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ATTRACTORS AND GLOBAL AVERAGING OF NON-AUTONOMOUS REACTION-DIFFUSION EQUATIONS IN $\mathbb{R}^N$

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Abstract. We consider a family of non-autonomous reaction-diffusion equations

$$(E_\omega) \qquad u_t = \sum_{i,j=1}^{N} a_{ij}(\omega t)\partial_i\partial_j u + f(\omega t, u) + g(\omega t, x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N$$

with almost periodic, rapidly oscillating principal part and nonlinear interactions. As $\omega \to +\infty$, we prove that the solutions of (E_ω) converge to the solutions of the averaged equation

$$(E_\infty) \qquad u_t = \sum_{i,j=1}^{N} \bar{a}_{ij}\partial_i\partial_j u + \bar{f}(u) + \bar{g}(x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N.$$

If f is dissipative, we prove existence and upper-semicontinuity of attractors for the family (E_ω) as $\omega \to +\infty$.

1. Introduction

In this paper we study a family of non-autonomous reaction-diffusion equations

$$(1.1) \qquad u_t = \sum_{i,j=1}^{N} a_{ij}(\omega t)\partial_i\partial_j u + f(\omega t, u) + g(\omega t, x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N$$

with almost periodic, rapidly oscillating principal part and nonlinear interactions. Under suitable hypothesis (see Section 2), the Cauchy problem for (1.1) is well-posed in $H^1(\mathbb{R}^N)$ and the equation generates a (global) process, that is, a two-parameter family of nonlinear operators $\Pi_\omega(t,s)$ from $H^1(\mathbb{R}^N)$ into itself such that

$$\begin{cases} \Pi_\omega(t,p)\Pi_\omega(p,s) = \Pi_\omega(t,s) & t \geq p \geq s \\ \Pi_\omega(t,t) = I & t \in \mathbb{R}, \end{cases}$$

where, for every $u_s \in H^1(\mathbb{R}^N)$, $\Pi_\omega(t,s)u_s$ is the solution of (1.1) with $u(s) = u_s$.

We are interested in the behaviour of the solutions of (1.1) as $\omega \to +\infty$. It is a well known fact that, given a Banach space $\mathcal{M}$, if a function $\sigma : \mathbb{R} \to \mathcal{M}$ is almost periodic, the mean value

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \sigma(p)\, dp =: \bar{\sigma}$$

exists. We observe that, for fixed $T > 0$,

$$\lim_{\omega\to+\infty} \int_{-T}^{T} (\sigma(\omega p) - \bar{\sigma})\, dp = 2T \lim_{\omega\to+\infty} \frac{1}{2\omega T} \int_{-\omega T}^{\omega T} (\sigma(p) - \bar{\sigma})\, dp = 0.$$

Even if this convergence is very weak, it suggests that the averaged equation

$$u_t = \sum_{i,j=1}^{N} \bar{a}_{ij} \partial_i \partial_j u + \bar{f}(u) + \bar{g}(x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N \tag{1.2}$$

should behave like a limit equation for (1.1) as $\omega \to +\infty$.

Results of this kind have been known for quite a long time for ordinary differential equations with almost periodic coefficients. For partial differential equations, local results in this direction have been obtained in an abstract setting (fit also for the study of functional equations) by Hale and Verduyn Lunel [8]. They consider an abstract semilinear parabolic equation

$$u_t = Lu + f(\omega p, u) + g(\omega p) \tag{1.3}$$

in a Banach space E, where L is the generator of a strongly continuous semigroup of linear operators and $f(\cdot, u)$ and $g(\cdot)$ are almost periodic. They show the convergence of local solutions of (1.3) to solutions of the averaged equation

$$u_t = Lu + \bar{f}(u) + \bar{g}; \tag{1.4}$$

moreover, they prove a continuation principle for strongly hyperbolic equilibria of (1.4) and obtain an upper-semicontinuity result for local attractors of the Poincaré map of (1.3).

In a recent paper ([11]), Ilyin proposes a *global* criterion for comparison between the process generated by (1.3) and the semigroup generated by the averaged problem (1.4). For autonomous equations like (1.4), it is well known that if $\bar{f}$ is dissipative and compact, then the semiflow generated by (1.4) possesses a compact global attractor in E. In this case, it is possible to express the concept of closeness of two semiflows in terms of the Hausdorff distance of their attractors. As Ilyin shows in [11], the same can be done in the non-autonomous case. Ilyin considers an abstract semilinear parabolic equation like (1.3), where now L is a sectorial linear operator, and the corresponding averaged equation (1.4). Using a notion of global attractor for families of processes introduced by Chepyzhov and Vishik in [5] (see Section 3), he shows that, under suitable dissipativeness and compactness hypotheses, the global attractor $\mathcal{A}_\omega$ of (1.3) converges in the Hausdorff metric to the global attractor $\mathcal{A}$ of (1.4). Then he applies the abstract results to reaction-diffusion, Navier-Stokes and damped wave equations on a bounded domain Ω.

The aim of our paper is to extend the results of [11] to reaction-diffusion equations on the whole $\mathbb{R}^N$ with time dependent principal part, like (1.1). To this end, we cannot apply directly the abstract results of [11]. Indeed, since we are working on the whole $\mathbb{R}^N$, the imbedding of H^1 into L^2 is not compact; this makes much more difficult to recover the asymptotic compactness of the processes generated by (1.1). Even in the autonomous case, establishing the existence of compact global attractors becomes then itself an interesting task. In [3] Babin and Vishik overcame the difficulties arising from the lack of compactness by introducing weighted Sobolev spaces. The choice of weighted spaces, however, imposes some severe conditions on the forcing term g and on the initial data. Very recently, Wang ([18]) established the asymptotic L^2-compactness of the semiflows and consequently the existence of global $(L^2 - L^2)$ attractors for reaction-diffusion equations on $\mathbb{R}^N$ (or, more generally, on unbounded subdomains of $\mathbb{R}^N$) avoiding the use of weighted spaces. Following Wang's pattern, we shall prove uniform asymptotic L^2-compactness of the processes generated by (1.1). Then we shall obtain the asymptotic H^1-compactness by a continuity argument similar to that of [1] and [15].

On the other hand, since we assume that the principal part is time-dependent, in the variation of constant formula the linear semigroup e^{-Lt} has to be replaced by the linear processes $V_\omega(t,s)$ generated by the linear equations

$$u_t = \sum_{i,j=1}^{N} a_{ij}(\omega t)\partial_i\partial_j u. \tag{1.5}$$

As a consequence, we have to prove also the convergence of $V_\omega(t,s)$ to $e^{-\bar{A}t}$ as $\omega \to +\infty$, where $e^{-\bar{A}t}$ denotes the linear semigroup generated by the averaged linear equation. This is done by mean of an explicit representation of the solutions of the linear equations in terms of their Fourier transforms.

The paper is organized as follows. In Section 2 we introduce notations and some necessary preliminaries; moreover, we obtain some a-priori estimates for equation (1.1) and we deduce the existence of uniformly absorbing sets for the corresponding process. In Section 3, we recall some basic properties of almost periodic functions and the notion of uniform attractor for a family of processes introduced by Chepyzhov and Vishik in [5]; then we prove the existence of compact global uniform attractors for the families of processes associated to (1.1). In Section 4 we investigate the behaviour of the solutions of (1.1) as $\omega \to +\infty$, proving that the solutions of (1.1) with initial datum $u_0 \in H^1(\mathbb{R}^N)$ converge, as $\omega \to +\infty$, to the solution of (1.2) with the same initial datum. Finally, we prove the upper-semicontinuity of the family of the uniform attractors of (1.1) as $\omega \to +\infty$, showing that the uniform attractor of (1.1) is H^1-close to that of (1.2) for sufficiently large ω.

We would like to remark that the same results hold for a family of reaction-diffusion equations of the form

$$u_t = \nu(\omega t) \sum_{i,j=1}^{N} \partial_i (a_{ij}(x) \partial_j u) + f(\omega t, u) + g(\omega t, x), \quad (t, x) \in \mathbb{R}^+ \times \Omega,$$

with Dirichlet or Neumann boundary conditions on a bounded domain $\Omega \subset R^N$. To this end, it suffices to replace the Fourier transform representation of the linear processes with their spectral representation on a basis of eigenfunctions of the linear operator $\sum_{i,j=1}^{N} \partial_i (a_{ij}(x) \partial_j)$ with the given boundary conditions.

2. Preliminaries

We consider the equation

$$u_t = \sum_{i,j=1}^{N} a_{ij}(\omega t) \partial_i \partial_j u - a_0(\omega t) u + f(\omega t, u) + g(\omega t, x), \quad x \in \mathbb{R}^N, \tag{2.1}$$

where ω is a positive constant.

We make the following assumptions: the functions a_{ij} and a_0 are Hölder continuous on $\mathbb{R}$ with exponent θ, $a_{ij}(\tau) = a_{ji}(\tau)$ for $i, j = 1, \ldots, N$ and for all $\tau \in \mathbb{R}$, and there exist positive constants $\nu_1 \geq \nu_0 > 0$ and $C > 0$ such that

$$\nu_0 |\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(\tau) \xi_i \xi_j \leq \nu_1 |\xi|^2 \quad \text{for all } \tau \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^N \tag{2.2}$$

and

$$|a_0(\tau)| \leq C \quad \text{for all } \tau \in \mathbb{R}.$$

Moreover,

$$\|g(\tau, \cdot)\|_{L^2} \leq C \quad \text{for all } \tau \in \mathbb{R} \tag{2.3}$$

and there exist $g_0 \in L^2(\mathbb{R}^N)$ and $0 < \theta \leq 1$ such that

$$|g(\tau_1, x) - g(\tau_2, x)| \leq g_0(x) |\tau_1 - \tau_2|^\theta \quad \text{for all } \tau_1, \tau_2 \in \mathbb{R} \text{ and for a.e. } x \in \mathbb{R}^N. \tag{2.4}$$

Finally,

$$f(\tau, 0) = 0, \quad |f_u(\tau, u)| \leq C(1 + |u|^\beta) \quad \text{for all } u, \tau \in \mathbb{R} \tag{2.5}$$

and

$$|f(\tau_1, u) - f(\tau_2, u)| \leq C(|u| + |u|^{\beta+1}) |\tau_1 - \tau_2|^\theta \quad \text{for all } u, \tau_1, \tau_2 \in \mathbb{R}, \tag{2.6}$$

where

$$0 \le \beta \quad \text{if } N \le 2; \quad 0 \le \beta \le 2^*/2 - 1 \quad \text{if } N \ge 3. \tag{2.7}$$

For $t \in \mathbb{R}$ and $\omega > 0$ we define the operator $A_\omega(t)\colon H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ by

$$A_\omega(t)u := -\sum_{i,j=1}^{N} a_{ij}(\omega t)\partial_i\partial_j u, \quad u \in H^2(\mathbb{R}^N).$$

Then $A_\omega(t)$ is a self-adjoint positive operator in $L^2(\mathbb{R}^N)$ and our assumptions on the coefficients $a_{ij}(\tau)$ imply that the abstract parabolic equation

$$\dot{u} = -A_\omega(t)u$$

generates a linear process

$$U_\omega(t,s)\colon L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \quad t \ge s,$$

such that

$$\|U_\omega(t,s)u\|_{L^2} \le M\|u\|_{L^2}, \quad u \in L^2(\mathbb{R}^N), \tag{2.8}$$

$$\|U_\omega(t,s)u\|_{H^1} \le M\|u\|_{H^1}, \quad u \in H^1(\mathbb{R}^N), \tag{2.9}$$

and

$$\|U_\omega(t,s)u\|_{H^1} \le M(1 + (t-s)^{-1/2})\|u\|_{L^2}, \quad u \in L^2(\mathbb{R}^N), \tag{2.10}$$

where M is a positive constant (see e.g. [14, Ch.5], [16]).

A useful explicit representation of $U_\omega(t,s)$ can be given in terms of its Fourier transform. We denote by $\mathcal{F}v$ the Fourier-Plancherel transform of $v \in L^2(\mathbb{R}^N)$, normalized in such a way that, for $v \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$,

$$(\mathcal{F}v)(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} v(x)\, dx.$$

It is well known that $\mathcal{F}$ is an isometry of $L^2(\mathbb{R}^N)$ onto itself, and $v \in H^k(\mathbb{R}^N)$ if and only if $(1+|\xi|^2)^{k/2}(\mathcal{F}v)(\xi) \in L^2(\mathbb{R}^N)$. Moreover,

$$\|u\|_{H^1}^2 = \int_{\mathbb{R}^N} (1+|\xi|^2)(\mathcal{F}u)(\xi)^2\, d\xi. \tag{2.11}$$

Then an easy computation gives

$$(\mathcal{F}(U_\omega(t,s)u))(\xi) = \exp\left\{ -\int_s^t \Big(\sum_{i,j=1}^{N} a_{ij}(\omega p)\xi_i\xi_j\Big)\, dp \right\} (\mathcal{F}u)(\xi). \tag{2.12}$$

An immediate consequence of (2.12) is that the constant M in (2.8) – (2.10) depends only on ν_0, so in particular is independent of ω.

As for the nonlinear term, conditions (2.5) and (2.6) and the Sobolev embedding Theorem imply that the Nemitski operator

$$\hat{f}: \mathbb{R} \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$$

is well defined and satisfies

$$\|\hat{f}(\tau, u)\|_{L^2} \leq \tilde{C}(\|u\|_{L^2} + \|u\|_{H^1}^{\beta+1}), \quad \tau \in \mathbb{R}, u \in H^1(\mathbb{R}^N) \tag{2.13}$$

and

$$\begin{aligned} \|\hat{f}(\tau_1, u_1) - \hat{f}(\tau_2, u_2)\|_{L^2} &\leq \tilde{C}(1 + \|u_1\|_{H^1}^{\beta+1} + \|u_2\|_{H^1}^{\beta+1})|\tau_1 - \tau_2|^\theta \\ &+ \tilde{C}(1 + \|u_1\|_{H^1}^{\beta} + \|u_2\|_{H^1}^{\beta})\|u_1 - u_2\|_{H^1}, \quad \tau_1, \tau_2 \in \mathbb{R}, \quad u_1, u_2 \in H^1(\mathbb{R}^N), \end{aligned} \tag{2.14}$$

where $\tilde{C}$ is a positive constant depending only on C, ν_0, ν_1 and β. By classical results of [6], [10] and [14], for every $s \in \mathbb{R}$ and for every $u_s \in H^1(\mathbb{R}^N)$ the semilinear Cauchy problem

$$\begin{cases} \dot{u} = -A_\omega(t)u - a_0(\omega t)u + \hat{f}(\omega t, u) + g(\omega t) \\ u(s) = u_s \end{cases} \tag{2.15}$$

is locally well-posed and hence possesses a unique maximal classical solution $u \in C^0([s, s+T[, H^1) \cap C^1(]s, s+T[, L^2)$, T depending on s and u_s. Moreover, u satisfies the variation of constant formula

$$u(t) = U_\omega(t,s)u_s + \int_s^t U_\omega(t,p)(-a_0(\omega p)u(p) + \hat{f}(\omega p, u(p)) + g(\omega p))\, dp, \quad t \geq s.$$

The following set of dissipativeness and monotonicity conditions ensures that the solutions of (2.15) are global and bounded:

$$a_0(\tau) \geq \lambda_0 > 0 \quad \text{for all } \tau \in \mathbb{R}; \tag{2.16}$$

$$f(\tau, u)u \leq 0; \quad f_u(\tau, u) \leq L \quad \text{for all } u, \tau \in \mathbb{R}. \tag{2.17}$$

We start with the following a-priori estimates in L^2:

Lemma 2.1. *Let $u_\omega: [s, s+T[\to H^1(\mathbb{R}^N)$ be the maximal solution of the Cauchy problem (2.15). If $\|u_s\|_{L^2} \leq R$, then, for $t \in [s, s+T[$,*

$$\|u_\omega(t)\|_{L^2}^2 \leq e^{-\lambda_0(t-s)}R^2 + \frac{C^2}{\lambda_0^2}$$

Proof. For $t \in]s, s+T[$, we have

$$\frac{d}{dt}\frac{1}{2}\|u_\omega(t)\|_{L^2}^2 = \langle u_\omega(t), \dot{u}_\omega(t)\rangle$$
$$= \langle u_\omega(t), -A_\omega(t)u_\omega(t) - a_0(\omega t)u_\omega(t) + \hat{f}(\omega t, u_\omega(t)) + g(\omega t)\rangle$$
$$\leq -\langle u_\omega(t), A_\omega(t)u_\omega(t)\rangle - \lambda_0\langle u_\omega(t), u_\omega(t)\rangle + \langle u_\omega(t), \hat{f}(\omega t, u_\omega(t))\rangle + \langle u_\omega(t), g(\omega t)\rangle.$$

By (2.17) and by Young's inequality, we get

$$\frac{d}{dt}\frac{1}{2}\|u_\omega(t)\|_{L^2}^2 + \langle u_\omega(t), A_\omega(t)u_\omega(t)\rangle + \lambda_0\|u_\omega(t)\|_{L^2}^2$$
$$\leq \langle u_\omega(t), g(\omega t)\rangle \leq \frac{\lambda_0}{2}\|u_\omega(t)\|_{L^2}^2 + \frac{1}{2\lambda_0}\|g(\omega t)\|_{L^2}^2.$$

By (2.3) it follows that

$$\frac{d}{dt}\|u_\omega(t)\|_{L^2}^2 + \lambda_0\|u_\omega(t)\|_{L^2}^2 \leq \frac{C^2}{\lambda_0}.$$

Multiplication by $e^{\lambda_0 t}$ and integration yields

$$\|u_\omega(t)\|_{L^2}^2 \leq e^{-\lambda_0(t-s)}\|u_\omega(s)\|_{L^2}^2 + \frac{C^2}{\lambda_0^2}, \tag{2.18}$$

and the conclusion follows. □

In order to get H^1-estimates, we need the following lemmas:

Lemma 2.2. *Let $u \in H^2(\mathbb{R}^N)$. Then $\langle \hat{f}(\omega t, u), -\Delta u\rangle \leq L\langle u, -\Delta u\rangle$ for all $t \in \mathbb{R}$, where L is the constant of condition (2.17).*

Proof. For $n \in \mathbb{N}$, choose a function $h_n \in C^\infty(\mathbb{R})$, with $0 \leq h_n'(u) \leq 1$ for all $u \in \mathbb{R}$, such that

$$h_n(u) = \begin{cases} u & \text{if } -n \leq u \leq n \\ n+1 & \text{if } 2n \leq u \\ -(n+1) & \text{if } u \leq -2n \end{cases}$$

Let us fix $t \in \mathbb{R}$ and define $f_n(\omega t, u) := f(\omega t, h_n(u))$. By (2.17), it follows that $f_n(\omega t, 0) = 0$, $|(f_n)_u(\omega t, u)|$ is bounded on $\mathbb{R}$ and $(f_n)_u(\omega t, u) \leq L$ for all $u \in \mathbb{R}$. By Proposition IX.5 in [4], it follows that $f_n(\omega t, u(\cdot)) \in H^1(\mathbb{R}^N)$ and $\nabla f_n(\omega t, u(\cdot)) = (f_n)_u(\omega t, u(\cdot)) \cdot \nabla u$. Then, for all $n \in \mathbb{N}$, we have

$$\langle \hat{f}_n(\omega t, u), -\Delta u\rangle = \int_{\mathbb{R}^N} (f_n)_u(\omega t, u(x))|\nabla u(x)|^2\,dx$$
$$\leq L\int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx = L\langle u, -\Delta u\rangle$$

The proof will be complete if we show that $f_n(\omega t, u(\cdot)) \to f(\omega t, u(\cdot))$ in $L^2(\Omega)$ as $n \to \infty$. Now, since $f_n(\omega t, u(x)) \to f(\omega t, u(x))$ almost everywhere in $\mathbb{R}^N$ and the estimates

$$|f_n(\omega t, u(x))| \le C(|u(x)| + |u(x)|^{\beta+1}),$$
$$|f(\omega t, u(x))| \le C(|u(x)| + |u(x)|^{\beta+1})$$

hold, the conclusion follows from the Lebesgue dominated convergence theorem. $\square$

Lemma 2.3. *For all $u \in H^2(\mathbb{R}^N)$ and for all $t \in \mathbb{R}$*

$$-\langle A_\omega(t)u, \Delta u\rangle \ge \nu_0 \|\Delta u\|^2_{L^2}.$$

Proof. Again denoting by $\mathcal{F}v$ the Fourier-Plancherel transform of $v \in L^2(\mathbb{R}^N)$, we have

$$\begin{aligned}
-\langle A_\omega(t)u, \Delta u\rangle &= \langle \mathcal{F}(A_\omega(t)u), \mathcal{F}(-\Delta u)\rangle \\
&= \int_{\mathbb{R}^N} (\sum_{i,j=1}^N a_{ij}(\omega t)\xi_i\xi_j)(\mathcal{F}u)(\xi)(\sum_{l=1}^N \xi_l^2)(\mathcal{F}u)(\xi)\, d\xi \\
&\ge \nu_0 \int_{\mathbb{R}^N} \left(\sum_{l=1}^N \xi_l^2 (\mathcal{F}u)(\xi)\right)^2 d\xi = \nu_0 \|\Delta u\|^2_{L^2}.
\end{aligned}$$

$\square$

Now we are able to prove

Lemma 2.4. *Let $u_\omega \colon [s, s+T[\to H^1(\mathbb{R}^N)$ be the maximal solution of the Cauchy problem (2.15). There exist two positive constants K_1 and K_2, depending only on C, ν_0, ν_1, λ_0 and L, such that, if $\|u_s\|_{H^1} \le R$, then, for $t \in [s, s+T[$,*

$$\|u_\omega(t)\|^2_{H^1} \le K_1 R^2 e^{-\lambda_0(t-s)} + K_2$$

Proof. For $t \in]s, s+T[$, by Lemma 2.3 and by (2.16) we have

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\|\nabla u_\omega(t)\|^2_{L^2} = -\langle \Delta u_\omega(t), \dot u_\omega(t)\rangle \\
&= \langle -\Delta u_\omega(t), -A_\omega(t)u_\omega(t) - a_0(\omega t)u_\omega(t) + \hat f(\omega t, u_\omega(t)) + g(\omega t)\rangle \\
&\le -\nu_0\|\Delta u_\omega(t)\|^2_{L^2} - \lambda_0\|\nabla u_\omega(t)\|^2_{L^2} - \langle \Delta u_\omega(t), \hat f(\omega t, u_\omega(t))\rangle - \langle \Delta u_\omega(t), g(\omega t)\rangle.
\end{aligned}$$

By (2.3), by Lemma 2.2 and by Young's inequality we obtain

$$\frac{d}{dt}\|\nabla u_\omega(t)\|^2_{L^2} \le -\nu_0\|\Delta u_\omega(t)\|^2_{L^2} - 2(\lambda_0 - L)\|\nabla u_\omega(t)\|^2_{L^2} + \frac{C^2}{\nu_0}. \tag{2.19}$$

Let $\delta > 0$ and let $v \in H^2(\mathbb{R}^N)$. We have

$$\|\nabla v\|_{L^2}^2 \leq \frac{\delta}{2}\|\Delta v\|_{L^2}^2 + \frac{1}{2\delta}\|v\|_{L^2}^2,$$

whence

$$-\|\Delta v\|_{L^2}^2 \leq -\frac{2}{\delta}\|\nabla v\|_{L^2}^2 + \frac{1}{\delta^2}\|v\|_{L^2}^2. \tag{2.20}$$

By (2.19) and (2.20), choosing $\delta := \nu_0/L$, we obtain

$$\frac{d}{dt}\|\nabla u_\omega(t)\|_{L^2}^2 \leq -2\lambda_0\|\nabla u_\omega(t)\|_{L^2}^2 + \frac{L^2}{\nu_0}\|u_\omega(t)\|_{L^2}^2 + \frac{C^2}{\nu_0}.$$

By (2.18),

$$\frac{d}{dt}\|\nabla u_\omega(t)\|_{L^2}^2 + 2\lambda_0\|\nabla u_\omega(t)\|_{L^2}^2 \leq \frac{L^2}{\nu_0}\|u_\omega(s)\|_{L^2}^2 e^{-\lambda_0(t-s)} + \left(\frac{L^2}{\nu_0\lambda_0^2} + \frac{1}{\nu_0}\right)C^2.$$

Multiplication by $e^{2\lambda_0 t}$ and integration yields

$$\begin{aligned}&\|\nabla u_\omega(t)\|_{L^2}^2\\ &\quad\leq e^{-2\lambda_0(t-s)}\|\nabla u_\omega(s)\|_{L^2}^2 + \frac{L^2}{\nu_0\lambda_0}\|u_\omega(s)\|_{L^2}^2 e^{-\lambda_0(t-s)} + \left(\frac{L^2}{2\nu_0\lambda_0^3} + \frac{1}{\lambda_0\nu_0}\right)C^2,\end{aligned}$$

and the conclusion follows. $\square$

As a consequence, we have the following result:

Proposition 2.5. *Let $u_\omega\colon [s, s+T[\to H^1(\mathbb{R}^N)$ be the maximal solution of the Cauchy problem (2.15). Then*

1. *$T = +\infty$;*
2. *if $\|u_s\|_{H^1} \leq R$, then, for every $t \geq s$, $\|u_\omega(t)\|_{H^1}^2 \leq K_1 R^2 + K_2$, where K_1 and K_2 are independent of R and ω;*
3. *there exists a positive constant K, and for every $R > 0$ there exists $T(R) > 0$ such that, whenever $\|\bar{u}_s\|_{H^1} \leq R$, $\|u_\omega(t)\|_{H^1} < K$ for all t such that $(t-s) \geq T(R)$. Both K and $T(R)$, besides R, depend only on C, ν_0, ν_1, λ_0 and L. In particular, they are independent of s and ω.*

$\square$

Proposition 2.5 says also that the global process generated by (2.15) possesses a bounded absorbing set in $H^1(\mathbb{R}^N)$ independent of ω.

We end this section with a result which will be useful in proving the asymptotic compactness of the processes generated by (2.15).

Lemma 2.6. *Let $u_\omega\colon [s,+\infty[\to H^1(\mathbb{R}^N)$ be the solution of the Cauchy problem (2.15), with $\|u_s\|_{H^1}\le R$. Assume moreover that the set $\{\, g(\tau,\cdot)\mid \tau\in\mathbb{R}\}$ is compact in $L^2(\mathbb{R}^N)$. Then, for every $\eta>0$, there exist two positive constants $\bar k(R)$ and $\bar T(R)$ such that, if $(t-s)\ge \bar T(R)$ and $k\ge \bar k(R)$,*

$$\int_{\{|x|>k\}} |u_\omega(t,x)|^2\,dx\le \eta.$$

The constants $\bar k(R)$ and $\bar T(R)$, besides R and η, depend only on C, ν_0, ν_1, λ_0 and L. In particular, they are independent of s and ω.

Proof. We adapt to the non-autonomous case the proof of Lemma 5 in [18], being careful that all the estimates involved are independent of ω.

Let $\theta\colon \mathbb{R}_+\to\mathbb{R}$ be a smooth function such that $0\le\theta(s)\le 1$ for $s\in\mathbb{R}_+$, $\theta(s)=0$ for $0\le s\le 1$ and $\theta(s)=1$ for $s\ge 2$. Let $D:=\sup_{s\in\mathbb{R}_+}|\theta'(s)|$. For $k\in\mathbb{N}$, let us define the multiplication operator

$$\Theta_k\colon H^1(\mathbb{R}^N)\to H^1(\mathbb{R}^N),\quad (\Theta_k u)(x):=\theta(|x|^2/k^2)u(x).$$

By (2.16) and (2.17), we have

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^N}\theta(|x|^2/k^2)|u_\omega(t,x)|^2\,dx=\frac{d}{dt}\frac{1}{2}\langle\Theta_k u_\omega(t),u_\omega(t)\rangle=\langle\Theta_k u_\omega(t),\dot u_\omega(t)\rangle\\
&=\langle\Theta_k u_\omega(t),-A_\omega(t)u_\omega(t)-a_0(\omega t)u_\omega(t)+\hat f(\omega t,u_\omega(t))+g(\omega t)\rangle\\
&\le\langle\Theta_k u_\omega(t),-A_\omega(t)u_\omega(t)\rangle-\lambda_0\langle\Theta_k u_\omega(t),u_\omega(t)\rangle+\langle\Theta_k u_\omega(t),g(\omega t)\rangle
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{d}{dt}\langle\Theta_k u_\omega(t),u_\omega(t)\rangle+2\lambda_0\langle\Theta_k u_\omega(t),u_\omega(t)\rangle\\
&\qquad\qquad\le -2\langle\Theta_k u_\omega(t),A_\omega(t)u_\omega(t)\rangle+2\langle\Theta_k u_\omega(t),g(\omega t)\rangle.
\end{aligned}$$

Since

$$\begin{aligned}
\langle\Theta_k u_\omega(t),A_\omega(t)u_\omega(t)\rangle&=\int_{\mathbb{R}^N}\theta(|x|^2/k^2)\sum_{i,j=1}^N a_{ij}(\omega t)\partial_i u_\omega(t,x)\partial_j u_\omega(t,x)\,dx\\
&+\int_{\mathbb{R}^N}\theta'(|x|^2/k^2)u_\omega(t,x)\frac{2}{k^2}\sum_{i,j=1}^N a_{ij}(\omega t)x_i\partial_j u_\omega(t,x)\,dx,
\end{aligned}$$

it follows that

$$\begin{aligned}
&-\langle\Theta_k u_\omega(t),A_\omega(t)u_\omega(t)\rangle\le\int_{\mathbb{R}^N}\theta'(|x|^2/k^2)|u_\omega(t,x)||\nabla u_\omega(t,x)|\frac{2}{k^2}|x|\,dx\\
&\le 2D\int_{\{k\le|x|\le\sqrt2 k\}}\frac{|x|}{k^2}|u_\omega(t,x)||\nabla u_\omega(t,x)|\,dx\\
&\le\frac{2\sqrt2 D}{k}\int_{\{k\le|x|\le\sqrt2 k\}}|u_\omega(t,x)||\nabla u_\omega(t,x)|\,dx\\
&\le\frac{2\sqrt2 D}{k}\|u_\omega(t)\|_{L^2}\|\nabla u_\omega(t)\|_{L^2}.
\end{aligned}$$

So, by Proposition 2.5, for $(t-s) \geq T(R)$, we have

$$-\langle \Theta_k u_\omega(t), A_\omega(t) u_\omega(t) \rangle \leq \frac{2\sqrt{2}DK^2}{k}$$

Let $\eta > 0$ and choose $k = k(\eta)$ such that

$$\frac{2\sqrt{2}DK^2}{k} < \eta.$$

Then for $(t-s) > T(R)$ and $k > k(\eta)$, we obtain

$$\frac{d}{dt}\langle \Theta_k u_\omega(t), u_\omega(t) \rangle + 2\lambda_0 \langle \Theta_k u_\omega(t), u_\omega(t) \rangle \leq 2\eta + 2\langle \Theta_k u_\omega(t), g(\omega t) \rangle.$$

By Young's inequality, we have

$$\langle \Theta_k u_\omega(t), g(\omega t) \rangle \leq \frac{\lambda_0}{2}\langle \Theta_k u_\omega(t), u_\omega(t) \rangle + \frac{1}{2\lambda_0}\int_{\mathbb{R}^N} \theta(|x|^2/k^2) g(\omega t, x)^2 \, dx.$$

Since we have assumed that $\{ g(\tau, \cdot) \mid \tau \in \mathbb{R} \}$ is compact in $L^2(\mathbb{R}^N)$, there exists $k' = k'(\eta)$ such that, if $k > k'(\eta)$,

$$\frac{1}{2\lambda_0}\int_{\mathbb{R}^N} \theta(|x|^2/k^2) g(\omega t, x)^2 \, dx \leq \eta \quad \text{for all } t \in \mathbb{R};$$

As a consequence, for $(t-s) > T(R)$ and for $k > \max\{k(\eta), k'(\eta)\}$

$$\frac{d}{dt}\langle \Theta_k u_\omega(t), u_\omega(t) \rangle + \lambda_0 \langle \Theta_k u_\omega(t), u_\omega(t) \rangle \leq 4\eta.$$

Multiplication by $e^{\lambda_0 t}$ and integration yields

$$e^{\lambda_0 t}\langle \Theta_k u_\omega(t), u_\omega(t) \rangle - e^{\lambda_0(s+T(R))}\langle \Theta_k u_\omega(s+T(R)), u_\omega(s+T(R)) \rangle \leq \frac{4\eta}{\lambda_0} e^{\lambda_0 t}$$

for $(t-s) > T(R)$. It follows that, for $(t-s) > T(R)$,

$$\begin{aligned}\langle \Theta_k u_\omega(t), u_\omega(t) \rangle &\leq e^{-\lambda_0((t-s)-T(R))}\langle \Theta_k u_\omega(s+T(R)), u_\omega(s+T(R)) \rangle + \frac{4\eta}{\lambda_0} \\ &\leq e^{-\lambda_0((t-s)-T(R))} K^2 + \frac{4\eta}{\lambda_0}.\end{aligned}$$

Finally, for $(t-s) \geq T(R) + \lambda_0^{-1}\log(\eta^{-1})$ and for $k > \max\{k(\eta), k'(\eta)\}$, we get

$$\int_{\{|x| > \sqrt{2}k\}} |u_\omega(t,x)|^2 \, dx \leq \int_{\mathbb{R}^N} \theta(|x|^2/k^2) |u_\omega(t,x)|^2 \, dx \leq \left(K^2 + \frac{4}{\lambda_0}\right)\eta,$$

and the proof is complete. □

3. Existence of the compact global attractors

It is well known (see e.g. [12], [7], [2] and [17]) that if a continuous semigroup $P(t)$, acting on a complete connected metric space X, is bounded, pointwise-dissipative and asymptotically compact, then it possesses a compact global attractor. The attractor is non-empty, connected, strictly invariant, and can be characterized as the union of all complete bounded trajectories of $P(t)$. This idea can be quite naturally extended to the class of processes generated by periodically time-dependent partial differential equations, since such systems undergo a discrete semigroup structure given by the period map. In more general situations, like the almost periodic case considered here, the leading property of invariance fails and new approaches had to be developed.

In [9], Haraux proposed a notion of attractor for a process $\Pi(t,s)$ based on the concept of minimality rather than invariance. However, as it was suggested by the same Haraux, the theory of skew-product flows, at the expense of introducing an extended phase space, provides the right extension of invariance. This alternative approach, developed by Chepyzhov and Vishik in [5], turns out to be particularly well suited if the process is generated by an almost periodic partial differential equation.

We shall describe this approach in the context of equation (2.1).

We define $\mathcal{M}_1$ as the space of $N\times N$ real symmetric matrices and $\mathcal{M}_2 := \mathbb{R}$; moreover, we denote by $\mathcal{M}_3$ the set

$$\mathcal{M}_3 := \{\, \Psi\colon \mathbb{R}\to\mathbb{R} \mid \Psi(0)=0,\ \|\Psi\|_{\mathcal{M}_3} < +\infty \,\},$$

where

$$\|\Psi\|_{\mathcal{M}_3} := \sup_{u\in\mathbb{R}} \frac{|\Psi_u(u)|}{1+|u|^\beta}.$$

Finally, we set $\mathcal{M}_4 := L^2(\mathbb{R}^N)$.

Besides conditions (2.2)–(2.7) and (2.16)–(2.17), from now on we assume that also the following condition is satisfied:

(AP) the functions $t\mapsto (a_{ij}(t))_{ij}\in\mathcal{M}_1$; $t\mapsto a_0(t)\in\mathcal{M}_2$; $t\mapsto f(t,\cdot)\in\mathcal{M}_3$ and $t\mapsto g(t,\cdot)\in\mathcal{M}_4$ are almost periodic.

By Bochner's criterion (see e.g. [13]), whenever $\sigma\colon\mathbb{R}\to\mathcal{M}$ is almost periodic, the set of all translations $\{\,\sigma(\cdot+h)\mid h\in\mathbb{R}\,\}$ is precompact in $C_b(\mathbb{R},\mathcal{M})$. The closure of this set in $C_b(\mathbb{R},\mathcal{M})$ is called the hull of σ and is usually denoted by $\mathcal{H}(\sigma)$; if $\zeta\in\mathcal{H}(\sigma)$, then ζ is almost periodic and $\mathcal{H}(\zeta)=\mathcal{H}(\sigma)$.

For an almost periodic function σ, the mean value

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\sigma(t)\,dt =: \bar\sigma\in\mathcal{M}$$

exists. More remarkably, (see again [13]) there exists a bounded decreasing function $\mu\colon\mathbb{R}_+\to\mathbb{R}_+$, $\mu(T)\to 0$ as $T\to\infty$, such that

$$\|(1/T)\int_s^{s+T}(\zeta(t)-\bar\sigma)\,dt\|_{\mathcal{M}} \le \mu(T) \quad \text{for all } s\in\mathbb{R} \text{ and all } \zeta\in\mathcal{H}(\sigma). \tag{3.1}$$

We will denote by Σ_1, Σ_2, Σ_3 and Σ_4 the hulls of the functions $t \mapsto (a_{ij}(t))_{ij}$, $t \mapsto a_0(t)$, $t \mapsto f(t,\cdot)$ and $t \mapsto g(t,\cdot)$ in $C_b(\mathbb{R},\mathcal{M}_1)$, $C_b(\mathbb{R},\mathcal{M}_2)$, $C_b(\mathbb{R},\mathcal{M}_3)$ and $C_b(\mathbb{R},\mathcal{M}_4)$ respectively. The corresponding mean values will be denoted by $(\bar{a}_{ij}) \in \mathcal{M}_1$, $\bar{a}_0 \in \mathcal{M}_2$, $\bar{f}(\cdot) \in \mathcal{M}_3$ and $\bar{g}(\cdot) \in \mathcal{M}_4$. Moreover, let us set $\Sigma := \Sigma_1 \times \Sigma_2 \times \Sigma_3 \times \Sigma_4$.

Remark. It is an easy exercise to check that properties (2.2)–(2.7) and (2.16)–(2.17) are satisfied by any element of Σ_1, Σ_2, Σ_3 and Σ_4, as well as by the corresponding mean values (with the same constants!). Hence, the results of Lemmas 2.1 and 2.4, of Proposition 2.5 and of Lemma 2.6 still hold true if we replace $a_{ij}(\tau)$, $a_0(\tau)$, $f(\tau,u)$ and $g(\tau,x)$ with arbitrary elements of Σ_1, Σ_2, Σ_3 and Σ_4 or with the corresponding mean values.

Remark. Applying (3.1) to $f(\tau,u)$, we have

$$|(1/T)\int_s^{s+T}(f(\tau,u)-\bar{f}(u))\,d\tau| \le \mu(T)(|u|+|u|^{\beta+1}) \quad \text{for all } s \text{ and } u \in \mathbb{R},$$

and integration yields

$$\|(1/T)\int_s^{s+T}(\hat{f}(\tau,u)-\hat{\bar{f}}(u))\,d\tau\|_{L^2} \le K\mu(T)(\|u\|_{L^2}+\|u\|_{H^1}^{\beta+1}). \tag{3.2}$$

As a consequence of Lemmas 2.1 and 2.4 and of Proposition 2.5, for any $\sigma = ((\alpha_{ij}),\alpha_0,\phi,\gamma) \in \Sigma$ and for any $\omega > 0$, the equation

$$u_t = \sum_{i,j=1}^{N} \alpha_{ij}(\omega t)\partial_i\partial_j u - \alpha_0(\omega t)u + \phi(\omega t,u) + \gamma(\omega t,x), \quad x \in \mathbb{R}^N$$

generates a global process $\Pi_\omega^\sigma(t,s)$ in the space $H^1(\mathbb{R}^N)$.

According to [5], now we are able to give the following

Definition 3.1(Chepyzhov and Vishik, '94). *A closed set $\mathcal{A}_\omega^\Sigma$ is said to be the Σ-uniform attractor of the family of processes $\{\,\Pi_\omega^\sigma \mid \sigma \in \Sigma\,\}$ iff*

(1) *For every bounded set $B \subset H^1(\mathbb{R}^N)$*

$$\lim_{t\to\infty}\sup_{\sigma\in\Sigma}\operatorname{dist}(\Pi_\omega^\sigma(t,s)B,\mathcal{A}_\omega^\Sigma) = 0 \quad \textit{for all } s \in \mathbb{R};$$

(2) *$\mathcal{A}_\omega^\Sigma$ is minimal among all closed subsets of $H^1(\mathbb{R}^N)$ satisfying property (1), i.e. $\mathcal{A}_\omega^\Sigma \subset \mathcal{A}'$ for every closed set $\mathcal{A}' \subset H^1(\mathbb{R}^N)$ which satisfies property (1).*

Our first goal is to prove that the almost periodic dissipative equation (2.1) possesses a Σ-uniform attractor in $H^1(\mathbb{R}^N)$. Following [5], we introduce the extended phase-space $\Sigma\times H^1(\mathbb{R}^N)$; for $\omega > 0$, we define on Σ the unitary group of translations

$$(T_\omega(h)\sigma)(\cdot) := \sigma(\cdot + \omega h).$$

One can easily prove the following translation identity:

$$\Pi_\omega^\sigma(t+h,s+h)=\Pi_\omega^{T_\omega(h)\sigma}(t,s),\quad h\in\mathbb{R}. \tag{3.3}$$

Thanks to (3.3), we can associate to the family of processes $\{\,\Pi_\omega^\sigma \mid \sigma\in\Sigma\,\}$ a (nonlinear) semigroup $P_\omega(t)$ acting on the extended phase-space $\Sigma\times H^1(\mathbb{R}^N)$, by the formula

$$P_\omega(t)(\sigma,u):=(T_\omega(t)\sigma,\Pi_\omega^\sigma(t,0)u).$$

In [5], Chepyzhov and Vishik proved that, if the semigroup $P_\omega(t)(\sigma,u)$ above is continuous, bounded, pointwise-dissipative and asymptotically compact (and hence possesses a compact global attractor $\mathcal{M}_\omega^\Sigma$), then the projection of $\mathcal{M}_\omega^\Sigma$ onto $H^1(\mathbb{R}^N)$ is the global Σ-uniform attractor of the family of processes $\{\,\Pi_\omega^\sigma\mid\sigma\in\Sigma\,\}$.

Let us describe in some detail the results of [5].

Definition 3.2. *A curve $t\to u(t)\in H^1(\mathbb{R}^N)$, $t\in\mathbb{R}$ is said to be a full solution of the process $\Pi_\omega^\sigma(t,s)$ iff*

$$\Pi_\omega^\sigma(t,s)u(s)=u(t)\quad \textit{for all } t\geq s,\ s\in\mathbb{R}.$$

Definition 3.3. *The kernel of the process $\Pi_\omega^\sigma(t,s)$ is by definition the set $\mathcal{K}_\omega^\sigma$ of all full bounded solutions of the process $\Pi_\omega^\sigma(t,s)$. We call the set*

$$\mathcal{K}_\omega^\sigma(s):=\{\,u(s)\mid u(\cdot)\in\mathcal{K}_\omega^\sigma\,\}\subset H^1(\mathbb{R}^N)$$

the kernel section at time s.

We introduce also the two projectors J_1 and J_2 from $\Sigma\times H^1(\mathbb{R}^N)$ onto Σ and $H^1(\mathbb{R}^N)$ respectively: $J_1(\sigma,u):=\sigma$, $J_2(\sigma,u):=u$. Then we have

Theorem 3.4 (Chepyzhov and Vishik, '94). *Assume that the semigroup $P_\omega(t)$ is continuous, bounded, pointwise-dissipative and asymptotically compact, so it possesses a compact global attractor $\mathcal{M}_\omega^\Sigma$. Then*

1. *$J_2\mathcal{M}_\omega^\Sigma=:\mathcal{A}_\omega^\Sigma$ is the global Σ-uniform attractor of the family of processes $\{\,\Pi_\omega^\sigma\mid\sigma\in\Sigma\,\}$;*
2. *$J_1\mathcal{M}_\omega^\Sigma=\Sigma$;*
3. *$\mathcal{M}_\omega^\Sigma=\bigcup_{\sigma\in\Sigma}\{\sigma\}\times\mathcal{K}_\omega^\sigma(0)$;*
4. *$\mathcal{A}_\omega^\Sigma=\bigcup_{\sigma\in\Sigma}\mathcal{K}_\omega^\sigma(0)$.*

□

As in [5], in order to apply Theorem 3.4, we need to check that $P_\omega(t)(\sigma,u)$ is continuous, bounded, pointwise-dissipative and asymptotically compact. Boundedness and pointwise-dissipativeness are a straightforward consequence of Proposition 2.5. For continuity and asymptotic compactness, we need some preliminary lemmas.

Lemma 3.5. *Let $(\alpha^1_{ij}(\cdot))$ and $(\alpha^2_{ij}(\cdot)) \in \Sigma_1$. For $k = 1, 2$, let $V^k_\omega(t,s)$ be the linear process in $L^2(\mathbb{R}^N)$ generated by the equation*

$$u_t = \sum_{i,j=1}^{N} \alpha^k_{ij}(\omega t)\partial_i\partial_j u.$$

Then

(1) *there exists a continuous function $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$, $\rho(q) \to 0$ as $q \to 0$, such that, for any $\omega > 0$, for $u \in L^2(\mathbb{R}^N)$ and for $t > s$,*

$$\|V^1_\omega(t,s)u - V^2_\omega(t,s)u\|_{H^1} \leq \left(1 + \frac{1}{(t-s)^{1/2}}\right)\rho(\|(a^1_{ij}) - (a^2_{ij})\|_\infty)\|u\|_{L^2};$$

(2) *there exists a continuous function $\chi\colon \mathbb{R}_+ \to \mathbb{R}_+$, $\chi(q) \to 0$ as $q \to 0$, such that, for any $\omega > 0$, for $u \in H^1(\mathbb{R}^N)$ and for $t \geq s$,*

$$\|V^1_\omega(t,s)u - V^2_\omega(t,s)u\|_{H^1} \leq \chi(\|(a^1_{ij}) - (a^2_{ij})\|_\infty)\|u\|_{H^1}.$$

Proof. Let $u \in L^2(\mathbb{R}^N)$. By (2.11) and (2.12), for any $R > 0$ we have

$$\begin{aligned}
&\|V^1_\omega(t,s)u - V^2_\omega(t,s)u\|^2_{H^1} \\
&= \int_{\mathbb{R}^N} (1+|\xi|^2)[\exp(-\int_s^t \sum_{i,j} \alpha^1_{ij}(\omega p)\xi_i\xi_j\,dp) \\
&\qquad\qquad - \exp(-\int_s^t \sum_{i,j} \alpha^2_{ij}(\omega p)\xi_i\xi_j\,dp)\,]^2(\mathcal{F}u)(\xi)^2\,d\xi \\
&\leq 2\int_{\{|\xi|\geq R\}} (1+|\xi|^2)\exp(-2\nu_0|\xi|^2(t-s))(\mathcal{F}u)(\xi)^2\,d\xi \\
&+ \int_{\{|\xi|\leq R\}} (1+|\xi|^2)\exp(-2\nu_0|\xi|^2(t-s)) \\
&\qquad\qquad [\,\exp(-\sum_{i,j}\int_s^t (\alpha^1_{ij}(\omega p) - \alpha^2_{ij}(\omega p))\,dp\,\xi_i\xi_j) - 1\,]^2(\mathcal{F}u)(\xi)^2\,d\xi \\
&=: S_1 + S_2.
\end{aligned}$$

Choose $R := k^{1/2}(t-s)^{-1/2}$, k to be determined. If $k \geq 1/(2\nu_0)$, we have

$$S_1 \leq 2\sup_{z\geq 2\nu_0 k}\left(1 + \frac{z}{2\nu_0(t-s)}\right)e^{-z}\|u\|^2_{L^2} \leq 2\left(1 + \frac{1}{(t-s)}\right)ke^{-2\nu_0 k}\|u\|^2_{L^2}.$$

On the other hand,

$$S_2 \leq \left(1 + \frac{1}{(t-s)}\right)k[\,e^{\|(\alpha^1_{ij})-(\alpha^2_{ij})\|_\infty k} - 1\,]^2\|u\|^2_{L^2}.$$

Choosing
$$k = \|(\alpha_{ij}^1) - (\alpha_{ij}^2)\|_\infty^{-1/3}$$
we obtain the desired result with
$$\rho(q) = \sqrt{2} q^{-1/6} e^{-\nu_0 q^{-1/3}} + q^{-1/6} [e^{q^{2/3}} - 1], \quad q \leq 8\nu_0^3.$$
If $u \in H^1(\mathbb{R}^N)$ we argue in the same way: we get
$$\|V_\omega^1(t,s)u - V_\omega^2(t,s)u\|_{H^1}^2 \leq S_1 + S_2,$$
where
$$S_1 = 2e^{-2\nu_0 k}\|u\|_{H^1}^2$$
and
$$S_2 = [\, e^{\|(\alpha_{ij}^1)-(\alpha_{ij}^2)\|_\infty k} - 1\,]^2 \|u\|_{H^1}^2.$$
Again, choosing k as above, we obtain the desired result. □

Lemma 3.6. *Let $\sigma \in \Sigma$ and let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence in Σ, such that $\sigma_n \to \sigma$ as $n \to \infty$. Let $(t_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ be two sequences of real numbers, with $t_n \geq s_n$ for all n and assume that $t_n \to t$ and $s_n \to s$ as $n \to \infty$. Finally, let $u \in H^1(\mathbb{R}^N)$ and let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R}^N)$. Then, for any $\omega > 0$,*

(1) *if $u_n \to u$ in $L^2(\mathbb{R}^N)$ and $t > s$,*
$$\|\Pi_\omega^{\sigma_n}(t_n, s_n)u_n - \Pi_\omega^\sigma(t,s)u\|_{H^1} \to 0 \quad \text{as } n \to \infty;$$
(2) *if $u_n \to u$ in $H^1(\mathbb{R}^N)$,*
$$\|\Pi_\omega^{\sigma_n}(t_n, s_n)u_n - \Pi_\omega^\sigma(t,s)u\|_{H^1} \to 0 \quad \text{as } n \to \infty.$$

Proof. First, let us notice that
$$\Pi_\omega^{\sigma_n}(t_n, s_n)u_n = \Pi_\omega^{T_\omega(s_n - s)\sigma_n}(t_n - (s_n - s), s)u_n.$$
Since $T_\omega(s_n - s)\sigma_n \to \sigma$ in Σ and $t_n - (s_n - s) \to t$ as $n \to \infty$, we can assume without loss of generality that $s_n = s$ for all n. Let's write
$$v_n(t) := \Pi_\omega^{\sigma_n}(t,s)u_n,$$
$$v(t) := \Pi_\omega^\sigma(t,s)u.$$
We introduce the following notations:
$$\sigma_n(\tau) =: ((\alpha_{ij}^n(\tau)), \alpha_0^n(\tau), \varphi_n(\tau), \gamma_n(\tau)),$$
$$\sigma(\tau) =: ((\alpha_{ij}(\tau)), \alpha_0(\tau), \varphi(\tau), \gamma(\tau)),$$

and

$$\begin{aligned}
A_n &:= \sup_{\tau \in \mathbb{R}} |(\alpha_{ij}^n(\tau)) - (\alpha_{ij}(\tau))|, \\
B_n &:= \sup_{\tau \in \mathbb{R}} |\alpha_0^n(\tau) - \alpha_0(\tau)|, \\
D_n &:= \sup_{\tau \in \mathbb{R}} \sup_{u \in \mathbb{R}} \frac{|(\varphi_n)_u(\tau, u) - \varphi_u(\tau, u)|}{1 + |u|^\beta}, \\
E_n &:= \sup_{\tau \in \mathbb{R}} \|\gamma_n(\tau) - \gamma(\tau)\|_{L^2}.
\end{aligned}$$

Notice that A_n, B_n, D_n and E_n tend to zero as $n \to +\infty$.
Moreover, let's observe that for every $\tau \in \mathbb{R}$ we have

$$\|\hat{\varphi_n}(\tau, u) - \hat{\varphi}(\tau, u)\|_{L^2} \le D_n(\|u\|_{L^2} + \|u\|_{H^1}^{\beta+1}).$$

Finally, in view of Proposition 2.5, there exists $\tilde{K} > 0$ such that, for every $t \ge s$,

$$\|v_n(t)\|_{H^1} \le \tilde{K} \quad \text{for all } n \in \mathbb{N} \ ,$$

$$\|v(t)\|_{H^1} \le \tilde{K}.$$

Let T be a positive number; for $0 < t - s < T$ we have

$$\begin{aligned}
v_n(t) - v(t) = &V_\omega^n(t, s)u_n - V_\omega(t, s)u \\
&+ \int_s^t V_\omega^n(t, p)[-\alpha_0^n(\omega p)v_n(p) + \hat{\varphi}_n(\omega p, v_n(p)) + \gamma_n(\omega p)]\, dp \\
&- \int_s^t V_\omega(t, p)[-\alpha(\omega p)v(p) + \hat{\varphi}(\omega p, v(p)) + \gamma(\omega p)]\, dp.
\end{aligned}$$

Hence

$$\begin{aligned}
\|v_n(t) - v(t)\|_{H^1} \le &\|V_\omega^n(t, s)[u_n - u]\|_{H^1} \\
&+ \|[V_\omega^n(t, s) - V_n(t, s)]u\|_{H^1} + I_1 + I_2 + I_3,
\end{aligned}$$

where

$$I_1 := \int_s^t \|(V_\omega^n(t, p) - V_\omega(t, p))[\alpha_0^n(\omega p)v_n(p) + \hat{\varphi}_n(\omega p, v_n(p)) + \gamma_n(\omega p)]\|_{H^1}\, dp,$$

$$\begin{aligned}
I_2 := \int_s^t &\|V_\omega(t, p)[(-\alpha_0^n(\omega p) + \alpha_0(\omega p))v_n(p) \\
&+ \hat{\varphi}_n(\omega p, v_n(p)) - \hat{\varphi}(\omega p, v_n(p)) + \gamma_n(\omega p) - \gamma(\omega p)]\|_{H^1}\, dp,
\end{aligned}$$

$$I_3 := \int_s^t \|V_\omega(t, p)[-\alpha_0(\omega p)(v_n(p) - v(p)) + \hat{\varphi}(\omega p, v_n(p)) - \hat{\varphi}(\omega p, v(p))]\|_{H^1}\, dp.$$

First of all, let's observe that, thanks to Lemma 3.5,

$$\|[V^n_\omega(p) - V_\omega(p)]u\|_{H^1} \le \chi(A_n)\|u\|_{H^1} \le \chi(A_n)\tilde{K}.$$

As for I_1, Lemma 3.5 implies that

$$\begin{aligned} I_1 &\le \int_s^t (1+(t-p)^{-1/2})\rho(A_n)\|\alpha_0^n(\omega p)v_n(p) + \hat{\varphi_n}(\omega p, v_n(p)) + \gamma_n(\omega p)\|_{L^2}\,dp \\ &\le \int_s^t (1+(t-p)^{-1/2})\rho(A_n)(C\tilde{K} + \tilde{C}(\tilde{K}+\tilde{K}^{\beta+1}) + C)\,dp \le Q_1\rho(A_n), \end{aligned}$$

where Q_1 is a positive constant depending on T. Analogously, (2.10) implies that

$$\begin{aligned} I_2 \le \int_s^t & M(1+(t-p)^{-1/2})\|(-\alpha_0^n(\omega p) + \alpha_0(\omega p))v_n(p) \\ &+ \hat{\varphi}_n(\omega p, v_n(p)) - \hat{\varphi}(\omega p, v_n(p)) + \gamma_n(\omega p) - \gamma(\omega p)\|_{L^2}\,dp \\ \le \int_s^t & M(1+(t-p)^{-1/2})(B_n\tilde{K} + D_n(\tilde{K}+\tilde{K}^{\beta+1}) + E_n)\,dp \le Q_2(B_n+D_n+E_n), \end{aligned}$$

where Q_2 is a positive constant depending on T. Finally, (2.10) implies

$$\begin{aligned} I_3 \le \int_s^t & M(1+(t-p)^{-1/2})\|-\alpha_0(\omega p)(v_n(p) - v(p)) \\ &+ \hat{\varphi}(\omega p, v_n(p)) - \hat{\varphi}(\omega p, v(p))\|_{L^2}\,dp \\ \le \int_s^t & M(1+(t-p)^{-1/2})[C\|v_n(p)-v(p)\|_{H^1} + \tilde{C}(1+2\tilde{K}^\beta)\|v_n(p)-v(p)\|_{H^1}]\,dp \\ &\le Q_3\int_s^t (t-p)^{-1/2}\|v_n(p)-v(p)\|_{H^1}\,dp, \end{aligned}$$

where Q_3 is a positive constant depending on T.

As a consequence,

$$\|v_n(t)-v(t)\|_{H^1} \le \|V^n_\omega(t,s)(u_n-u)\|_{H^1} + Z_n + Q_3\int_s^t (t-p)^{-1/2}\|v_n(p)-v(p)\|_{H^1}\,dp,$$

where $Z_n \to 0$ as $n \to +\infty$.

In case (1) we have, due to (2.10),

$$\|V^n_\omega(t,s)(u_n-u)\|_{H^1} \le M(1+(t-s)^{-1/2})\|u_n-u\|_{L^2} \le \tilde{M}(t-s)^{-1/2}\|u_n-u\|_{L^2},$$

hence

$$\|v_n(t)-v(t)\|_{H^1} \le (t-s)^{-1/2}F_n + Q_3\int_s^t (t-p)^{-1/2}\|v_n(p)-v(p)\|_{H^1}\,dp,$$

where $F_n \to 0$ as $n \to +\infty$, and by the singular version of Gronwall's inequality (see [10, Th. 7.1.1])

$$\|v_n(t) - v(t)\|_{H^1} \leq QF_n(t-s)^{-1/2},$$

where Q is a positive constant. This implies $v_n(t) \to v(t)$ in H^1 uniformly on $[s+\delta, s+T]$ for every $T > \delta > 0$, and proves (1).

In case (2), (2.9) implies that

$$\|V_\omega^n(t,s)(u_n - u)\|_{H^1} \leq M\|u_n - u\|_{H^1},$$

hence

$$\|v_n(t) - v(t)\|_{H^1} \leq \tilde{F}_n + \tilde{Q}_3 \int_s^t (t-p)^{-1/2}\|v_n(p) - v(p)\|_{H^1}\, dp,$$

where $\tilde{Q}_3$ is a positive constant and $\tilde{F}_n \to 0$ as $n \to +\infty$. Again by the singular version of Gronwall's inequality

$$\|v_n(t) - v(t)\|_{H^1} \leq \tilde{Q}\tilde{F}_n,$$

where $\tilde{Q}$ is a positive constant. This implies $v_n(t) \to v(t)$ in H^1 uniformly on $[s, s+T]$, and proves (2). □

We recall the following

Definition 3.7. *A bounded semigroup $P(t)$ acting on a complete metric space X is said to be asymptotically compact iff for every bounded sequence $(u_n)_{n\in\mathbb{N}}$ and for every sequence $(t_n)_{n\in\mathbb{N}}$, $t_n \to +\infty$ as $n \to \infty$, there exists $u_\infty \in X$ such that, up to a subsequence, $P(t_n)u_n \to u_\infty$ as $n \to \infty$.*

Now we can prove

Proposition 3.8. *The semigroup $P_\omega(t)$ is continuous and asymptotically compact on $\Sigma \times H^1(\mathbb{R}^N)$.*

Proof. The continuity of $P_\omega(t)$ is a straightforward consequence of Lemma 3.6 and we omit the easy proof.

In order to prove the asymptotic compactness of $P_\omega(t)$, we take a bounded sequence $((\sigma_n, u_n))_{n\in\mathbb{N}}$ in $\Sigma \times H^1(\mathbb{R}^N)$. Let $R > 0$ be such that $\|u_n\|_{H^1} \leq R$ for all $n \in \mathbb{N}$. We seek for $(\sigma_\infty, u_\infty) \in \Sigma \times H^1(\mathbb{R}^N)$ such that, up to a subsequence,

$$P_\omega(t_n)(\sigma_n, u_n) = (T_\omega(t_n)\sigma_n, \Pi_\omega^{\sigma_n}(t_n, 0)u_n) \to (\sigma_\infty, u_\infty) \quad \text{in } \Sigma \times H^1(\mathbb{R}^N)$$

as $n \to \infty$. First of all, since Σ is compact, we can assume, without loss of generality, that there exists $\bar{\sigma}_\infty \in \Sigma$ such that $T_\omega(t_n - 1)\sigma_n \to \bar{\sigma}_\infty$ and $T_\omega(t_n)\sigma_n \to T_\omega(1)\bar{\sigma}_\infty =: \sigma_\infty$ as $n \to \infty$. Moreover, since the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$, by Proposition 2.5 the set

$$\{\, \Pi_\omega^{\sigma_n}(t_n, 0)u_n \mid n \in \mathbb{N} \,\}$$

is bounded, and hence weakly compact in $H^1(\mathbb{R}^N)$. So, passing to a subsequence if necessary, we can assume that there exists $u_\infty \in H^1(\mathbb{R}^N)$ such that

$$\Pi_\omega^{\sigma_n}(t_n,0)u_n \rightharpoonup u_\infty \quad \text{in } H^1(\mathbb{R}^N) \text{ as } n \to \infty.$$

We must show that the convergence is actually strong in $H^1(\mathbb{R}^N)$.

We claim first that $\Pi_\omega^{\sigma_n}(t_n,0)u_n \to u_\infty$ in the strong L^2-topology. To this end, it is enough to show that the set

$$\{\, \Pi_\omega^{\sigma_n}(t_n,0)u_n \mid n \in \mathbb{N} \,\}$$

is relatively compact in the strong L^2 topology, or equivalently that it is totally bounded. This is a consequence of Lemma 2.6 and of Rellich theorem. Let $\eta > 0$; by Lemma 2.6, there exists $k > 0$ and $\bar{n} \in \mathbb{N}$, depending on R and η, such that

$$\int_{\{|x|>k\}} |\Pi_\omega^{\sigma_n}(t_n,0)u_n(x)|^2\,dx \le \eta \quad \text{for all } n \ge \bar{n}.$$

We introduce the operator $\Xi\colon L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$,

$$(\Xi u)(x) := \begin{cases} u(x) & \text{if } |x| \le k \\ 0 & \text{if } |x| > k \end{cases}$$

Then we have

$$\begin{aligned} \{\, \Pi_\omega^{\sigma_n}(t_n,0)u_n \mid n \in \mathbb{N} \,\} &= \{\, \Xi\,\Pi_\omega^{\sigma_n}(t_n,0)u_n + (I-\Xi)\Pi_\omega^{\sigma_n}(t_n,0)u_n \mid n \in \mathbb{N} \,\} \\ &\subset \{\, \Xi\,\Pi_\omega^{\sigma_n}(t_n,0)u_n \mid n \in \mathbb{N} \,\} + \{\, (I-\Xi)\Pi_\omega^{\sigma_n}(t_n,0)u_n \mid n \in \mathbb{N} \,\} \\ &\subset B_\eta(0) + \{\, \Xi\,\Pi_\omega^{\sigma_n}(t_n,0)u_n \mid n \in \mathbb{N} \,\} \end{aligned}$$

where $B_\eta(0)$ is the ball of radius η centered at 0 in $L^2(\mathbb{R}^N)$. The set

$$\{\, \Xi\,\Pi_\omega^{\sigma_n}(t_n,0)u_n \mid n \in \mathbb{N} \,\}$$

consists of functions of $L^2(\mathbb{R}^N)$ which are equal to zero outside the ball of radius k in $\mathbb{R}^N$ and whose restriction to the same ball is in H^1. On the other hand, the H^1-norm of these functions is uniformly bounded. Then, by Rellich Theorem, we deduce that the set $\{\, \Xi\,\Pi_\omega^{\sigma_n}(t_n,0)u_n \mid n \in \mathbb{N} \,\}$ is precompact in $L^2(\mathbb{R}^N)$. Hence we can cover it by a finite number of balls of radius η in $L^2(\mathbb{R}^N)$. This implies that the set $\{\, \Pi_\omega^{\sigma_n}(t_n,0)u_n \mid n \in \mathbb{N} \,\}$ is totally bounded and hence precompact in $L^2(\mathbb{R}^N)$. The claim is proved.

The same conclusions obviously hold also for the set

$$\{\, \Pi_\omega^{\sigma_n}(t_n-1,0)u_n \mid n \in \mathbb{N} \,\};$$

so there exists $\bar{u}_\infty \in H^1(\mathbb{R}^N)$ such that, up to a subsequence,

$$\Pi_\omega^{\sigma_n}(t_n-1,0)u_n \to \bar{u}_\infty \quad \text{in } L^2(\mathbb{R}^N) \text{ as } n \to \infty.$$

Finally, by Lemma 3.6, we have

$$\begin{aligned} &\Pi_\omega^{\sigma_n}(t_n,0)u_n = \Pi_\omega^{\sigma_n}(t_n,t_n-1)\Pi_\omega^{\sigma_n}(t_n-1,0)u_n \\ &= \Pi_\omega^{T_\omega(t_n-1)\sigma_n}(1,0)\Pi_\omega^{\sigma_n}(t_n-1,0)u_n \to \Pi_\omega^{\bar{\sigma}_\infty}(1,0)\bar{u}_\infty \quad \text{in } H^1(\mathbb{R}^N) \text{ as } n \to \infty. \end{aligned}$$

It follows that $u_\infty = \Pi_\omega^{\bar{\sigma}_\infty}(1,0)\bar{u}_\infty$ and $\Pi_\omega^{\sigma_n}(t_n,0)u_n \to u_\infty$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. The proof is complete. □

Finally, combining Theorem 3.4 and Proposition 3.8, we have:

Theorem 3.9. *The family of processes* $\{\Pi_\omega^\sigma \mid \sigma \in \Sigma\}$ *in* $H^1(\mathbb{R}^N)$ *possesses a compact* Σ*-uniform attractor* $\mathcal{A}_\omega^\Sigma$. *As a point set,*

$$\mathcal{A}_\omega^\Sigma = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\omega^\sigma(0),$$

where $\mathcal{K}_\omega^\sigma(0)$ *is the kernel section introduced in Definition 3.3. In other words,* $\mathcal{A}_\omega^\Sigma$ *is the union of all the full bounded trajectories of* Π_ω^σ, $\sigma \in \Sigma$. □

4. Behaviour as $\omega \to +\infty$

In this section we shall investigate the behaviour of the solutions of (2.1) as $\omega \to +\infty$. As we explained in the Introduction, we expect that the averaged equation

$$u_t = \sum_{i,j=1}^N \bar{a}_{ij}\partial_i\partial_j u - \bar{a}_0 u + \bar{f}(u) + \bar{g}(x) \tag{4.1}$$

behaves like a 'limit' equation of (2.1). Roughly speaking, this means that the solutions of (2.1) with initial datum $u_0 \in H^1(\mathbb{R}^N)$, as $\omega \to +\infty$, converge in some sense to the solution of (4.1) with the same initial datum. Moreover, we claim that the attractor of (2.1) is H^1-close to that of (4.1) for sufficiently large ω.

Let us denote by $\bar{A}\colon H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ the self-adjoint positive operator defined by

$$\bar{A}u := -\sum_{i,j=1}^N \bar{a}_{ij}\partial_i\partial_j u, \quad u \in H^2(\mathbb{R}^N);$$

we denote by $e^{-\bar{A}t}$ the analytic semigroup generated by $\bar{A}$. Then equation (4.1) can be written as an abstract parabolic equation in $L^2(\mathbb{R}^N)$, namely

$$\dot{u} = -\bar{A}u - \bar{a}_0 u + \hat{\bar{f}}(u) + \bar{g}. \tag{4.2}$$

Equation (4.2) defines a global semiflow π in $H^1(\mathbb{R}^N)$: in fact, as we already observed in Section 3, all a-priori estimates of Section 2 are independent of ω and $\sigma \in \Sigma$, and are valid also for the averaged equation (4.1). So the semiflow π possesses a compact global attractor $\mathcal{A}$.

We begin with a convergence result for the linear problems associated to (2.1) and (4.1):

Proposition 4.1. *For* $(\alpha_{ij}(\cdot)) \in \Sigma_1$ *and* $\omega > 0$, *let* $V_\omega^\alpha(t,s)$ *be the linear process in* $L^2(\mathbb{R}^N)$ *generated by the equation*

$$u_t = \sum_{i,j=1}^N \alpha_{ij}(\omega t)\partial_i\partial_j u.$$

Moreover, let $e^{-\bar{A}t}$ *be the linear semigroup in* $L^2(\mathbb{R}^N)$ *generated by the equation*

$$u_t = \sum_{i,j=1}^{N} \bar{a}_{ij}\partial_i\partial_j u.$$

There exists a bounded, continuous and decreasing function $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$, $\theta(q) \to 0$ *as* $q \to \infty$, *such that, for* $u \in L^2(\mathbb{R}^N)$ *and for* $t > s$,

$$\|V_\omega^\alpha(t,s)u - e^{-\bar{A}(t-s)}u\|_{H_1} \leq \left(1 + \frac{1}{(t-s)^{1/2}}\right)\theta(\omega(t-s))\|u\|_{L^2}$$

for any $(\alpha_{ij}(\cdot)) \in \Sigma_1$ *and* $\omega > 0$.

Proof. Let $u \in L^2(\mathbb{R}^N)$ and $t > s$. By (2.11) and (2.12), for any $R > 0$ we have

$$\begin{aligned}
&\|V_\omega^\alpha(t,s)u - e^{-\bar{A}(t-s)}u\|_{H^1}^2 \\
&= \int_{\mathbb{R}^N} (1+|\xi|^2)\big[\exp(-\int_s^t \sum_{i,j} \alpha_{ij}(\omega p)\xi_i\xi_j\,dp) \\
&\qquad\qquad - \exp(-\sum_{i,j} \bar{a}_{ij}\xi_i\xi_j(t-s))\big]^2 (\mathcal{F}u)(\xi)^2\,d\xi \\
&\leq 2\int_{\{|\xi|\geq R\}} (1+|\xi|^2)\exp(-2\nu_0|\xi|^2(t-s))(\mathcal{F}u)(\xi)^2\,d\xi \\
&+ \int_{\{|\xi|\leq R\}} (1+|\xi|^2)\exp(-2\nu_0|\xi|^2(t-s)) \\
&\qquad\qquad \big[\exp(-\sum_{i,j}\int_s^t (\alpha_{ij}(\omega p) - \bar{a}_{ij})\,dp\,\xi_i\xi_j) - 1\big]^2 (\mathcal{F}u)(\xi)^2\,d\xi \\
&=: S_1 + S_2.
\end{aligned}$$

Choose $R := k^{1/2}(t-s)^{-1/2}$, k to be determined. If $k \geq 1/2\nu_0$,

$$S_1 \leq 2\sup_{z\geq k}\left(1 + \frac{z}{(t-s)}\right)e^{-2\nu_0 z}\|u\|_{L^2}^2 \leq 2\left(1 + \frac{1}{(t-s)}\right)ke^{-2\nu_0 k}\|u\|_{L^2}^2.$$

In order to estimate S_2, we observe that

$$\begin{aligned}
&|\int_s^t (\alpha_{ij}(\omega p) - \bar{a}_{ij})\,dp| \\
&\qquad = (t-s)\left|\frac{1}{\omega(t-s)}\int_{\omega s}^{\omega s+\omega(t-s)} (\alpha_{ij}(p) - \bar{a}_{ij})\,dp\right| \leq (t-s)\mu(\omega(t-s)).
\end{aligned}$$

Then

$$S_2 \le \left(1 + \frac{k}{(t-s)}\right) \int_{\{|\xi|^2 \le k(t-s)^{-1}\}} [\, e^{Nk\mu(\omega(t-s))} - 1\,]^2 (\mathcal{F}u)(\xi)^2 \, d\xi$$
$$\le \left(1 + \frac{1}{(t-s)}\right) k[\, e^{Nk\mu(\omega(t-s))} - 1\,]^2 \|u\|_{L^2}^2$$

By the mean value theorem, we get

$$S_2 \le \left(1 + \frac{1}{(t-s)}\right) k[\, e^{Nk\mu(\omega(t-s))} Nk\mu(\omega(t-s))\,]^2 \|u\|_{L^2}^2$$
$$= \left(1 + \frac{1}{(t-s)}\right) N^2 k^3 e^{2Nk\mu(\omega(t-s))} \mu(\omega(t-s))^2 \|u\|_{L^2}^2$$

Now, set $\mu_\infty := \sup_{q \ge 0} \mu(q)$ and take

$$k := \frac{1}{2\nu_0} \left(\frac{\mu_\infty}{\mu(\omega(t-s))}\right)^{1/2}$$

With this choice of k, we obtain that there exist positive constants $\bar{\kappa}$, μ_1 and μ_2, depending only on N, μ_∞ and ν_0, such that

$$S_1 + S_2 \le \bar{\kappa} \left(1 + \frac{1}{(t-s)}\right)$$
$$\left(\mu(\omega(t-s))^{-1/2} e^{-\mu_1 \mu(\omega(t-s))^{-1/2}} + \mu(\omega(t-s))^{1/2} e^{\mu_2 \mu(\omega(t-s))^{1/2}}\right) \|u\|_{L^2}^2$$

The conclusion follows if we define

$$\theta(q) := \bar{\kappa}^{1/2} \left(\mu(q)^{-1/2} e^{-\mu_1 \mu(q)^{-1/2}} + \mu(q)^{1/2} e^{\mu_2 \mu(q)^{1/2}}\right)^{1/2}$$

and, if necessary, modify it on some bounded interval in order to make it decreasing on $\mathbb{R}_+$. The proof is complete. $\square$

Corollary 4.2. *Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers, $\omega_n \to +\infty$ as $n \to \infty$. Let $(\alpha_{ij}^n(\cdot))_{n\in\mathbb{N}}$ be a sequence in Σ_1 and let $V_{\omega_n}^n(t,s)$ be the linear process in $L^2(\mathbb{R}^N)$ generated by the equation*

$$u_t = \sum_{i,j=1}^N \alpha_{ij}^n(\omega_n t) \partial_i \partial_j u.$$

Fix $0 < \delta < T$ and take a sequence $(u_n)_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^N)$, $u_n \to u$ in $L^2(\mathbb{R}^N)$. Then

$$\sup_{s\in\mathbb{R}} \sup_{t\in[s+\delta, s+T]} \|V_{\omega_n}^n(t,s)u_n - e^{-\bar{A}(t-s)}u\|_{H^1} \to 0 \quad \textit{as } n \to \infty.$$

Proof. We have

$$
\begin{aligned}
&\|V^n_{\omega_n}(t,s)u_n - e^{-\bar{A}(t-s)}u\|_{H^1} \\
&\qquad \le \|V^n_{\omega_n}(t,s)u_n - V^n_{\omega_n}(t,s)u\|_{H^1} + \|V^n_{\omega_n}(t,s)u - e^{-\bar{A}(t-s)}u\|_{H^1} \\
&\qquad \le M(1+(t-s)^{-1/2})\,(\|u_n-u\|_{L^2} + \theta(\mu(\omega_n(t-s)))\|u\|_{L^2}) \\
&\qquad \le M(1+\delta^{-1/2})\,(\|u_n-u\|_{L^2} + \theta(\mu(\omega_n\delta))\|u\|_{L^2}) \to 0 \quad \text{as } n\to\infty,
\end{aligned}
$$

and the corollary is proved. □

If we deal with a fixed $u \in H^1(\mathbb{R}^N)$, we obtain uniform convergence on the whole interval $[s, s+T]$. Indeed, we have the following

Proposition 4.3. *Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers, $\omega_n \to \infty$ as $n\to\infty$. Let $(\alpha^n_{ij}(\cdot))_{n\in\mathbb{N}}$ be a sequence in Σ_1 and let $V^n_{\omega_n}(t,s)$ be the linear process in $L^2(\mathbb{R}^N)$ generated by the equation*

$$u_t = \sum_{i,j=1}^N \alpha^n_{ij}(\omega_n t)\partial_i\partial_j u.$$

Finally, let $u \in H^1(\mathbb{R}^N)$. Then, for any $T>0$,

$$\sup_{s\in\mathbb{R}}\ \sup_{t\in[s,s+T]} \|V^n_{\omega_n}(t,s)u - e^{-\bar{A}(t-s)}u\|_{H^1} \to 0 \quad \textit{as } n\to\infty.$$

Proof. Let $u \in H^1(\mathbb{R}^N)$ and $t>s$. Arguing like in the proof of Proposition 4.1, for any $R>0$ we have

$$
\begin{aligned}
&\|V^n_{\omega_n}(t,s)u - e^{-\bar{A}(t-s)}u\|^2_{H^1} \le 2\int_{\{|\xi|\ge R\}} (1+|\xi|^2)(\mathcal{F}u)(\xi)^2\,d\xi \\
&\quad + \int_{\{|\xi|\le R\}} (1+|\xi|^2)[\exp(-\sum_{i,j}\int_s^t (\alpha^n_{ij}(\omega_n p) - \bar{a}_{ij})\,dp\,\xi_i\xi_j) - 1]^2(\mathcal{F}u)(\xi)^2\,d\xi.
\end{aligned}
$$

Since

$$|\int_s^t (\alpha_{ij}(\omega p) - \bar{a}_{ij})\,dp| \le (t-s)\mu(\omega(t-s)),$$

we obtain

$$
\begin{aligned}
&\|V^n_{\omega_n}(t,s)u - e^{-\bar{A}(t-s)}u\|^2_{H^1} \\
&\qquad \le 2\int_{\{|\xi|\ge R\}} (1+|\xi|^2)(\mathcal{F}u)(\xi)^2\,d\xi + \left(e^{N(t-s)\mu(\omega_n(t-s))R^2} - 1\right)^2 \|u\|^2_{H^1}
\end{aligned}
$$

Now, given $\epsilon>0$, we choose $R>0$ (depending on u and ϵ) such that

$$\int_{\{|\xi|\ge R\}} (1+|\xi|^2)(\mathcal{F}u)(\xi)^2\,d\xi \le \epsilon.$$

Let δ be a positive number, depending on R, $\|u\|_{H^1}$ and ϵ, such that $\delta < T$ and

$$\left(e^{N\delta\mu_\infty R^2} - 1\right)^2 \|u\|_{H^1}^2 \leq \epsilon.$$

Then, for $t - s < \delta$,

$$\left(e^{N(t-s)\mu(\omega_n(t-s))R^2} - 1\right)^2 \|u\|_{H^1}^2 \leq \left(e^{N\delta\mu_\infty R^2} - 1\right)^2 \|u\|_{H^1}^2 \leq \epsilon.$$

On the other hand, if $\delta \leq (t-s) \leq T$, we have

$$\left(e^{N(t-s)\mu(\omega_n(t-s))R^2} - 1\right)^2 \leq \left(e^{NT\mu(\omega_n\delta)R^2} - 1\right)^2.$$

As a consequence, given $\epsilon > 0$, we can find R and δ (depending on ϵ) such that, for all $n \in \mathbb{N}$,

$$\sup_{s\in\mathbb{R}} \sup_{t\in[s,s+T]} \|V_{\omega_n}^n(t,s)u - e^{-\bar{A}(t-s)}u\|_{H^1}^2 \leq 3\epsilon + \left(e^{NT\mu(\omega_n\delta)R^2} - 1\right)^2 \|u\|_{H^1}^2.$$

The conclusion follows by letting $n \to \infty$. $\square$

Remark. The convergence in Proposition 4.3 is not uniform with respect to u in a bounded subset of $H^1(\mathbb{R}^N)$. As a matter of fact, if we try to repeat the arguments of Proposition 4.1, we see that there exists a bounded, continuous and decreasing function $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$, $\theta(q) \to 0$ as $q \to \infty$, such that, for $u \in H^1(\mathbb{R}^N)$ and for $t > s$,

$$\|V_\omega^\alpha(t,s)u - e^{-\bar{A}(t-s)}u\|_{H_1} \leq \theta(\omega(t-s))\|u\|_{H^1}$$

for any $(\alpha_{ij}(\cdot)) \in \Sigma_1$ and $\omega > 0$. It is clear that this is not enough to detect uniform convergence up to $t = s$, since there is still an initial layer one cannot get rid of. This is due to the microlocal effect of the rapid oscillations of the coefficients $a_{ij}(\omega p)$.

Now we can state our first 'local' averaging result for the nonlinear equation (2.1):

Theorem 4.4. *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence in Σ. Let $(t_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ be two sequences of real numbers, with $t_n > s_n$ for all n and assume that $t_n \to t$ and $s_n \to s$ as $n \to \infty$, with $t > s$. Let $u \in H^1(\mathbb{R}^N)$ and let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R}^N)$ and assume that $u_n \to u$ in $L^2(\mathbb{R}^N)$. Finally let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers, $\omega_n \to +\infty$ as $n \to \infty$. Then*

$$\|\Pi_{\omega_n}^{\sigma_n}(t_n,s_n)u_n - \pi(t-s)u\|_{H^1} \to 0 \quad \textit{as } n \to \infty.$$

In order to prove Theorem 4.4, we need the following

Lemma 4.5. *Let $(\alpha_0^n, \phi_n, \gamma_n)_{n\in\mathbb{N}}$ be a sequence in $\Sigma_2 \times \Sigma_3 \times \Sigma_4$. Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers, $\omega_n \to +\infty$ as $n \to \infty$. Finally, let $v\colon [s, s+T] \to H^1(\mathbb{R}^N)$ be a continuous function. For $t \in [s, s+T]$ set*

$$G_n^1(t) := \int_s^t e^{-\bar{A}(t-p)} \left[\alpha_0^n(\omega_n p) - \bar{a}_0\right] v(p)\, dp;$$

$$G_n^2(t) := \int_s^t e^{-\bar{A}(t-p)} \left[\hat{\phi}_n(\omega_n p, v(p)) - \hat{\bar{f}}(v(p))\right]\, dp;$$

$$G_n^3(t) := \int_s^t e^{-\bar{A}(t-p)} \left[\gamma_n(\omega_n p) - \bar{g}\right]\, dp.$$

Then $G_n^j(t) \to 0$ in $H^1(\mathbb{R}^N)$ uniformly on $[s, s+T]$ for $j = 1, 2, 3$.

Proof. The proof of this lemma is essentially contained in [11, Th. 1.1] and, in a more general setting, in [10, Th. 3.4.7]. We give the details for sake of completeness.

We start by considering G_n^3. First of all, we observe that, for $s < p < t$,

$$\frac{d}{dp}\left(e^{-\bar{A}(t-p)} \int_p^t \left[\gamma_n(\omega_n q) - \bar{g}\right]\, dq\right)$$
$$= \bar{A} e^{-\bar{A}(t-p)} \int_p^t \left[\gamma_n(\omega_n q) - \bar{g}\right]\, dq + e^{-\bar{A}(t-p)} \left[\gamma_n(\omega_n p) - \bar{g}\right].$$

Since

$$\begin{aligned} &\|\bar{A} e^{-\bar{A}(t-p)} \int_p^t \left[\gamma_n(\omega_n q) - \bar{g}\right]\, dq\|_{H^1} \\ &\le M(t-p)^{-3/2}(t-p)\left\|(\omega_n(t-p))^{-1} \int_{\omega_n p}^{\omega_n p + \omega_n(t-p)} \left[\gamma_n(q) - \bar{g}\right]\, dq\right\|_{L^2} \\ &\le M(t-p)^{-1/2}\mu(\omega_n(t-p)) \in L^1(]\, p, t\, [) \end{aligned} \tag{4.3}$$

and

$$\|e^{-\bar{A}(t-p)} \left[\gamma_n(\omega_n p) - \bar{g}\right]\|_{H^1} \le 2C(t-p)^{-1/2} \in L^1(]\, p, t\, [),$$

the 'integration-by-part' formula

$$\int_s^t e^{-\bar{A}(t-p)} \left[\gamma_n(\omega_n p) - \bar{g}\right]\, dp$$
$$= -e^{-\bar{A}(t-s)} \int_s^t \left[\gamma_n(\omega_n p) - \bar{g}\right]\, dp - \int_s^t \bar{A} e^{-\bar{A}(t-p)} \int_p^t \left[\gamma_n(\omega_n q) - \bar{g}\right]\, dq\, dp$$

is valid. In view of (4.3), we get

$$\begin{aligned} \|G_n^3(t)\|_{H^1} &\le \left(1 + \frac{1}{(t-s)^{1/2}}\right)(t-s)\mu(\omega_n(t-s)) + \int_s^t (t-p)^{-1/2}\mu(\omega_n(t-p))\, dp \\ &\le (t-s)^{1/2}\mu(\omega_n(t-s)) + \int_s^t (t-p)^{-1/2}\mu(\omega_n(t-p))\, dp. \end{aligned}$$

Now let $\epsilon > 0$. If $t - s \leq \epsilon$, a simple integration yields

$$\|G_n^3(t)\|_{H^1} \leq 3\mu_\infty \epsilon^{1/2}.$$

If $t - s \geq \epsilon$, we have

$$\begin{aligned}\|G_n^3(t)\|_{H^1} \leq T^{1/2}\mu(\omega_n\epsilon) + \mu(\omega_n\epsilon)\int_s^{t-\epsilon}(t-p)^{-1/2}\,dp + \mu_\infty\int_{t-\epsilon}^t (t-p)^{-1/2}\,dp \\ \leq 3T^{1/2}\mu(\omega_n\epsilon) + 2\mu_\infty\epsilon^{1/2}.\end{aligned}$$

It follows that, for all $n \in \mathbb{N}$,

$$\sup_{t\in[s,s+T]} \|G_n^3(t)\|_{H^1} \leq 3\mu_\infty\epsilon^{1/2} + 3T^{1/2}\mu(\omega_n\epsilon).$$

The conclusion follows by letting $n \to \infty$.

Next we consider G_n^1. We assume first that $v(t) \equiv \bar{v} \in H^1(\mathbb{R}^N)$. Then, arguing as above, we see that

$$\begin{aligned}G_n^1(t) = -e^{-\bar{A}(t-s)}\int_s^t [\alpha_0^n(\omega_n p) - \bar{a}_0]\,\bar{v}\,dp \\ - \int_s^t \bar{A}e^{-\bar{A}(t-p)}\int_p^t [\alpha_0^n(\omega_n q) - \bar{a}_0]\,\bar{v}\,dq\,dp,\end{aligned}$$

hence

$$\|G_n^1(t)\|_{H^1} \leq (t-s)\mu(\omega_n(t-s))\|\bar{v}\|_{H^1} + \int_s^t \mu(\omega_n(t-p))\,dp\|\bar{v}\|_{H^1}.$$

The same argument used for estimating G_n^3 shows that $G_n^1(t) \to 0$ in $H^1(\mathbb{R}^N)$ uniformly on $[s, s+T]$. One can easily see that the same is true if $v(t)$ is an arbitrary bounded step function. The conclusion then follows by a density argument.

Finally, we consider G_n^2. Again we assume first that $v(t) \equiv \bar{v} \in H^1(\mathbb{R}^N)$. Then we have

$$\begin{aligned}G_n^2(t) = -e^{-\bar{A}(t-s)}\int_s^t \left[\hat{\phi}(\omega_n p, \bar{v}) - \hat{\bar{f}}(\bar{v})\right]\,dp \\ - \int_s^t \bar{A}e^{-\bar{A}(t-p)}\int_p^t \left[\hat{\phi}(\omega_n q, \bar{v}) - \hat{\bar{f}}(\bar{v})\right]\,dq\,dp.\end{aligned}$$

By (3.2), we obtain

$$\begin{aligned}\|G_n^2(t)\|_{H^1} \leq K\left(1 + \frac{1}{(t-s)^{1/2}}\right)(t-s)\mu(\omega_n(t-s))(\|\bar{v}\|_{L^2} + \|\bar{v}\|_{H^1}^{\beta+1}) \\ + K\int_s^t (t-p)^{-1/2}\mu(\omega_n(t-p))(\|\bar{v}\|_{L^2} + \|\bar{v}\|_{H^1}^{\beta+1})\,dp \\ \leq K\left((t-s)^{1/2}\mu(\omega_n(t-s)) + \int_s^t (t-p)^{-1/2}\mu(\omega_n(t-p))\,dp\right)(\|\bar{v}\|_{L^2} + \|\bar{v}\|_{H^1}^{\beta+1}).\end{aligned}$$

Arguing as before, $G_n^2(t) \to 0$ in $H^1(\mathbb{R}^N)$ uniformly on $[s, s+T]$, and the same is true if $v(t)$ is an arbitrary bounded step function. The conclusion then follows again by a density argument. $\square$

Proof of Theorem 4.4. First, let us notice that

$$\Pi_{\omega_n}^{\sigma_n}(t_n, s_n)u_n = \Pi_{\omega_n}^{T_{\omega_n}(s_n - s)\sigma_n}(t_n - (s_n - s), s)u_n.$$

Since $t_n - (s_n - s) \to t$ as $n \to \infty$, we can assume without loss of generality that $s_n = s$ for all n.

Let's write

$$v_n(t) := \Pi_{\omega_n}^{\sigma_n}(t, s)u_n,$$
$$v(t) := \pi(t - s)u.$$

We recall that, in view of Proposition 2.5, there exists $\tilde{K} > 0$ such that, for every $t \geq s$,

$$\|v_n(t)\|_{H^1} \leq \tilde{K} \quad \text{for all } n \in \mathbb{N}\ ,$$
$$\|v(t)\|_{H^1} \leq \tilde{K}.$$

By the variation of constant formula we get

$$\|v_n(t) - v(t)\|_{H^1} \leq \|V_{\omega_n}^n(t, s)u_n - e^{-\bar{A}(t-s)}u\|_{H^1} + I_1(t) + I_2(t) + I_3(t),$$

where

$$I_1(t) := \int_s^t \|(V_{\omega_n}^n(t, p) - e^{-\bar{A}(t-p)})[-\alpha_0^n(\omega_n p)v_n(p) \\ + \hat{\varphi}_n(\omega_n p, v_n(p)) + \gamma_n(\omega_n p)]\|_{H^1}\, dp,$$

$$I_2(t) := \int_s^t \|e^{-\bar{A}(t-p)}[-\alpha_0^n(\omega_n p)(v(p) - v_n(p)) \\ + \hat{\varphi}_n(\omega_n p, v(p)) - \hat{\varphi}_n(\omega_n p, v_n(p))]\|_{H^1}\, dp,$$

$$I_3(t) := \| \int_s^t e^{-\bar{A}(t-p)}[(\bar{a}_0 - \alpha_0^n(\omega_n p))v(p) \\ + \hat{\bar{f}}(v(p)) - \hat{\varphi}_n(\omega_n p, v(p)) + \gamma_n(\omega_n p) - \bar{g}]\, dp\|_{H^1}.$$

First of all, we have

$$\|V_{\omega_n}^n(t, s)u_n - e^{-\bar{A}(t-s)}u\|_{H^1} \leq M(t - s)^{-1/2}\|u_n - u\|_{L^2} \\ + \sup_{t \in [s, s+T]} \|(V_{\omega_n}^n(t, s) - e^{\bar{A}(t-s)})u\|_{H^1}.$$

As for $I_1(t)$, by Proposition 4.1

$$\begin{aligned}
&I_1(t)\\
&\le \int_s^t (1+(t-p)^{-1/2})\theta(\omega_n(t-p))\|-\alpha_0^n(\omega_n p)v_n(p)+\hat{\varphi}_n(\omega_n p, v_n(p))+\gamma_n(\omega_n p)\|_{L^2}\,dp\\
&\qquad\le (C\tilde{K}+\tilde{C}(\tilde{K}+\tilde{K}^{\beta+1}))\int_s^t (t-p)^{-1/2}\theta(\omega_n(t-p))\,dp.
\end{aligned}$$

By the same argument used in the proof of Lemma 4.5, we find that $I_1(t)\to 0$ as $n\to\infty$ uniformly on $[s,s+T]$.

Next we consider $I_2(t)$:

$$\begin{aligned}
I_2(t) &\le \int_s^t M(1+(t-p)^{-1/2})\|-\alpha_0^n(\omega_n p)(v(p)-v_n(p))\\
&\qquad + \hat{\varphi}_n(\omega_n p, v(p)) - \hat{\varphi}_n(\omega_n p, v_n(p))\|_{L^2}\,dp\\
&\le \int_s^t M(1+(t-p)^{-1/2})(C\|v(p)-v_n(p)\|_{H^1}+\tilde{C}(1+2\tilde{K}^{\beta})\|v(p)-v_n(p)\|_{H^1})\,dp\\
&\qquad \le Q\int_s^t (t-p)^{-1/2}\|v_n(p)-v(p)\|_{H^1}\,dp,
\end{aligned}$$

where Q is a positive constant.

Finally,

$$\begin{aligned}
I_3(t) &\le \Big\|\int_s^t e^{-\bar{A}(t-p)}[(\bar{a}_0-\alpha_0^n(\omega_n p))v(p)]\,dp\Big\|_{H^1}\\
&\qquad + \Big\|\int_s^t e^{-\bar{A}(t-p)}[\hat{\bar{f}}(v(p))-\hat{\varphi}_n(\omega_n p, v(p))]\,dp\Big\|_{H^1}\\
&\qquad\qquad + \Big\|\int_s^t e^{-\bar{A}(t-p)}[\gamma_n(\omega_n p)-\bar{g}]\,dp\Big\|_{H^1},
\end{aligned}$$

and, by Lemma 4.5, $I_3(t)\to 0$ as $n\to\infty$ uniformly on $[s,s+T]$.

Summing up, for $t\in]s,s+T]$ we get

$$\|v_n(t)-v(t)\|_{H^1}\le (t-s)^{-1/2}F_n + Q\int_s^t (t-p)^{-1/2}\|v_n(p)-v(p)\|_{H^1}\,dp,$$

where $F_n\to 0$ as $n\to\infty$.

By the singular version of Gronwall's inequality (see [10, Th. 7.1.1]),

$$\|v_n(t)-v(t)\|_{H^1}\le \tilde{Q}F_n(t-s)^{-1/2},$$

where $\tilde{Q}$ is a positive constant. This implies that $v_n(t)\to v(t)$ in $H^1(\mathbb{R}^N)$ uniformly on $[s+\delta,s+T]$ for every $\delta>0$, and completes the proof. □

Remark. If in Theorem 4.4 we assume that $u_n\to u$ in $H^1(\mathbb{R}^N)$, by the same techniques we can show that $v_n(t)\to v(t)$ in $H^1(\mathbb{R}^N)$ uniformly on $[s,s+T]$.

The following lemma provides a kind of joint asymptotic compactness of $\Pi_\omega^\sigma(t,s)$ with respect to t and ω.

Lemma 4.6. *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R}^N)$, $(\sigma_n)_{n\in\mathbb{N}}$ an arbitrary sequence in Σ, $(t_n)_{n\in\mathbb{N}}$ and $(\omega_n)_{n\in\mathbb{N}}$ two sequences of positive real numbers, $t_n \to +\infty$ and $\omega_n \to +\infty$ as $n \to \infty$. Then there exists $u_\infty \in H^1(\mathbb{R}^N)$ such that, up to a subsequence,*

$$\Pi^{\sigma_n}_{\omega_n}(t_n, 0)u_n \to u_\infty \quad \text{in } H^1(\mathbb{R}^N) \text{ as } n \to \infty.$$

Proof. The proof is analogous to that of Proposition 3.8: from the boundedness of $(u_n)_{n\in\mathbb{N}}$ in $H^1(\mathbb{R}^N)$ it follows that there exists $u_\infty \in H^1(\mathbb{R}^N)$ such that, up to a subsequence,

$$\Pi^{\sigma_n}_{\omega_n}(t_n - 1, 0)u_n \rightharpoonup \bar{u}_\infty \quad \text{in } H^1(\mathbb{R}^N) \text{ as } n \to \infty.$$

Again by Lemma 2.6

$$\Pi^{\sigma_n}_{\omega_n}(t_n - 1, 0)u_n \to \bar{u}_\infty \quad \text{in } L^2(\mathbb{R}^N) \text{ as } n \to \infty,$$

and by Theorem 4.4

$$\begin{aligned}\Pi^{\sigma_n}_{\omega_n}(t_n, 0)u_n &= \Pi^{\sigma_n}_{\omega_n}(t_n, t_n - 1)\Pi^{\sigma_n}_{\omega_n}(t_n - 1, 0)u_n \\ &= \Pi^{T_{\omega_n}(t_n-1)\sigma_n}_{\omega_n}(1, 0)\Pi^{\sigma_n}_{\omega_n}(t_n - 1, 0)u_n \to \pi(1)\bar{u}_\infty \quad \text{in } H^1(\mathbb{R}^N) \text{ as } n \to \infty.\end{aligned}$$

This completes the proof. $\square$

Finally, we can prove the upper-semicontinuity result announced in the Introduction:

Theorem 4.7. *For $\omega > 0$, let $\mathcal{A}^\Sigma_\omega$ be the Σ-uniform attractor of the family of processes $\{\,\Pi^\sigma_\omega \mid \sigma \in \Sigma\,\}$. Moreover, let $\mathcal{A}$ be the attractor of the semiflow π. Then for every $\delta > 0$ there exists $\bar{\omega} > 0$ such that if $\omega \geq \bar{\omega}$*

$$\mathrm{d}_{H^1}(\mathcal{A}^\Sigma_\omega, \mathcal{A}) := \max_{u\in\mathcal{A}^\Sigma_\omega} \mathrm{d}_{H^1}(u, \mathcal{A}) < \delta.$$

Proof. Let's assume, by contradiction, that the thesis is not true: then there exist $\bar{\delta} > 0$, a sequence $(\omega_n)_{n\in\mathbb{N}}$ of positive numbers, $\omega_n \to +\infty$, and a sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \in \mathcal{A}^\Sigma_{\omega_n}$ for all $n \in \mathbb{N}$, such that

$$\mathrm{d}_{H^1}(u_n, \mathcal{A}) \geq \bar{\delta} \quad \text{for all } n \in \mathbb{N}.$$

Since $u_n \in \mathcal{A}^\Sigma_{\omega_n}$, by Theorem 3.4 for every $n \in \mathbb{N}$ there exists $\sigma_n \in \Sigma$ and $v_n \in \mathcal{K}^{\sigma_n}_{\omega_n}$ such that $u_n = v_n(0)$. Since

$$(t \mapsto v_n(t + h)) \in \mathcal{K}^{T_{\omega_n}(h)\sigma_n}_{\omega_n}$$

for all $h \in \mathbb{R}$, it follows that $v_n(t) \in \mathcal{A}^{\Sigma}_{\omega_n}$ for all $t \in \mathbb{R}$. Hence, by Proposition 2.5, there exists $K > 0$, independent of n, such that $\|v_n(t)\|_{H^1} \le K$ for $t \in \mathbb{R}$.

Let k be a positive integer and let $(h_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers, $h_n \to +\infty$. We have

$$v_n(-k) = \Pi^{\sigma_n}_{\omega_n}(-k, -h_n - k)v(-h_n - k) = \Pi^{T_{\omega_n}(-k-h_n)\sigma_n}_{\omega_n}(h_n, 0)v(-h_n - k),$$

so, by Proposition 4.6, there exists $\bar{u}_k \in H^1(\mathbb{R}^N)$ such that, up to a subsequence,

$$v_n(-k) \to \bar{u}_k \quad \text{in } H^1(\mathbb{R}^N) \text{ as } n \to +\infty. \tag{4.4}$$

By a Cantor diagonal procedure, we can assume that (4.4) holds for positive integer k. By Theorem 4.4, for every $t > -k$

$$\begin{aligned} v_n(t) = \Pi^{\sigma_n}_{\omega_n}(t, -k)v_n(-k) &= \Pi^{T_{\omega_n}(-k)\sigma_n}_{\omega_n}(t + k, 0)v_n(-k) \\ &\to \pi(t + k)\bar{u}_k \quad \text{in } H^1(\mathbb{R}^N) \text{ as } n \to \infty. \end{aligned} \tag{4.5}$$

In particular, choosing $t = 0$ we get

$$u_n \to \pi(k)\bar{u}_k \quad \text{in } H^1(\mathbb{R}^N) \text{ as } n \to \infty.$$

Notice that $\pi(k)\bar{u}_k$ is independent of k, so we can define $u_\infty := \pi(k)\bar{u}_k$. The proof will be complete if we show that $u_\infty \in \mathcal{A}$. So we must prove that there exists a full bounded solution $v_\infty(t)$ of the semiflow π, such that $v_\infty(0) = u_\infty$ To this end, we just have to define $v_\infty(t) := \pi(t + k)\bar{u}_k$, $t > -k$. By (4.5) it follows that $\pi(t + k)\bar{u}_k$ is independent of k and therefore $v_\infty(t)$ is unambiguously defined for every $t \in \mathbb{R}$. Moreover, $v_\infty(t)$ is by construction a full bounded solution of π, with $v_\infty(0) = u_\infty$. This finally implies that $u_\infty \in \mathcal{A}$, a contradiction. □

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