Ordinals, Cardinals and New Ideas for Convergence

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Abstract

We recall main facts about ordinal and cardinal numbers and a classification of topological spaces based on the convergence of sequences. We give an example of a R-monolithic and weakly Whyburn space neither radial nor sequential nor Whyburn. We introduce the new notion of essential sequence, giving some results.

1 Ordinal numbers

The concept of an ordinal number arises from the process of enumerating things *taking memory of the order followed in counting.* In the realm of ordinal numbers, even infinite ones, there is always the successor of every number, but there will not always be a predecessor for infinite ordinals (and 0). According to von Neumann [vN], in ZF, an ordinal is the set of all its predecessors. A common definition of an ordinal number is a set which is transitive (a set A is transitive if $a \in A$ implies $a \subseteq A$) and well ordered (every nonempty subset has a least element) by the relation $\in$. For example

$$\begin{aligned} 0 &= \emptyset \\ 1 &= \{0\} \\ 5 &= \{0,1,2,3,4\} \\ \omega &= \{0,1,\ldots,n,\ldots\} \\ \omega+1 &= \omega \cup \{\omega\} = \{0,1,\ldots,n,\ldots,\omega\}. \end{aligned}$$

The ordinal number ω is denoted also by ω_0. Given an ordinal β its immediate successor is $\beta + 1 := \beta \cup \{\beta\}$. Every ordinal number α is of one of these two types: successor (if $\alpha = \beta + 1$) or limit (if lacking of an immediate predecessor, as ω). Between ordinals operations of addition, multiplication

and exponentiation are defined (see for example [K]). Given a set A of ordinals we define $\sup A := \cup\{\alpha : \alpha \in A\}$. One proves that $\sup A$ is an ordinal number. Addition and multiplication are not commutative: for example $(\omega + 1) + 1 = \omega + 2 \neq 2 + \omega = \sup\{2 + n : n \in \omega\} = \omega$. And $\omega \cdot 2 = \omega + \omega = \sup\{\omega + n : n \in \omega\}$; instead $2 \cdot \omega = \sup\{2 \cdot n : n \in \omega\} = \omega \neq \omega \cdot 2$. And $\omega^2 = \omega \cdot \omega = \sup\{\omega \cdot n : n \in \omega\} \neq 2^\omega = \sup\{2^n : n \in \omega\} = \omega$. Remark that $\sup\{n \cdot \omega : n \in \omega\} = \omega$, since $n \cdot \omega = \omega$ for all $n \in \omega$; and $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$ but [K] not always $(\beta + \gamma) \cdot \alpha = \beta \cdot \alpha + \gamma \cdot \alpha$.
In Figure 1, taken from [W], one can recognize $0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots, \omega \cdot 3, \omega \cdot 3 + 1, \ldots, \omega^2, \omega^2 + 1, \ldots, \omega^2 + \omega, \omega^2 + \omega + 1, \ldots, \omega^2 + \omega \cdot 2, \omega^2 + \omega \cdot 2 + 1, \cdots, \omega^3, \cdots, \omega^\omega$. The next ordinal, not shown in figure, will be $\omega^\omega + 1$; and so on: $\ldots, \omega^\omega + \omega, \omega^\omega + \omega + 1, \ldots, \omega^\omega + \omega^2, \ldots, \omega^{\omega^\omega}, \ldots, \omega^{\omega^{\omega^{\cdots^{\omega}}}}, \ldots, \epsilon_0, \epsilon_0 + 1, \ldots$ This ϵ_0 is the first ordinal α such that $\alpha = \omega^\alpha$.
Every ordinal number $\alpha > 0$ has [S] a unique *Cantor normal form*

$$\alpha = \omega^{\beta_1} \cdot b_1 + \omega^{\beta_2} \cdot b_2 + \ldots \omega^{\beta_n} \cdot b_n,$$

where $b_1, b_2, \ldots, b_n$ are positive integers numbers and $\alpha \geq \beta_1 > \beta_2 > \ldots > \beta_n \geq 0$ are ordinal numbers.

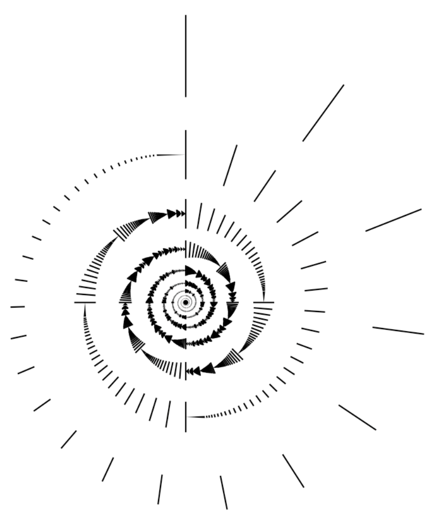

Figure 1: Ordinals up to ω^ω.

2 Cardinal numbers

Also cardinal numbers extend natural numbers beyond finite sets, but the idea of a cardinal number arises from the process of counting sets *altogether, without taking memory of order.* The original idea of Cantor was to compare two different sets by maps: two sets A, B have the same cardinality if there is a one-to-one and surjective map $f : A \to B$ and in this case we say that $\text{card}\,(A) = \text{card}\,(B)$. Obviously this notion does not define the cardinality of a set A. Naively, if we suppose the existence of the set of all sets it would be sufficient to consider the equivalence relation of equal cardinality and define the cardinality of a set A as the equivalence class of sets with the same cardinality of A. But the naive point of view of considering the set of all sets leads [R] to Russel's paradox (or Barber paradox).

In the theory ZFC of Zermelo - Fraenkel with the Axiom of Choice, for example, the problem is solved by means of ordinal numbers. Using the Axiom of Choice (AC) every set D can be well ordered, and every well ordered set is isomorphic to some ordinal number δ (i.e. there is an order preserving bijection $h : D \to \delta$). So under AC every set has a bijection with some ordinal number. The least ordinal δ_0 which is in bijection with a given set D is called the cardinality or cardinal number of D, denoted by $\text{card}\,(D)$ or by $|D|$. The first cardinal and ordinal numbers are natural numbers: $0 = \text{card}\,(\emptyset), 1 = \text{card}\,(\{0\}), 2 = \text{card}\,(\{0,1\}), \ldots$ The first infinite cardinal is the cardinality of $\mathbb{N}$, denoted by ω or ω_0 or $\aleph_0$. As is well known, one can prove that, though $\mathbb{N} \subsetneq \mathbb{Q}$, it is $\text{card}\,(\mathbb{Q}) = \omega_0$, essentially by considering the "weight" $n + m$ of $\dfrac{n}{m}$, with some attention to take care of negative rational numbers and to avoid to enumerate twice rationals with different representations. Already in 1638 Galilei observed [Ga] that $\mathbb{N}$ is in bijection with its proper subset of squares.

By the Diagonal Argument of Cantor [C] one can show that there are infinite sets of cardinality greater that ω_0; in particular that $\text{card}\,(\mathbb{R}) > \text{card}\,(\mathbb{N})$. There exists [H] a minimal cardinal bigger than ω_0 denoted by ω_1 (or $\aleph_1$), similarly the next cardinal is denoted by ω_2 (or $\aleph_2$) and so on. Since Cantor showed that $\mathfrak{c} := \text{card}\,(\mathbb{R})$ is bigger than ω_0, and so $\mathfrak{c} \geq \omega_1$, the problem arises of where in the sequence of the ω_α the cardinal $\mathfrak{c}$ has to be placed. Cantor tried to prove that $\mathfrak{c} = \omega_1$, without success. Gödel [G] proved in 1937, by exhibiting the minimal model of ZFC, L, of *constructible sets*, that $\mathfrak{c}$ can be ω_1 and in 1963 Cohen [Co], inventing the *Method of Forcing*, showed that it can be greater than ω_1. So the Cantor's Continuum Hypothesis (noted CH), $\mathfrak{c} = \omega_1$, is independent from axioms of ZFC. The current view is that $\mathfrak{c} = \omega_2$ could be a good choice [Wo]. There are infinite cardinals since by Cantor's Theorem $\mathrm{card}\,(\mathcal{P}(E)) > \mathrm{card}\,(E)$, for every set E, where $\mathcal{P}(E) = \{G : G \subseteq E\}$ is the powerset of E.

Operations among cardinals are rather trivial if they concern addition and multiplication. In fact, if $\kappa := \mathrm{card}\,(K), \mu := \mathrm{card}\,(M)$, with $K \cap M = \emptyset$, it can be proved that $\kappa \oplus \mu := \mathrm{card}\,(K \cup M) = \max(\kappa, \mu) = \mu \oplus \kappa$. Analogously, if $\kappa := \mathrm{card}\,(K), \mu := \mathrm{card}\,(M)$, $\kappa \otimes \mu := \mathrm{card}\,(K \times M) = \max(\kappa, \mu) = \mu \otimes \kappa$, and distributivity holds. Note that $\omega \oplus 1 = 1 \oplus \omega = \omega = \omega \oplus \omega = \omega \otimes \omega$. But exponentiation of cardinals is highly non-trivial. If $\kappa := \mathrm{card}\,(K)$, $2^\kappa := \mathrm{card}\,(\mathcal{P}(K))$. In general, given two sets K and M, consider the set of all maps $f : M \to K$, which is denoted by K^M or ${}^M K$. If $\kappa := \mathrm{card}\,(K), \mu := \mathrm{card}\,(M)$, then $\kappa^\mu := \mathrm{card}\,({}^M K)$. Since $\mathbb{R}$ is in bijection with $\mathcal{P}(\mathbb{N})$, $\mathfrak{c} = \mathrm{card}\,(\mathbb{R}) = 2^\omega$.

The Generalized Continuum Hypothesis (GCH) affirms that $\omega_{\alpha+1} = 2^{\omega_\alpha}$. Every ordinal is a set and so has a cardinality but even "enormous" ordinals like ϵ_0 are countable. Note that μ^κ is a very different operation if intended as an ordinal or as a cardinal. The ordinal ω^ω is countable, but in cardinal sense $\omega^\omega = \aleph_0^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c} > \omega = \aleph_0$.

In the last decades the theory of cardinal numbers have been widely expanded and employed in Set Theory (see for example [K, J]) and in Set-Theoretic Topology (see for example [Av]). In Topology, the so called *small cardinals*, $\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \ldots$, play a very important role. The seminal article in this field is *The integers and topology* by E. K. van Douwen in [Av].

3 Long Sequences and New Ideas

A sequence is a map $S : \mathbb{N} \to X$, where X is a set. To consider the concept of convergence, X will be supposed a topological space. We can think of $\mathbb{N}$ as represented by the ordinal ω. From this point of view it is immediate to step from the ordinal ω to an arbitrary ordinal α. A *long sequence* is a map $S : \alpha \to X$. If X is a topological space it is natural to say that the (long) sequence S converges to a point $x \in X$ if given a neighborhood U of x, there is an ordinal δ such that for all ordinals $\eta > \delta$ we have $x_\eta \in U$. It can be interesting to find which (long) sequences converge in a given topological space X. There are also spaces in which no sequence converges, as $\beta\omega$ (see for example [E]). Topological spaces in which usual sequences suffice to determine the topology are called *sequential spaces*. Sequential spaces belonged to the folklore of topology since its beginning, but were first formalized in the Sixties in the papers of Dolcher [D], Franklin [F1, F2]

and Novák [N]. In a sequential space X a notion of sequential closure can be defined; given $A \subseteq X$, its sequential closure $\hat{A}$ is $\{x \in X$: there is a sequence $\langle x_n \rangle_{n\in\omega}$ of elements of A which converges to $x\}$. The sequential closure of A can be iterated; $\hat{A}^0 := A, \hat{A}^{\eta+1} := \widehat{\hat{A}^\eta}$ and $\hat{A}^\eta := \bigcup_{\beta<\eta} \hat{A}^\beta$ if η is limit. In sequential spaces the least ordinal η such that $\hat{A}^\eta = \overline{A}$ (the topological closure of A) is called the sequential order of the sequential space. If $\eta = 1$ then the space is called a Fréchet-Urysohn space. In general it can be shown that $\eta \leq \omega_1$. There are [AF] spaces that have every sequential order up to ω_1. Examples of *compact Haudorff sequential spaces* of every sequential order up to ω_1 where described [B, Ba, Ka] under CH, but it is an open problem to find such example in ZFC or even with extra axioms and the negation of CH. Some progress was obtained by Dow [D1, D2] and the Authors [ST], under the Martin's Axiom (MA) or at least $\mathfrak{b} = \mathfrak{c}$. We recall that $\mathfrak{b}$ is the least cardinal number of an unbounded subset of ${}^\omega\omega$.
In the sequel, topological spaces will be assumed to be Hausdorff. We remember a classification of topological spaces concerning convergent (long) sequences. X is said to be *radial* if for every subset $A \subseteq X$ and for every $x \in \overline{A}$ there is a sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$, with $x_\alpha \in A$ which converges to x. The space X is *R-monolithic* if for every subset $A \subseteq X$ if $B \subseteq \overline{A}$ is closed for sequences of length not greater that $|A|$, then B is closed. It is *pseudoradial* if for every non-closed subset $A \subseteq X$ there is a point $x \in \overline{A} \setminus A$ and a sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$, with range in A, converging to x. Radial and pseudoradial spaces are generalizations of Fréchet-Urysohn and sequential spaces respectively. A space X is *Whyburn* if for every subset $A \subseteq X$, for every $x \in \overline{A}$ there is a subset $B \subseteq A$ such that $\overline{B} = B \cup \{x\}$; it is *weakly Whyburn* if for any non-closed subset $A \subseteq X$ there is some $x \in \overline{A} \setminus A$ and a subset $B \subseteq A$ such that $\overline{B} = B \cup \{x\}$.

Proposition 1 *There is a space which is R-monolythic, weakly Whyburn and neither radial nor sequential nor Whyburn.*

Proof. Let $X := \omega_1 \times (\omega_1 + 1) \cup \{\infty\}$. The topology is the following: all points of $\omega_1 \times \omega_1$ are isolated, points of type $\langle \eta, \omega_1 \rangle$ have a neighborhood base of type $U_\delta := \{\langle \eta, \omega_1 \rangle : \delta < \eta < \omega_1\}$ while a base of neighborhoods of the point ∞ are sets

$$U_{\epsilon,f} := \{\langle \gamma, \delta \rangle : \epsilon < \gamma < \omega_1, f(\gamma) < \delta \leq \omega_1\} \cup \{\infty\}, \tag{1}$$

where $f(\gamma) :]\epsilon, \omega_1[\to \omega_1$ is an arbitrary function. The space is not radial, since if $A := \omega_1 \times \omega_1$, then $\infty \in \overline{A}$, but any sequence in A cannot converge to the point ∞; in fact given any sequence $S := \langle x_\alpha, y_\alpha \rangle_{\alpha<\omega_1}$, if for any $\beta < \omega_1$ the cardinality of $Y_\beta := \{y_\alpha : x_\alpha = \beta\}$ is not bigger than ω, let $f(\beta) := \sup Y_\beta$ and then $U_{0,f} \cap \operatorname{im} S = \emptyset$; if there is some $\beta < \omega_1$ such that $|Y_\beta| = \omega_1$ there is in Y_β a subsequence of S which converges to $\langle \beta, \omega_1 \rangle \neq \infty$, a contradiction. By a similar proof it is easy to check that X is Whyburn. Obviously the space is not sequential because if $S := \langle x_n, y_n \rangle_{n\in\omega}$ is a sequence in A, let $\epsilon := \sup \pi_1(S)$ and then $U_{\epsilon,0} \cap \operatorname{im} S = \emptyset$. The space is R-monolythic; in fact let $A \subseteq \omega_1 \times (\omega_1 + 1)$ and suppose that there is some $\eta < \omega_1$ such that $y \leq \eta$ for all $\langle x, y \rangle \in A$, then $\overline{A} = A$ and if $B \subseteq \overline{A} = A$ it is closed; if there is not such an η, there are two cases: or there is $\alpha < \omega_1$ such that $\langle \alpha, \omega_1 \rangle \in \overline{A}$ or there is not such an α and A is closed. Observe that if $\infty \in \overline{A}$ then there must be ω_1 many $\alpha < \omega_1$ such that $\langle \alpha, \omega_1 \rangle \in \overline{A}$. If not there is $\delta < \omega_1$ such that $]\delta, \omega_1[\times \{\omega_1\} \cap \overline{A} = \emptyset$ and so $\infty \notin \overline{A}$. If $B \subseteq \overline{A}$ and $\infty \notin \overline{A}$, if B is ω_1-sequentially closed, since the only converging ω_1-sequences are the one converging to some point $\langle \alpha, \omega_1 \rangle$ it must be closed. If $\infty \in \overline{A}$ and $B \subseteq \overline{A}$ is ω_1 is sequentially closed and $\infty \in \overline{B}$ then an ω_1 sequence $\langle \alpha_\gamma, \omega_1 \rangle$ converging to ∞ must be contained in B. So any $B \subseteq \overline{A}$ if it is ω_1-sequentially closed must be closed. This shows that the space is R-monolythic. A similar proof shows that the space is weakly Whyburn. $\square$

Remark 1 Theorem 3.8 in [BY] gives an example of a space which is R-monolythic, weakly Whyburn and neither Whyburn nor radial (nor sequential).The space considered there is $C_p(\kappa)$ where κ is an ω-inaccessible cardinal (i.e. for every $\lambda < \kappa$, we have $\lambda^\omega < \kappa$). Our example is simpler.

Definition 1 *We say that a κ-sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$ in a Hausdorff space is* essential *if it is injective, converging and*

$$\overline{\{x_\alpha : \alpha < \kappa\}} = \{x_\alpha : \alpha < \kappa\} \cup \{x\}, \tag{2}$$

where $x := \lim x_\alpha$. (It is not excluded $x = x_\alpha$ for one α).

Proposition 2 *Usual injective and converging ω-sequences are essential.*

Proof. In fact if $\langle x_n \rangle_{n\in\omega}$ converges to a point x and $y \notin \{x\} \cup \{x_n : n \in \omega\}$, any neighborhood W of y disjoint from a neighborhood U of x contains only a finite number of points of the sequence, and then $y \notin \overline{\{x_n : n \in \omega\}}$. $\square$

Remark 2 Then the notion of essential sequence does not add anything new to Fréchet-Uryson and sequential spaces. However, if the space is not Hausdorff and sequences are usual countable sequences the notion of Definition 1 was introduced with the name of SC-spaces in [AW] and considered, for example, in [Be, BC].

Recall that a κ-sequence $\langle x_\alpha\rangle_{\alpha<\kappa}$ converging to a point x is said a *thin* sequence if for every $\lambda < \kappa$, $x \notin \overline{\{x_\alpha : \alpha < \lambda\}}$[AIT, Ny].

Proposition 3 *There are injective convergent sequences that are: (i) essential and thin; (ii) essential and not thin; (iii) not essential but thin; (iv) neither essential nor thin.*

Proof. (i) Let $x_n := \frac{1}{n+1}$, for $n < \omega$. (ii) Let $x_0 := 0$, $x_n := \frac{1}{n}$, for $0 < n < \omega$. (iii) Let $\langle x_\beta\rangle_{\beta<\omega_1}$ the sequence of successor ordinals in the set $\omega_1 + 1$. (iv) The same as in (iii) except $x_0 := \omega_1$. $\square$

Definition 2 *A topological space X is said to be* essentially radial *if for every subset $A \subseteq X$ and for every $x \in \overline{A}$ there is an essential sequence $\langle x_\alpha\rangle_{\alpha<\kappa}$, with $x_\alpha \in A$, which converges to x.*

Proposition 4 *(i) All Fréchet-Urysohn spaces are essentially radial spaces. (ii) $X := \omega_1 + 1$, endowed with the topology where all points $\alpha < \omega_1$ are isolated and $]\beta, \omega_1]$ are fundamental neighborhoods of ω_1, is an essentially radial space. (iii) $\omega_1 + 1 = \omega_1 \cup \{\omega_1\}$ with the usual topology is a radial but not essentially radial space.*

Proof. (i) and (ii) are obvious. For (iii) let $A := \{\beta + 1 : \beta < \omega_1\}$. We have $\omega_1 \in \overline{A} \setminus A$, but if $\langle\beta_\delta\rangle_{\delta<\omega_1}$ is a sequence in A converging to ω_1, the closure of its image contains all limit ordinals $\gamma < \omega_1$. So no sequence in A converging to ω_1 can be essential. $\square$

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