WHEN IS A TOPOLOGY ON A PRODUCT A PRODUCT TOPOLOGY?

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ABSTRACT. We study the following problem: when is a given topology $\mathcal{T}$ on a cartesian product $X \times Y$ a product topology of two topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$ on its factors? We are interested in particular in the case when $\mathcal{T}$ is a weak topology.

1. INTRODUCTION

By investigating properties of a family $\mathcal{F}$ of closed convex subsets of a linear space X, in the attempt to construct a hypertopology on this family so that the hyperspace becomes complete and suitable to the construction of a Debreu integral on it, A. Martellotti [MA] had the idea to consider the cartesian product $X \times \mathcal{F}$ and the weak topology on it induced by special real valued functions. To achieve what she needed, however, she wanted this weak topology on $X \times \mathcal{F}$ to be a (Tychonoff) product topology of two topologies on X and $\mathcal{F}$. This raised the following question:

Problem 1.1. Let X and Y be two sets and consider a topology $\mathcal{T}$ on the cartesian product $X \times Y$. Are there any topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$ on X and Y respectively such that $\mathcal{T}$ is the product (Tychonoff) topology $\mathcal{T}_X \times \mathcal{T}_Y$ on $X \times Y$?

As we shall see soon, trivial examples show that this is not true in general.

Observation 1.2. Assume $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$. Then the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are quotient maps, hence $\mathcal{T}_X$ and $\mathcal{T}_Y$ are the quotient topologies induced by π_X and π_Y respectively.

Problem 1.1 can therefore be stated as follows:

Problem 1.3. Let X and Y be two sets and consider a topology $\mathcal{T}$ on the cartesian product $X \times Y$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the quotient topologies on X and Y induced respectively by the projections π_X and π_Y. When is $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$?

Observation 1.4. For any $\mathcal{T}$ on $X \times Y$ we always have $\mathcal{T}_X \times \mathcal{T}_Y \subseteq \mathcal{T}$. In fact a set $A \subseteq X$ is in $\mathcal{T}_X$ if and only if $\pi_X^{-1}(A)$ is in $\mathcal{T}$, and a set $B \subseteq Y$ is in $\mathcal{T}_Y$ if and only if $\pi_Y^{-1}(B)$ is in $\mathcal{T}$. Therefore all rectangles $A \times B = \pi_X^{-1}(A) \cap \pi_Y^{-1}(B)$, with $A \in \mathcal{T}_X$ and $B \in \mathcal{T}_Y$ are in $\mathcal{T}$.

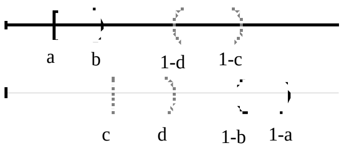
Proposition 1.5. *Let $\mathcal{T}$ be a compact Hausdorff topology on the set $X \times Y$. Then $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$ if and only if $\mathcal{T}_X \times \mathcal{T}_Y$ is Hausdorff.*

Proof. By Observation 1.4 $\mathcal{T}_X \times \mathcal{T}_Y \subseteq \mathcal{T}$. Then the identity $f : (X \times Y, \mathcal{T}) \to (X \times Y, \mathcal{T}_X \times \mathcal{T}_Y)$ is closed and continuous, hence a homeomorphism. $\square$

1991 *Mathematics Subject Classification.* 54A10, 54B10.
Key words and phrases. Product, partial function, weak topology.

An example of a compact Hausdorff topology $\mathcal{T}$ on $X \times Y$ where $\mathcal{T} \neq \mathcal{T}_X \times \mathcal{T}_Y$ is easily given.

Example 1.6. By $]a, b[$ ($[a, b[$) we denote the open (right-open) interval $\{x \in \mathbb{R} : a < x < b\}$ ($\{x \in \mathbb{R} : a \leq x < b\}$). Let $X = [0, 1[$, $Y = \{0, 1\}$. If $a < b$ we define $U_{a,b} :=]1-b, 1-a[\times \{0\} \cup [a, b[\times \{1\}$, and $V_{c,d} := [c, d[\times \{0\} \cup]1-d, 1-c[\times \{1\}$.



We consider the topology $\mathcal{T}$ on $X \times Y$ generated by all sets of the form $U_{a,b}$ and $V_{a,b}$ for $0 \leq a < b \leq 1$.

The space $(X \times Y, \mathcal{T})$ is compact Hausdorff. However $\mathcal{T}_Y$ is the trivial topology on Y, hence $\mathcal{T}_X \times \mathcal{T}_Y$ is not Hausdorff and $\mathcal{T} \neq \mathcal{T}_X \times \mathcal{T}_Y$.

Let us remark that in this example the projections are open maps.

We would like to find conditions so that $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$. One peculiarity of a product topology is that in the space $(X \times Y, \mathcal{T}_X \times \mathcal{T}_Y)$ all fibers of each projection are homeomorphic to each other, i.e., $\pi_X^{-1}(x_1) \simeq \pi_X^{-1}(x_2) \simeq Y$ for all x_1, $x_2 \in X$ and $\pi_Y^{-1}(y_1) \simeq \pi_Y^{-1}(y_2) \simeq X$ for all y_1, $y_2 \in Y$. This property can be described in several ways, in particular, at the end of this section, we will give a description in the case $\mathcal{T}$ is seen as a weak topology generated by a family of functions.

Let X be any set, $\{(Z_\alpha, \mathcal{T}_\alpha) : \alpha \in A\}$ a family of topological spaces, $\{f_\alpha : X \to Z_\alpha : \alpha \in A\}$ a family of functions. The weak topology on X generated by $\{f_\alpha : \alpha \in A\}$ is the coarsest topology on X for which all functions f_α are continuous (see for example [E] Proposition 1.4.8).

Notation 1.7. Let $\mathcal{T}$ be a topology defined on a cartesian product $X \times Y$. For any point $x \in X$ we denote by $i^x_{\langle Y \rangle}$ the function $i^x_{\langle Y \rangle} : X \times Y \to X \times Y$ defined by $i^x_{\langle Y \rangle}(u, v) = (x, v)$, and by i^x_Y the function $i^x_Y : Y \to X \times Y$ defined by $i^x_Y(v) = (x, v)$. Similarly for any point $y \in Y$ we denote by $i^y_{\langle X \rangle}$ the function $i^y_{\langle X \rangle} : X \times Y \to X \times Y$ defined by $i^y_{\langle X \rangle}(u, v) = (u, y)$, and by i^y_X the function $i^y_X : X \to X \times Y$ defined by $i^y_X(u) = (u, y)$.

Let X, Y, Z be sets, $f : X \times Y \to Z$ a function. For any point $x \in X$ we denote by $f^x_{\langle Y \rangle}$ the function $f^x_{\langle Y \rangle} : X \times Y \to Z$ defined by $f^x_{\langle Y \rangle}(u, v) = f(x, v)$ and by f^x_Y the function $f^x_Y : Y \to Z$ defined by $f^x_Y(v) = f(x, v)$ (the function f^x_Y is often denoted by $f(x, \cdot)$). Similarly for any point $y \in Y$ we denote by $f^y_{\langle X \rangle}$ the function $f^y_{\langle X \rangle} : X \times Y \to Z$ defined by $f^y_{\langle X \rangle}(u, v) = f(u, y)$, and by f^y_X $(= f(\cdot, y))$ the function $f^y_X : X \to Z$ defined by $f^y_X(u) = f(u, y)$. By *partial functions* we will mean functions of the form $f^y_{\langle X \rangle}$, $f^x_{\langle Y \rangle}$, f^y_X or f^x_Y.

Proposition 1.8. *Let X and Y be sets and $\mathcal{T}$ the weak topology induced on $X \times Y$ by a family of partial functions $\{(f_\alpha)^x_{\langle Y \rangle} : X \times Y \to Z_\alpha$, $(f_\beta)^y_{\langle X \rangle} : X \times Y \to Z_\beta :$ $\alpha \in A$, $\beta \in B\}$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the quotient topologies induced on X and Y respectively by the projections π_X and π_Y. Then $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$.*

Proof. We only need to show $\mathcal{T} \subseteq \mathcal{T}_X \times \mathcal{T}_Y$. Since $\mathcal{T}$ is the coarsest topology on $X \times Y$ for which all partial functions $(f_\alpha)^x_{\langle Y \rangle}$ and $(f_\beta)^y$ are continuous it will be sufficient to show that all such functions are continuous for $\mathcal{T}_X \times \mathcal{T}_Y$. However this is clear since for any open set $U \subseteq Z_\gamma$, $\gamma \in A$ (respectively $\gamma \in B$) the preimage $((f_\gamma)^x_{\langle Y \rangle})^{-1}(U)$ (respectively $((f_\gamma)^y_{\langle X \rangle})^{-1}(U)$) are strips on $X \times Y$, i.e. of the form $X \times E$ (respectively $F \times Y$) with $E \in \mathcal{T}_Y$ ($F \in \mathcal{T}_X$). □

Remark 1.9. Any topology $\mathcal{T}$ on a set S can be considered as the weak topology with respect to the identity map $id : S \to (S, \mathcal{T})$. If $S = X \times Y$, with the Notation 1.7 we have $id^x_{\langle Y \rangle} = i^x_{\langle Y \rangle}$, $id^x_Y = i^x_Y$, $id^y_{\langle X \rangle} = i^y_{\langle X \rangle}$ and $id^y_X = i^y_X$ for all $(x, y) \in S$.

Lemma 1.10. *Let X and Y be sets, $(Z, \mathcal{S})$ a topological space, $\mathcal{T}$ a topology on $X \times Y$, $f : X \times Y \to Z$ a continuous function. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the quotient topologies induced on X and Y respectively by the projections π_X and π_Y. Then, for all $x \in X$ ($y \in Y$) the function $f^x_{\langle Y \rangle}$ ($f^y_{\langle X \rangle}$) is continuous if and only if the function f^x_Y (f^y_X) is continuous; the function $i^x_{\langle Y \rangle}$ ($i^y_{\langle X \rangle}$) is continuous if and only if the function i^x_Y (i^y_X) is continuous. Moreover all such functions are continuous if $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$.*

Proof. The first claim derives from the well-known universal property of quotients (see for example [E] Proposition 2.4.2) and the fact that $f^y_{\langle X \rangle} = f^y_X \circ \pi_X$. The rest is clear. □

Lemma 1.11. *Let X and Y be sets, $\{(Z_\alpha, \mathcal{T}_\alpha) : \alpha \in A\}$ a family of topological spaces, $\{f_\alpha : X \times Y \to Z_\alpha : \alpha \in A\}$ a family of functions. Let $\mathcal{T}$ be the weak topology on $X \times Y$ induced by the functions $\{f_\alpha : \alpha \in A\}$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be topologies on X and Y respectively. Then, for all $x \in X$ ($y \in Y$) the function i^x_Y (i^y_X) is continuous if and only if the functions $(f_\alpha)^x_Y$ ($(f_\alpha)^y_X$) are continuous for all $\alpha \in A$. Moreover all such functions are continuous if $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$.*

Proof. The claim derives from the well-known universal property of weak topologies (see for example [E] Proposition 1.4.9) and the fact that $(f_\alpha)^y_X = f_\alpha \circ i^y_X$. □

Lemmas 1.10 and 1.11 say that continuity of partial functions is a necessary condition for $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$. We can observe that this condition is equivalent to the fact that the projections π_X and π_Y, restricted to any fiber $\pi_Y^{-1}(x)$ ($x \in X$) and $\pi_X^{-1}(y)$ ($y \in Y$) respectively, are open maps. This is also equivalent to the fact that all fibers of each projection are homeomorphic to each other. Expecially if we are interested in weak topologies, it is natural to ask for the continuity of partial functions. In next section we will see when this condition is sufficient to get the equality $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$. In what follows, unless special properties of the functions generating $\mathcal{T}$ as a weak topology are required, we will consider any topology $\mathcal{T}$ on $X \times Y$, and the partial functions to be considered will be of the form $i^y_{\langle X \rangle}$ and $i^x_{\langle Y \rangle}$. By Lemma 1.11 this can be done without loss of generality.

2. When continuity of partial functions is sufficient

The following proposition is practically a rewriting of the definition of product topology:

Proposition 2.1. *Let X and Y be sets, and let $\mathcal{T}$ be a topology on $X \times Y$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the quotient topologies induced on X and Y respectively by the projections π_X and π_Y. Then $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$ if and only if the following conditions are satisfied:*

1. *For all* $(x, y) \in X \times Y$ *the functions* $i^x_{\langle Y \rangle}$ *and* $i^y_{\langle X \rangle}$ *are continuous.*
2. *For any* $(x, y) \in X \times Y$*, any* $\mathcal{T}$*-neighbourhood* U *of* (x, y)*, there exists a* $\mathcal{T}$*-neighbourhood* V *of* (x, y) *such that* $(x, y) \in (i^y_{\langle X \rangle})^{-1}(V) \cap (i^x_{\langle Y \rangle})^{-1}(V) \subseteq U$.

As we already remarked, the first condition in Proposition 2.1 is natural to be considered, expecially if we are interested in weak topologies. On the other hand the second condition is too complicated and difficult to handle with. Our goal is to drop the second condition at least in some special cases. It is clear however that both conditions are necessary in a general setting.

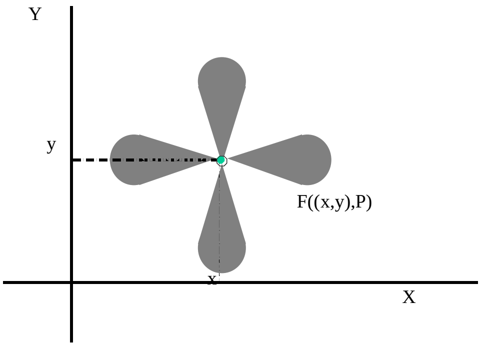
Example 2.2. Consider the Niemytzki plane N (see for example [E] Example 1.2.4). It is easily checked that the quotient topologies $\mathcal{T}_X$ on $X = \mathbb{R}$ and $\mathcal{T}_Y$ on $Y = \mathbb{R}^+$ are the Euclidean topologies.

Clearly this is an example where the product topology $\mathcal{T}_X \times \mathcal{T}_Y$ is strictly coarser than $\mathcal{T}$.

Observe that the second condition of Proposition 2.1 is satisfied. However the functions $i^x_{\langle Y \rangle}$ and $i^y_{\langle X \rangle}$ are not continuous.

Let us remark that in this example the projections π_X and π_Y are open maps.

Example 2.3. Let $X = Y = \mathbb{R}$. Let P_{upper} be an upper *petal*, i.e. a convex domain contained in the region $\{(x, y) \in \mathbb{R}^2 : y > |x|\}$ whose boundary is a regular curve except in $(0, 0)$ where it has a cuspid. In a similar way we can define lower, left and right petals. We can consider the *flower at* $(0, 0)$ defined as the set $F((0, 0), P) = \{(0, 0)\} \cup \text{int}(P_{\text{upper}}) \cup \text{int}(P_{\text{lower}}) \cup \text{int}(P_{\text{left}}) \cup \text{int}(P_{\text{right}})$ (where int denotes the interior in the Euclidean topology of the plane). Let $(x, y) \in X \times Y$; a *flower at* (x, y) is defined as $F((0, 0), P) + (x, y)$.



The family of all flowers at (x, y), for all $(x, y) \in X \times Y$ is a base for a topology $\mathcal{T}$ on $X \times Y$, strictly finer than the Euclidean topology.

It is easily checked that the quotient topologies $\mathcal{T}_X$ on X and $\mathcal{T}_Y$ on Y are both the Euclidean topology on $\mathbb{R}$, hence the product topology $\mathcal{T}_X \times \mathcal{T}_Y$ is strictly coarser than $\mathcal{T}$.

Observe that in this example the functions $i^x_{\langle Y \rangle}$ and $i^y_{\langle X \rangle}$ are all continuous. However the second condition of Proposition 2.1 is not satisfied.

Remark 2.4. All examples of topologies $\mathcal{T}$ on a product $X \times Y$ where the first condition of Proposition 2.1 is satisfied while the second condition is not satisfied are constructed essentially as in Example 2.3. The assumption implies that any $\mathcal{T}$-neighbourhood Ω of a point (x, y) contains a "cross" of the form $\{x\} \times B \cup A \times \{y\}$ where $A = (i^x_{\langle Y \rangle})^{-1}(\Omega)$ and $B = (i^y_{\langle X \rangle})^{-1}(\Omega)$. In particular the family of all "crosses" of this form is a network for $\mathcal{T}$.

If we remember (see the end of the previous section) that the continuity of partial functions is equivalent to the fact that any fiber of the projection π_X is homeomorphic to Y and any fiber of the projection π_Y is homeomorphic to X, and moreover we observe that the fibers are closed sets in $(X \times Y, \mathcal{T})$ if $\mathcal{T}$ is T_1 we immediately obtain the following:

Proposition 2.5. *Let X and Y be sets and $\mathcal{T}$ a topology on $X \times Y$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the quotient topologies induced on X and Y respectively by the projections π_X and π_Y. Assume that all partial functions $i^x_{\langle Y \rangle}$ and $i^y_{\langle X \rangle}$ are continuous and that $(X \times Y, \mathcal{T})$ has property P where P is any topological property preserved by (closed if $\mathcal{T}$ is T_1) subspaces (in particular T_i for $i = 0, 1, 2, 3, 3\frac{1}{2}$ and metrizability), or by continuous open images (for instance separabillity, sequentiality, connectedeness...). Then the spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ have property P.*

Let us remark that the assumption of continuity of partial functions in Proposition 2.5 cannot be replaced by the condition that requires the projections π_X and π_Y to be open maps (see Example 1.6).

Corollary 2.6. *Let X and Y be sets and $\mathcal{T}$ a topology on $X \times Y$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the quotient topologies induced on X and Y respectively by the projections π_X and π_Y. Assume that $(X \times Y, \mathcal{T})$ is Hausdorff and compact. Then $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$ if and only if all partial functions $i^x_{\langle Y \rangle}$ and $i^y_{\langle X \rangle}$ are continuous.*

Proof. This follows immediately from Proposition 1.5 and Proposition 2.5. □

The compactness assumption cannot be weakened to local compactness, even in the realm of metric spaces, or countably compactness as next examples show. By Remark 2.4 these spaces must be constructed in a way similar to the space in Example 2.3.

Example 2.7. We construct a metrizable locally compact space $(X \times Y, \mathcal{T})$ for which all functions $i^x_{\langle Y \rangle}$ and $i^y_{\langle X \rangle}$ are continuous but $\mathcal{T} \neq \mathcal{T}_X \times \mathcal{T}_Y$.

Let $X = Y = \omega + 1$, endowed with the natural order topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$. Let us consider the product space $(X \times Y, \mathcal{T}_X \times \mathcal{T}_Y)$. We define a finer topology $\mathcal{T}$ on $X \times Y$ obtained by adding, for any $n \in \omega$, the following new neighbourhoods S_n of the point (ω, ω): $S_n := \{(\omega, \omega)\} \cup \{(i, j) : i > n, n \leq j < i\} \cup \{(i, j) : i \geq n, i < j\}$ (S_n is a square with a deleted diagonal on the bisector).

The topology we obtain in this way is finer than the product topology, the functions $i^x_{\langle Y \rangle}$ and $i^y_{\langle X \rangle}$ are all continuous and the quotient topologies are $\mathcal{T}_X$ and $\mathcal{T}_Y$.

Let us show that $(X \times Y, \mathcal{T})$ is locally compact. Clearly we only need to worry about neighbourhoods of the point (ω, ω). Let us show that any set S_n is compact. Let $\mathcal{U}$ be an open cover of S_n. There must be some $U_0 \in \mathcal{U}$ with $(\omega, \omega) \in U_0$, hence there is an integer k with $S_k \subseteq U_0$. Only finitely many points of the form (i, ω) or (ω, j) can belong to $S_n \setminus S_k$; let $U_1, U_2, \ldots U_r \in \mathcal{U}$ be such that all such points are contained in $\bigcup_{p=1}^r U_p$. Since the set $S_n \setminus \bigcup_{p=0}^r U_p$ is finite we have completed the proof.

The space $(X \times Y, \mathcal{T})$ is regular and second countable, hence it is metrizable.

Example 2.8. We construct an example of a (non regular) countably compact space $(X \times Y, \mathcal{T})$ for which all functions $i^x_{\langle Y \rangle}$ and $i^y_{\langle X \rangle}$ are continuous but $\mathcal{T} \neq$

$\mathcal{T}_X \times \mathcal{T}_Y$. The construction is the same as in Example 2.7 except that we replace ω with ω_1.

Let $X = Y = \omega_1 + 1$ endowed with the natural order topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$. Let us consider the product space $(X \times Y, \mathcal{T}_X \times \mathcal{T}_Y)$. We define a finer topology $\mathcal{T}$ on $X \times Y$ obtained by adding, for any non-limit ordinal $\alpha \in \omega_1$, the following new neighbourhoods S_α of the point (ω_1, ω_1): $S_\alpha := \{(\omega_1, \omega_1)\} \cup \{(\delta, \gamma) : \alpha < \delta \leq \omega_1, \alpha \leq \gamma < \delta\} \cup \{(\delta, \gamma) : \delta \geq \alpha, \delta < \gamma \leq \omega_1\}$ (S_α is a square with a deleted diagonal on the bisector).

The topology $\mathcal{T}$ we obtain in this way is clearly strictly finer than the product topology $\mathcal{T}_X \times \mathcal{T}_Y$; in particular it is not compact any more (and even not regular since no set S_α can contain the closure of any neighbourhood of (ω_1, ω_1)). We note that the functions $i^x_{\langle Y \rangle}$ and $i^y_{\langle X \rangle}$ are all continuous and the quotient topologies are $\mathcal{T}_X$ and $\mathcal{T}_Y$.

We observe that the topology in all points except (ω_1, ω_1) has not been altered, since all sets S_α are open in $(X \times Y \setminus \{(\omega_1, \omega_1)\}, \mathcal{T}_X \times \mathcal{T}_Y)$.

Let us show that $(X \times Y, \mathcal{T})$ is countably compact. Let $\mathcal{U}$ be a countable open cover of $X \times Y$. Let $U_0 \in \mathcal{U}$ be such that $(\omega_1, \omega_1) \in U_0$. Let $\alpha \in \omega_1$ be a non-limit ordinal such that $S_\alpha \subseteq U_0$. The space $X \times Y \setminus U_0$ can be written as the union of the two compact subspaces $K_1 := X \times (\alpha+1)$, $K_2 := (\alpha+1) \times Y$ and the countably compact subspace $D := \{(\delta, \delta) : \alpha < \delta < \omega_1\}$, homeomorphic to ω_1. The family $\mathcal{U}$ is an open covering of the countably compact $K_1 \cup K_2 \cup D$, hence we can pick a finite subcover $\tilde{\mathcal{U}} \subset \mathcal{U}$. Clearly $U_0 \cup \tilde{\mathcal{U}}$ is the required finite subfamily of $\mathcal{U}$ that covers $X \times Y$.

Question 2.9. Can we find a regular (Tychonoff, normal, locally compact...) example ?

The space in Example 2.7 is zero dimensional; we see in particular that a locally connected example of this kind does not exist.

Theorem 2.10. *Let X and Y be sets (with more than one point) and $\mathcal{T}$ a topology on $X \times Y$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the quotient topologies induced on X and Y respectively by the projections π_X and π_Y. Assume that $(X \times Y, \mathcal{T})$ is Hausdorff locally compact and locally connected. Then $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$ if and only if all partial functions $i^x_{\langle Y \rangle}$ and $i^y_{\langle X \rangle}$ are continuous.*

Proof. We show $\mathcal{T} \subseteq \mathcal{T}_X \times \mathcal{T}_Y$. By Proposition 2.5 the spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are Hausdorff, locally compact and locally connected. Assume by contradiction that $\mathcal{T} \not\subseteq \mathcal{T}_X \times \mathcal{T}_Y$. Then we can pick a point $(x_0, y_0) \in X \times Y$ and a connected open neighbourhood $\Omega \in \mathcal{T}$ of (x_0, y_0) such that $\overline{\Omega}$ is compact, $\overline{\Omega} \setminus \Omega \neq \emptyset$, and such that for any open sets $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ we never have $(x_0, y_0) \in U \times V \subseteq \Omega$. Let $A = \pi_X (X \times \{y_0\} \cap \Omega)$ and $B = \pi_Y (\{x_0\} \times Y \cap \Omega)$. By Remark 2.4 the set $A \times \{y_0\} \cup \{x_0\} \times B$ is contained in Ω. Let $\{U_\delta\}_{\delta \in \Delta}$ and $\{V_\gamma\}_{\gamma \in \Gamma}$ be families of connected open sets (in $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ respectively) such that $\{x_0\} = \bigcap_{\delta \in \Delta} U_\delta$, $\{y_0\} = \bigcap_{\gamma \in \Gamma} V_\gamma$, $V_1 \neq Y$, $U_\delta \subseteq A$ for all $\delta \in \Delta$, $V_\gamma \subseteq B$ for all $\gamma \in \Gamma$. For $(\delta, \gamma) \in \Delta \times \Gamma$ let us pick a point $x_{\delta,\gamma} \in U_\delta$ such that $\{x_{\delta,\gamma}\} \times V_\gamma \not\subseteq \Omega$ (this can be done since $U_\delta \times V_\gamma \not\subseteq \Omega$) and let $B_{\delta,\gamma} = \pi_Y (\{x_{\delta,\gamma}\} \times V_\gamma \cap \Omega)$. The set $B_{\delta,\gamma}$ is a proper open subset of the connected set V_γ, therefore we can pick a point $y_{\delta,\gamma} \in \mathrm{cl}_{V_\gamma}(B_{\delta,\gamma}) \setminus B_{\delta,\gamma}$ (here $\mathrm{cl}_F(G)$ denotes as usual the closure in the subspace F of the set G).

Let us note that $(x_{\delta,\gamma}, y_{\delta,\gamma}) \in \overline{\Omega} \setminus \Omega$ (closure in $\mathcal{T}$). In fact, if O is a $\mathcal{T}$-neighbourhood of $(x_{\delta,\gamma}, y_{\delta,\gamma})$, then $\pi_Y(\{x_{\delta,\gamma}\} \times V_\gamma \cap O)$ is a neighbourhood of $y_{\delta,\gamma}$ in V_γ, hence there is a point $w \in B_{\delta,\gamma} \cap \pi_Y(\{x_{\delta,\gamma}\} \times V_\gamma \cap O)$. Clearly $(x_{\delta,\gamma}, w) \in O \cap \Omega$. Moreover $(x_{\delta,\gamma}, y_{\delta,\gamma}) \notin \Omega$ since $y_{\delta,\gamma} \notin B_{\delta,\gamma}$.

By compactness of $\overline{\Omega}$ there is a $\mathcal{T}$-cluster point $(\tilde{x}, \tilde{y})$ of the net $\mathcal{N} := \{(x_{\delta,\gamma}, y_{\delta,\gamma}); (\delta, \gamma) \in (\Delta, \Gamma)\}$. Since $\mathcal{T}_X \times \mathcal{T}_Y \subseteq \mathcal{T}$ the point $(\tilde{x}, \tilde{y})$ is also a $\mathcal{T}_X \times \mathcal{T}_Y$-cluster point, but the net $\mathcal{N}$ $\mathcal{T}_X \times \mathcal{T}_Y$-converges to (x_0, y_0) and the product topology $\mathcal{T}_X \times \mathcal{T}_Y$ is Hausdorff, hence $(x_0, y_0) = (\tilde{x}, \tilde{y})$. However Ω is a neighbourhood of (x_0, y_0) that misses $\mathcal{N}$, a contradiction. □

Question 2.11. Can we replace the property "locally connected" with "connected" in Theorem 2.10 ?

Example 2.12. We cannot replace the property " all partial functions $i^x_{\langle Y \rangle}$ and $i^y_{\langle X \rangle}$ are continuous " with "the projections π_X and π_Y are open" in Theorem 2.10. Consider for example the unit square $S = [0, 1] \times [0, 1]$. Define new neighbourhoods for points (x, x) on the diagonal as open intervals on the diagonal $\{(u, u) \in X \times X : |x - u| < \varepsilon\}$. We obtain a locally compact, locally connected topology $\mathcal{T}$ on S, finer than the Euclidean topolgy. The quotient topology on each factor is the Euclidean topology and the projections are open.

We consider now topologies $\mathcal{T}$ compatible with some algebraic structure on $X \times Y$.

Theorem 2.13. *Let $(X, \cdot, 1)$ and $(Y, \cdot, 1)$ be unitary semigroups and $\mathcal{T}$ a topology on $X \times Y$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the quotient topologies induced on X and Y respectively by the projections π_X and π_Y. Assume that the natural (componentwise) multiplication $\cdot$ induced on the product $X \times Y$ is a $\mathcal{T}$-continuous $\mathcal{T}$-open map. Then $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$ if and only if all partial functions i^x_Y and i^y_X are continuous.*

Proof. We show $\mathcal{T} \subseteq \mathcal{T}_X \times \mathcal{T}_Y$. Let W be a neighbourhood of a point $(x, y) \in X \times Y$. By the continuity of the operation $\cdot$ there exists an open neighbourhood Ω of $(1, 1)$ such that $(x, y) \in (x, y) \cdot \Omega \subseteq W$. Since $\mathcal{T}$ is a semigroup topology there exists a $\mathcal{T}$-neighbourhood V of $(1, 1)$ such that $V \cdot V \subseteq \Omega$.

The sets $A = (i^x_{\langle Y \rangle})^{-1}(V) = \pi_X(V \cap X \times \{1\})$ and $B = (i^y_{\langle X \rangle})^{-1}(V) = \pi_Y(V \cap \{1\} \times Y)$ are open in $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ respectively. Clearly $(1, 1) \in A \times B$ and, since $(u, v) = (u, 1) \cdot (1, v)$, we have $A \times B \subseteq V \cdot V \subseteq \Omega$. Since $\cdot$ is an open function, the sets $x \cdot A$ and $y \cdot B$ are open in $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ respectively. We have $(x, y) \in (x \cdot A) \times (y \cdot B) \subseteq W$, and the theorem is proved. □

Theorem 2.13 is trivially not true for arbitrary topological semigroups, since any topological space can be considered as a topological semigroup with a constant operation. However, I have not been able to find an example of a unitary topological semigroup with continuous partial functions where $\mathcal{T} \neq \mathcal{T}_X \times \mathcal{T}_Y$.

Question 2.14. Can we drop the condition that the operation is an open map in Theorem 2.13 ?

Corollary 2.15. *Let X and Y be groups and $\mathcal{T}$ a group topology on $X \times Y$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the quotient topologies induced on X and Y respectively by the projections π_X and π_Y. Then $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$ if and only if all partial functions i^x_Y and i^y_X are continuous.*

Corollary 2.16. *Let X, Y be groups, $\{(Z_\alpha, \mathcal{T}_\alpha) : \alpha \in A\}$ a family of topological groups and $\mathcal{T}$ the weak topology induced on $X \times Y$ by a family of homomorphisms $\{f_\alpha : X \times Y \to Z_\alpha : \alpha \in A\}$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the quotient topologies induced on X and Y respectively by the projections π_X and π_Y. Then $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$ if and only if for all $(x, y) \in X \times Y$ and for all $\alpha \in A$ the functions $(f_\alpha)^x_{\langle Y \rangle}$ and $(f_\alpha)^y_{\langle X \rangle}$ are continuous.*

Proof. The weak topology induced by homomorphisms is compatible with the group structure (see for example [MU]). Therefore $\mathcal{T}$ is a group topology for $X \times Y$ and we can apply Theorem 2.13. $\square$

Let X be any set endowed with some algebraic structure, $\{(Z_\alpha, \mathcal{T}_\alpha) : \alpha \in A\}$ a family of topological spaces, $\{f_\alpha : X \to Z_\alpha : \alpha \in A\}$ a family of functions. The linear weak topology on X with respect to $\{f_\alpha : \alpha \in A\}$ is the coarsest topology on X compatible with the algebraic structure on X for which all functions f_α are continuous (see [MU]).

Corollary 2.17. *Let X, Y be groups, $\{(Z_\alpha, \mathcal{T}_\alpha) : \alpha \in A\}$ a family of topological spaces and $\mathcal{T}$ the linear weak topology induced on $X \times Y$ by a family of functions $\{f_\alpha : X \times Y \to Z_\alpha : \alpha \in A\}$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the quotient topologies induced on X and Y respectively by the projections π_X and π_Y. Then $\mathcal{T} = \mathcal{T}_X \times \mathcal{T}_Y$ if and only if for all $(x, y) \in X \times Y$ and for all $\alpha \in A$ the functions $(f_\alpha)^x_{\langle Y \rangle}$ and $(f_\alpha)^y_{\langle X \rangle}$ are continuous.*

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