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Non-linear σ-models in noncommutative geometry: fields with values in finite spaces

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Abstract

We study σ-models on noncommutative spaces, notably on noncommutative tori. We construct instanton solutions carrying a nontrivial topological charge q and satisfying a Belavin-Polyakov bound. The moduli space of these instantons is conjectured to consists of an ordinary torus endowed with a complex structure times a projective space $\mathbb{CP}^{q-1}$.

Dedicated to A.P. Balachandran on the occasion of his 65th birthday.

1 Introduction

In [8] we have constructed some noncommutative analogues of two dimensional non-linear σ-models. Since these models exhibit rich and easily accessible geometrical structures, their noncommutative counterparts are very useful to probe the interplay between noncommutative geometry and field theory. We proposed three classes of models: an analogue of the Ising model which admits instanton solutions, the analogue of the principal chiral model together with its infinite number of conserved currents and the noncommutative Wess-Zumino-Witten model together with its modified conformal invariance.

In this short report we extend the first model by constructing instanton solutions for any value of the topological charge (restricted to be 1 in [8]) and for an arbitrary complex structure parametrized by a complex number $\tau \in \mathbb{C}$, $\Im\tau > 0$.

2 The general construction

Ordinary non-linear σ-models are field theories whose configuration space consists of maps X from the *source* space, a Riemannian manifold (Σ, g), which we assume to be compact and orientable, to a *target* space, an other Riemannian manifold $(\mathcal{M}, G)$. The corresponding action functional is given, in local coordinates, by

$$S[X] = \frac{1}{2\pi} \int_\Sigma \sqrt{g}\; g^{\mu\nu}\, G_{ij}(X) \partial_\mu X^i\, \partial_\nu X^j \,, \tag{1}$$

where as usual $g = \det g_{\mu\nu}$ and $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$. The stationary points of this functional are harmonic maps from Σ to $\mathcal{M}$ and describe minimal surfaces embedded in $\mathcal{M}$. Different choices of the source and target spaces lead to different field theories, some of them playing a major role in physics.

When Σ is two dimensional, the action S is conformally invariant, that is it is left invariant by any rescaling of the metric $g \to g e^\sigma$, where σ is any map from Σ to $\mathbb{R}$. As a consequence, the action only depends on the conformal class of the metric and may be rewritten using a complex structure on Σ as

$$S[X] = \frac{i}{\pi} \int_\Sigma G_{ij}(X)\, \partial X^i \wedge \bar\partial X^j \,, \tag{2}$$

where $\partial = \partial_z dz$ and $\bar\partial = \partial_{\bar z} d\bar z$ and z is a suitable local complex coordinate.

In order to A noncommutative generalization is constructed by a dualization and reformulation it in terms of the $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ of complex valued smooth functions defined respectively on Σ and $\mathcal{M}$. Then, embeddings X of Σ into $\mathcal{M}$ correspond to $*$-algebra morphisms π_X from $\mathcal{B}$ to $\mathcal{A}$, the correspondence being simply given by the pullback, $f \mapsto \pi_X(f) = f \circ X$.
Now, all this makes perfectly sense also for noncommutative algebras $\mathcal{A}$ and $\mathcal{B}$, and we take as configuration space the space of all $*$-algebra morphisms from $\mathcal{B}$ to $\mathcal{A}$. Both algebras are over $\mathbb{C}$ and for simplicity we take them to be unital. The definition of the action functional involves noncommutative generalizations of the conformal and Riemannian geometries. According to Connes [5, 6], the former can be understood within the framework of positive

Hochschild cohomology. Indeed, in the commutative two dimensional situation, the trilinear map $\phi : \mathcal{A}^{\otimes 3} \to \mathbb{R}$ defined by

$$\phi(f_0, f_1, f_2) = \frac{i}{\pi} \int_\Sigma f_0 \partial f_1 \wedge \bar{\partial} f_2 \tag{3}$$

is an extremal element of the space of positive Hochschild cocycles that belongs to the Hochschild cohomology class of the cyclic cocycle ψ defined by

$$\psi(f_0, f_1, f_2) = \frac{i}{2\pi} \int_\Sigma f_0 df_1 \wedge df_2. \tag{4}$$

Now, expressions (3) and (4) still make perfectly sense for a general noncommutative algebra $\mathcal{A}$. One can say that ψ allows to integrate 2-forms in dimension 2, so that it is a metric independent object, whereas ϕ defines a suitable scalar product

$$\langle a_0 da_1, b_0 db_1 \rangle = \phi(b_0^* a_0, a_1, b_1^*) \tag{5}$$

on the space of 1-forms and thus depends on the conformal class of the metric. Moreover, this scalar product is positive and invariant with respect to the action of the unitary elements of $\mathcal{A}$ on 1-forms, and its relation to the cyclic cocycle ψ allows to prove various inequalities involving topological quantities [5].

We can compose such a cocycle ϕ with a morphism $\pi : \mathcal{B} \to \mathcal{A}$ to obtain a positive cocycle on $\mathcal{B}$ defined by $\phi_\pi = \phi \circ (\pi \otimes \pi \otimes \pi)$. In order to build an action functional, which assigns a number to any morphism π, we have to evaluate the cocycle ϕ_π on a suitably chosen element of $\mathcal{B}^{\otimes 3}$. Such an element provides the noncommutative analogue of the metric on the target, and we take it as a positive element $G = \sum_i b_0^i \delta b_1^i \delta b_2^i$ of the space of universal 2-forms $\Omega^2(\mathcal{B})$. Thus, the quantity

$$S[\pi] = \phi_\pi(G) \tag{6}$$

is well defined and positive. We shall consider it to be a noncommutative analogue of the action functional of the non linear σ-model.

Clearly, we consider π as the dynamical variable (the embedding) whereas ϕ (the conformal structure on the source) and G (the metric on the target) are background structures that have been fixed. Alternatively, one could take only the metric G on the target as a background field and use the morphism $\pi : \mathcal{B} \to \mathcal{A}$ to define the induced metric $\pi_* G$ on the source as $\pi_* G = \sum_i \pi(b_0^i) \delta\pi(b_1^i) \delta\pi(b_2^i)$, which is obviously a positive universal 2-form on $\mathcal{A}$. To such an object one can associate, by means of a variational problem [5, 6], a positive Hochschild cocycle that stands for the conformal class of the induced metric.

The critical points of the σ-model corresponding to the action functional (6) are noncommutative generalizations of harmonic maps and describe "minimally embedded surfaces" in the noncommutative space associated with $\mathcal{B}$.

3 Two points as a target space

The simplest example of a target space is that of a finite space made of two points $\mathcal{M} = \{1, 2\}$, like in the Ising model. Now, any continuous map from a connected surface

to a discrete space is constant and the resulting (commutative) theory would be trivial. However, this is not the case if the source is a noncommutative space and one has, in general, lots of algebra morphisms. The algebra of functions over $\mathcal{M} = \{1, 2\}$ is just $\mathcal{B} = \mathbb{C}^2$ and any element $f \in \mathcal{M}$ is a couple of complex numbers (f_1, f_2) with $f_i = f(i)$, the value of f at the point i. As a vector space $\mathcal{B}$ is generated by the function e defined by $e(1) = 1, e(2) = 0$. Clearly, e is a hermitian projection, $e^2 = e^* = e$, and $\mathcal{B}$ can be thought of as the unital *-algebra generated by such an element e. As a consequence, any $*$-algebra morphism π from $\mathcal{B}$ to $\mathcal{A}$ is given by a hermitian projection $p = \pi(e)$ in $\mathcal{A}$. The configuration space is then the collection of all such projections and our noncommutative σ-model provides a dynamics of projections. Choosing the metric $G = \delta e \delta e$ on the space $\mathcal{M}$ of two points, the action functional (6) simply becomes

$$S[p] = \phi(1, p, p), \tag{7}$$

where ϕ is a given Hochschild cocycle corresponding to the conformal structure.
As we have already mentioned, from general consideration of positivity in Hochschild cohomology this action is bounded by a topological term [5]. In the following, we shall explicitly prove this fact when taking the noncommutative torus as source space.

4 The noncommutative torus as a source space

We recall the very basic aspects of the noncommutative torus that we shall need in the following. The algebra $\mathcal{A}_\theta$ of smooth functions on the noncommutative torus is the unital $*$-algebra made of power series,

$$a = \sum_{m,n \in \mathbb{Z}^2} a_{mn}\ (U_1)^m (U_2)^n\ , \tag{8}$$

with a_{mn} a complex-valued Schwarz function on $\mathbb{Z}^2$ that is, the sequence of complex numbers $\{a_{mn} \in \mathbb{C}\ ,\ (m, n) \in \mathbb{Z}^2\}$ decreases rapidly at 'infinity'. The two unitary elements U_1, U_2 satisfy the commutation relations

$$U_2\ U_1 = e^{2\pi i \theta} U_1\ U_2\ . \tag{9}$$

There exist on $\mathcal{A}_\theta$ a unique normalized positive definite trace, denoted by the integral symbol $\int\!\!\!\!\!\!-\, : \mathcal{A}_\theta \to \mathbb{C}$, which is given by

$$\int\!\!\!\!\!\!-\, \Big(\sum_{(m,n) \in \mathbb{Z}^2} a_{mn}\ (U_1)^m (U_2)^n \Big) := a_{00}\ . \tag{10}$$

This trace is invariant under the action of the commutative torus $\mathbb{T}^2$ on $\mathcal{A}_\theta$ whose infinitesimal form is given by two commuting derivations ∂_1, ∂_2 acting by

$$\partial_\mu(U_\nu) = 2\pi i\ \delta^\nu_\mu U_\nu\ ,\quad \mu, \nu = 1, 2 \tag{11}$$

and the invariance means just that $\int\!\!\!\!\!\!-\, \partial_\mu(a) = 0\ ,\quad \mu = 1, 2$ for any $a \in \mathcal{A}_\theta$.
The cyclic 2-cocycle allowing the integration of 2-forms is simply given by

$$\psi(a_0, a_1, a_2) = -\frac{1}{2\pi i} \int\!\!\!\!\!\!-\, \epsilon_{\mu\nu} a_0 \partial_\mu a_1 \partial_\nu a_2, \tag{12}$$

where $\epsilon_{\mu\nu}$ is the standard antisymmetric tensor. Its normalization ensures that for any hermitian projection $p \in \mathcal{A}_\theta$, the quantity $\psi(p,p,p)$ is an integer: it is indeed the index of a Fredholm operator [4].

The conformal class of a general constant metric is parametrized by a complex number $\tau \in \mathbb{C}$, $\Im\tau > 0$. Then, up to a conformal factor, the metric is given by

$$g = (g_{\mu\nu}) = \begin{pmatrix} 1 & \Re\tau \\ \Re\tau & |\tau|^2 \end{pmatrix} , \tag{13}$$

with inverse given by $g^{-1} = (g^{\mu\nu}) = \frac{1}{(\Im\tau)^2}\begin{pmatrix} |\tau|^2 & -\Re\tau \\ -\Re\tau & 1 \end{pmatrix}$ and $\sqrt{detg} = \Im\tau$.
By using the two derivations ∂_1, ∂_2 defined in (11) we still think of 'the complex torus' $\mathbb{T}^2$ as acting on the noncommutative torus $\mathcal{A}_\theta$ and construct two associated derivations of $\mathcal{A}_\theta$ given by $\partial_{(\tau)} = \frac{1}{(\tau-\bar\tau)}\ (-\bar\tau\partial_1 + \partial_2)$ and $\bar\partial_{(\tau)} = \frac{1}{(\tau-\bar\tau)}\ (\tau\partial_1 - \partial_2)$. One easily finds that $\partial_{(\tau)}\bar\partial_{(\tau)} = \bar\partial_{(\tau)}\partial_{(\tau)} = \frac{1}{4}g^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{4}\Delta$, and the operator $\Delta = g^{\mu\nu}\partial_\mu\partial_\nu$ is just the Laplacian of the metric (13).

By working with the metric (13), the positive Hochschild cocycle ϕ associated with the cyclic one (12) will be given by

$$\phi(a_0,a_1,a_2) = \frac{2}{\pi}\int\!\!\!\!\!\!- \ a_0\partial_{(\tau)}a_1\bar\partial_{(\tau)}a_2 \ . \tag{14}$$

A construction of the cocycle (14) as the conformal class of a general constant metric on the noncommutative torus can be found in [5, 6].

4.1 The action and the field equations

With $\mathcal{P}_\theta = Proj(\mathcal{A}_\theta)$ denoting the collection of all projections in the algebra $\mathcal{A}_\theta$, we construct an action functional $S : \mathcal{P}_\theta \to \mathbb{R}^+$ by

$$S_{(\tau)}(p) = \phi(1,p,p) = \frac{2}{\pi}\int\!\!\!\!\!\!- \ \sqrt{detg}\ \partial_{(\tau)}p\bar\partial_{(\tau)}p \ . \tag{15}$$

The action functional can also be written as

$$S_{(\tau)}(p) = \frac{1}{2\pi}\int\!\!\!\!\!\!- \ \sqrt{detg}\ g^{\mu\nu}\partial_\mu p\partial_\nu p = \frac{1}{\pi}\int\!\!\!\!\!\!- \ \sqrt{detg}\ g^{\mu\nu}p\partial_\mu p\partial_\nu p \ . \tag{16}$$

Here the two derivations ∂_μ are the ones defined in (11) while the metric g is the one in (13) which carries also the dependence on the complex parameter τ.

By taking into account the nonlinear nature of the space $\mathcal{P}_\theta$, one finds that the most general infinitesimal variation of its elements (the tangent vectors) is of the form $\delta p = (1-p)zp + pz^*(1-p)$, with z an arbitrary elements in $\mathcal{A}_\theta$. Then, simple algebraic manipulations give the equations of motion,

$$p\ \Delta(p)\ -\ \Delta(p)\ p = 0 \ , \tag{17}$$

where $\Delta = g^{\mu\nu}\partial_\mu\partial_\nu$ is the Laplacian. These are non linear second order 'differential' equations and it is not simple to give their solutions in a closed form. We shall show that the absolute minima of (16) in a given connected component of $\mathcal{P}_\theta$ actually fulfill first order equations which are easier to solve.

4.2 Topological charges and self-duality equations

The cyclic 2-cocycle (12) assigns to any projection $p \in \mathcal{P}_\theta$ a 'topological charge' (the first Chern number)

$$\psi(p) := -\frac{1}{2\pi i} \int\!\!\!\!\!\!- \, p\Big[\partial_1(p)\partial_2(p) - \partial_2(p)\partial_1(p)\Big] \;\in\; \mathbb{Z} \; . \tag{18}$$

Then, the following inequality holds

$$S_{(\tau)}(p) \geq 2|\psi(p)| \; . \tag{19}$$

Indeed, due to positivity of the trace $\int\!\!\!\!\!\!-$ and its cyclic properties, we have that

$$0 \leq \int\!\!\!\!\!\!- \, \sqrt{det g}\; g^{\mu\nu} \Big[\partial_\mu(p)\; p \pm i\epsilon_\mu{}^\alpha \partial_\alpha(p)\; p\Big]^* \Big[\partial_\nu(p)\; p \pm i\epsilon_\nu{}^\beta \partial_\beta(p)\; p\Big] \; . \tag{20}$$

By expanding the LHS and comparing with (16) and (18) we get inequality (19). In (20) the symbol $\epsilon_{\mu,\nu}$ stands for the volume form of the metric $g_{\mu,\nu}$. The inequality (19), which gives a lower bound for the action, is the analogue of the one for ordinary σ-models [2]. A similar bound for a model on the fuzzy 2-sphere was obtained in [1].

It is clear that the equality in (19) occurs when the projection p satisfies the following *self-dual* or *anti-self duality* equations

$$\Big[\partial_\mu p \pm i\epsilon_\mu{}^\alpha \partial_\alpha p\Big]\; p = 0 \qquad \text{and/or} \qquad p\Big[\partial_\mu p \mp i\epsilon_\mu{}^\alpha \partial_\alpha p\Big] = 0 \tag{21}$$

The self duality equations (21) can be written as

$$\bar\partial_{(\tau)}(p)\; p = 0 \qquad \text{and/or} \qquad p\; \partial_{(\tau)}(p) = 0 \; , \tag{22}$$

while the anti-self duality one is

$$\partial_{(\tau)}(p)\; p = 0 \qquad \text{and/or} \qquad p\; \bar\partial_{(\tau)}(p) = 0 \; . \tag{23}$$

It is straightforward to check that either of the equations (22) and (23) implies the field equations (17) as it should.

5 The instantons

The connected components of $\mathcal{P}_\theta$ are parametrized by two integers r and q such that $r + q\theta > 0$. [12] When θ is irrational, the corresponding projections have trace $r + q\theta$ and the topological charge $\psi(p)$ appearing in (18) is just q.

Thus our task is to find projections that belongs to the previous homotopy classes and satisfy (say) the self-duality equation $\partial(p)\; p = 0$ or, equivalently, $p\partial p = 0$. These equations are still non linear and to solve them our next step will be a reduction to a linear problem. The key point is to identify the algebra $\mathcal{A}_\theta$ as the endomorphism algebra of a suitable bundle and to think of any projection in it as an operator on such a bundle. The bundle in question will be a projective module of finite type on a different copy $\mathcal{A}_\alpha$ of the noncommutative torus, the two algebras $\mathcal{A}_\theta$ and $\mathcal{A}_\alpha$ being related by Morita equivalence.

5.1 The modules

Let us then consider another copy $\mathcal{A}_\alpha$ of the noncommutative torus with generators Z_1, Z_2 obeying the relation $Z_2 Z_1 = e^{2\pi i\alpha} Z_1 Z_2$. When α is not rational, every finitely generated projective module over the algebra $\mathcal{A}_\alpha$ which is not free is isomorphic to a Heisenberg module [7, 12]. Any such a module $\mathcal{E}_{r,q}$ is characterized by two integers r, q which can be taken to be relatively coprime with $q > 0$, or $r = 0$ and $q = 1$. We shall briefly describe them. As a vector space

$$\mathcal{E}_{r,q} = \mathcal{S}(\mathbb{R} \times \mathbb{Z}_q) \simeq \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^q \ , \tag{24}$$

the space of Schwarz functions of one continuous variable $s \in \mathbb{R}$ and a discrete one $k \in \mathbb{Z}_q$ (we shall implicitly understand that such a variable is defined modulo q). By denoting $\varepsilon = r/q - \alpha$, the space $\mathcal{E}_{r,q}$ is made into a *right* module over $\mathcal{A}_\alpha$ by

$$(\xi \ Z_1)(s,k) := \xi(s - \varepsilon, k - r) \ , \qquad (\xi \ Z_2)(s,k) := e^{2\pi i(s - k/q)} \xi(s,k) \ , \tag{25}$$

with $\xi \in \mathcal{E}_{r,q}$; the relations $Z_2 \ Z_1 = e^{2\pi i\alpha} Z_1 \ Z_2$ for the torus $\mathcal{A}_\alpha$ are easily verified.
On $\mathcal{E}_{r,q}$ one defines an $\mathcal{A}_\alpha$-valued hermitian structure $\langle \cdot, \cdot \rangle_\alpha : \mathcal{E}_{r,q} \times \mathcal{E}_{r,q} \to \mathcal{A}_\alpha$, which is antilinear in the first factor.

It is proven in [7] that the endomorphism algebra $End_{\mathcal{A}_\alpha}(\mathcal{E}_{r,q})$, which acts on the left on $\mathcal{E}_{r,q}$, can be identified with another copy of the noncommutative torus $\mathcal{A}_\theta$ where the parameter θ is 'uniquely' determined by α in the following way. Since r and q are coprime, there exist integer numbers $a, b \in \mathbb{Z}$ such that $ar + bq = 1$. Then, the transformed parameter is given by $\theta = (a\alpha + b)/(-q\alpha + r)$. Notice that, given any two other integers $a', b' \in \mathbb{Z}$ such that $a'r + b'q = 1$, one would find that $\theta' - \theta \in \mathbb{Z}$ so that $\mathcal{A}_{\theta'} \simeq \mathcal{A}_\theta$. Thus, we are saying that the algebra $End_{\mathcal{A}_\alpha}(\mathcal{E}_{r,q})$ is generated by two operators U_1, U_2 acting on the *left* on $\mathcal{E}_{r,q}$ by

$$(U_1 \xi)(s,k) := \xi(s - 1/q, k - 1) \ , \qquad (U_2 \xi)(s,k) := e^{2\pi i \ (s/\varepsilon - ak)/q} \ \xi(s,k) \ , \tag{26}$$

and one verifies that $U_2 \ U_1 = e^{2\pi i\theta} U_1 \ U_2$, the defining relations of the algebra $\mathcal{A}_\theta$.

The crucial fact is that the $\mathcal{A}_\theta$-$\mathcal{A}_\alpha$-bimodule $\mathcal{E}_{r,q}$ is a *Morita equivalence* between the two algebras $\mathcal{A}_\theta$ and $\mathcal{A}_\alpha$: there exists also a $\mathcal{A}_\theta$-valued hermitian structure on $\mathcal{E}_{r,q}$, $\langle \cdot, \cdot \rangle_\theta : \mathcal{E}_{r,q} \times \mathcal{E}_{r,q} \to \mathcal{A}_\theta$, which is compatible with the $\mathcal{A}_\alpha$-valued one $\langle \cdot, \cdot \rangle_\alpha$,

$$\langle \xi, \eta \rangle_\theta \ \ \zeta = \xi \ \ \langle \eta, \zeta \rangle_\alpha \ \ , \tag{27}$$

for all $\xi, \eta, \zeta \in \mathcal{E}_{r,q}$. The hermitian structure $\langle \cdot, \cdot \rangle_\theta$ is antilinear in the second factor.

5.2 The constant curvature connection

A gauge connection on the right $\mathcal{A}_\alpha$-module $\mathcal{E}_{r,q}$ is given by two covariant derivative operators $\nabla_\mu : \mathcal{E}_{r,q} \to \mathcal{E}_{r,q}$, $\mu = 1, 2$ which satisfy a right Leibniz rule

$$\nabla_\mu(\xi \ a) = (\nabla_\mu \xi) a + \xi(\partial_\mu a) \ , \quad \mu = 1, 2 \ . \tag{28}$$

One also requires compatibility with the $\mathcal{A}_\alpha$-valued hermitian structure

$$\partial_\mu(\langle \xi, \eta \rangle_\alpha) = \langle \nabla_\mu \xi, \eta \rangle_\alpha + \langle \xi, \nabla_\mu \eta \rangle_\alpha \ \ , \quad \mu = 1, 2 \ . \tag{29}$$

A particular connection on the right $\mathcal{A}_\alpha$-module $\mathcal{E}_{r,q}$ is given by the operators

$$(\nabla_1\xi)(s,k) := \frac{2\pi i}{\varepsilon} s\ \xi(s,k)\ , \quad (\nabla_2\xi)(s,k) := \frac{d\xi}{ds}(s,k) \tag{30}$$

(the discrete index k is not touched); this connection is of constant curvature,

$$F_{1,2} := [\nabla_1, \nabla_2] - \nabla_{[\partial_1,\partial_2]} = -\frac{2\pi i}{\varepsilon}\ \mathbb{I}_{\mathcal{E}_{r,q}}\ , \tag{31}$$

with $\mathbb{I}_{\mathcal{E}_{r,q}}$ the identity operator on $\mathcal{E}_{r,q}$.

Given any connection ∇_μ on $\mathcal{E}_{r,q}$, one can define derivations on the endomorphism algebra $End_{\mathcal{A}_\alpha}(\mathcal{E}_{r,q})$ by commutators: $\hat{\delta}_\mu(T) := \nabla_\mu \circ T - T \circ \nabla_\mu$, $\mu = 1,2$, for any $T \in End_{\mathcal{A}_\alpha}(\mathcal{E}_{r,q})$. Then, by remembering that $End_{\mathcal{A}_\alpha}(\mathcal{E}_{r,q}) \simeq \mathcal{A}_\theta$, one finds that the derivations $\hat{\delta}_\mu$ on $End_{\mathcal{A}_\alpha}(\mathcal{E}_{r,q})$ determined by the particular connection (30) are proportional to the generators of the infinitesimal action of the torus $\mathbb{T}^2$ on $\mathcal{A}_\theta$, that is the canonical derivations (11): $\hat{\delta}_\mu(U_\nu) = \frac{2\pi i}{q\varepsilon}\ \delta^\nu_\mu U_\nu$, $\mu,\nu = 1,2$.

The holomorphic and anti-holomorphic connections $\nabla_{(\tau)}, \bar{\nabla}_{(\tau)}$ will be the lift of the derivations $\partial_{(\tau)}, \bar{\partial}_{(\tau)}$ with respect to the connection (30).

5.3 Instantons from Gaussians

We shall look for solutions of the self-dual equations (22) of the form

$$p_\psi := |\psi\rangle \langle\psi,\psi\rangle^{-1} \langle\psi|\ , \tag{32}$$

with $|\psi\rangle$ a 'section of a suitable vector bundle' over the noncommutative torus $\mathcal{A}_\theta$ and $\langle\psi,\psi\rangle$ an invertible element in another noncommutative torus which cannot be $\mathcal{A}_\theta$ itself but rather is Morita equivalent to it, being indeed $\mathcal{A}_\alpha$. We shall take $|\psi\rangle$ to be an element of the Schwarz space $\mathcal{E}_{r,q}$ on which $\mathcal{A}_\theta$ acts on the *left* as the endomorphism algebra. Thus, $\mathcal{E}_{r,q}$ will be though of as a *right* module over the algebra $\mathcal{A}_\alpha$ and $\langle\psi,\psi\rangle = \langle\psi,\psi\rangle_\alpha$ will be required to be an invertible element of $\mathcal{A}_\alpha$.

Than, let us suppose that $|\psi\rangle \in \mathcal{E}_{r,q}$ be such that $\langle\psi,\psi\rangle_\alpha$ is invertible. A simple computation shows that the projection $p_\psi := |\psi\rangle (\langle\psi,\psi\rangle_\alpha)^{-1} \langle\psi|$ is a solution of the self-duality equation (22) if and only if the element $|\psi\rangle \in \mathcal{E}_{r,q}$ obeys the equation

$$\bar{\nabla}\psi - \psi\lambda = 0\ , \tag{33}$$

with λ a suitable element in $\mathcal{A}_\alpha$ and $\bar{\nabla}$ the anti-holomorphic connection.

For $\lambda \in \mathbb{C}$ there are simple solutions to (33) given by generalized Gaussians

$$\psi_\lambda(s,k) = A_k e^{i\tau\pi s^2/\varepsilon + \lambda(\bar{\tau}-\tau)s}\ . \tag{34}$$

Any such a function is an element of $\mathcal{E}_{r,q}$ since $\Im\tau > 0$. The vector $A = (A_1, \ldots, A_q) \in \mathbb{C}^q$ can be taken in the complex projective space $\mathbb{CP}^{q-1}$ by removing an inessential normalization. Restrictions on the possible values of the constant parameter λ will be discussed presently.

A projection corresponding to a Gaussian was constructed in [3] for the lowest value of the charge $q = 1$ and for $\tau = i$ and the invertibility of the corresponding element $\langle\psi_\lambda, \psi_\lambda\rangle_\alpha$

(now $\alpha = -1/\theta$) was proved for any values of $0 < \theta < 1$ in [13]. By using the methods of [13] one should be able to prove invertibility of $\langle \psi_\lambda, \psi_\lambda \rangle_\alpha \in \mathcal{A}_\alpha$ for the most general situation.

Gauge transformations (in fact complexfied ones) are provided by invertible elements in $\mathcal{A}_\alpha$ acting on the right on $\mathcal{E}_{r,q}$,

$$\mathcal{E}_{r,q} \ni |\psi\rangle \to |\psi^g\rangle = |\psi\rangle\, g \in \mathcal{E}_{r,q} \ , \quad \forall\ g \in \mathrm{GL}(\mathcal{A}_\alpha) \ . \tag{35}$$

It is clear that projections of the form (32) are invariant under gauge transformations. Notice that we do not require g to be unitary.

Now, let $|\psi\rangle$ be a solution of (33), $\bar{\nabla}\psi - \psi\lambda = 0$; and let $g \in \mathrm{GL}(\mathcal{A}_\alpha)$. Then a simple computation shows that the gauge transformed vector $|\psi^g\rangle$ will be a solution of an equation of the form (33): $\bar{\nabla}\psi^g - \psi^g\lambda_g = 0$ with λ_g given by

$$\lambda_g = g^{-1}\lambda g + g^{-1}\bar{\partial}_{(\tau)}g \ . \tag{36}$$

The moduli space of Gaussians is simply described. It turns out that two Gaussians ψ_λ and $\psi_{\lambda'}$ are related by a gauge transformation if an only if there exist integers $(m, n) \in \mathbb{Z}^2$ such that

$$\lambda' = \lambda_g \ , \qquad \text{with} \qquad g = (Z_1)^m (Z_2)^n \ . \tag{37}$$

Furthermore,

$$\lambda_g - \lambda = \frac{2\pi i \tau}{\tau - \bar{\tau}} \left(m - \frac{1}{\tau} n \right) \tag{38}$$

Thus, gauge nonequivalent constant parameters λ form a complex ordinary torus $\mathbb{T}^2_\tau$ and the moduli space of Gaussians (34) is $\mathbb{CP}^{q-1} \times \mathbb{T}^2_\tau$.

It is an open problem to prove whether it is possible to gauge to a Gaussian any solution of the self-duality equation (33). Equivalently, this problem can be stated as follows: given any $\rho \in \mathcal{A}_\alpha$, there exists an element $g \in \mathrm{GL}(\mathcal{A}_\alpha)$ such that $\rho = \lambda + g^{-1}\bar{\partial}_{(\tau)}g$, with $\lambda \in \mathbb{C}$. Such a gauge transformation can be found if the deformation parameter θ is small enough.

As a final remark, me mention that for special values of the deformation parameter, $\theta = 1/N$, with $N \in \mathbb{Z}$, the Gaussian (Boca) projection can be mapped, in the large torus limit [10], to the basic GMS soliton [9].

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