 5.2. *Let $\mathscr{F}$ be a nondefective bounded family of real $n\times n$-matrices and $\mathscr{E}$ be the set of its standard leading eigenvectors. The family $\mathscr{F}$ is said to be* asymptotically simple *if $\mathscr{E}$ is finite (modulo scalar nonzero factors) and the following properties hold:*

- *If $\mathscr{E}$ is a real set, then:*
  (i-r) $\mathscr{E}$ is $\mathscr{F}$-cyclic.
  (ii-r) If $\tilde{Q}$ is an l.s.m.p. of $\hat{\mathscr{F}}$, then the set of its leading eigenvectors is included in $\mathscr{E}$.
- *If $\mathscr{E}$ is not a real set, then:*

(i) $\mathscr{E}$ *is self-conjugate, that is* $\mathscr{E} = \mathscr{E}_1 \cup \overline{\mathscr{E}}_1$*, and* $\mathscr{E}_1$ *is* $\mathscr{F}$*-cyclic.*

(ii) If $\tilde{Q}$ *is an l.s.m.p. of* $\hat{\mathscr{F}}$*, then the set of its leading eigenvectors lies in a subspace of dimension* 2 *and includes a complex conjugate pair of elements of* $\mathscr{E}$*.*

Observe that, when the leading eigenvectors are all real, there is no change with respect to Definition 4.9.

REMARK 5.1. *If a family* $\mathscr{F}$ *of real* $n\times n$*-matrices is asymptotically simple and* $\tilde{P}$ *is a minimal s.m.p., then* $\tilde{P}$ *has a unique leading eigenvector if the set* $\mathscr{E}$ *is real, whereas* $\tilde{P}$ *has a unique pair of complex conjugate leading eigenvectors if the set* $\mathscr{E}$ *is not real.*

REMARK 5.2. *For a nondefective asymptotically simple normalized family* $\hat{\mathscr{F}}$ *such that the set* $\mathscr{E}$ *is not real, there exists an l.s.m.p.* $\tilde{Q}$*, obtained as an infinite power (possibly, a finite power) of a minimal s.m.p.* $\tilde{P}$*, with an eigenspace of dimension* 2 *related to the eigenvalue* $\lambda = 1$*.*

The following characterization of asymptotically simple real families holds.

PROPOSITION 5.3. *A nondefective bounded family* $\mathscr{F}$ *of real* $n\times n$*-matrices is asymptotically simple if and only if one of the following situations occurs:*

1) $\mathscr{F}$ *fulfils property (ii-r) of Definition 5.2 and has a minimal s.m.p.* $\tilde{P}$ *with a unique real leading eigenvector such that the set* $\mathscr{E}$ *of the standard leading eigenvectors of* $\mathscr{F}$ *is equal to the set of the leading eigenvectors of* $\tilde{P}$ *and of its cyclic permutations.*

2) $\mathscr{F}$ *fulfils property (ii) of Definition 5.2 and has a minimal s.m.p.* $\tilde{P}$ *with a unique pair of complex conjugate leading eigenvectors such that the set* $\mathscr{E}$ *of the standard leading eigenvectors of* $\mathscr{F}$ *is equal to the set of the leading eigenvectors of* $\tilde{P}$ *and of its cyclic permutations.*

Proof.

1) The proof of this occurrence is given in [GWZ05].

2) *Necessity.* Let $x_1,\ldots,x_s,\overline{x}_1,\ldots,\overline{x}_s \in \mathbb{C}^n$ form a set of distinct representatives (modulo scalar nonzero factors) of all the standard leading eigenvectors of $\mathscr{F}$. Since $\mathscr{F}$ is asymptotically simple, they are finitely many and, for any $i = 1,\ldots,s$, there exist $\alpha_i,\beta_i \in \mathbb{C}$ with $|\alpha_i|\cdot|\beta_i| = 1$ and two (finite) normalized products $\hat{P}_i,\hat{Q}_i \in \Sigma(\hat{\mathscr{F}})$ such that

$$x_{i+1} = \alpha_i \hat{P}_i x_i \quad \text{and} \quad x_i = \beta_i \hat{Q}_i x_{i+1}$$

and (by reality of $\mathscr{F}$),

$$\overline{x}_{i+1} = \overline{\alpha}_i \hat{P}_i \overline{x}_i \quad \text{and} \quad \overline{x}_i = \overline{\beta}_i \hat{Q}_i \overline{x}_{i+1}$$

where, conventionally, $x_{s+1} = x_1$. Therefore, we obtain that $x_1 = \alpha\hat{P}x_1 = \beta\hat{Q}x_1$ and $\overline{x}_1 = \overline{\alpha}\hat{P}\overline{x}_1 = \overline{\beta}\hat{Q}\overline{x}_1$, where $\alpha = \alpha_1\ldots\alpha_s$, $\beta = \beta_1\ldots\beta_s$ and $\hat{P} = \hat{P}_s\ldots\hat{P}_1$, $\hat{Q} = \hat{Q}_1\ldots\hat{Q}_s$, with $|\alpha|\cdot|\beta| = 1$. Now, since $\rho(\hat{P}) \le 1$ and $\rho(\hat{Q}) \le 1$, it follows that $|\alpha| = |\beta| = 1$, which implies $\rho(\hat{P}) = 1$ and $\rho(\hat{Q}) = 1$.

So we can conclude that the matrix $\tilde{P} = \hat{P}_s\ldots\hat{P}_1$ is an s.m.p. of $\hat{\mathscr{F}}$ such that the set of the leading eigenvectors of $\tilde{P}$ and of its cyclic permutations includes (and thus is equal to) the set $\mathscr{E}$ of the standard leading eigenvectors of $\mathscr{F}$. Since it is not restrictive assuming $\tilde{P}$ to be minimal, the proof is concluded.

Sufficiency. Assume that there is a (minimal) s.m.p. $\tilde{P}$ with only 2 complex conjugate leading eigenvectors and that the set $\mathscr{V}$ of the leading eigenvectors of $\tilde{P}$ and of its cyclic permutations (modulo scalar nonzero factors) coincides with the set $\mathscr{E}$ of standard leading eigenvectors of $\mathscr{F}$. Therefore, since $\mathscr{V}$ may be generated (through multiplication by nonzero complex numbers) by $2p$ elements, where p denotes the number of factors of $\tilde{P}$, and since $\mathscr{V} = \mathscr{V}_1 \cup \overline{\mathscr{V}_1}$, where the sets $\mathscr{V}_1$ and $\overline{\mathscr{V}_1}$ are clearly $\mathscr{F}$-cyclic, the proof is complete. ∎

LEMMA 5.4. *Let the nondefective bounded family* $\mathscr{F}$ *of real* $n\times n$*-matrices be asymptotically simple and let the set* $\mathscr{E}$ *of its standard leading eigenvectors be non-real. Then the set* $\mathscr{L}$ *of all the leading eigenvectors of* $\mathscr{F}$ *is given by*

$$\mathscr{L} = \bigcup_{i=1}^{p} \operatorname{span}(\{x_i,\overline{x}_i\}),$$

where $\{x_1,\dots,x_p\}$ is a set of linearly independent representatives of $\mathscr{E}_1$, introduced in Definition 5.2 - (i).

Proof. As observed in Remark 5.2, for a nondefective asymptotically simple normalized family $\hat{\mathscr{F}}$, there exists an l.s.m.p. $\tilde{Q}$, which is an infinite power (possibly, a finite power) of a minimal s.m.p. $\tilde{P}$, with an eigenspace of dimension 2 related to the eigenvalue $\lambda = 1$. Such eigenspace is clearly spanned by the pair of complex conjugate leading eigenvectors of the minimal s.m.p. $\tilde{P}$. Therefore, by virtue of Proposition 5.3, the proof is complete. $\square$

The foregoing result implies that the set of the distinct representatives (modulo scalar nonzero factors) of all the leading eigenvectors of $\mathscr{F}$ is not finite.

The definition of *absolutely asymptotically simple* family remains formally unchanged (see Definition 4.12) and, again, for such kind of families, the unique minimal s.m.p. $\tilde{P}$ coincides with the minimal s.m.p. given by the characterizing Proposition 5.3.

Moreover, the cardinality of the set $\mathscr{E}$ of the standard leading eigenvectors (modulo scalar nonzero factors) is either the number of factors of $\tilde{P}$ (in the case that $\mathscr{E}$ is real) or twice such a number (in the case that $\mathscr{E}$ is not real).

In the sequel, for a real family $\mathscr{F}$ and for a non-zero complex conjugate pair of vectors $x,\overline{x} \in \mathbb{C}^n$ (with $\Im(x) \neq 0$), we will consider the *trajectory* obtained by applying all the normalized products $\hat{P} \in \hat{\mathscr{F}}$ to x and $\overline{x}$, that is

$$\mathscr{T}[\hat{\mathscr{F}},x,\overline{x}] = \mathscr{T}[\hat{\mathscr{F}},x] \cup \mathscr{T}[\hat{\mathscr{F}},\overline{x}] = \{x,\overline{x}\} \cup \Big\{ \{\hat{P}x, \hat{P}\overline{x}\} \;\mid\; \hat{P} \in \Sigma(\hat{\mathscr{F}}) \Big\}, \tag{5.4}$$

and the set

$$\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}] = \overline{\mathrm{absco}\Big(\mathscr{T}[\hat{\mathscr{F}},x,\overline{x}] \Big)}. \tag{5.5}$$

Similarly to Proposition 4.5, we have that, if $\mathscr{F}$ is nondefective and if

$$\mathrm{span}\Big(\mathscr{T}[\hat{\mathscr{F}},x,\overline{x}] \Big) = \mathbb{C}^n, \tag{5.6}$$

the set $\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}]$ is the unit ball of an extremal norm. Therefore, our aim is to find conditions under which $\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}]$ is generated by a finite number of points of the trajectory $\mathscr{T}[\hat{\mathscr{F}},x,\overline{x}]$, in order that the set $\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}]$ be a b.c.p..

In any case, since $\mathscr{F}$ is real, the set $\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}]$ is self-conjugate. Some specific properties of self-conjugate b.c.p.'s may be found in [VZss].

The proof of the following lemma is analogous to that of Lemma 5.19 in [GWZ05].

LEMMA 5.5. *Let $\mathscr{F} = \{A^{(i)}\}_{1\le i\le m}$ be a nondefective finite family of real $n\times n$-matrices and, given a non-real non-zero vector $x \in \mathbb{C}^n$, assume that (5.6) holds and that the (bounded) set $\partial\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}] \bigcap \mathscr{T}[\hat{\mathscr{F}},x,\overline{x}]$ is not finite (modulo scalar factors of unitary modulus). Then there exist two conjugate sequences of distinct vectors $\{x^{(k)}\}$ and $\{\overline{x}^{(k)}\}$ with $x^{(1)} = x$ and $\overline{x}^{(1)} = \overline{x}$ such that, for all $k \ge 1$,*

$$x^{(k)}, \overline{x}^{(k)} \in \partial\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}] \bigcap \mathscr{T}[\hat{\mathscr{F}},x,\overline{x}]$$

and

$$x^{(k+1)} = \hat{A}^{(\ell_k)} x^{(k)} \quad \text{and} \quad \overline{x}^{(k+1)} = \hat{A}^{(\ell_k)} \overline{x}^{(k)} \quad \text{for some } \ell_k \in \{1,\dots,m\},$$

where $\hat{A}^{(i)} = A^{(i)}/\rho(\mathscr{F}) \in \hat{\mathscr{F}}$, $1 \le i \le m$, and such that, whenever $k \ne h$,

$$x^{(k)} \neq u x^{(h)} \quad \text{and} \quad \overline{x}^{(k)} \neq u \overline{x}^{(h)} \quad \text{for all } u \in \mathbb{C} \text{ with } |u| = 1.$$

Eventually, we are in a position to state the main result of this paper, which extends the validity of the Small CPE Theorem proved in [GWZ05]. However, so far we were not able to prove it unless under the following technical hypothesis, even if we strongly believe that it holds even without assuming it (see also the forthcoming Example 5.2).

HYPOTHESIS 5.1. *If the set $\mathscr{E}$ of the standard leading eigenvectors of the family $\mathscr{F}$ is non-real, then the pair of leading eigenvalues $(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta})$ of any minimal s.m.p. of the normalized family $\hat{\mathscr{F}}$ is such that the numbers θ and π are rationally independent.*

The next proposition illustrates the practical meaning of Hypothesis 5.1.

PROPOSITION 5.6. *Let the finite family $\mathscr{F}$ of real $n \times n$-matrices be asymptotically simple and let the set $\mathscr{E}$ of its standard leading eigenvectors be non-real. Then Hypothesis 5.1 is equivalent to requiring that every s.m.p. of $\mathscr{F}$ (even if non-minimal) only has a pair of complex conjugate standard leading eigenvectors.*

Proof. Given a minimal s.m.p. $\hat{P}$ of $\hat{\mathscr{F}}$, we have that θ and π are rationally dependent if and only if there exists a positive integer k such that the power $\hat{P}^k$, which is an s.m.p. of $\hat{\mathscr{F}}$, has the eigenvalue $\lambda = 1$ of multiplicity 2 with infinitely many corresponding eigenvectors (i.e., all the vectors belonging to $\mathrm{span}(\{x, \overline{x}\})$, x and $\overline{x}$ being the leading eigenvectors of $\hat{P}$). □

The following lemma plays an important role in the proof of the subsequent main result (Theorem 5.8).

LEMMA 5.7. *Assume that a nondefective finite family $\mathscr{F} = \{A^{(i)}\}_{1 \le i \le m}$ of real $n \times n$-matrices is asymptotically simple, let Hypothesis 5.1 hold and let $\{x, \overline{x}\}$ be a pair of complex conjugate standard leading eigenvectors of $\mathscr{F}$. Moreover, assume that, for some matrix $\hat{S} \in \overline{\Sigma\left(\hat{\mathscr{F}}\right)}$, the vectors $y = \hat{S}x$ and $\overline{y} = \hat{S}\overline{x}$ are leading eigenvectors of $\mathscr{F}$.*

Then y and $\overline{y}$ are leading eigenvectors of standard type.

Proof. We observe that $y, \overline{y} \in \partial \mathscr{S}[\hat{\mathscr{F}}, x, \overline{x}]$ and that, by Lemma 5.4, $y, \overline{y} \in \mathrm{span}(\{z, \overline{z}\})$ for some standard leading eigenvector z, that is

$$y = \alpha z + \beta \overline{z}, \quad \overline{y} = \overline{\beta} z + \overline{\alpha}\overline{z}, \tag{5.7}$$

for suitable $\alpha, \beta \in \mathbb{C}$. Moreover, we can assume that $z, \overline{z} \in \partial \mathscr{S}[\hat{\mathscr{F}}, x, \overline{x}]$ and, by virtue of $\mathscr{F}$-ciclicity, that there exists $\hat{Q} \in \Sigma_k(\hat{\mathscr{F}})$ (for some integer k) such that $x = \hat{Q}z$, so that

$$y = \hat{S}\hat{Q}z, \qquad \overline{y} = \hat{S}\hat{Q}\overline{z}. \tag{5.8}$$

Then consider a linear transformation (represented by the non-singular matrix $T \in \mathbb{C}^{n,n}$) which transforms z into $e^{(1)}$ and $\overline{z}$ into $e^{(2)}$, $e^{(1)}$ and $e^{(2)}$ being the first two vectors of the canonical basis of $\mathbb{C}^n$. After setting

$$C^{(i)} = T A^{(i)} T^{-1},$$

we introduce the family of complex matrices $\mathscr{G} = \{C^{(i)}\}_{1 \le i \le m}$, which is such that $\rho(\mathscr{G}) = \rho(\mathscr{F})$, and denote by $\hat{\mathscr{G}}$ the associated normalized family. By formulae (5.7), we have that

$$T y = \begin{bmatrix} \alpha & \beta & 0 & \dots & 0 \end{bmatrix}^{\mathrm{T}} \quad \text{and} \quad T \overline{y} = \begin{bmatrix} \overline{\beta} & \overline{\alpha} & 0 & \dots & 0 \end{bmatrix}^{\mathrm{T}}$$

and, therefore, by setting $\hat{M} = T \hat{S} \hat{Q} T^{-1}$, formulæ (5.8) imply

$$\hat{M} = \begin{bmatrix} \alpha & \overline{\beta} & \cdot & \cdots & \cdot \\ \beta & \overline{\alpha} & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

If $|\alpha| + |\beta| < 1$, then $y, \overline{y}$ are strictly internal to $\mathrm{absco}(\{z, \overline{z}\}) \subseteq \mathscr{S}[\hat{\mathscr{F}}, x, \overline{x}]$, which gives a contradiction. Thus

$$|\alpha| + |\beta| \ge 1. \tag{5.9}$$

According to Proposition 5.3, z and $\overline{z}$ are the leading eigenvectors of a minimal s.m.p. of $\hat{\mathscr{F}}$, say $\hat{P}_\circ \in \Sigma_p(\hat{\mathscr{F}})$, with leading eigenvalues $\mathrm{e}^{\pm \mathrm{i}\,\theta_0}$. Now consider the corresponding product $\hat{N} \in \Sigma_p(\hat{\mathscr{G}})$ defined as $\hat{N} = T\,\hat{P}_\circ\,T^{-1}$,

$$\hat{N} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\,\theta_0} & 0 & 0 & \cdots & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\,\theta_0} & 0 & \cdots & 0 \\ 0 & 0 & \cdot & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdot & \cdots & \cdot \end{bmatrix},$$

and the power sequence $\hat{N}^\ell$. By Hypothesis 5.1, we can conclude (see, e.g., [HW79]) that there exists a suitable subsequence $\hat{N}^{\ell_m}$ which converges to the matrix

$$\hat{N}_\infty = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\arg(\alpha)} & 0 & 0 & \cdots & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\arg(\alpha)} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then we consider the matrix

$$\hat{O} = \hat{N}_\infty \hat{M} = \begin{bmatrix} |\alpha| & \mathrm{e}^{-\mathrm{i}\arg(\alpha)}\,\overline{\beta} & \cdot & \cdots & \cdot \\ \mathrm{e}^{\mathrm{i}\arg(\alpha)}\,\beta & |\alpha| & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

which belongs to $\overline{\Sigma(\hat{\mathscr{G}})}$ and whose eigenvalues are $|\alpha| \pm |\beta|$. Since $\hat{\mathscr{G}}$ is normalized, it follows that

$$|\alpha| + |\beta| \leq 1. \tag{5.10}$$

In conclusion, combining (5.9) and (5.10), we get $|\alpha| + |\beta| = 1$. Therefore, $\hat{O}$ is an l.s.m.p. for $\mathscr{G}$ and, hence, the corresponding matrix $O = T^{-1}\hat{O}T$ is an l.s.m.p. for $\mathscr{F}$. Finally, property (ii) in Definition 5.2 yields

$$|\alpha| = 1 \quad \text{and} \quad \beta = 0.$$

Therefore, by (5.7), y and $\overline{y}$ are standard leading eigenvectors. ☐

THEOREM 5.8 (extended small CPE theorem). *Assume that a nondefective finite family $\mathscr{F} = \{A^{(i)}\}_{1\leq i\leq m}$ of real $n\times n$-matrices is asymptotically simple. If the set $\mathscr{E}$ of its standard leading eigenvectors is real, then Theorem 4.14 applies. Otherwise, let Hypothesis 5.1 hold and let $\{x,\overline{x}\}$ be a pair of complex conjugate standard leading eigenvectors of $\mathscr{F}$ such that (5.6) holds. Then the set*

$$\partial\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}] \bigcap \mathscr{T}[\hat{\mathscr{F}},x,\overline{x}] \tag{5.11}$$

is finite modulo scalar factors of unitary modulus. As a consequence, there exist a finite number of normalized products $\hat{P}^{(1)},\ldots,\hat{P}^{(s)} \in \Sigma(\hat{\mathscr{F}})$ such that

$$\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}] = \operatorname{absco}\Big(\{x,\overline{x},\hat{P}^{(1)}x,\hat{P}^{(1)}\overline{x},\ldots,\hat{P}^{(s)}x,\hat{P}^{(s)}\overline{x}\}\Big), \tag{5.12}$$

so that $\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}]$ is a b.c.p.

Proof. Assume that the set $\mathscr{E}$ of the standard leading eigenvectors is not real and let $\{x,\overline{x}\}$ be a complex conjugate pair of them. According to Proposition 5.3, x and $\overline{x}$ are the leading eigenvectors of a minimal s.m.p. $\hat{P}$ of $\hat{\mathscr{F}}$. Now consider

$$\Xi = \mathscr{E} \bigcap \partial\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}]. \tag{5.13}$$

Since the family $\mathscr{F}$ is asymptotically simple and $\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}]$ is the unit ball of a norm, the set Ξ is finite modulo scalar factors of unitary modulus and not empty. Now assume, by contradiction, that the set

$$\partial\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}] \bigcap \mathscr{T}[\hat{\mathscr{F}},x,\overline{x}],$$

even if considered modulo scalar factors of unitary modulus, is not finite, so that Lemma 5.5 can be applied to obtain the sequences $x^{(k)},\overline{x}^{(k)} \in \partial\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}] \bigcap \mathscr{T}[\hat{\mathscr{F}},x,\overline{x}]$ with $x^{(1)} = x$ and $\overline{x}^{(1)} = \overline{x}$. Therefore, there exists $j \geq 1$ such that

$$x^{(j+1)} \notin \Xi \quad \text{and} \quad x^{(i)} \in \Xi \bigcap \mathscr{T}[\hat{\mathscr{F}},x] \quad \text{for all } i \leq j, \tag{5.14}$$

and, similarly,

$$\overline{x}^{(j+1)} \notin \Xi \quad \text{and} \quad \overline{x}^{(i)} \in \Xi \bigcap \mathscr{T}[\hat{\mathscr{F}},\overline{x}] \quad \text{for all } i \leq j.$$

Since $\Sigma(\hat{\mathscr{F}})$ is bounded, the resulting sequence of normalized matrix products $\hat{B}^{(k)} = \hat{A}^{(\ell_{k-1})} \cdots \hat{A}^{(\ell_{j+1})}$ such that $x^{(k)} = \hat{B}^{(k)} x^{(j+1)}$ has a subsequence $\{\hat{B}^{(k_s)}\}_{s\geq 1}$ that converges to a limit point $\hat{B}$ in $\overline{\Sigma(\hat{\mathscr{F}})}$. Therefore, also the subsequences of vectors $\{x^{(k_s)}\}_{s\geq 1}$ and $\{\overline{x}^{(k_s)}\}_{s\geq 1}$ have limit points

$$v = \hat{B}x^{(j+1)} = \hat{B}\hat{A}^{(\ell_j)} \cdots \hat{A}^{(\ell_1)} x \qquad \text{and} \qquad \overline{v} = \hat{B}\overline{x}^{(j+1)} = \hat{B}\hat{A}^{(\ell_j)} \cdots \hat{A}^{(\ell_1)} \overline{x}. \tag{5.15}$$

For each $s \geq 1$ there exists a matrix $\hat{R}^{(s)} \in \Sigma(\hat{\mathscr{F}})$ such that

$$\hat{B}^{(k_{s+1})} = \hat{R}^{(s)} \hat{B}^{(k_s)}.$$

Again by the boundedness of $\Sigma(\hat{\mathscr{F}})$, the sequence $\{\hat{R}^{(s)}\}_{s\geq 1}$ has a limit point $\hat{R}$ in $\overline{\Sigma(\hat{\mathscr{F}})}$. By passing to the limit, we can conclude that $\hat{B} = \hat{R}\hat{B}$ and, thus, $v = \hat{R}v$ and $\overline{v} = \hat{R}\overline{v}$.

In other words, $\hat{R}$ is an l.s.m.p. of $\hat{\mathscr{F}}$ and v and $\overline{v}$ are leading eigenvectors of $\mathscr{F}$. Thus, with $y = v$ and $\hat{S} = \hat{B}\hat{A}^{(\ell_j)} \cdots \hat{A}^{(\ell_1)}$, Lemma 5.7 implies that v and $\overline{v}$ are standard leading eigenvectors. Therefore, by the assumed $\mathscr{F}$-ciclicity and since $v,\overline{v} \in \partial\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}]$, there exists $\hat{Q} \in \Sigma_k$ (for some integer k) and some $\varphi \in (0,2\pi]$ such that

$$x = \mathrm{e}^{\mathrm{i}\varphi} \hat{Q} v \qquad \text{or} \qquad x = \mathrm{e}^{\mathrm{i}\varphi} \hat{Q} \overline{v}.$$

Consequently, by (5.15), we obtain

$$x^{(j+1)} = \mathrm{e}^{\mathrm{i}\varphi} \hat{A}^{(\ell_j)} \cdots \hat{A}^{(\ell_1)} \hat{Q}\hat{B} x^{(j+1)} \qquad \text{or} \qquad x^{(j+1)} = \left(\hat{A}^{(\ell_j)} \cdots \hat{A}^{(\ell_1)} \hat{Q}\hat{B}\right)^2 x^{(j+1)}.$$

Therefore, since $\hat{A}^{(\ell_j)} \cdots \hat{A}^{(\ell_1)} \hat{Q}\hat{B} \in \overline{\Sigma(\hat{\mathscr{F}})}$, in both cases the vector $x^{(j+1)}$ is a leading eigenvector of $\mathscr{F}$. Thus, with $y = x^{(j+1)}$ and $\hat{S} = \hat{A}^{(\ell_j)} \cdots \hat{A}^{(\ell_1)}$, Lemma 5.7 can be applied again to conclude that $x^{(j+1)}$ is a standard leading eigenvector, which contradicts (5.14). ☐

Observe that the family $\mathscr{F} = \{A,B\}$ of Example 5.1 fits perfectly the hypotheses of Theorem 5.8 for all $\beta < 1$, but not for $\beta = 1$. In fact, for $\beta = 1$ the set (5.11) is not finite modulo scalar factors of unitary modulus, even if (5.12) holds all the same.

The next example illustrates the case of a family that satisfies the hypotheses of Lemma 5.7 and Theorem 5.8 but Hypothesis 5.1. It shows that, whereas the thesis of Lemma 5.7 fails to hold, that of Theorem 5.8

is satisfied. Therefore, we suspect that the thesis of Theorem 5.8 could be obtained in another way, without passing necessarily through Lemma 5.7.

EXAMPLE 5.2. Consider the real 2×2-matrix family $\mathscr{F} = \{A, B\}$, where

$$A = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \frac{11}{20} & \frac{11}{20} \\ -\frac{11}{20} & -\frac{11}{20} \end{bmatrix}.$$

The eigenvalues of A are $\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$ and $\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ with corresponding eigenvectors $x = [1 \ \mathrm{i}]^{\mathrm{T}}$ and $\overline{x} = [1 \ -\mathrm{i}]^{\mathrm{T}}$, respectively, so that Hypothesis 5.1 clearly does not hold. However, it is easy to see that $\mathscr{F}$ is absolutely asymptotically simple with $\rho(\mathscr{F}) = 1$, A being the unique minimal s.m.p., that $\partial\mathscr{S}[\mathscr{F}, x] \bigcap \mathscr{T}[\mathscr{F}, x]$ is finite (modulus scalar factors of unitary modulus) and that

$$\mathscr{S}[\mathscr{F}, x, \overline{x}] = \operatorname{absco}\Big(\{x, \overline{x}, Bx, ABx, A^2Bx\}\Big).$$

For this purpose, note that $A^3 = I$ (identity matrix) and $B^2 = O$ (zero matrix) and that the range of B is a one-dimensional subspace proportional to the real vector $[\frac{11}{20} \ -\frac{11}{20}]^{\mathrm{T}}$.

Thus, since all nonzero vectors of $\mathbb{C}^2$ are leading eigenvectors of $\mathscr{F}$ and since Bx, ABx and A^2Bx are not of standard type, we can conclude that Lemma 5.7 is not true and that, on the contrary, the thesis of Theorem 5.8 does hold all the same. $\Diamond$

The next results, analogous to Theorem 4.15 and to Corollary 4.16, respectively, provide a deeper understanding of the structure of the b.c.p. $\mathscr{S}[\hat{\mathscr{F}}, x, \overline{x}]$ obtained under the hypotheses of Theorem 5.8.

THEOREM 5.9. *Let the hypotheses of Theorem 5.8 hold and let the set $\mathscr{E}$ of the standard leading eigenvectors of $\mathscr{F}$ be non-real. Then each eigenvector $\xi \in \Xi = \mathscr{E} \bigcap \partial\mathscr{S}[\hat{\mathscr{F}}, x, \overline{x}]$ satisfies one of the following two statements:*

(a) *ξ is a vertex of the b.c.p. $\mathscr{S}[\hat{\mathscr{F}}, x, \overline{x}]$.*

(b) *There exist $s \geq 2$ vertices $\xi_1, \ldots, \xi_s$ of the b.c.p. $\mathscr{S}[\hat{\mathscr{F}}, x, \overline{x}]$ such that*

$$\xi_1, \ldots, \xi_s \in \Xi \quad \text{and} \quad \xi \in \operatorname{absco}\Big(\{\xi_1, \ldots, \xi_s\}\Big). \tag{5.16}$$

Proof. Consider a standard leading eigenvector $\xi \in \Xi$ and assume that it is not a vertex of the b.c.p. $\mathscr{S}[\hat{\mathscr{F}}, x, \overline{x}]$. Then there must exist $s \geq 2$ vertices $\xi_1, \ldots, \xi_s$ of $\mathscr{S}[\hat{\mathscr{F}}, x, \overline{x}]$ such that

$$\xi = \sum_{i=1}^{s} \lambda_i \xi_i \quad \text{with} \quad \lambda_i \neq 0, \ i = 1, \ldots, s, \quad \text{and} \quad \sum_{i=1}^{s} |\lambda_i| = 1. \tag{5.17}$$

Since ξ is a standard leading eigenvector of $\mathscr{F}$, there exists an s.m.p. $\tilde{P}$ of $\hat{\mathscr{F}}$ such that $\tilde{P}\xi = u\xi$ with $|u| = 1$. Thus, denoting by $\|\cdot\|$ the complex polytope norm determined by $\mathscr{S}[\hat{\mathscr{F}}, x, \overline{x}]$, for any $k \geq 1$ we have

$$1 = \|\xi\| = \|\tilde{P}^k\xi\| \leq \sum_{i=1}^{s} |\lambda_i| \cdot \|\tilde{P}^k\xi_i\|.$$

Since $\|\tilde{P}^k\xi_i\| \leq 1$, in view of (5.17) we can claim that $\|\tilde{P}^k\xi_i\| = 1$, that is

$$\tilde{P}^k\xi_i \in \partial\mathscr{S}[\hat{\mathscr{F}}, x, \overline{x}], \quad i = 1, \ldots, s.$$

By Theorem 5.8, for any fixed i, the set of vectors $\{\tilde{P}^k\xi_i\}_{k=0}^{\infty}$ is finite (modulo scalar factors of unitary modulus). Hence there exist integers ℓ, m such that

$$\tilde{P}^{\ell+m}\xi_i = \gamma\tilde{P}^{\ell}\xi_i, \qquad \text{with } |\gamma| = 1.$$

This implies that $\tilde{P}^\ell \xi_i$ is a standard leading eigenvector of $\tilde{P}^m$.

Moreover, since all the vertices of the b.c.p. $\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}]$ obviously belong to $\mathscr{T}[\hat{\mathscr{F}},x,\overline{x}]$ (modulo scalar factors of unitary modulus), there exist finite normalized products $\hat{S}_i \in \Sigma(\hat{\mathscr{F}})$ such that

$$\xi_i = \hat{S}_i x.$$

This assumption is not restrictive because the other possibility, that is $\xi_i = \hat{S}_i \overline{x}$, would lead to the same conclusions.

Furthermore, $\mathscr{F}$-cyclicity implies that there exist finite normalized products $\hat{R}_i, \hat{U}_i \in \Sigma(\hat{\mathscr{F}})$ and complex numbers r_i and u_i with $|r_i| = |u_i| = 1$ such that

$$\left\{ \begin{array}{lcl} \xi_i & = & r_i \hat{S}_i \hat{R}_i \xi \\ \text{or} & & \\ \xi_i & = & r_i \hat{S}_i \hat{R}_i \overline{\xi}, \end{array} \right. \tag{5.18}$$

$$\left\{ \begin{array}{lcl} \xi & = & u_i \hat{U}_i \tilde{P}^\ell \xi_i \\ \text{or} & & \\ \xi & = & u_i \hat{U}_i \tilde{P}^\ell \overline{\xi_i}. \end{array} \right. \tag{5.19}$$

Using the first of (5.18) and the first of (5.19) we obtain that ξ_i is a standard leading eigenvector of $\mathscr{F}$ whereas, using the second of (5.18) and the second of (5.19), we obtain that $\overline{\xi_i}$ (and thus ξ_i) is a standard leading eigenvector of $\mathscr{F}$. Similarly, using the first of (5.18) and the second of (5.19) (or, specularly, the second of (5.18) and the first of (5.19)) we obtain $\xi_i = \gamma_i \hat{S}_i \hat{R}_i \hat{U}_i \tilde{P}^\ell \overline{\xi_i}$ for some γ_i with $|\gamma_i| = 1$ and then, in turn, $\xi_i = (\hat{S}_i \hat{R}_i \hat{U}_i \tilde{P}^\ell)^2 \xi_i$, which means again that ξ_i is a standard leading eigenvector of $\mathscr{F}$. Therefore, we conclude that $\xi_i \in \Xi$. $\square$

COROLLARY 5.10. *Let the hypotheses of Theorem 5.8 hold and let the set $\mathscr{E}$ of the standard leading eigenvectors of $\mathscr{F}$ be non-real. Moreover, let the family $\mathscr{F}$ be absolutely asymptotically simple. Then each eigenvector $\xi \in \Xi = \mathscr{E} \bigcap \partial \mathscr{S}[\hat{\mathscr{F}},x,\overline{x}]$ is a vertex of the b.c.p. $\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}]$.*

Proof. Assume, by contradiction, that there exists a standard leading eigenvector $\xi \in \Xi$ which is not a vertex of the b.c.p. $\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}]$. Then it necessarily satisfies statement (b) of Theorem 5.9.

On the other hand, there exists a unique normalized minimal s.m.p. $\tilde{P}$ such that $\tilde{P}\xi = u\xi$ with $|u| = 1$ (see Proposition 5.3). Therefore, for each ξ_i appearing in statement (b) there exists a proper normalized right factor $\tilde{P}_i$ of the s.m.p. $\tilde{P}$ such that $\xi_i = u_i \tilde{P}_i \xi$ or $\xi_i = u_i \tilde{P}_i \overline{\xi}$ with $|u_i| = 1$.

Thus we obtain

$$\xi_i \in \text{absco}\Big(\{\tilde{P}_i \xi_1, \ldots, \tilde{P}_i \xi_s\}\Big) \quad \text{or} \quad \xi_i \in \text{absco}\Big(\{\tilde{P}_i \overline{\xi_1}, \ldots, \tilde{P}_i \overline{\xi_s}\}\Big).$$

Now, since the essential system of vertices of a b.c.p. is unique modulo scalar factors of unitary modulus and since ξ_i is a vertex of $\mathscr{S}[\hat{\mathscr{F}},x,\overline{x}]$, it necessarily holds that, for all $j = 1, \ldots, s$,

$$\xi_i = v_j \tilde{P}_i \xi_j \quad \text{or} \quad \xi_i = v_j \tilde{P}_i \overline{\xi}_j \qquad \text{with } |v_j| = 1.$$

In particular, for $j = i$, we obtain

$$\xi_i = v_i \tilde{P}_i \xi_i \quad \text{with } |v_i| = 1 \quad \text{or} \quad \xi_i = \tilde{P}_i^2 \xi_i,$$

which, in both cases, implies that the proper normalized right factor $\tilde{P}_i$ of $\tilde{P}$ is itself an s.m.p., against the uniqueness of $\tilde{P}$ (modulo cyclic permutations). $\square$

6. Applications of the extended small CPE theorem. Although our results are mostly theoretical in nature, they have potential impact on applications. One of these would lie in the fact that, if there is prior knowledge that a certain set $\mathscr{F}$ has an extremal complex polytope norm, then one could devise algorithms for the computation of $\rho(\mathscr{F})$ that rely on the computation of the extremal points of the unit ball of the norm.

Now we propose a suitable modification of the algorithm presented in [GZ08], which was based on the small CPE theorem proved in [GWZ05]. This new version allows us to construct the unit ball of an extremal complex polytope norm for a nondefective finite real family $\mathscr{F} = \{A^{(i)}\}_{1\le i\le m}$ which satisfies the hypotheses of Theorem 5.8 in the case that the set $\mathscr{E}$ of the standard leading eigenvectors is not real.

ALGORITHM 6.1.

(Step 1) Choose a candidate s.m.p. $P_k \in \Sigma_k(\mathscr{F})$ (for some k), which we assume to have a pair $\{x, \bar{x}\}$ of complex conjugate leading eigenvectors.

(Step 2) Compute $\rho = \rho(P_k)^{1/k}$ and define the scaled family

$$\mathscr{F}^* = \rho^{-1}\mathscr{F} \quad \text{(which is such that } \rho(\mathscr{F}^*) \ge 1).$$

(Step 3) Set $\mathscr{W}^{(0)} = \mathscr{V}^{(0)} = \mathscr{X}^{(0)} = \{x, \bar{x}\}$, $\mathscr{P}^{(0)} = \text{absco}\left(\mathscr{X}^{(0)}\right)$ and $s = 1$.

(Step 4) Compute the set of vectors

$$\mathscr{V}^{(s)} = \mathscr{F}^*\left(\mathscr{W}^{(s-1)}\right).$$

(Step 5) If $\mathscr{V}^{(s)} \subset \mathscr{P}^{(s-1)}$ then STOP

(Step 6) Set $\mathscr{P}^{(s)} = \text{absco}\left(\mathscr{X}^{(s-1)} \cup \mathscr{V}^{(s)}\right)$ and compute an essential system of vertices $\mathscr{X}^{(s)}$ of $\mathscr{P}^{(s)}$ such that

$$\mathscr{X}^{(s)} \subseteq \mathscr{X}^{(s-1)} \cup \mathscr{V}^{(s)}.$$

(Step 7) Set $\mathscr{W}^{(s)} = \mathscr{X}^{(s)} \cap \mathscr{V}^{(s)}$, $s = s+1$ and Goto (Step 4).

The procedure produces a (possibly finite) sequence of self-conjugate absolutely convex sets $\mathscr{P}^{(s)}$. If it halts at (Step 5) for some s^* and if $\text{span}(\mathscr{X}^{(s^*-1)}) = \mathbb{C}^n$, then necessarily $\rho(\mathscr{F}^*) = 1$, so that $\mathscr{F}^*$ is nothing but the normalized family $\hat{\mathscr{F}}$. Moreover, the self-conjugate b.c.p. $\mathscr{P}^{(s^*-1)}$ is equal to $\mathscr{S}[\hat{\mathscr{F}}, x, \bar{x}]$, which determines an extremal norm for $\mathscr{F}$, and we have that

$$\rho(\mathscr{F}) = \rho(P_k)^{1/k}.$$

We conclude the paper by illustrating the foregoing algorithm with an example.

7. An illustrative example. We consider the real 4×4-matrix family $\mathscr{F} = \{A, B\}$, where

$$A = \begin{bmatrix} -3 & -2 & 1 & 2 \\ -2 & 0 & -2 & 1 \\ 1 & 3 & -1 & -5 \\ -3 & -3 & -2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -3 & -1 \\ -4 & -2 & -1 & -4 \\ -1 & 0 & -1 & 2 \\ -1 & -2 & -1 & 2 \end{bmatrix}.$$

Step 1. On the basis of a preliminary computational investigation, we guess that $P_2 = AB$ is a reasonable candidate s.m.p. for the family $\mathscr{F}$. We compute the two leading complex conjugate eigenpairs of P_2, that is

$$\lambda_{1,2} = 10.874286706162435289 \pm 17.646835410096406617\,\mathrm{i}$$

and

$$x = \begin{bmatrix} 0.72792977781254290782 - 0.34062961167742906135\,\mathrm{i} \\ 0.20390080201908339851 + 0.41457724740357478799\,\mathrm{i} \\ -0.56613846958271645283 + 1.2278515180664638032\,\mathrm{i} \\ 1 \end{bmatrix},$$

$$\bar{x} = \begin{bmatrix} 0.72792977781254290782 + 0.34062961167742906135\,\mathrm{i} \\ 0.20390080201908339851 - 0.41457724740357478799\,\mathrm{i} \\ -0.56613846958271645283 - 1.2278515180664638032\,\mathrm{i} \\ 1 \end{bmatrix}.$$

Step 2. We scale the family $\mathscr{F}$ by $\rho(P_2)^{1/2} = |\lambda_{1,2}|^{1/2} = 4.5528302832023213335$, so as to obtain

$$\mathscr{F}^* = \{A^*, B^*\} = \left\{ \frac{A}{\rho(P_2)^{1/2}}, \frac{B}{\rho(P_2)^{1/2}} \right\}.$$

Step 3. The two starting vectors are $v_1 = x$ and $\overline{v}_1 = \overline{x}$.
We set $\mathscr{W}^{(0)} = \mathscr{V}^{(0)} = \mathscr{X}^{(0)} = \{v_1, \overline{v}_1\}$ and $\mathscr{P}^{(0)} = \text{absco}\left(\mathscr{X}^{(0)}\right)$.
Step 4. By applying $\mathscr{F}^*$ to $\mathscr{W}^{(0)}$ we obtain

$$\mathscr{V}^{(1)} = \{v_2, v_3, \overline{v}_2, \overline{v}_3\},$$

where $v_2 = A^* v_1, v_3 = B^* v_1$. A 4-digit approximation of the computed vectors follows:

$$\begin{aligned} v_2 &= \left[\ -0.2543 + 0.3120\,\mathrm{i} \quad 0.1486 - 0.3897\,\mathrm{i} \quad -0.6796 - 0.07133\,\mathrm{i} \quad -0.5850 - 0.5881\,\mathrm{i}\ \right]^{\mathrm{T}}, \\ v_3 &= \left[\ 0.3133 - 0.8839\,\mathrm{i} \quad -1.483 - 0.1530\,\mathrm{i} \quad 0.4038 - 0.1949\,\mathrm{i} \quad 0.3142 - 0.3770\,\mathrm{i}\ \right]^{\mathrm{T}}. \end{aligned}$$

Step 5. It is immediate to see that $\mathscr{V}^{(1)}$ is not included in $\mathscr{P}^{(0)}$.
Step 6. We set $\mathscr{P}^{(1)} = \text{absco}\left(\mathscr{X}^{(0)} \cup \mathscr{V}^{(1)}\right)$ and compute the essential system of vertices

$$\mathscr{X}^{(1)} = \{v_1, v_2, v_3, \overline{v}_1, \overline{v}_2, \overline{v}_3\} = \mathscr{X}^{(0)} \cup \mathscr{V}^{(1)}.$$

Step 7. We set

$$\mathscr{W}^{(1)} = \mathscr{X}^{(1)} \cap \mathscr{V}^{(1)} = \{v_2, v_3, \overline{v}_2, \overline{v}_3\}$$

and go back to (Step 4).
Step 4. By applying $\mathscr{F}^*$ to $\mathscr{W}^{(1)}$ we obtain

$$\mathscr{V}^{(2)} = \{v_4, v_5, v_6, v_7, \overline{v}_4, \overline{v}_5, \overline{v}_6, \overline{v}_7\},$$

where $v_4 = A^* v_2, v_5 = A^* v_3, v_6 = B^* v_2, v_7 = B^* v_3$. A 4-digit approximation of the computed vectors follows:

$$\begin{aligned} v_4 &= \left[\ -0.3039 - 0.3084\,\mathrm{i} \quad 0.2818 - 0.2349\,\mathrm{i} \quad 0.8337 + 0.4733\,\mathrm{i} \quad 0.4967 + 0.2117\,\mathrm{i}\ \right]^{\mathrm{T}}, \\ v_5 &= \left[\ 0.6719 + 0.4410\,\mathrm{i} \quad -0.2460 + 0.3911\,\mathrm{i} \quad -1.3423 + 0.1622\,\mathrm{i} \quad 0.5246 + 0.8513\,\mathrm{i}\ \right]^{\mathrm{T}}, \\ v_6 &= \left[\ 0.5205 + 0.2447\,\mathrm{i} \quad 0.8214 + 0.4294\,\mathrm{i} \quad -0.0518 - 0.3112\,\mathrm{i} \quad -0.1171 - 0.1400\,\mathrm{i}\ \right]^{\mathrm{T}}, \\ v_7 &= \left[\ -0.2662 + 0.0171\,\mathrm{i} \quad 0.0117 + 1.2176\,\mathrm{i} \quad -0.01948 + 0.07134\,\mathrm{i} \quad 0.6321 + 0.1383\,\mathrm{i}\ \right]^{\mathrm{T}}. \end{aligned}$$

Step 5. By computing the norms $\|\cdot\|_{\mathscr{P}^{(1)}}$ of the elements of $\mathscr{V}^{(2)}$, $\|v_4\|_{\mathscr{P}^{(1)}} > 1$, $\|v_5\|_{\mathscr{P}^{(1)}} = 1$, $\|v_6\|_{\mathscr{P}^{(1)}} > 1$, $\|v_7\|_{\mathscr{P}^{(1)}} > 1$, we see that $\mathscr{V}^{(2)}$ is not included in $\mathscr{P}^{(1)}$.
Step 6. We set $\mathscr{P}^{(2)} = \text{absco}\left(\mathscr{X}^{(1)} \cup \mathscr{V}^{(2)}\right)$ and compute the essential system of vertices

$$\mathscr{X}^{(2)} = \{v_1, v_2, v_3, v_4, v_6, v_7, \overline{v}_1, \overline{v}_2, \overline{v}_3, \overline{v}_4, \overline{v}_6, \overline{v}_7\} \subset \mathscr{X}^{(1)} \cup \mathscr{V}^{(2)}.$$

Step 7. We set

$$\mathscr{W}^{(2)} = \mathscr{X}^{(2)} \cap \mathscr{V}^{(2)} = \{v_4, v_6, v_7, \overline{v}_4, \overline{v}_6, \overline{v}_7\}$$

and go back to (Step 4).

Step 4. By applying $\mathscr{F}^*$ to $\mathscr{W}^{(2)}$ we obtain

$$\mathscr{V}^{(3)} = \{v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, \overline{v}_8, \overline{v}_9, \overline{v}_{10}, \overline{v}_{11}, \overline{v}_{12}, \overline{v}_{13}\},$$

where $v_8 = A^*v_4, v_9 = A^*v_6, v_{10} = A^*v_7, v_{11} = B^*v_4, v_{12} = B^*v_6, v_{13} = B^*v_7$. A 4-digit approximation of the computed vectors follows:

$$\begin{aligned}
v_8 &= \begin{bmatrix} 0.4778+0.5034\,\mathrm{i} & -0.12363-0.02591\,\mathrm{i} & -0.6097-0.5590\,\mathrm{i} & -0.4607+0.1036\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_9 &= \begin{bmatrix} -0.7666-0.4798\,\mathrm{i} & -0.2316-0.0015\,\mathrm{i} & 0.7955+0.5588\,\mathrm{i} & -0.8357-0.2768\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{10} &= \begin{bmatrix} 0.4437-0.4697\,\mathrm{i} & 0.2644-0.0084\,\mathrm{i} & -0.7407+0.6385\,\mathrm{i} & 0.0375-0.8753\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{11} &= \begin{bmatrix} -0.7252-0.4261\,\mathrm{i} & -0.4762+0.0842\,\mathrm{i} & 0.10183+0.05680\,\mathrm{i} & -0.0220+0.1600\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{12} &= \begin{bmatrix} 0.1742+0.2896\,\mathrm{i} & -0.7038-0.2123\,\mathrm{i} & -0.1544-0.0469\,\mathrm{i} & -0.5152-0.2355\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{13} &= \begin{bmatrix} -0.1845-0.0736\,\mathrm{i} & -0.3223-0.6871\,\mathrm{i} & 0.3404+0.0414\,\mathrm{i} & 0.3353-0.4935\,\mathrm{i} \end{bmatrix}^{\mathrm{T}}.
\end{aligned}$$

Step 5. By computing the norms $\|\cdot\|_{\mathscr{P}^{(2)}}$ of the elements of $\mathscr{V}^{(3)}$, $\|v_8\|_{\mathscr{P}^{(2)}} > 1$, $\|v_9\|_{\mathscr{P}^{(2)}} > 1$, $\|v_{10}\|_{\mathscr{P}^{(2)}} > 1$, $\|v_{11}\|_{\mathscr{P}^{(2)}} > 1$, $\|v_{12}\|_{\mathscr{P}^{(2)}} > 1$, $\|v_{13}\|_{\mathscr{P}^{(2)}} < 1$, we see that $\mathscr{V}^{(3)}$ is not included in $\mathscr{P}^{(2)}$.

Step 6. We set $\mathscr{P}^{(3)} = \mathrm{absco}\left(\mathscr{X}^{(2)} \cup \mathscr{V}^{(3)}\right)$ and compute the essential system of vertices

$$\mathscr{X}^{(3)} = \{v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, \overline{v}_1, \overline{v}_2, \overline{v}_3, \overline{v}_4, \overline{v}_6, \overline{v}_7, \overline{v}_8, \overline{v}_9, \overline{v}_{10}, \overline{v}_{11}, \overline{v}_{12}\} \subset \mathscr{X}^{(2)} \cup \mathscr{V}^{(3)}.$$

Step 7. We set

$$\mathscr{W}^{(3)} = \mathscr{X}^{(3)} \cap \mathscr{V}^{(3)} = \{v_8, v_9, v_{10}, v_{11}, v_{12}, \overline{v}_8, \overline{v}_9, \overline{v}_{10}, \overline{v}_{11}, \overline{v}_{12}\}$$

and go back to (Step 4).

Step 4. By applying $\mathscr{F}^*$ to $\mathscr{W}^{(3)}$ we obtain

$$\mathscr{V}^{(4)} = \{v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, \overline{v}_{14}, \overline{v}_{15}, \overline{v}_{16}, \overline{v}_{17}, \overline{v}_{18}, \overline{v}_{19}, \overline{v}_{20}, \overline{v}_{21}, \overline{v}_{22}, \overline{v}_{23}\},$$

where $v_{14} = A^*v_8, v_{15} = A^*v_9, v_{16} = A^*v_{10}, v_{17} = A^*v_{11}, v_{18} = A^*v_{12}, v_{19} = B^*v_8, v_{20} = B^*v_9, v_{21} = B^*v_{10}, v_{22} = B^*v_{11}, v_{23} = B^*v_{12}$. A 4-digit approximation of the computed vectors follows:

$$\begin{aligned}
v_{14} &= \begin{bmatrix} -0.5968-0.3976\,\mathrm{i} & -0.04327+0.04719\,\mathrm{i} & 0.6634+0.1025\,\mathrm{i} & 0.1356-0.0918\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{15} &= \begin{bmatrix} 0.4145+0.3180\,\mathrm{i} & -0.1963-0.0955\,\mathrm{i} & 0.4220+0.0748\,\mathrm{i} & 0.4918+0.1324\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{16} &= \begin{bmatrix} -0.5548+0.0689\,\mathrm{i} & 0.1387-0.2664\,\mathrm{i} & 0.3932+0.7123\,\mathrm{i} & -0.1494+0.2268\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{17} &= \begin{bmatrix} 0.6998+0.3265\,\mathrm{i} & 0.2690+0.1974\,\mathrm{i} & -0.4714-0.2263\,\mathrm{i} & 0.7518+0.1652\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{18} &= \begin{bmatrix} -0.0658-0.2113\,\mathrm{i} & -0.1219-0.1583\,\mathrm{i} & 0.1742+0.1927\,\mathrm{i} & 0.5299+0.0214\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{19} &= \begin{bmatrix} 0.6079+0.4561\,\mathrm{i} & 0.1732-0.3991\,\mathrm{i} & -0.1734+0.0577\,\mathrm{i} & -0.11912+0.06911\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{20} &= \begin{bmatrix} -0.5090-0.4128\,\mathrm{i} & 1.335+0.5430\,\mathrm{i} & -0.3735-0.1389\,\mathrm{i} & -0.2717-0.1383\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{21} &= \begin{bmatrix} 0.5773-0.3316\,\mathrm{i} & -0.3762+1.0451\,\mathrm{i} & 0.0817-0.4216\,\mathrm{i} & -0.0344-0.4179\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{22} &= \begin{bmatrix} -0.2216-0.1662\,\mathrm{i} & 0.8433+0.1843\,\mathrm{i} & 0.1273+0.1514\,\mathrm{i} & 0.3365+0.1144\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{23} &= \begin{bmatrix} 0.2531+0.1462\,\mathrm{i} & 0.6427+0.0561\,\mathrm{i} & -0.2307-0.1568\,\mathrm{i} & 0.07851-0.06352\,\mathrm{i} \end{bmatrix}^{\mathrm{T}}.
\end{aligned}$$

Step 5. By computing the norms $\|\cdot\|_{\mathscr{P}^{(3)}}$ of the elements of $\mathscr{V}^{(4)}$, $\|v_{14}\|_{\mathscr{P}^{(3)}} > 1$, $\|v_{15}\|_{\mathscr{P}^{(3)}} < 1$, $\|v_{16}\|_{\mathscr{P}^{(3)}} < 1$, $\|v_{17}\|_{\mathscr{P}^{(3)}} > 1$, $\|v_{18}\|_{\mathscr{P}^{(3)}} < 1$, $\|v_{19}\|_{\mathscr{P}^{(3)}} > 1$, $\|v_{20}\|_{\mathscr{P}^{(3)}} > 1$, $\|v_{21}\|_{\mathscr{P}^{(3)}} > 1$, $\|v_{22}\|_{\mathscr{P}^{(3)}} < 1$, $\|v_{23}\|_{\mathscr{P}^{(3)}} < 1$, we see that $\mathscr{V}^{(4)}$ is not included in $\mathscr{P}^{(3)}$.

Step 6. We set $\mathscr{P}^{(4)} = \text{absco}\left(\mathscr{X}^{(3)} \cup \mathscr{V}^{(4)}\right)$ and compute the essential system of vertices

$$\begin{aligned}\mathscr{X}^{(4)} = \{&v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{14}, v_{17}, v_{19}, v_{20}, v_{21},\\ &\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_6, \bar{v}_7, \bar{v}_8, \bar{v}_9, \bar{v}_{10}, \bar{v}_{11}, \bar{v}_{12}, \bar{v}_{14}, \bar{v}_{17}, \bar{v}_{19}, \bar{v}_{20}, \bar{v}_{21}\} \subset \mathscr{X}^{(3)} \cup \mathscr{V}^{(4)}.\end{aligned}$$

Step 7. We set

$$\mathscr{W}^{(4)} = \mathscr{X}^{(4)} \cap \mathscr{V}^{(4)} = \{v_{14}, v_{17}, v_{19}, v_{20}, v_{21}, \bar{v}_{14}, \bar{v}_{17}, \bar{v}_{19}, \bar{v}_{20}, \bar{v}_{21}\}$$

and go back to (Step 4).

Step 4. By applying $\mathscr{F}^*$ to $\mathscr{W}^{(4)}$ we obtain

$$\mathscr{V}^{(5)} = \{v_{24}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}, v_{30}, v_{31}, v_{32}, v_{33}, \bar{v}_{24}, \bar{v}_{25}, \bar{v}_{26}, \bar{v}_{27}, \bar{v}_{28}, \bar{v}_{29}, \bar{v}_{30}, \bar{v}_{31}, \bar{v}_{32}, \bar{v}_{33}\},$$

where $v_{24} = A^*v_{14}, v_{25} = A^*v_{17}, v_{26} = A^*v_{19}, v_{27} = A^*v_{20}, v_{28} = A^*v_{21}, v_{29} = B^*v_{14}, v_{30} = B^*v_{17}, v_{31} = B^*v_{19}, v_{32} = B^*v_{20}, v_{33} = B^*v_{21}$. A 4-digit approximation of the computed vectors follows:

$$\begin{aligned}
v_{24} &= \begin{bmatrix} 0.6176+0.2234\mathrm{i} & 0.00056+0.10946\mathrm{i} & -0.4543+0.0221\mathrm{i} & 0.1006+0.2060\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{25} &= \begin{bmatrix} -0.3526-0.2790\mathrm{i} & 0.06477-0.00775\mathrm{i} & -0.3911+0.0701\mathrm{i} & -0.5965-0.2821\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{26} &= \begin{bmatrix} -0.5671-0.0822\mathrm{i} & -0.2170-0.2106\mathrm{i} & 0.4166-0.2514\mathrm{i} & -0.4123-0.0781\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{27} &= \begin{bmatrix} -0.4523-0.0576\mathrm{i} & 0.3280+0.2120\mathrm{i} & 1.1481+0.4492\mathrm{i} & -0.3203+0.0059\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{28} &= \begin{bmatrix} -0.2123-0.5168\mathrm{i} & -0.2971+0.2391\mathrm{i} & -0.1012+1.1673\mathrm{i} & -0.1609-0.1932\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{29} &= \begin{bmatrix} -0.5980-0.1347\mathrm{i} & 0.2785+0.3867\mathrm{i} & 0.04497+0.02448\mathrm{i} & 0.06398+0.00375\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{30} &= \begin{bmatrix} 0.2992+0.1845\mathrm{i} & -1.2900-0.4690\mathrm{i} & 0.2801+0.0505\mathrm{i} & 0.1619-0.0361\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{31} &= \begin{bmatrix} 0.2740+0.0470\mathrm{i} & -0.4674-0.2988\mathrm{i} & -0.1478-0.0825\mathrm{i} & -0.2238+0.0928\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{32} &= \begin{bmatrix} 0.1940+0.0313\mathrm{i} & 0.1817+0.2763\mathrm{i} & 0.07446+0.06045\mathrm{i} & -0.5118-0.1779\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{33} &= \begin{bmatrix} 0.0805+0.2967\mathrm{i} & -0.3297+0.2919\mathrm{i} & -0.1599-0.0181\mathrm{i} & 0.0054-0.4772\mathrm{i} \end{bmatrix}^{\mathrm{T}}.
\end{aligned}$$

Step 5. By computing the norms $\|\cdot\|_{\mathscr{P}^{(4)}}$ of the elements of $\mathscr{V}^{(5)}$, $\|v_{24}\|_{\mathscr{P}^{(4)}} < 1$, $\|v_{25}\|_{\mathscr{P}^{(4)}} < 1$, $\|v_{26}\|_{\mathscr{P}^{(4)}} < 1$, $\|v_{27}\|_{\mathscr{P}^{(4)}} > 1$, $\|v_{28}\|_{\mathscr{P}^{(4)}} > 1$, $\|v_{29}\|_{\mathscr{P}^{(4)}} < 1$, $\|v_{30}\|_{\mathscr{P}^{(4)}} < 1$, $\|v_{31}\|_{\mathscr{P}^{(4)}} < 1$, $\|v_{32}\|_{\mathscr{P}^{(4)}} < 1$, $\|v_{33}\|_{\mathscr{P}^{(4)}} < 1$, we see that $\mathscr{V}^{(5)}$ is not included in $\mathscr{P}^{(4)}$.

Step 6. We set $\mathscr{P}^{(5)} = \text{absco}\left(\mathscr{X}^{(4)} \cup \mathscr{V}^{(5)}\right)$ and compute the essential system of vertices

$$\begin{aligned}\mathscr{X}^{(5)} = \{&v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{14}, v_{17}, v_{19}, v_{20}, v_{21}, v_{27}, v_{28}\\ &\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_6, \bar{v}_7, \bar{v}_8, \bar{v}_9, \bar{v}_{10}, \bar{v}_{11}, \bar{v}_{12}, \bar{v}_{14}, \bar{v}_{17}, \bar{v}_{19}, \bar{v}_{20}, \bar{v}_{21}, \bar{v}_{27}, \bar{v}_{28}\} \subset \mathscr{X}^{(4)} \cup \mathscr{V}^{(5)}.\end{aligned}$$

Step 6. We set

$$\mathscr{W}^{(5)} = \mathscr{X}^{(5)} \cap \mathscr{V}^{(5)} = \{v_{27}, v_{28}, \bar{v}_{27}, \bar{v}_{28}\}$$

and go back to (Step 4).

Step 4. By applying $\mathscr{F}^*$ to $\mathscr{W}^{(5)}$ we obtain

$$\mathscr{V}^{(6)} = \{v_{34}, v_{35}, v_{36}, v_{37}, \bar{v}_{34}, \bar{v}_{35}, \bar{v}_{36}, \bar{v}_{37}\},$$

where $v_{34} = A^* v_{27}, v_{35} = A^* v_{28}, v_{36} = B^* v_{27}, v_{37} = B^* v_{28}$. A 4-digit approximation of the computed vectors follows:

$$\begin{aligned}
v_{34} &= \begin{bmatrix} 0.2654 + 0.0461\,\mathrm{i} & -0.3760 - 0.1707\,\mathrm{i} & 0.2164 + 0.0219\,\mathrm{i} & -0.3521 - 0.3004\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{35} &= \begin{bmatrix} 0.1775 + 0.4070\,\mathrm{i} & 0.1024 - 0.3282\,\mathrm{i} & -0.04349 - 0.00020\,\mathrm{i} & 0.4155 - 0.2874\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{36} &= \begin{bmatrix} -0.7855 - 0.3099\,\mathrm{i} & 0.2826 - 0.1464\,\mathrm{i} & -0.2935 - 0.0834\,\mathrm{i} & -0.4376 - 0.1766\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{37} &= \begin{bmatrix} 0.0554 - 0.8403\,\mathrm{i} & 0.4806 + 0.2623\,\mathrm{i} & -0.0018 - 0.2278\,\mathrm{i} & 0.1287 - 0.3328\,\mathrm{i} \end{bmatrix}^{\mathrm{T}}.
\end{aligned}$$

Step 5. By computing the norms $\|\cdot\|_{\mathscr{P}^{(5)}}$ of the elements of $\mathscr{V}^{(6)}$, $\|v_{34}\|_{\mathscr{P}^{(5)}} > 1$, $\|v_{35}\|_{\mathscr{P}^{(5)}} < 1$, $\|v_{36}\|_{\mathscr{P}^{(5)}} < 1$, $\|v_{37}\|_{\mathscr{P}^{(5)}} < 1$, we see that $\mathscr{V}^{(6)}$ is not included in $\mathscr{P}^{(5)}$.

Step 6. We set $\mathscr{P}^{(6)} = \operatorname{absco}\left(\mathscr{X}^{(5)} \cup \mathscr{V}^{(6)}\right)$ and compute the essential system of vertices

$$\begin{aligned}
\mathscr{X}^{(6)} = \{&v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{14}, v_{17}, v_{19}, v_{20}, v_{21}, v_{27}, v_{28}, v_{34}\\
&\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_6, \bar{v}_7, \bar{v}_8, \bar{v}_9, \bar{v}_{10}, \bar{v}_{11}, \bar{v}_{12}, \bar{v}_{14}, \bar{v}_{17}, \bar{v}_{19}, \bar{v}_{20}, \bar{v}_{21}, \bar{v}_{27}, \bar{v}_{28}, \bar{v}_{34}\} \subset \mathscr{X}^{(5)} \cup \mathscr{V}^{(6)}.
\end{aligned}$$

Step 7. We set

$$\mathscr{W}^{(6)} = \mathscr{X}^{(6)} \cap \mathscr{V}^{(6)} = \{v_{34}, \bar{v}_{34}\}$$

and go back to (Step 4).

Step 4. By applying $\mathscr{F}^*$ to $\mathscr{W}^{(6)}$ we obtain

$$\mathscr{V}^{(7)} = \{v_{38}, v_{39}, \bar{v}_{38}, \bar{v}_{39}\},$$

where $v_{38} = A^* v_{34}, v_{39} = B^* v_{34}$. A 4-digit approximation of the computed vectors follows:

$$\begin{aligned}
v_{38} &= \begin{bmatrix} -0.1168 - 0.0825\,\mathrm{i} & -0.2890 - 0.0958\,\mathrm{i} & 0.1496 + 0.2227\,\mathrm{i} & 0.0552 + 0.1385\,\mathrm{i} \end{bmatrix}^{\mathrm{T}},\\
v_{39} &= \begin{bmatrix} -0.00696 + 0.06167\,\mathrm{i} & 0.1938 + 0.2936\,\mathrm{i} & -0.2605 - 0.1469\,\mathrm{i} & -0.09530 - 0.07188\,\mathrm{i} \end{bmatrix}^{\mathrm{T}}.
\end{aligned}$$

Step 5. By computing the norms $\|\cdot\|_{\mathscr{P}^{(6)}}$ of the elements of $\mathscr{V}^{(7)}$, $\|v_{38}\|_{\mathscr{P}^{(6)}} < 1$, $\|v_{39}\|_{\mathscr{P}^{(6)}} < 1$, we see that

$$\mathscr{V}^{(7)} \subset \mathscr{P}^{(6)}.$$

Hence, the algorithm halts.

Since $\operatorname{span}\left(\mathscr{P}^{(6)}\right) = \mathbb{C}^4$, we can conclude that $\mathscr{P}^{(6)} = \operatorname{absco}\left(\mathscr{X}^{(6)}\right)$ is a self-conjugate b.c.p. which determines an extremal norm for $\mathscr{F}$ and that $\rho(\mathscr{F}) = |\lambda_{1,2}|^{1/2} = 4.5528302832023213335$.

The computations have been performed using the software *Mathematica* (see [Wol05]) with a 30-digit accuracy. For the computation of the polytope norms at (Step 5) we used the function *NMinimize*.

8. Concluding discussion. In this section we discuss the results of this paper in the light of the existing literature.

First of all, we observe that there exist families of real matrices which have an extremal complex polytope norm but do not admit any extremal real polytope norm. For example, any family $\mathscr{F} = \{A_\theta\}$ consisting of a single rotation matrix

$$A_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix},$$

with θ rationally independent on π, does not admit any extremal real polytope norm (see [BW92]). However, it admits a complex polytope norm with unit ball given by $\mathscr{P} = \mathrm{absco}\left(\{z_\theta, \bar{z}_\theta\}\right)$, z_θ being a leading eigenvector of A_θ. This justifies the use of complex polytope norms and related algorithms also for families of real matrices.

Concerning our main result, that is Theorem 5.8, we remark that the proof is based on Lemma 5.5, which makes use of Hypothesis 5.1 in an essential way. Since the lemma is not valid if Hypothesis 5.1 does not hold (see Example 5.2), it seems that, in order to extend the validity of Theorem 5.8 to the this case, a different proof should be found. On the other hand, Hypothesis 5.1 appears to be generic in the sense that the measure of the set of excluded cases is zero. Nevertheless, we consider such an extension of the theorem an important theoretical issue.

Our approach for the computation of the joint spectral radius passes through the construction of the unit ball of a polytope extremal norm. In order to illustrate its theoretical relevance, we recall its applications to the zero-stability analysis of variable stepsize BDF formulæ[GZ01b] and to the asymptotic stability analysis of one-step methods for the numerical approximation of delay differential equations [GZ03b]. Finally, in the recent paper [GCCZ], we have proved that every pair of 2×2 binary sign-matrices, that is, with entries in $\{-1, 0, 1\}$, has the finiteness property. This supports the conjecture by Blondel and Jüngers [JBss] that this holds in general for all pairs of sign-matrices of any dimension. In turn, this fact would have the consequence that the finiteness property holds true for all families of rational matrices.

Now we discuss the computational relevance of Algorithm 6.1. It is clear that, at its current state, the algorithm is designed to verify that a candidate product $P \in \Sigma(\mathscr{F})$ is an s.m.p. for a non-defective family $\mathscr{F}$. This is possible under certain assumptions, the main of which is asymptotic simplicity, which cannot be checked a priori. Nevertheless, if P is an s.m.p., the algorithm always converges. Indeed, the mentioned assumptions provide sufficient conditions for convergence in a finite number of steps.

We conclude by remarking that a different, quite interesting, approach in order to approximate the joint spectral radius of a finite family $\mathscr{F}$ of real matrices has been proposed recently by Protasov [Pro96, Pro05]. The main idea is still based on the property that any irreducible family $\mathscr{F}$ has an extremal norm and that its unit ball is a centrally symmetric invariant compact set for $\mathscr{F}$. The algorithm constructs an *almost invariant* set $\mathscr{R}$, that is, such that

$$\min_{\lambda > 0} \mathrm{dist}\left(\mathrm{co}\left(\mathscr{F}\left(\mathscr{R}\right)\right), \lambda\,\mathscr{R}\right)$$

be sufficiently small, where dist denotes some distance between sets and $\mathrm{co}\left(\mathscr{F}\left(\mathscr{R}\right)\right)$ denotes the convex hull of the union of the sets $A\mathscr{R}$ for all $A \in \mathscr{F}$. If this holds, the minimizer λ^* provides a good approximation of $\rho(\mathscr{F})$.

Protasov's algorithm is able to reach a given accuracy ε in polynomial time with respect to $1/\varepsilon$. Its implementation is based on the recursive application of $\mathscr{F}$ to a sequence of real polytopes which are defined in the following way. The first polytope $\mathscr{R}_0$ is chosen, for example, as the unit ball of the 1-norm. Then the polytope $\mathscr{R}_{m+1}$ is given either by $\mathrm{co}\left(\mathscr{F}\left(\mathscr{R}_m\right)\right)$ if the number of its vertices does not exceed a certain bound $\nu(\varepsilon)$) or, otherwise, by a polytope with at most $\nu(\varepsilon)$ vertices chosen in such a way that

$$(1+\varepsilon)\,\mathrm{co}\left(\mathscr{F}\left(\mathscr{R}_m\right)\right) \subset \mathscr{R}_{m+1} \subset \mathrm{co}\left(\mathscr{F}\left(\mathscr{R}_m\right)\right).$$

At the m-th step the approximation to $\rho(\mathscr{F})$ is given by $\sqrt[m]{d_m}$, where d_m is the radius of the smallest circle including $\mathscr{R}_m$.

Differently from our approach, in general the algorithm proposed by Protasov does not show convergence in a finite number of steps but, on the other hand, it does not require a guess for an s.m.p. (which is crucial for Algorithm 6.1) and also provides estimates of the accuracy obtained at every iteration.

REFERENCES

[Bar88] N. E. Barabanov. Lyapunov indicator of discrete inclusions. I–III. *Autom. Remote Control*, 49(2,3,5):152–157, 283–287, 558–565, 1988.

[BM02] T. Bousch and J. Mairesse. Asymptotic height optimization for topical IFS, Tetris heaps and the finiteness conjecture. *J. Amer. Math. Soc.*, 15:77–111, 2002.

[BN05a] V. D. Blondel and Yu. Nesterov. Fast and precise computation of the joint spectral radius. *SIAM J. Matrix Analysis Appl.*, 2005. to appear.

[BN05b] V.D. Blondel and Y. Nesterov. Computationally efficient approximations of the joint spectral radius. *SIAM J. Matrix Analysis Appl.*, 27(1):256–272, 2005.

[BNT05] V.D. Blondel, Y. Nesterov, and J. Theys. On the accuracy of the ellipsoid norm approximation of the joint spectral radius. *Linear Algebra Appl.*, 394(1):91–107, 2005.

[BTV03] V.D. Blondel, J. Theys, and A.A. Vladimirov. An elementary counterexample to the finiteness conjecture. *SIAM J. Matrix Analysis Appl.*, 24(4):963–970, 2003.

[BW92] M.A. Berger and Y. Wang. Bounded semigroups of matrices. *Linear Algebra Appl.*, 166:21–27, 1992.

[DL92] I. Daubechies and J.C. Lagarias. Sets of matrices all infinite products of which converge. *Linear Algebra Appl.*, 161:227–263, 1992.

[DL01] I. Daubechies and J.C. Lagarias. Corrigendum/addendum to: "Sets of matrices all infinite products of which converge". *Linear Algebra Appl.*, 327(1-3):69–83, 2001.

[Els95] L. Elsner. The generalized spectral-radius theorem: an analytic-geometric proof. *Linear Algebra Appl.*, 220:151–159, 1995.

[GCCZ] N. Guglielmi, A. Cicone, S. Serra Capizzano, and M. Zennaro. Finiteness properties of pairs of 2×2 sign-matrices via polytope norms. *in preparation*.

[Gri96] G. Gripenberg. Computing the joint spectral radius. *Linear Algebra Appl.*, 234:43–60, 1996.

[GWZ05] N. Guglielmi, F. Wirth, and M. Zennaro. Complex polytope extremality results for families of matrices. *SIAM J. Matrix Analysis Appl.*, 27(3):721–743, 2005.

[GZ01a] N. Guglielmi and M. Zennaro. On the asymptotic properties of a family of matrices. *Linear Algebra Appl.*, 322:169–192, 2001.

[GZ01b] N. Guglielmi and M. Zennaro. On the zero-stability of variable stepsize multistep methods: the spectral radius approach. *Numer. Math.*, 88:445–458, 2001.

[GZ03a] N. Guglielmi and M. Zennaro. On the limit products of a family of matrices. *Linear Algebra Appl.*, 362:11–27, 2003.

[GZ03b] N. Guglielmi and M. Zennaro. Stability of one-leg Θ-methods for the variable coefficient pantograph equation on the quasi-geometric mesh. *IMA J. Numer. Anal.*, 23:421–438, 2003.

[GZ07] N. Guglielmi and M. Zennaro. Balanced complex polytopes and related vector and matrix norms. *J. Convex Anal.*, 14:729–766, 2007.

[GZ08] N. Guglielmi and M. Zennaro. An algorithm for finding extremal polytope norms of matrix families. *Linear Algebra Appl.*, 428:2265–2282, 2008.

[HW79] G.H. Hardy and E.M. Wright. *An introduction to the theory of numbers*. Clarendon Press, Oxford, 1979.

[JBss] R. Jüngers and V. Blondel. On the finiteness properties for rational matrices. *Linear Algebra Appl.*, in press.

[Koz90] V. S. Kozyakin. Algebraic unsolvability of problem of absolute stability of desynchronized systems. *Autom. Remote Control*, 51(6):754–759, 1990.

[LW95] J.C. Lagarias and Y. Wang. The finiteness conjecture for the generalized spectral radius of a set of matrices. *Linear Algebra Appl.*, 214:17–42, 1995.

[Mae95] M. Maesumi. Optimum unit ball for joint spectral radius: an example from four-coefficient MRA. In "Approximation Theory VIII: Wavelets and Multilevel Approximation". C.K. Chui and L.L. Schumaker (eds.). 2:267–274, 1995.

[Mae98] M. Maesumi. Calculating joint spectral radius of matrices and Holder exponent of wavelets. In "Approximation Theory IX". C.K. Chui and L.L. Schumaker (eds.). pages 1–8, 1998.

[MAS01] B.E. Moision, A.Orlitsky, and P.H. Siegel. On codes that avoid specified differences. *IEEE Trans. Inform. Theory*, 47(1):433–442, 2001.

[Pro96] V. Protasov. The joint spectral radius and invarian sets of linear operators. *Fundamentalnaya i prikladnaya matematika*, 2(1):205–231, 1996.

[Pro05] V. Protasov. The geometric approach for computing the joint spectral radius. *Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference CDC-ECC'05*, pages 3001–3006, 2005.

[RR00] H. Radjavi and P. Rosenthal. *Simultaneous triangularization*. Universitext. Springer-Verlag, New York, 2000.

[RS60] G.C. Rota and G. Strang. A note on the joint spectral radius. *Indag. Math.*, 22:379–381, 1960.

[Shi99] M.H. Shih. Simultaneous Schur stability. *Linear Algebra Appl.*, 287:323–336, 1999.

[SWP97] M.H. Shih, J.W. Wu, and C.T. Pang. Asymptotic stability and generalized Gelfand spectral radius formula. *Linear Algebra Appl.*, 252:61–70, 1997.

[TB97] J. N. Tsitsiklis and V. Blondel. The Lyapunov exponent and joint spectral radius of pairs of matrices are hard - when not impossible - to compute and to approximate. *Math. Contr., Sign., and Syst.*, 10:31–40, 1997.

[VZss] C. Vagnoni and M. Zennaro. The analysis and the representation of balanced complex polytopes in 2d. *Found. Comput. Math.*, in press.

[Wir98] F. Wirth. On the calculation of time-varying stability radii. *Internat. J. Robust Nonlinear Control*, 8(12):1043–1058, 1998.

[Wir02] F. Wirth. The generalized spectral radius and extremal norms. *Linear Algebra Appl.*, 342:17–40, 2002.

[Wol05] S. Wolfram. *The Mathematica book*. Addison Wesley, New York, 2005.

[Zie95] G.M. Ziegler. *Lectures on polytopes*. Springer-Verlag, New York, 1995.