Noncommutative second quantization

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Abstract

We considered a quatum oscillator in a Minkowski space-time endowed of noncommuting spatial coordinate operators, namely $[x_i, x_j] = i\lambda^2\Theta_{ij}$. Time is still retained commutative. We reproduced the noncommutative hamiltonian in terms of usual commutative coordinates. Having in mind the scalar field theory we adopt the raising/lowering operator formalism. We show that the noncommutativity induces energy level splitting.

Quantum Field Theory (QFT) and General Relativity (GR) can be considered among the most successful physical theories for the agreement between theoretical predictions and experimental data [1]. Nevertheless they are not free of problems and in particular that regarding the irregularity of their short distance behavior is still unsolved. Various attempts of solutions one can find in literature are mostly unsatisfactory for the introduction of further (and even worse) complications, while on the other hand people advance the hypothesis that, mimicking quantum mechanics, a correct 'noncommutative' formulation of these theories could furnish a natural cut-off i.e. a minimal length λ able to cover QFT ultraviolet divergences and GR singularities. This is an explanation for the increasing interest towards the so called noncommutative geometry, an old theme [3] common to several sectors of theoretical physics [4] and recently reappeared in a context of string theory [5].

The noncommutative formulation of QFT can be achieved through two main different (but equivalent) ways [6]: one by the introduction of the Moyal product on the space of ordinary functions, the other by deforming the coordinate operator algebra into a noncommutative one. Central at this stage is the study of the harmonic oscillator, as first step towards the noncommutative quantization of field theories [7]: recently some authors have put in evidence that noncommutativity manifests itself through redefinitions of various parameters and produces additional terms in the Hamiltonian of the equivalent commutative description [8]. In this paper we follow an equivalent approach, but using the 'language' of the second quantization: the noncommutative raising/lowering operators ξ_i, $\xi_i^\dagger$ are built from a set of noncommutative coordinates x_i and noncommutative momenta y_i. Furthermore we try to build a noncommutative scalar field theory, using what found for the oscillator, in order to shed a bit of light in a formulation affected by the controversial appearance of IR/UV mixing in NC Feynman diagrams.

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[1] The relative error of GR prediction for the measure of the revolution period of the binary pulsar Hulse Taylor PSR 1913+16 [1] is 10^{-14}. Experimental data and QFT prediction of the value of the anomalous magnetic momentum of the electron agree within a relative error of 10^{-11} [2].

We start from the description of the *noncommutative world* through the coordinates x_i and the momenta y_i satisfying extended commutator relations[2]

$$\begin{aligned} [x_i, x_j] &= i\lambda^2\Theta_{ij} \\ [y_i, y_j] &= i\lambda^{-2}B_{ij} \\ [x_i, y_j] &= i\mathbf{1}_{ij} \end{aligned} \tag{1}$$

where λ is the minimal length and Θ_{ij}, B_{ij} are antisymmetric matrices. Having in mind the noncommutative field theory it is useful to define the counterpart of the raising and lowering operators

$$\begin{aligned} \xi_i &= \frac{m\omega x_i - iy_i}{\sqrt{2m\omega}} \\ \xi_i^\dagger &= \frac{m\omega x_i + iy_i}{\sqrt{2m\omega}}. \end{aligned} \tag{2}$$

subject to the following commutation relations one can get from (1)

$$\begin{aligned} [\xi_i, \xi_j] &= iT_{ij} \\ \left[\xi_i^\dagger, \xi_j^\dagger\right] &= iT_{ij} \\ \left[\xi_i, \xi_j^\dagger\right] &= i\mathbf{1}_{ij} + iS_{ij} \end{aligned} \tag{3}$$

with

$$T_{ij} = \frac{m\omega\lambda^2\Theta_{ij}}{2} - \frac{B_{ij}}{2m\omega\lambda^2} \tag{4}$$

and

$$S_{ij} = \frac{m\omega\lambda^2\Theta_{ij}}{2} + \frac{B_{ij}}{2m\omega\lambda^2}. \tag{5}$$

At this stage we may try to define states and observables for physical systems defined through these variables: the procedure requires a complete set of commuting observables, thing that is in contradiction with the noncommutative structure a priori deliberately introduced. A reasonable solution to bypass this contradiction consists in postulating the existence of a transformation between the set of noncommutative operators (ξ, $\xi^\dagger$) and an ordinary set of (η, $\eta^\dagger$), defined through a set of canonically conjugate variables q_i and p_i as follows

$$\begin{aligned} \eta_i &= \frac{1}{\sqrt{2m\omega}}\left(m\omega q_i - ip_i\right) \\ \eta_i^\dagger &= \frac{1}{\sqrt{2m\omega}}\left(m\omega q_i + ip_i\right). \end{aligned} \tag{6}$$

and subject to the following relations

$$\begin{aligned} [\eta_i, \eta_j] &= 0 \\ \left[\eta_i^\dagger, \eta_j^\dagger\right] &= 0 \\ \left[\eta_i, \eta_j^\dagger\right] &= \mathbf{1}_{ij}. \end{aligned} \tag{7}$$

[2]For sake of clarity we set $\hbar = 1$.

Let's suppose that the transformation law between those algebras is given by

$$\begin{aligned} \xi_i &= a_{il}\eta_l + b_{il}\eta_l^\dagger \\ \xi_i^\dagger &= a_{il}^\dagger\eta_l^\dagger + b_{il}^\dagger\eta_l \end{aligned} \tag{8}$$

where $\mathbf{a}$, $\mathbf{b}$ are $N \times N$ transformation matrices, whose form is restricted by conditions they have to satisfy i.e. by demanding that the commutation relations (3) and (7) be contemporary true. The leads to the following conditions, written in matrix form

$$\mathbf{a}\mathbf{b}^T - \mathbf{b}\mathbf{a}^T = i\mathbf{T} \tag{9}$$

$$\mathbf{a}^T\mathbf{b} - \mathbf{b}^T\mathbf{a} = i\mathbf{T} \tag{10}$$

$$\mathbf{a}\mathbf{a}^* - \mathbf{b}\mathbf{b}^* = \mathbf{1} + i\mathbf{S} \tag{11}$$

These equation determine the structure of the transformation law, which is established by fixing the form of $\mathbf{a}$ and $\mathbf{b}$.

As a specific model let's choose noncommutative harmonic oscillator described by the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{2}\omega\left(\xi_i^\dagger\xi_i + \xi_i\xi_i^\dagger\right) \tag{12}$$

written in the so called 'abstract' notation. Using eq.ns (3) one obtains

$$H = \omega\left(1 + \sum_{i=1}^{2}\xi_i^\dagger\xi_i\right). \tag{13}$$

Inserting (8) in the above Hamiltonian one finds its equivalent commutative form as

$$H = \omega\left[1 + \sum_i\left(a_{il}^\dagger a_{ij}\eta_l^\dagger\eta_j + b_{il}^\dagger a_{ij}\eta_l\eta_j + a_{il}^\dagger b_{ij}\eta_l^\dagger\eta_j^\dagger + b_{il}^\dagger b_{ij}\eta_l\eta_j^\dagger\right)\right]. \tag{14}$$

Noncommutativity is responsible of the appearance of extra terms $\eta\eta$, $\eta^\dagger\eta^\dagger$ and $\eta\eta^\dagger$. 2D representations for Θ_{ij} and B_{ij} are

$$\Theta_{ij} = \theta\epsilon_{ij} \tag{15}$$

$$B_{ij} = \beta\epsilon_{ij} \tag{16}$$

so that $\mathbf{T}$ and $\mathbf{S}$ are

$$\begin{aligned} T_{ij} &= T\epsilon_{ij} \\ S_{ij} &= S\epsilon_{ij} \end{aligned} \tag{17}$$

with S, T, θ and β ordinary real numbers and ϵ_{ij} the antisymmetric tensor. The number of elements of the transformation matrices is eight while (9-11) furnishes six conditions; thus we are left with 2 unknown parameter;the simplest solution for the aformentioned equation is on the other hand and it is necessary to impose the following form

$$a_{il} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \tag{18}$$

$$b_{il} = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix} \tag{19}$$

The solutions of the complete set of equations can be obtained putting

$$\begin{aligned} a_{11} &= i\alpha \\ a_{22} &= -i\alpha \end{aligned} \tag{20}$$

where α is a real number. Substituting back into (9-11) one is able to determine the complete set of solutions

$$a_{il} = \begin{pmatrix} i\alpha & S/2\alpha \\ S/2\alpha & -i\alpha \end{pmatrix} \tag{21}$$

$$b_{il} = \begin{pmatrix} 0 & T/2\alpha \\ T/2\alpha & 0 \end{pmatrix} \tag{22}$$

with the condition on α

$$4\alpha^4 - 4\alpha^2 + \left(S^2 - T^2\right) = 0. \tag{23}$$

The Hamiltonian (12) can now be written in terms of commutative operators

$$H = \omega \left[\left(1 + \frac{T^2}{2\alpha^2}\right) \left(1 + \sum_{i=1}^{2} \eta_i^\dagger \eta_i\right) + \frac{ST}{4\alpha^2} \sum_{i=1}^{2} \left(\eta_i^2 + \eta_i^{\dagger 2}\right) \right] \tag{24}$$

where H_0 is formally the commutative counterpart of 12 (apart of an overall factor)

$$H_0 = \omega \left(1 + \frac{T^2}{2\alpha^2}\right) \left(1 + \sum_{i=1}^{2} \eta_i^\dagger \eta_i\right) \tag{25}$$

while H_1 is the perturbation due to the noncommutative structure of ξ_k's

$$H_1 = \frac{\omega ST}{4\alpha} \sum_{k=1}^{2} \left(\eta_k^2 + \eta_k^{\dagger 2}\right) \tag{26}$$

The eigenvalues' problem can be solved imposing the following relations

$$A = \frac{1}{\sqrt{2}} \left(\eta_1 + i\eta_2\right) \tag{27}$$

$$B = \frac{1}{\sqrt{2}} \left(\eta_1 - \eta_2\right). \tag{28}$$

The new operators obey the same algebra as in (7), that is

$$\left[A, A^\dagger\right] = \left[B, B^\dagger\right] = 1, \tag{29}$$

all other commutators being zero. The Hamiltonian (24)in terms of A and B is

$$h = h_0 + \epsilon h_1 \tag{30}$$

where the small letters are for the previous quantities divided by the factor $\left(1 + \frac{T^2}{2\alpha^2}\right)$ and

$$h_0 = \omega + \omega \left(A^\dagger A + B^\dagger B\right) \tag{31}$$

$$h_1 = 2\omega \left(A^\dagger B^\dagger + AB\right) \tag{32}$$

with

$$\epsilon = \frac{ST}{4\alpha^2}\left(1+\frac{T^2}{2\alpha^2}\right)^{-1} \tag{33}$$

The eigenvalues of $A^\dagger A$ and $B^\dagger B$ are common n_A, $n_B = 0, 1, 2\ldots$, so that the eigenvalues of h_0 are $\omega\left(n_A + n_B + 1\right)$, with eigenstates $|n_A, n_B\rangle$ belonging to Hilbert space. In the regime for which $T = S \simeq 0$ we get from (23) that

$$\alpha \simeq 1 \tag{34}$$

$$\epsilon \ll 1 \tag{35}$$

and h_1 can be properly called 'perturbation'. Such conditions for T and S can be obtained from the supposed smallness of the minimal length λ i.e.

$$m\omega\lambda^2\theta \ll 1 \tag{36}$$

and from the further condition

$$\frac{\beta}{m\omega\lambda^2} \ll 1. \tag{37}$$

The latter can be relaxed, without loss of generality, in favor of a stronger one i.e.

$$\beta = 0 \tag{38}$$

which leads to a commuting relations for the momenta y_i. Having in mind the scalar field theory it is well-known that one can achieve it from oscillators by means of the following substitution [9]

$$i \longrightarrow k_i, \tag{39}$$

where k_i is the wave number vector component. Then the Hamiltonian (12) becomes the Hamiltonian of a scalar field

$$H = \sum_{\mathbf{k}} \omega\left(\frac{1}{2} + \xi^\dagger_{\mathbf{k}}\xi_{\mathbf{k}}\right) \tag{40}$$

where the sum now is intended as

$$\sum_{\mathbf{k}} \longrightarrow (L/2\pi)^{(N-1)} \int d^{N-1}k \tag{41}$$

with N is the space-time dimension Γ is the Euler function, L is the side of a space-like (N-1)-torus, $\omega = \left(k^2 + m^2\right)^{1/2}$, while the measure is defined onto the spatial components as follows

$$d^{N-1}k = \prod_{i=1}^{N-1} dk_i. \tag{42}$$

Noncommutativity generates ambiguity in the definition of the aforementioned measure and so the condition (38) turns to be useful for a dual interpretation (oscillator/field) of the Hamiltonian (12). In order to investigate the eigenspectrum of the transformed Hamiltonian (24), it is of advantage to introduce [10]

$$\phi_x = \frac{1}{2}\left(A^\dagger B^\dagger + AB\right) \tag{43}$$

$$\phi_y = \frac{1}{2}i\left(A^\dagger B^\dagger - AB\right) \tag{44}$$

$$\phi_z = \frac{1}{2}\left(A^\dagger A + BB^\dagger\right). \tag{45}$$

and to define

$$\phi_0 \equiv \frac{1}{2}\left(A^\dagger A - B^\dagger B\right). \tag{46}$$

Notice that

$$h_0 = 2\omega\phi_z \tag{47}$$
$$h_1 = 4\omega\phi_x. \tag{48}$$

The algebra of this spin-like operators is given by

$$[\phi_x, \phi_y] = i\phi_z \tag{49}$$
$$[\phi_z, \phi_y] = i\phi_x \tag{50}$$
$$[\phi_x, \phi_z] = i\phi_y \tag{51}$$

while

$$[\phi_0, \phi_x] = [\phi_0, \phi_y] = [\phi_0, \phi_z] = 0. \tag{52}$$

Be aware of the difference in sign in (50) and (51) with respect to the usual spin algebra. One easily computes now that

$$\phi_z^2 - \left(\phi_x^2 + \phi_y^2\right) = \phi_0^2 - \frac{1}{4}. \tag{53}$$

Of course ϕ_0^2 commutes with ϕ_x, ϕ_y and ϕ_z. It is the only Casimir operator for the present algebra (as the rank is unity). Further, let us conventionally introduce

$$j = \frac{1}{2}\left(n_A - n_B\right), \tag{54}$$
$$m = \frac{1}{2}\left(n_A + n_B\right), \tag{55}$$

and label the eigenstates of h_0, ϕ_0 and ϕ_z as

$$|jm\rangle \tag{56}$$

rather than $|n_A, n_B\rangle$. For the above relations one can see that these eigenstates are able to diagonalize the free Hamiltonian h_0,

$$h_0|jm\rangle = 2\omega\left(m + \frac{1}{2}\right)|jm\rangle \tag{57}$$

$$E_m^{(0)} \equiv 2\omega\left(m + \frac{1}{2}\right) \tag{58}$$

while on the contrary do not coincide with the eigenvectors of h_1, that can be studied only by means of the perturbation theory. Therefore let assume

$$|jm\rangle \equiv |\psi_m^{(0)}\rangle \tag{59}$$

for the unperturbed hamiltonian h_0, while let $|\psi_m\rangle$ be the eigenstates for the full hamiltonian h

$$\left(h_0 + \epsilon h_1\right)|\psi_m\rangle = E_m|\psi_m\rangle. \tag{60}$$

Since h depends on ϵ, the eigenfunctions and eigenstates of h must also depend on ϵ. Thus $|\psi_m\rangle$ and E_m can be expanded in power series with respect to ϵ

$$E_m \equiv E_m^{(0)} + \epsilon E_m^{(1)} + \epsilon^2 E_m^{(2)} + \dots \tag{61}$$

$$|\psi_m\rangle = |\psi_m^{(0)}\rangle + \epsilon|\psi_m^{(1)}\rangle\epsilon^2|\psi_m^{(2)}\rangle + \dots \tag{62}$$

The following abbreviations were employed in the expansion.

$$E_n^{(k)} \equiv \frac{1}{k!}\frac{\partial^k E_n}{\partial\lambda^k} \tag{63}$$

$$|\psi_n^k\rangle \equiv \frac{1}{k!}\frac{\partial^k|\psi_n\rangle}{\partial\lambda^k} \tag{64}$$

To obtain the energy E_n or wave function $|\psi_n\rangle$ of a system you merely sum the corrections. Obviously the convergence of these power series becomes a key issue since the summations must be truncated in practice.

In this paper we face the problem of formulating noncommutativity by means of coordinate operator algebra deformation. We studied the consequence of this deformation on a simple but crucial quantum system: the harmonic oscillator. We show, in the laguage of second quantization, how the commutative free hamiltonian is perturbed by noncommutativity of the background space-time manifold. Finally, we gave prescription for the study of noncommutative oscillator by means od perturbation theory.

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