Topological Methods in Nonlinear Analysis
Vol. 19, No. 2 (2002), 199–220

RECENT RESULTS ON THIN DOMAIN PROBLEMS II

Martino Prizzi
and
Krzysztof P. Rybakowski

Dedicated to Professor Andrzej Granas

Abstract. In this paper we survey some recent results on parabolic equations on curved squeezed domains. More specifically, consider the family of semilinear Neumann boundary value problems

$$(E_\varepsilon) \qquad \begin{aligned} u_t &= \Delta u + f(u), && t > 0,\, x \in \Omega_\varepsilon, \\ \partial_{\nu_\varepsilon} u &= 0, && t > 0,\, x \in \partial\Omega_\varepsilon \end{aligned}$$

where, for $\varepsilon > 0$ small, the set Ω_ε is a thin domain in $\mathbb{R}^\ell$, possibly with holes, which collapses, as $\varepsilon \to 0^+$, onto a (curved) k-dimensional submanifold $\mathcal{M}$ of $\mathbb{R}^\ell$. If f is dissipative, then equation (E_ε) has a global attractor $\mathcal{A}_\varepsilon$. We identify a 'limit' equation for the family (E_ε), establish an upper semicontinuity result for the family $\mathcal{A}_\varepsilon$ and prove an inertial manifold theorem in case $\mathcal{M}$ is a k-sphere.

1. Squeezing transformations

In this paper we report on some recent results on the qualitative dynamics of parabolic equations on curved thin domains. More detailed statements and proofs of these results will appear elsewhere (cf [17], [20] and [21]).

Consider an evolution equation on a spatial domain $\Omega \subset \mathbb{R}^\ell$, and assume that Ω is 'small' in some direction. A natural question arises whether it is possible to approximate this equation by an equation defined on a lower dimensional spatial domain. In their paper [8, Reaction-diffusion equations on thin domains] J. Hale and G. Raugel treat in detail the case of the reaction-diffusion equation

$$(1.1) \qquad \begin{aligned} u_t &= \Delta u + f(u), && t > 0,\, x \in \Omega_\varepsilon, \\ \partial_{\nu_\varepsilon} u &= 0, && t > 0,\, x \in \partial\Omega_\varepsilon, \end{aligned}$$

when the domain Ω_ε has the form

$$\Omega_\varepsilon = \{\, (x, y) \mid x \in \omega \text{ and } 0 < y < \varepsilon g(x) \,\},$$

1991 *Mathematics Subject Classification.* Primary. 35k57, 35K90 Secondary. 37L25, 35B25.
Key words and phrases. Thin domains, curved squeezing, evolution equations, inertial manifolds.

where g is a smooth positive function defined on a set $\omega \subset \mathbb{R}^{\ell-1}$ and f is a dissipative nonlinearity. They prove that, as $\varepsilon \to 0$, the limit equation is the $(\ell-1)$-dimensional boundary value problem

$$\begin{aligned} u_t &= (1/g)\operatorname{div}(g\nabla u) + f(u), && t > 0,\ x \in \omega, \\ \partial_\nu u &= 0, && t > 0,\ x \in \partial\omega. \end{aligned}$$

They compare the semiflows of these equations and establish an important upper-semicontinuity result for the corresponding family of attractors. If $\ell = 2$, they also prove existence of inertial manifolds. A much more general class of thin domains, including domains with holes, was considered by the present authors in [18]. Let $N, M \geq 1$ and let Ω be an arbitrary smooth bounded domain in $\mathbb{R}^\ell := \mathbb{R}^N \times \mathbb{R}^M$. Write (x, y) for a generic point of $\mathbb{R}^N \times \mathbb{R}^M$. Given $\varepsilon > 0$, we squeeze Ω by the factor ε in the y-direction to obtain the squeezed domain Ω_ε. More precisely, let $T_\varepsilon: \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N \times \mathbb{R}^M$, $(x, y) \mapsto (x, \varepsilon y)$ and let $\Omega_\varepsilon := T_\varepsilon(\Omega)$. It was proved in [18] that in this case the family of equations (1.1) singularly converges as $\varepsilon \to 0$ to an abstract parabolic equation on a closed subspace of $H^1(\Omega)$. The equation reads

$$\dot{u} = -A_0 u + \hat{f}(u), \tag{1.2}$$

where $\hat{f}$ is the Nemitski operator induced by f and A_0 is a positive self-adjoint operator. The phase space consists of all H^1 functions whose distributional derivative in the y-direction vanishes. All the results of Hale and Raugel are still valid in this rather more general situation. Under additional assumptions it is possible to charachterize the limit equation (1.2) as a concrete reaction diffusion system of N-dimensional equations, coupled by compatibility and balance boundary conditions (see [18]).

The above mentioned papers deal with the *flat* squeezing of a domain onto a lower dimensional subspace of $\mathbb{R}^\ell$. In the present paper we consider the effect of *curved* squeezing upon the behavior of the solutions of reaction-diffusion equations. The main difference consists in the global nature of the curved squeezing, as opposed to the essentially local nature of the flat squeezing considered in [8] and [18].

Let us briefly describe the geometry of the problem considered here. Let ℓ, k and r be positive integers with $r \geq 2$, $\ell \geq 2$ and $k < \ell$. Let $\mathcal{M} \subset \mathbb{R}^\ell$ be an arbitrary imbedded k-dimensional submanifold of $\mathbb{R}^\ell$ of class C^r. Note that, in the general case considered here, the manifold is *global*, i.e. $\mathcal{M}$ need not be included in a single coordinate chart.

By the Tubular neighborhood theorem (cf e.g. [1]) there exists an open set $\mathcal{U}$ in $\mathbb{R}^\ell$ and a map $\phi: \mathcal{U} \to \mathcal{M}$ of class C^{r-1} such that whenever $x \in \mathcal{U}$ and $p \in \mathcal{M}$ then $\phi(x) = p$ if and only if the vector $x - p$ is orthogonal to $T_p\mathcal{M}$; moreover, $\varepsilon x + (1-\varepsilon)\phi(x) \in \mathcal{U}$ for all $x \in \mathcal{U}$ and all $\varepsilon \in [0, 1]$.

For $\varepsilon \in [0, 1]$ let us define the *curved squeezing* transformation

$$\begin{gathered} \Phi_\varepsilon: \mathcal{U} \to \mathbb{R}^\ell, \\ \Phi_\varepsilon(x) := \varepsilon x + (1-\varepsilon)\phi(x) = \phi(x) + \varepsilon(x - \phi(x)). \end{gathered}$$

Now let Ω be an arbitrary nonempty bounded domain in $\mathbb{R}^\ell$ with Lipschitz boundary and such that $\overline{\Omega} \subset \mathcal{U}$. For $\varepsilon \in\ ]0, 1]$, define the *curved squeezed domain* $\Omega_\varepsilon := \Phi_\varepsilon(\Omega)$.

Let $\varepsilon \in]0,1]$ be arbitrary and consider the Neumann boundary value problem

$$
\begin{aligned}
u_t &= \Delta u + f(u), && t > 0,\ x \in \Omega_\varepsilon, \\
\partial_\nu u &= 0, && t > 0,\ x \in \partial\Omega_\varepsilon
\end{aligned}
\tag{1.3}
$$

on Ω_ε. Here, ν is the exterior normal vector field on $\partial\Omega_\varepsilon$. Suppose that $f \in C^1(\mathbb{R} \to \mathbb{R})$ is dissipative in the sense that

$$
\limsup_{|s|\to\infty} f(s)/s \le -\delta_0 \quad \text{for some } \delta_0 > 0. \tag{1.4}
$$

Furthermore, let f satisfy the growth estimate

$$
|f'(s)| \le C(1 + |s|^\beta) \quad \text{for } s \in \mathbb{R}, \tag{1.5}
$$

where C and $\beta \in [0, \infty[$ are arbitrary real constants. If $\ell > 2$, assume, in addition, that $\beta \le (2^*/2) - 1$, where $2^* = 2\ell/(\ell - 2)$.

This equation can be described in abstract terms as the equation

$$
\dot{u} + \widetilde{A}_\varepsilon u = \hat{f}(u) \tag{1.6}
$$

on $H^1(\Omega_\varepsilon)$. Here, the operator $\widetilde{A}_\varepsilon$ is induced by the pair $(\widetilde{a}_\varepsilon, \widetilde{b}_\varepsilon)$ of bilinear forms, where

$$
\widetilde{a}_\varepsilon(u, v) = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, \mathrm{d}x, \quad u, v \in H^1(\Omega_\varepsilon)
$$

and

$$
\widetilde{b}_\varepsilon(u, v) = \int_{\Omega_\varepsilon} uv \, \mathrm{d}x, \quad u, v \in L^2(\Omega_\varepsilon),
$$

in the sense that

$$
\widetilde{A}_\varepsilon u = w \text{ if and only if } \widetilde{a}_\varepsilon(u, v) = \widetilde{b}_\varepsilon(u, v) \text{ for all } v \in H^1(\Omega_\varepsilon).
$$

Furthermore, $\hat{f}(u) := f \circ u$ is the Nemitski operator defined by f. We can now use the change of variables $u(x) \mapsto u(\widetilde{x})$, where $\widetilde{x} = \Phi_\varepsilon(x)$, to transform equation (1.6) to the equivalent problem

$$
\dot{u} + A_\varepsilon u = \hat{f}(u) \tag{1.7}
$$

on the fixed phase space $H^1(\Omega)$. Equation (1.7) defines a semiflow π_ε on $H^1(\Omega)$, which possesses a global attractor $\mathcal{A}_\varepsilon$. Here, the operator A_ε is defined by the formula

$$
A_\varepsilon(u \circ \Phi_\varepsilon) = (\widetilde{A}_\varepsilon u) \circ \Phi_\varepsilon.
$$

We need a more precise characterization of A_ε. For $x \in \mathcal{U}$ denote by $Q(x)\colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ the orthogonal projection of $\mathbb{R}^\ell \cong T_p\mathbb{R}^\ell$ onto $T_p\mathcal{M}$, where $p := \phi(x)$. Let $P(x) = I - Q(x)$. Note that $P(x)$ is the orthogonal projection of $\mathbb{R}^\ell \cong T_p\mathbb{R}^\ell$ onto the orthogonal complement of $T_p\mathcal{M}$ in $T_p\mathbb{R}^\ell \cong \mathbb{R}^\ell$. For $\varepsilon \in]0,1]$ define

$$
S_\varepsilon(x) := D\Phi_\varepsilon^{-1}(\Phi_\varepsilon(x)) - (1/\varepsilon)P(x)
$$

and

$$J_\varepsilon := \varepsilon^{-(\ell-k)/2} |\det D\Phi_\varepsilon(x)|.$$

Then, as it is proved in [17],

$$\widetilde{a}_\varepsilon(u,v) = \varepsilon^{(\ell-k)/2} a_\varepsilon(u\circ\Phi_\varepsilon, v\circ\Phi_\varepsilon), \quad u,v\in H^1(\Omega_\varepsilon)$$

and

$$\widetilde{b}_\varepsilon(u,v) = \varepsilon^{(\ell-k)/2} b_\varepsilon(u\circ\Phi_\varepsilon, v\circ\Phi_\varepsilon), \quad u,v\in L^2(\Omega_\varepsilon),$$

where $a_\varepsilon\colon H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ is given by

$$a_\varepsilon(u,v) := \int_\Omega J_\varepsilon(x)\langle S_\varepsilon(x)^T\nabla u(x), S_\varepsilon(x)^T\nabla v(x)\rangle\,\mathrm{d}x \\ + \frac{1}{\varepsilon^2}\int_\Omega J_\varepsilon(x)\langle P(x)\nabla u(x), P(x)\nabla v(x)\rangle\,\mathrm{d}x$$

and

$$b_\varepsilon\colon L^2(\Omega)\times L^2(\Omega)\to\mathbb{R}, \quad b_\varepsilon(u,v) := \int_\Omega J_\varepsilon(x)u(x)v(x)\,\mathrm{d}x.$$

It follows that

$$A_\varepsilon u = w \text{ if and only if } a_\varepsilon(u,v) = b_\varepsilon(u,v) \text{ for all } v\in H^1(\Omega).$$

It is proved in [17] that, as $\varepsilon\to 0$, the linear map $S_\varepsilon(x)$ converges (uniformly on compact subsets of $\mathcal{U}$) to a linear map $S_0(x)$ satisfying the property

$$(S_0(x)|_{T_{\phi(x)}\mathcal{M}})^{-1} = D\Phi(x)|_{T_{\phi(x)}\mathcal{M}}$$

Moreover, the function J_ε converges (uniformly on compact subsets of $\mathcal{U}$) to the function

$$J_0(x) := |\det(D\Phi(x)|_{T_{\phi(x)}\mathcal{M}})|.$$

Notice that, for $u\in H^1(\Omega)$,

$$\lim_{\varepsilon\to 0} a_\varepsilon(u,u) = \int_\Omega J_0(x)\langle S_0(x)^T\nabla u(x), S_0(x)^T\nabla v(x)\rangle\,\mathrm{d}x \tag{1.8}$$

if $P(x)\nabla u(x) = 0$ a.e., and $\lim_{\varepsilon\to 0} a_\varepsilon(u,u) = \infty$ otherwise. Let us define the space

$$H^1_s(\Omega) := \{\, u\in H^1(\Omega) \mid P(x)\nabla u(x) = 0 \ \text{ a.e.}\,\}. \tag{1.9}$$

Note that $H^1_s(\Omega)$ is a closed infinite dimensional linear subspace of the Hilbert space $H^1(\Omega)$. On $H^1_s(\Omega)$, define the 'limit' bilinear form

$$a_0\colon H^1_s(\Omega)\times H^1_s(\Omega)\to\mathbb{R}, \quad a_0(u,v) := \int_\Omega J_0(x)\langle S_0(x)^T\nabla u(x), S_0(x)^T\nabla v(x)\rangle\,\mathrm{d}x.$$

Let $L^2_s(\Omega)$ be the closure of $H^1_s(\Omega)$ in $L^2(\Omega)$, and define

$$b_0\colon L^2_s(\Omega)\times L^2_s(\Omega)\to\mathbb{R}, \quad b_0(u,v) := \int_\Omega J_0(x)u(x)v(x)\ \mathrm{d}x.$$

We will denote by A_0 the self-adjoint positive operator generated by the pair (a_0, b_0), i.e.

$$A_0 u = w \text{ if and only if } a_0(u,v) = b_0(u,v) \text{ for all } v \in H^1_s(\Omega).$$

For $\varepsilon \in [0,1]$ and $u \in L^2(\Omega)$ set

$$|u|_\varepsilon := b_\varepsilon(u,u)^{1/2}.$$

The norms $|\cdot|_\varepsilon$ are all equivalent to the L^2 norm on Ω, with equivalence constants independent of ε. For $\varepsilon \in]0,1]$ and for $u \in H^1(\Omega)$ set

$$\|u\|_\varepsilon := (a_\varepsilon(u,u) + b_\varepsilon(u,u))^{1/2}.$$

There exists a constant γ, independent of ε, such that $\gamma\|u\|_{H^1} \leq \|u\|_\varepsilon$ for all $u \in H^1(\Omega)$. Finally, for $\varepsilon = 0$ and $u \in H^1_s(\Omega)$ set

$$\|u\|_0 := (a_0(u,u) + b_0(u,u))^{1/2}.$$

The norm $\|\cdot\|_0$ is equivalent to the H^1-norm restricted to $H^1_s(\Omega)$.

As it is shown in the paper [17], the family of operators $(A_\varepsilon)_{\varepsilon\in]0,1]}$ converges in a strong spectral sense to the operator A_0 in $L^2_s(\Omega)$. One can now consider the abstract parabolic equation

$$\dot{u} + A_0 u = \hat{f}(u). \tag{1.10}$$

on the space $H^1_s(\Omega)$, where $H^1_s(\Omega)$ is defined in (1.9). Equation (1.10) defines a semiflow π_0 on $H^1_s(\Omega)$, which possesses a global attractor $\mathcal{A}_0$. It is proved in [17] that, as $\varepsilon \to 0^+$, the linear semigroups e^{-tA_ε} converge in a singular sense to the semigroup e^{-tA_0} and the semiflows π_ε singularly converge to π_0. Furthermore, an upper semicontinuity result is established for the family $(\mathcal{A}_\varepsilon)_{\varepsilon\in[0,1]}$ of attractors. In the next section we will explain the main features of this limiting procedure and give a description of the limit problem.

2. The limit problem

We start by recalling that (λ, w) is an eigenvalue-eigenvector pair of $(a_\varepsilon, b_\varepsilon)$, $\varepsilon \in [0,1]$ if and only if $a_\varepsilon(w,u) = \lambda b_\varepsilon(w,u)$ for all $u \in H^1(\Omega)$ ($u \in H^1_s(\Omega)$ if $\varepsilon = 0$). This is equivalent to saying that (λ, w) is an eigenvalue-eigenvector pair of the operator A_ε in (L^2, b_ε), i.e. $b_\varepsilon(A_\varepsilon w, u) = \lambda b_\varepsilon(w,u)$ for all $u \in L^2(\Omega)$ ($u \in L^2_s(\Omega)$ if $\varepsilon = 0$).

We now have the following strong spectral convergence result:

Theorem 2.1. *For $\varepsilon \in]0,1]$ let $\lambda_{\varepsilon,1} \leq \lambda_{\varepsilon,2} \leq \lambda_{\varepsilon,3} \leq \dots$ be the repeated sequence of eigenvalues of the pair $(a_\varepsilon, b_\varepsilon)$ and $w_{\varepsilon,1}, w_{\varepsilon,2}, w_{\varepsilon,3}, \dots$ be a corresponding complete (L^2, b_ε)-orthonormal sequence of eigenvectors. Moreover, let $\lambda_{0,1} \leq \lambda_{0,2} \leq \lambda_{0,3} \leq \dots$ be the repeated sequence of eigenvalues of (a_0, b_0). Then the following properties hold:*

1. *For every $j \in \mathbb{N}$, $\lambda_{0,j} = \lim_{\varepsilon\to 0^+} \lambda_{\varepsilon,j}$.*
2. *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers converging to 0. Then there is a subsequence of $(\varepsilon_n)_{n\in\mathbb{N}}$, again denoted by $(\varepsilon_n)_{n\in\mathbb{N}}$, and there exists a complete (L^2_s, b_0)-orthonormal system $(w_{0,j})_{j\in\mathbb{N}}$ of eigenvectors of (a_0, b_0) corresponding to $(\lambda_{0,j})_{j\in\mathbb{N}}$ such that, for every $j \in \mathbb{N}$,*

$$\|w_{\varepsilon_n,j} - w_{0,j}\|_{\varepsilon_n} \to 0 \qquad \text{as } n \to \infty. \quad \square$$

Now we consider the abstract parabolic equation (1.10) on the space $H^1_s(\Omega)$. Equation (1.10) defines a semiflow π_0 on $H^1_s(\Omega)$, which possesses a global attractor $\mathcal{A}_0$. The linear singular convergence result alluded to in Section 1 reads as follows:

Theorem 2.2. *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers converging to zero. Moreover, assume that $u \in L^2_s(\Omega)$ and $(u_n)_{n\in\mathbb{N}}$ is a sequence in $L^2(\Omega)$ such that $|u_n - u|_{L^2} \to 0$ as $n \to \infty$. Then for all β and $\gamma \in]0,\infty[$ with $\beta < \gamma$*

$$\sup_{t\in[\beta,\gamma]} \left\|e^{-tA_{\varepsilon_n}}u_n - e^{-tA_0}u\right\|_{\varepsilon_n} \to 0 \quad \textit{as } n \to \infty. \quad \square$$

Now we can state the following nonlinear singular convergence result:

Theorem 2.3. *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers converging to zero. Write $\pi_n := \pi_{\varepsilon_n}$ and $\pi := \pi_0$. Assume that $u \in H^1_s(\Omega)$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $H^1(\Omega)$ such that $|u_n - u|_{L^2} \to 0$ as $n \to \infty$. Let $b \in]0,\infty[$ and suppose that $u_n\pi_n s$ and $u\pi s$ are defined for all $s \in [0,b]$, all $n \in \mathbb{N}$ and*

$$\sup_{n\in\mathbb{N}} \sup_{s\in[0,b]} \|u_n\pi s\|_{\varepsilon_n} \le M, \quad \sup_{n\in\mathbb{N}} \sup_{s\in[0,b]} \|u\pi s\|_{\varepsilon_n} \le M$$

for some constant $M \in [0, infty[$. Finally let $t_0 \in]0,b]$ and $(t_n)_{n\in\mathbb{N}}$ is a sequence in $]0,b]$ converging to t_0. Under these assumptions

$$\left\|u_n\pi_n t_n - u\pi t_0\right\|_{\varepsilon_n} \to 0 \quad \textit{as } n \to \infty. \quad \square$$

The upper-semicontinuity result for the family $(\mathcal{A}_\varepsilon)_{\varepsilon\in[0,1]}$ of attractors reads as follows:

Theorem 2.4. *Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a C^1-function satisfying the growth condition (1.5). For $\varepsilon \in [0,1]$ let $\mathcal{A}_\varepsilon$ be the union of all full bounded orbits of π_ε. Then, for all $\varepsilon \in [0,1]$, π_ε is a global semiflow and the set $\mathcal{A}_\varepsilon$ is nonempty, compact, connected in $H^1(\Omega)$. Furthermore, the set $\mathcal{A}_\varepsilon$ attracts every set B which is bounded in $H^1(\Omega)$ for $\varepsilon \in]0,1]$ and in $H^1_s(\Omega)$ for $\varepsilon = 0$. In other words, for every such B*

$$\lim_{t\to\infty} \sup_{u\in B} \inf_{v\in\mathcal{A}_\varepsilon} \|u\pi_\varepsilon t - v\|_\varepsilon = 0.$$

The family $(\mathcal{A}_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ is upper-semicontinuous at $\varepsilon = 0$ with respect to the family $\|\cdot\|_\varepsilon$ of norms, i.e.

$$\lim_{\varepsilon\to 0^+} \sup_{u\in\mathcal{A}_\varepsilon} \inf_{v\in\mathcal{A}_0} \|u - v\|_\varepsilon = 0. \quad \square$$

The proofs of Theorems 2.2, 2.3 and 2.4 can be found in [17] (cf also [18]).

The spaces $H^1_s(\Omega)$ and $L^2_s(\Omega)$ and the operator A_0 are defined in a very abstract way. It is therefore worthwhile providing a more direct description of $H^1_s(\Omega)$ and, at least in some special cases, giving a complete characterization of the operator A_0.

For $p \in \mathcal{M}$ define the *normal section Ω_p of Ω at p* to be the set of all $x \in \Omega$ with $\phi(x) = p$. We say that Ω has *connected normal sections* if the set Ω_p is connected for all $p \in \mathcal{M}$.

Given an arbitrary positive integer m, we denote by $\mathcal{H}^m$ the m-dimensional Hausdorff measure on $\mathbb{R}^\ell$ induced by the Euclidean metric. let $S \subset \mathcal{M}$ be open in $\mathcal{M}$. We denote by $L^2(S)$ (resp. $L^2_{\mathrm{loc}}(S)$) the set of all square integrable (resp. locally square integrable) $\mathcal{H}^k$-measurable functions defined on S. Besides, we denote by $\mathbb{L}^2(S)$ (resp. $\mathbb{L}^2_{\mathrm{loc}}(S)$) the space of all $\mathcal{H}^k$-measurable tangent vector fields X on S such that the function $p \mapsto \langle X(p), X(p)\rangle$ is integrable (resp. locally integrable) on S.

The first of our results shows that functions in $H^1_s(\Omega)$ are a.e. (relative to the corresponding Hausdorff measures) constant along the connected components of Ω_p:

Theorem 2.5. *For $u \in H^1(\Omega)$ the following conditions are equivalent:*

(1) *$P(x)\nabla u(x) = 0$ a.e. in Ω.*
(2) *There exists a set $Z \subset \mathcal{M}$, $\mathcal{H}^k(Z) = 0$, and for all $p \in \mathcal{M} \setminus Z$ there exists a set $S_p \subset \phi^{-1}(p)$, $\mathcal{H}^{\ell-k}(S_p) = 0$, such that the following property holds:*
for all $p \in \mathcal{M} \setminus Z$ and for all $\overline{x} \in \Omega_p$ there exists a constant $v(p, \overline{x}) \in \mathbb{R}$ such that $u(x) = v(p, \overline{x})$ for all $x \in \Omega_p(\overline{x}) \setminus S_p$. □

If Ω has *connected normal sections*, then we can completely characterize the spaces $L^2_s(\Omega)$ and $H^1_s(\Omega)$. In fact, set $\mathcal{G} := \phi(\Omega)$ and define $\mu(p) := \mathcal{H}^{\ell-k}(\Omega_p)$ for $p \in \mathcal{G}$. The set $\mathcal{G}$ is open in $\mathcal{M}$ by the surjective mapping theorem, since $D\phi(x): \mathbb{R}^\ell \to T_{\phi(x)}\mathcal{M}$ is surjective for all $x \in \mathcal{U}$. Moreover, by the coarea formula

$$\int_{\mathcal{U}} J_0(x) g(x)\,\mathrm{d}x = \int_{\mathcal{M}} \left(\int_{\phi^{-1}\{p\}} g(x)\,\mathrm{d}\mathcal{H}^{\ell-k}(x) \right) \mathrm{d}\mathcal{H}^k(p),$$

the function $\mu: \mathcal{G} \to \mathbb{R}$ is $\mathcal{H}^k$-measurable and, in fact, integrable on $\mathcal{G}$.

We now have the following result.

Theorem 2.6. *Assume that Ω has connected normal sections. Let $u \in L^2_s(\Omega)$. Then there exists a null set S in $\mathbb{R}^\ell$ and a function $v \in L^2_{\mathrm{loc}}(\mathcal{G})$ such that $u(x) = v(\phi(x))$ for all $x \in \Omega \setminus S$; moreover $\mu^{1/2} v \in L^2(\mathcal{G})$. If $u \in H^1_s(\Omega)$, then $v \in H^1_{\mathrm{loc}}(\mathcal{G})$,*

$$\nabla u(x) = D\phi(x)^T \nabla v(\phi(x)) \text{ a.e. in } \Omega \tag{2.1}$$

and $\mu^{1/2}\nabla v \in \mathbb{L}^2(\mathcal{G})$. Conversely, let $v \in L^2_{\mathrm{loc}}(\mathcal{G})$ be such that $\mu^{1/2} v \in L^2(\mathcal{G})$ and set $u(x) := v(\phi(x))$. Then $u \in L^2_s(\Omega)$. If $v \in H^1_{\mathrm{loc}}(\mathcal{G})$ and $\mu^{1/2}\nabla v \in \mathbb{L}^2(\mathcal{G})$, then $u \in H^1_s(\Omega)$. □

The proofs of Theorems 2.5 and 2.6 can be found in [21]. Under some additional regularity hypotheses, we can also give a simple description of the limit operator A_0 and of the corresponding limit equation. In particular, it turns out that A_0 is equivalent to a relatively bounded perturbation of the Laplace-Beltrami operator on an open subset of $\mathcal{M}$. Suppose that Ω has connected normal sections. Define

$$L^2(\mu, \mathcal{G}) := \{\, v \in L^2_{\mathrm{loc}}(\mathcal{G}) \mid \mu^{1/2} v \in L^2(\mathcal{G}) \,\}.$$

Then $L^2(\mu, \mathcal{G})$, endowed with the scalar product

$$b_\mu(v_1, v_2) := \int_{\mathcal{G}} \mu(p) v_1(p) v_2(p)\,\mathrm{d}\mathcal{H}^k(p),$$

is a Hilbert space. Moreover, define

$$H^1(\mu,\mathcal{G}) := \{\, v \in H^1_{\mathrm{loc}}(\mathcal{G}) \mid \mu^{1/2} v \in L^2(\mathcal{G}),\ \mu^{1/2}\nabla v \in \mathbb{L}^2(\mathcal{G}) \,\}.$$

and

$$a_\mu(v_1,v_2) := \int_{\mathcal{G}} \mu(p)\langle \nabla v_1(p), \nabla v_2(p)\rangle \,\mathrm{d}\mathcal{H}^k(p) \quad \text{for } v_1 \text{ and } v_2 \in H^1(\mu,\mathcal{G}).$$

Then $H^1(\mu,\mathcal{G})$, endowed with the scalar product $a_\mu(\cdot,\cdot)+b_\mu(\cdot,\cdot)$, is a Hilbert space. Let $\jmath$ be the linear map

$$\jmath\colon L^2_s(\Omega) \to L^2(\mu,\mathcal{G}), \quad u \mapsto v,$$

where v is the function given by Theorem 2.6. Then $\jmath$ is an isometry of the Hilbert space $(L^2_s(\Omega), b_0(\cdot,\cdot))$ onto $L^2(\mu,\mathcal{G})$. Furthermore, the restriction of the map $\jmath$ to $H^1_s(\Omega)$ is an isometry of the Hilbert space $(H^1_s(\Omega), a_0(\cdot,\cdot)+b_0(\cdot,\cdot))$ onto $H^1(\mu,\mathcal{G})$.

Let A_μ be the self-adjoint operator in $L^2(\mu,\mathcal{G})$ generated by the pair (a_μ,b_μ). Then $\jmath$ restricts to an isometry of $D(A_0)$ onto $D(A_\mu)$ and $A_0 = \jmath^{-1}A_\mu\,\jmath$.

Denote by $\partial\mathcal{G}$ the topological boundary of $\mathcal{G}$ in $\mathcal{M}$. Suppose that $\mathcal{G}$ is orientable (as a submanifold of $\mathcal{M}$), $\partial\mathcal{G}=\emptyset$ and the function μ is of class C^1 on $\mathcal{G}$. By the regularity theory for elliptic equations and by the divergence formula on Riemannian manifolds, we finally obtain

$$\begin{aligned} D(A_\mu) &= H^2(\mathcal{G}) \\ (A_\mu u)(p) &= -(1/\mu(p))\,\mathrm{div}(\mu(p)\nabla u(p)) \end{aligned} \tag{2.2}$$

As a consequence, the limit equation (1.10) is equivalent to the following reaction-diffusion equation on $\mathcal{G}$:

$$u_t = (1/\mu(p))\,\mathrm{div}(\mu(p)\nabla u) + f(u(p)), \quad t>0,\ p\in\mathcal{G}.$$

Instead of assuming $\partial\mathcal{G}=\emptyset$ we may alternatively assume that $\partial\mathcal{G}$ is a $(k-1)$-dimensional C^2-submanifold of $\mathcal{M}$ and that the function μ can be extended to a strictly positive C^1-function on $\overline{\mathcal{G}}$. In this case it not difficult to see that the domain of the operator A_μ is the set of all functions $u\in H^2(\mathcal{G})$ satisfying the boundary condition

$$\langle \nabla u(p), \nu(p)\rangle = 0 \quad \mathcal{H}^{k-1}\text{-a.e. on } \partial\mathcal{G}$$

in the sense of traces. Here $\nu(p)\in T_p\mathcal{M}$, $p\in\partial\mathcal{G}$, is the outward normal vector field on $\partial\mathcal{G}$. Again, for $u\in D(A_\mu)$, one has

$$(A_\mu u)(p) = -(1/\mu(p))\,\mathrm{div}(\mu(p)\nabla u(p)) \quad \text{a.e. in } \mathcal{G}.$$

Thus the limit equation (1.10) takes the form

$$\begin{aligned} u_t &= (1/\mu(p))\,\mathrm{div}(\mu(p)\nabla u) + f(u(p)), && t>0,\ p\in\mathcal{G}, \\ \langle \nabla u(p), \nu(p)\rangle &= 0, && t>0,\ p\in\partial\mathcal{G}. \end{aligned}$$

For the theory of Sobolev spaces on Riemannian manifolds the reader is referred to [11] and [24].

Using the above characterization of A_0 we will show in Section 4 that for thin domains close to spheres, a spectral gap condition is satisfied, which can be used to prove existence of inertial manifolds.

3. Inertial manifolds

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a C^1-function satisfying the growth estimate (1.5) together with the dissipativeness condition (1.4). Then, by Theorem 2.4, for every $\varepsilon \geq 0$ the semiflow $\pi_\varepsilon := \pi_{\varepsilon,\hat{f}}$ possesses a global attractor $\mathcal{A}_\varepsilon$. Moreover, the family $(\mathcal{A}_\varepsilon)_{\varepsilon\geq 0}$ of attractors is upper semicontinuous at $\varepsilon = 0$. If the eigenvalues of the limit operator A_0 satisfy the gap condition

$$\limsup_{\nu\to\infty} \frac{\lambda_{0,\nu+1} - \lambda_{0,\nu}}{{\lambda_{0,\nu}}^{1/2}} > 0, \tag{3.1}$$

then, as we shall see in Theorem 3.3 below, there exists an $\varepsilon_0 > 0$ and a family $\mathcal{I}_\varepsilon$, $0 \leq \varepsilon \leq \varepsilon_0$ of C^1 inertial manifolds of some finite dimension ν such that, whenever $0 \leq \varepsilon \leq \varepsilon_0$, then $\mathcal{A}_\varepsilon \subset \mathcal{I}_\varepsilon$ and the manifold $\mathcal{I}_\varepsilon$ is locally attracting and locally invariant relative to the semiflow π_ε on a neighborhood of the attractor $\mathcal{A}_\varepsilon$. Moreover, the flows on the inertial manifolds $\mathcal{I}_\varepsilon$ converge in the (regular) C^1-sense to the flow on $\mathcal{I}_0$.

The proof of Theorem 3.3 is based on the method of functions of exponential growth, used before by a number of researchers (cf. [5], [23] and the references contained in these papers). First one chooses an open set U in $H^1(\Omega)$ which includes all the attractors $\mathcal{A}_\varepsilon$, $\varepsilon \in [0,\varepsilon_0]$, $\varepsilon_0 > 0$ small. Then one modifies the Nemitski operator $\hat{f}$ by finding a globally Lipschitzian map $g\colon H^1(\Omega) \to L^2(\Omega)$ with $\hat{f}(u) = g(u)$ for $u \in U$. For fixed $\varepsilon \in [0,\varepsilon_0]$, one seeks an invariant manifold $\mathcal{I}_\varepsilon$ for the modified semiflow $\pi_{\varepsilon,g}$ in the form $\mathcal{I}_\varepsilon = \Lambda_\varepsilon(\mathbb{R}^\nu)$, where $\Lambda_\varepsilon\colon \mathbb{R}^\nu \to H^1(\Omega)$ is a map obtained from the contraction mapping principle applied to a properly defined nonlinear operator Γ_ε defined on a certain space of maps $y\colon]-\infty,0] \to H^1(\Omega)$ of exponential growth. If the operator Γ_ε is a contraction, then the map Λ_ε is well-defined and $\mathcal{A}_\varepsilon \subset \mathcal{I}_\varepsilon$. It follows that $\mathcal{I}_\varepsilon$ is invariant with respect to solutions of the original semiflow $\pi_{\varepsilon,\hat{f}}$ as long as these solutions stay in the open set U. One can even find an open set $V \subset \mathbb{R}^\nu$ such that for $\varepsilon \in [0,\varepsilon_0]$ the set $\Lambda_\varepsilon(V)$ is positively invariant with respect to $\pi_{\varepsilon,\hat{f}}$ and $\mathcal{A}_\varepsilon \subset \Lambda_\varepsilon(V) \subset U$.

In order to describe in detail this procedure, we need to introduce some notation. Given $\mu \in \mathbb{R}$, a Banach space $(Y, |\cdot|_Y)$ and a function $y\colon]-\infty,0] \to Y$ we write

$$|y|_{\mu,|\cdot|_Y} := \sup_{t\in]-\infty,0]} e^{\mu t}|y(t)|_Y$$

and we denote by $BC^\mu(Y, |\cdot|_Y)$ the set of all continuous functions $y\colon]-\infty,0] \to Y$ such that $|y|_{\mu,|\cdot|_Y} < \infty$. The space $BC^\mu(Y, |\cdot|_Y)$ is a Banach space with respect to the norm $y \mapsto |y|_{\mu,|\cdot|_Y}$. In particular, we write

(1) $BC^\mu(L^2(\Omega),\varepsilon) := BC^\mu(L^2(\Omega), |\cdot|_\varepsilon)$, with the norm $|y|_{\mu,\varepsilon} := |y|_{\mu,|\cdot|_\varepsilon}$ for $\varepsilon \geq 0$,
(2) $BC^\mu(L^2_s(\Omega),0) := BC^\mu(L^2_s(\Omega), |\cdot|_0)$, with the norm $|y|_{\mu,0} := |y|_{\mu,|\cdot|_0}$,
(3) $BC^\mu(H^1(\Omega),\varepsilon) := BC^\mu(H^1(\Omega), \|\cdot\|_\varepsilon)$ with the norm $\|y\|_{\mu,\varepsilon} := \|y\|_{\mu,\|\cdot\|_\varepsilon}$ for $\varepsilon > 0$,
(4) $BC^\mu(H^1_s(\Omega),0) := BC^\mu(H^1_s(\Omega), \|\cdot\|_0)$ with the norm $\|y\|_{\mu,\varepsilon} := \|y\|_{\mu,\|\cdot\|_0}$ for $\varepsilon = 0$.

For every $\varepsilon \in [0,1]$ and every $\nu \in \mathbb{N}$ let $X_{\varepsilon,\nu,1}$ be the span of the vectors $w_{\varepsilon,j}$, $j = 1, \ldots, \nu$ and let $X_{\varepsilon,\nu,2}$ be the orthogonal complement of $X_{\varepsilon,\nu,1}$ in $L^2(\Omega)$ if

$\varepsilon > 0$ and in $L^2_s(\Omega)$ if $\varepsilon = 0$. Let $A_{\varepsilon,\nu,i}$ be the restriction of A_ε to $X_{\varepsilon,\nu,i}$ for $i = 1, 2$. Let $E_{\varepsilon,\nu}\xi := \sum_{j=1}^{\nu} \xi_j w_{\varepsilon,j}$, $\xi \in \mathbb{R}^\nu$ and $P_{\varepsilon,\nu,i}$ be the orthogonal projection of $L^2(\Omega)$ onto $X_{\varepsilon,\nu,i}$, $i = 1, 2$ if $\varepsilon > 0$ and $P_{\varepsilon,\nu,i}$ be the orthogonal projection of $L^2_s(\Omega)$ onto $X_{\varepsilon,\nu,i}$, $i = 1, 2$ if $\varepsilon = 0$.

Now we try to define the operator $\Gamma_\varepsilon = \Gamma_{\varepsilon,\nu}$ in the following way: for $\xi \in \mathbb{R}^\nu$ and y in a suitable space of functions of exponential growth ζ with values in $H^1(\Omega)$,

$$\begin{aligned}\Gamma_{\varepsilon,\nu}(\xi, y)(t) = e^{-A_{\varepsilon,\nu,1}t} E_{\varepsilon,\nu}\xi + \int_0^t e^{-A_{\varepsilon,\nu,1}(t-s)} P_{\varepsilon,\nu,1} g(y(s))\, \mathrm{d}s \\ + \int_{-\infty}^t e^{-A_{\varepsilon,\nu,2}(t-s)} P_{\varepsilon,\nu,2} g(y(s))\, \mathrm{d}s.\end{aligned}$$

If $\phi_\varepsilon(\xi)\colon]-\infty, 0] \to H^1(\Omega)$ is a fixed point of $\Gamma_{\varepsilon,\nu}(\xi, \cdot)$, then $\phi_\varepsilon(\xi)$ can be extended to a full trajectory of π_ε, with exponential growth ζ at $-\infty$. The map Λ_ε will be defined as $\Lambda_\varepsilon\colon \xi \mapsto \phi_\varepsilon(\xi)(0)$.

First of all, we must choose ζ in such a way that the operator $\Gamma_{\varepsilon,\nu}$ is at least well defined. To this end, we recall that, for every $\varepsilon \in [0, 1]$ and every $\nu \in \mathbb{N}$

$$\begin{aligned}|e^{-A_{\varepsilon,\nu,1}t} u|_\varepsilon &\le e^{-\lambda_{\varepsilon,\nu}t} |u|_\varepsilon \quad u \in X_{\varepsilon,\nu,1},\ t \le 0,\\ |e^{-A_{\varepsilon,\nu,2}t} u|_\varepsilon &\le e^{-\lambda_{\varepsilon,\nu+1}t} |u|_\varepsilon \quad u \in X_{\varepsilon,\nu,2},\ t > 0,\\ \|e^{-A_{\varepsilon,\nu,1}t} u\|_\varepsilon &\le (\lambda_{\varepsilon,\nu} + 1)^{1/2} e^{-\lambda_{\varepsilon,\nu}t} |u|_\varepsilon \quad u \in X_{\varepsilon,\nu,1},\ t \le 0,\\ \|e^{-A_{\varepsilon,\nu,2}t} u\|_\varepsilon &\le \left((\lambda_{\varepsilon,\nu+1} + 1)^{1/2} + C_{1/2} t^{-1/2}\right) e^{-\lambda_{\varepsilon,\nu+1}t} |u|_\varepsilon \quad u \in X_{\varepsilon,\nu,2},\ t > 0.\end{aligned}$$

Thus we are led to choose the exponent $\zeta \in]\lambda_{\varepsilon,\nu}, \lambda_{\varepsilon,\nu+1}[$. In that case we can write $\Gamma_{\varepsilon,\nu}(\xi, y) = \Xi_{\varepsilon,\nu}(\xi, g \circ y)$, where

$$\Xi_{\varepsilon,\nu}(\xi, y)(t) = e^{-A_{\varepsilon,\nu,1}t} E_{\varepsilon,\nu}\xi + K_{\varepsilon,\nu} y(t)$$

and, for $\xi \in \mathbb{R}^\nu$, $y\colon]-\infty, 0] \to L^2(\Omega)$ and $t \le 0$,

$$K_{\varepsilon,\nu} y(t) = \int_0^t e^{-A_{\varepsilon,\nu,1}(t-s)} P_{\varepsilon,\nu,1} y(s)\, \mathrm{d}s + \int_{-\infty}^t e^{-A_{\varepsilon,\nu,2}(t-s)} P_{\varepsilon,\nu,2} y(s)\, \mathrm{d}s, \tag{3.2}$$

whenever the right hand side of (3.2) makes sense. The main properties of the operator $\Xi_{\varepsilon,\nu}$, deduced from the estimates given above and from Theorem 2.2, are stated in the following

Lemma 3.1. *Let $\varepsilon \in [0, 1]$, $\nu \in \mathbb{N}$ and $\zeta \in]\lambda_{\varepsilon,\nu}, \lambda_{\varepsilon,\nu+1}[$ be arbitrary. Then $\Xi_{\varepsilon,\nu}$ maps $\mathbb{R}^\nu \times BC^\zeta(L^2(\Omega))$ into $BC^\zeta(H^1(\Omega), \varepsilon)$ for $\varepsilon > 0$ and $\Xi_{\varepsilon,\nu}$ maps $BC^\zeta(L^2_s(\Omega))$ into $BC^\zeta(H^1_s(\Omega), \varepsilon)$ for $\varepsilon = 0$. Moreover, for $\varepsilon > 0$ and $y \in BC^\zeta(L^2(\Omega))$ (resp. for $\varepsilon = 0$ and $y \in BC^\zeta(L^2_s(\Omega))$)*

$$|K_{\varepsilon,\nu} y|_{\zeta,\varepsilon} \le \left(\frac{1}{\zeta - \lambda_{\varepsilon,\nu}} + \frac{1}{\lambda_{\varepsilon,\nu+1} - \zeta}\right) |y|_{\zeta,\varepsilon},$$

and

$$\|K_{\varepsilon,\nu} y\|_{\zeta,\varepsilon} \le \left(\frac{(\lambda_{\varepsilon,\nu} + 1)^{1/2}}{\zeta - \lambda_{\varepsilon,\nu}} + \frac{(\lambda_{\varepsilon,\nu+1} + 1)^{1/2}}{\lambda_{\varepsilon,\nu+1} - \zeta} + C'_{1/2}(\lambda_{\varepsilon,\nu+1} - \zeta)^{-1/2}\right) |y|_{\zeta,\varepsilon}.$$

If $\lambda_{0,\nu} < \zeta < \lambda_{0,\nu+1}$, $\varepsilon_n \to 0^+$, $\xi_n \to \xi_0$ in $\mathbb{R}^\nu$ and $y_n \to y_0$ in $BC^\zeta(L^2(\Omega))$, where $y_0 \in BC^\zeta(L^2_s(\Omega))$, then, for all n large enough, $\lambda_{\varepsilon_n,\nu} < \zeta < \lambda_{\varepsilon_n,\nu+1}$ and

$$\|\Xi_{\varepsilon_n,\nu}(\xi_n, y_n) - \Xi_{0,\nu}(\xi_0, y_0)\|_{\zeta,\varepsilon_n} \to 0 \quad \text{as } n \to \infty. \quad \square$$

Now, as g is globally lipschitzian with a Lipschitz constant $\widetilde{L}$, it follows that, for a fixed $\xi \in \mathbb{R}^\nu$, the map $\Gamma_{\varepsilon,\nu}(\xi, \cdot)$ is globally lipschitzian on $BC^\zeta(H^1(\Omega), \varepsilon)$. Its Lipschitz constant is

$$\widetilde{L}\left(\frac{(\lambda_{\varepsilon,\nu}+1)^{1/2}}{\zeta - \lambda_{\varepsilon,\nu}} + \frac{(\lambda_{\varepsilon,\nu+1}+1)^{1/2}}{\lambda_{\varepsilon,\nu+1} - \zeta} + C'_{1/2}(\lambda_{\varepsilon,\nu+1} - \zeta)^{-1/2}\right).$$

The problem is that the operator $\Gamma_{\varepsilon,\nu}$ is **not** a contraction. In fact, its Lipschitz constant **cannot** be made small by letting ν tend to $+\infty$. In order to overcome this difficulty, one usually procedes in the following way (see e.g. [8] or [16]): one first finds some L^∞-estimates for the attractors $\mathcal{A}_\varepsilon$, e.g. by using comparison principles; then one cuts off the *nonlinearity* f outside a large interval of $\mathbb{R}$. In such a way the modified nonlinearity induces a globally lipschitzian Nemitski operator from $H^1(\Omega)$ into $H^1(\Omega)$. Since it can be shown that

$$\|K_{\varepsilon,\nu}y\|_{\zeta,\varepsilon} \le \left(\frac{1}{\zeta - \lambda_{\varepsilon,\nu}} + \frac{1}{\lambda_{\varepsilon,\nu+1} - \zeta}\right)\|y\|_{\zeta,\varepsilon},$$

the gap condition (3.1) easily implies that the Lipschitz constant of $\Gamma_{\varepsilon,\nu}$ can be made small by letting ν tend to $+\infty$. The contraction principle then yields the desired existence result. However, we do not this approach completely satisfactory, since the manifolds constructed in this way are inertial manifolds only with respect to the modified semiflows, and the latter are equal to the original semiflows π_ε only on the attractor $\mathcal{A}_\varepsilon$ but are different from π_ε on every neighborhood of it. We point out that these kind of difficulties are not due to the particular technique we have chosen (the method of function of exponential growth). In fact, even if alternative methods are used, e.g. the cone-squeezing technique developed by J. Mallet Paret, G. Sell and other authors (see e.g. [16]), the same difficulties appear as soon as one tries to exploit the contraction principle.

Fortunately, these difficulties can be overcome by the use of an ingenious idea due to Brunovský and Tereščák (see Theorem 4.1 in [3] and its proof). This idea simply consists in working with a different, though equivalent, norm on $H^1(\Omega)$. More precisely, given positive numbers l and L, we introduce the following equivalent norm

$$\|\!|u|\!\|_\varepsilon = L|u|_\varepsilon + l\|u\|_\varepsilon \tag{3.3}$$

on $H^1(\Omega)$. Similarly as in [3] we now seek to choose the constants l and L in such a way that the operator Γ_ε is a uniform contraction with respect to the norm $\|\!|\cdot|\!\|_\varepsilon$. That this is possible follows from the following C^1-cut-off-result for Nemitski operators:

Proposition 3.2. *Let Ω be an open, bounded subset of $\mathbb{R}^\ell$, with Lipschitz boundary. Let $f \in C^1(\mathbb{R} \to \mathbb{R})$ satisfy the growth estimate (1.5). If $\ell > 2$, assume, in addition, that the growth exponent β in (1.5) is subcritical, i.e. $\beta < (2^*/2) - 1$, where, as* *before,* $2^* = 2\ell/(\ell-2) > 2$. *Let* l *be an arbitrary positive real number and* B *be an arbitrary bounded subset of* $H^1(\Omega)$. *Then there exists an open set* $U = U(l,B) \subset H^1(\Omega)$ *including* B*, a positive real number* $L = L(l,B)$ *and a map* $g = g(l,B) \in C^1(H^1(\Omega) \to L^2(\Omega))$ *with* $\hat{f}(u) = g(u)$ *for* $u \in U$ *and such that* g *maps* $H^1_s(\Omega)$ *into* $L^2_s(\Omega)$ *and satisfies the estimates*

$$\sup_{u \in H^1(\Omega)} |g(u)|_\varepsilon < \infty \tag{3.4}$$

$$|g(u) - g(v)|_\varepsilon \le L|u-v|_\varepsilon + l\|u-v\|_\varepsilon \quad \textit{for } u,\, v \in H^1(\Omega) \tag{3.5}$$

and

$$|Dg(u)v|_\varepsilon \le L|v|_\varepsilon + l\|v\|_\varepsilon \quad \textit{for } u,\, v \in H^1(\Omega). \quad \square \tag{3.6}$$

The key point in the proof of Proposition 3.2 is that, for all $q > 2$ such that $0 < \ell/2 - \ell/q < 1$ and $q \ge 2(\beta+1)$, $\hat{f}$ is a C^1-map from $L^q(\Omega)$ to $L^2(\Omega)$. Moreover, both $|\hat{f}(u)|_{L^2}$ and $|D\hat{f}(u)|_{L(L^q(\Omega)\to L^2(\Omega))}$ are bounded on bounded subsets of $L^q(\Omega)$. Thus there is a real positive constant M with $|u|_{L^q} < M$ for $u \in B$. Now we can define the function g by $g(u) := h(u)\hat{f}(u)$ for $u \in L^q(\Omega)$, where $h(u) := \phi(|u|_{L^q}{}^q)$ and $\phi \in C^1(\mathbb{R} \to \mathbb{R})$ is such that $\phi(x) = 1$ if $|x| \le M^q$ and $\phi(x) = 0$ if $|x| > (2M)^q$. Estimates (3.5) and (3.6) are an easy consequence of the Gagliardo-Nirenberg and Young inequalities.

With this in mind, we can state our main existence result on inertial manifolds.

Theorem 3.3. *Suppose that* $f \in C^1(\mathbb{R} \to \mathbb{R})$ *satisfies the growth condition* (1.5) *and the dissipativness condition* (1.4). *Moreover, suppose that the eigenvalues of* A_0 *satisfy the gap condition* (3.1). *Then there are an* $\varepsilon_0 > 0$ *and an open bounded set* $U \subset H^1(\Omega)$ *such that for every* $\varepsilon \in [0, \varepsilon_0[$ *the attractor* $\mathcal{A}_\varepsilon$ *of the semiflow* $\pi_{\varepsilon,\hat{f}}$ *lies in* U.

Furthermore, there exists a globally Lipschitzian map $g \in C^1(H^1(\Omega) \to L^2(\Omega))$ *with* $g(u) = \hat{f}(u)$ *for* $u \in U$.

Besides, there is a positive integer ν *and for every* $\varepsilon \in [0, \varepsilon_0[$ *there is a map* $\Lambda_\varepsilon \in C^1(\mathbb{R}^\nu \to H^1(\Omega))$ *if* $\varepsilon > 0$ *and* $\Lambda_\varepsilon \in C^1(\mathbb{R}^\nu \to H^1_s(\Omega))$ *if* $\varepsilon = 0$ *such that*

$$P_{\varepsilon,\nu,1} \circ \Lambda_\varepsilon = E_{\varepsilon,\nu} \tag{3.7}$$

and $\Lambda_\varepsilon(\mathbb{R}^\nu)$ *is an invariant manifold with respect to the semiflow* $\pi_{\varepsilon,g}$.

Finally, there is an open set $V \subset \mathbb{R}^\nu$ *such that for every* $\varepsilon \in [0, \varepsilon_0[$

$$\mathcal{A}_\varepsilon \subset \Lambda_\varepsilon(V) \subset U$$

and the set $\Lambda_\varepsilon(V)$ *is positively invariant with respect to the semiflow* $\pi_{\varepsilon,\hat{f}}$.

The reduced equation on $\Lambda_\varepsilon(\mathbb{R}^\nu)$ *takes the form*

$$\dot{\xi} = v_\varepsilon(\xi), \quad \xi \in \mathbb{R}^\nu, \tag{3.8}$$

where

$$v_\varepsilon \colon \mathbb{R}^\nu \to \mathbb{R}^\nu, \quad \xi \mapsto -A_\varepsilon E_{\varepsilon,\nu}\xi + P_{\varepsilon,\nu,1}g(\Lambda_\varepsilon(\xi)).$$

Moreover, whenever $\varepsilon_n \to 0^+$ *and* $\xi_n \to \xi_0$ *in* $\mathbb{R}^\nu$*, then*

$$\|\Lambda_{\varepsilon_n}(\xi_n) - \Lambda_0(\xi_0)\|_{\varepsilon_n} + \sum_{j=1}^{\nu} \|\partial_j \Lambda_{\varepsilon_n}(\xi_n) - \partial_j \Lambda_0(\xi_0)\|_{\varepsilon_n} \to 0 \tag{3.9}$$

and

$$|v_{\varepsilon_n}(\xi_n) - v_0(\xi_0)|_{\mathbb{R}^\nu} + \sum_{j=1}^{\nu} |\partial_j v_{\varepsilon_n}(\xi_n) - \partial_j v_0(\xi_0)|_{\mathbb{R}^\nu} \to 0. \tag{3.10}$$

Sketch of the proof. For $\nu \in \mathbb{N}$ with $\lambda_{0,\nu+1} - \lambda_{0,\nu} > 0$ define $\eta_\nu = (\lambda_{0,\nu+1} - \lambda_{0,\nu})/5$ and $I_\nu = [\lambda_{0,\nu} + 2\eta_\nu, \lambda_{0,\nu} + 3\eta_\nu]$. It follows that

$$\sup_{\zeta \in I_\nu} \left(\frac{1}{\zeta - \lambda_{0,\nu}} + \frac{1}{\lambda_{0,\nu+1} - \zeta} \right) < C_{\nu,1} := 6 \frac{1}{\lambda_{0,\nu+1} - \lambda_{0,\nu}} \tag{3.11}$$

and

$$\sup_{\zeta \in I_\nu} \left(\frac{(\lambda_{0,\nu} + 1)^{1/2}}{\zeta - \lambda_{0,\nu}} + \frac{(\lambda_{0,\nu+1} + 1)^{1/2}}{\lambda_{0,\nu+1} - \zeta} + C'_{1/2}(\lambda_{0,\nu+1} - \zeta)^{-1/2} \right) < C_{\nu,2}, \tag{3.12}$$

where

$$C_{\nu,2} := 3 \frac{(\lambda_{0,\nu} + 1)^{1/2}}{\lambda_{0,\nu+1} - \lambda_{0,\nu}} + 3 \frac{(\lambda_{0,\nu+1} + 1)^{1/2}}{\lambda_{0,\nu+1} - \lambda_{0,\nu}} + 3^{1/2} C'_{1/2}(\lambda_{0,\nu+1} - \lambda_{0,\nu})^{-1/2}.$$

In view of (3.1) there is a $C_1 \in [0, \infty[$ and a strictly increasing sequence $(\nu_k)_{k \in \mathbb{N}}$ in $\mathbb{N}$ such that

$$C_{\nu_k,1} \to 0 \quad \text{and } C_{\nu_k,2} \to C_1 \text{ as } k \to \infty. \tag{3.13}$$

Choose l such that $0 < l(C_1 + 1) < 1/4$. By Theorem 2.4 there is an ε with $0 < \varepsilon_0 < 1$ and a bounded set B_1 in $H^1(\Omega)$ such that for every $\varepsilon \in [0, \varepsilon_0]$ the attractor $\mathcal{A}_\varepsilon$ lies in B_1. Let V_0 be the Liapunov function of $\pi_{0,\hat{f}}$ defined by

$$V_0 \colon H^1_s(\Omega) \to \mathbb{R}, \quad u \mapsto (1/2) a_0(u, u) - \int_\Omega J_0(x) F(u(x))\, dx.$$

Here, as usual, $F(s) := \int_0^s f(p)\, dp$, $s \in \mathbb{R}$. Choose $M_0 \in]0, \infty[$ so that $V_0(u) < M_0$ for all $u \in \mathcal{A}_0$. By (1.4), it follows that

$$B_2 := \{\, u \in H^1_s(\Omega) \mid V_0(u) \le M_0 \,\}$$

is bounded. Define $B := B_1 \cup B_2$ and let $L = L(l, B)$ and $U = U(l, B)$ and $g = g(l, B)$ be as in Proposition 3.2. By (3.13) there exists a $k \in \mathbb{N}$ such that

$$L C_{\nu_k,1} < \frac{1}{4} \quad \text{and } l C_{\nu_k,2} < \frac{1}{4}. \tag{3.14}$$

Fix such a k and set $\nu := \nu_k$. Since $\lambda_{\varepsilon,\nu} \to \lambda_{0,\nu}$ and $\lambda_{\varepsilon,\nu+1} \to \lambda_{0,\nu+1}$ as $\varepsilon \to 0^+$ and using (3.11) and (3.12), we may assume, by taking ε_0 smaller if necessary, that for every $\varepsilon \in [0, \varepsilon_0]$

$$\sup_{\zeta\in I_\nu}\left(\frac{1}{\zeta-\lambda_{\varepsilon,\nu}}+\frac{1}{\lambda_{\varepsilon,\nu+1}-\zeta}\right)<C_{\nu,1} \tag{3.15}$$

and

$$\sup_{\zeta\in I_\nu}\left(\frac{(\lambda_{\varepsilon,\nu}+1)^{1/2}}{\zeta-\lambda_{\varepsilon,\nu}}+\frac{(\lambda_{\varepsilon,\nu+1}+1)^{1/2}}{\lambda_{\varepsilon,\nu+1}-\zeta}+C'_{1/2}(\lambda_{\varepsilon,\nu+1}-\zeta)^{-1/2}\right)<C_{\nu,2}. \tag{3.16}$$

If $\varepsilon > 0$ endow $H^1(\Omega)$ with the equivalent norm

$$|\!|\!| u |\!|\!|_\varepsilon := L|u|_\varepsilon + l\|u\|_\varepsilon.$$

Write $Z_\varepsilon^\zeta := BC^\zeta(H^1(\Omega), |\!|\!| \cdot |\!|\!|_\varepsilon)$ with the corresponding norm

$$|\!|\!| y |\!|\!|_{\zeta,\varepsilon} = \sup_{t\in]-\infty,0]} e^{\zeta t}|\!|\!| y(t) |\!|\!|_\varepsilon.$$

If $\varepsilon = 0$ endow $H^1_s(\Omega)$ with the equivalent norm

$$|\!|\!| u |\!|\!|_0 := L|u|_0 + l\|u\|_0.$$

Write $Z_0^\zeta := BC^\zeta(H^1_s(\Omega), |\!|\!| \cdot |\!|\!|_0)$ with the corresponding norm

$$|\!|\!| y |\!|\!|_{\zeta,0} = \sup_{t\in]-\infty,0]} e^{\zeta t}|\!|\!| y(t) |\!|\!|_0.$$

It follows that for $\varepsilon \in [0, \varepsilon_0]$, $\zeta \in I_\nu$ and $y \in BC^\zeta(L^2(\Omega), \varepsilon)$ (resp. for $\varepsilon = 0$ and $y \in BC^\zeta(L^2_s(\Omega), 0)$)

$$|\!|\!| K_{\varepsilon,\nu} y |\!|\!|_{\zeta,\varepsilon} \le (1/2)|y|_{\zeta,\varepsilon}. \tag{3.17}$$

Now fix ζ and $\mu \in I_\nu$ with $\zeta < \mu$.

Since $g\colon H^1(\Omega) \to L^2(\Omega)$ is globally bounded, it follows that $g \circ y$ is globally bounded for every $y \in BC^\zeta(H^1(\Omega), \varepsilon)$, $\varepsilon \ge 0$. Moreover, since g maps $H^1_s(\Omega)$ into $L^2_s(\Omega)$ it follows that the nonlinear operator $y \mapsto g \circ y$ maps Z_ε^ζ into $BC^\zeta(L^2(\Omega), \varepsilon)$ for $\varepsilon > 0$ and it maps Z_0^ζ into $BC^\zeta(L^2_s(\Omega), 0)$. Moreover,

$$|g(u) - g(v)|_\varepsilon \le L|u - v|_\varepsilon + l\|u - v\|_\varepsilon = |\!|\!| u - v |\!|\!|_\varepsilon \quad \text{for } u,\, v \in H^1(\Omega)$$

so

$$|g \circ y - g \circ w|_{\zeta,\varepsilon} \le |\!|\!| y - w |\!|\!|_{\zeta,\varepsilon} \quad \text{for } y,\, w \in Z_\varepsilon^\zeta. \tag{3.18}$$

It follows that the operator

$$\Gamma_\varepsilon\colon \mathbb{R}^\nu \times Z_\varepsilon^\zeta \to Z_\varepsilon^\zeta, \quad (\xi, y) \mapsto \Xi_{\varepsilon,\nu}(\xi, g \circ y)$$

is well-defined. If $0 \le \varepsilon \le \varepsilon_0$ then, by (3.17) and (3.18), Γ_ε is a uniform contraction in the second variable with contraction constant $1/2$. It follows that for every such ε there is a uniquely defined map $\phi_\varepsilon\colon \mathbb{R}^\nu \to Z^\zeta_\varepsilon$, such that

$$\phi_\varepsilon(\xi) = \Gamma_\varepsilon(\xi, \phi_\varepsilon(\xi)) \quad \text{for } \xi \in \mathbb{R}^\nu.$$

Either proceeding directly, or by using the fiber contraction theorem as in [5] or else by using the abstract results of [23] one proves that the map ϕ_ε is of class C^1 as a map from $\mathbb{R}^\nu$ into Z^μ_ε. Using Lemma 3.1. and the recursive formulas for the derivatives of ϕ_ε (cf [23]), one can show that, if $\xi_n \to \xi_0$ in $\mathbb{R}^\nu$ and $\varepsilon_n \to 0^+$, then

$$\|\phi_{\varepsilon_n}(\xi_n) - \phi_0(\xi_0)\|_{\zeta,\varepsilon_n} \to 0 \quad \text{for } n \to \infty, \tag{3.19}$$

and for every $\xi' \in \mathbb{R}^\nu$

$$\|D\phi_{\varepsilon_n}(\xi_n)(\xi') - D\phi_0(\xi_0)(\xi')\|_{\mu,\varepsilon_n} \to 0 \quad \text{for } n \to \infty. \tag{3.20}$$

Now define, for $0 < \varepsilon \le \varepsilon_0$,

$$\Lambda_\varepsilon\colon \mathbb{R}^\nu \to H^1(\Omega), \quad \xi \mapsto \phi_\varepsilon(\xi)(0),$$

and for $\varepsilon = 0$,

$$\Lambda_\varepsilon\colon \mathbb{R}^\nu \to H^1_s(\Omega), \quad \xi \mapsto \phi_\varepsilon(\xi)(0)$$

By what we have proved so far, Λ_ε is well-defined, of class C^1 and (3.7) and (3.9) hold. It is well-known and easily proved that $\Lambda_\varepsilon(\mathbb{R}^\nu)$ is an invariant manifold of the semiflow $\pi_{\varepsilon,g}$ which includes all orbits of solutions of $\pi_{\varepsilon,g}$ defined for $t \le 0$ and lying in Z^ζ_ε. Since g equals $\hat f$ on U, it follows that every point in $\mathcal{A}_\varepsilon$ is contained in $\Lambda_\varepsilon(\mathbb{R}^\nu)$. The reduced equation on the manifold $\Lambda_\varepsilon(\mathbb{R}^\nu)$ clearly takes the form (3.8) and (3.9) implies (3.10). Now let

$$K := \{\, \xi \in \mathbb{R}^\nu \mid V_0(\Lambda_0(\xi)) \le M_0 \,\} = \{\, \xi \in \mathbb{R}^\nu \mid \Lambda_0(\xi) \in B_2 \,\}.$$

Since B_2 is bounded and closed, it follows from (3.7) that K is bounded and closed, i.e. compact. Define

$$V := \{\, \xi \in \mathbb{R}^\nu \mid V_0(\Lambda_0(\xi)) < M_0 \,\}.$$

Thus $V \subset K$ and V is open in $\mathbb{R}^\nu$. Since $\Lambda(K) \subset U$ and K is compact and U is open in $H^1(\Omega)$, it follows from (3.9), by choosing $\varepsilon_0 > 0$ smaller, if necessary, that

$$\Lambda_\varepsilon(K) \subset U, \quad \varepsilon \in [0, \varepsilon_0].$$

Moreover, if $\varepsilon_0 > 0$ is small enough, then

$${\Lambda_\varepsilon}^{-1}(\mathcal{A}_\varepsilon) \subset V, \quad \varepsilon \in [0, \varepsilon_0].$$

Set $W := V_0 \circ \Lambda_0\colon \mathbb{R}^\nu \to \mathbb{R}$. Then, for every $\xi \in {\Lambda_\varepsilon}^{-1}(U)$, we have

$$\nabla W(\xi) \cdot v_0(\xi) = DV_0(\Lambda_0(\xi)) D\Lambda_0(\xi)(v_0(\xi)) = -|D\Lambda_0(\xi)(v_0(\xi))|_0^2.$$

Since there are no equilibria u of $\pi_{0,\hat f}$ with $V_0(u) = M_0$, it follows that whenever $W(\xi) = M_0$, then $D\Lambda_0(\xi)(v_0(\xi)) \ne 0$. By the compactness of K we now obtain that there is a $\delta > 0$ such that $\nabla W(\xi) \cdot v_0(\xi) < -\delta$, whenever $W(\xi) = M_0$. Therefore (3.10) implies that, if $\varepsilon_0 > 0$ is small enough, then $\nabla W(\xi) \cdot v_\varepsilon(\xi) < -\delta$, whenever $W(\xi) = M_0$. This shows, that, for $\varepsilon \in [0, \varepsilon_0]$, the set V is positively invariant for the equation (3.8) so the set $\Lambda_\varepsilon(V) \subset U$ is positively invariant for the semiflow $\pi_{\varepsilon,\hat f}$ and $\mathcal{A}_\varepsilon \subset \Lambda_\varepsilon(V)$. The theorem is proved. The complete details can be found in [20]. $\square$

4. An example: the sphere

We will now see that, for thin domains close to spheres, the spectral gap condition (3.1) is satisfied, so we can prove existence of inertial manifolds.

Theorem 4.1. *Suppose Ω has connected normal sections, regard $\mathbb{R}^{k+1}$ as isometrically imbedded into $\mathbb{R}^\ell$, let $r \in [0, \infty[$ be arbitrary and assume that*

$$\mathcal{G} = \mathbb{S}^k(r) := \{\, x \in \mathbb{R}^{k+1} \mid \langle x, x\rangle = r^2 \,\}$$

(i.e. $\mathcal{G}$ the k-dimensional sphere in $\mathbb{R}^\ell$ of radius r centered at 0). Assume that

$$C_\mu := \sup_{p \in \mathbb{S}^k(r)} (1/\mu(p)) \langle \nabla \mu(p), \nabla \mu(p)\rangle^{1/2} \leq 1/(4r)^2.$$

Under these assumptions the repeated sequence $(\lambda_{0,j})_{j\in\mathbb{N}}$ of the eigenvalues of the limit operator A_0 satisfies the following 'gap' condition:

$$\limsup_{\nu\to\infty} \frac{\lambda_{0,\nu+1} - \lambda_{0,\nu}}{(\lambda_{0,\nu})^{1/2}} > 0. \tag{4.1}$$

Sketch of the proof. Set $n := k+1$. By the results of Section 2, A_0 is equivalent to the operator A_μ defined by

$$\begin{aligned} D(A_\mu) &= H^2(\mathbb{S}^{n-1}(r)) \\ A_\mu u &= -(1/\mu)\operatorname{div}(\mu \nabla u) = -\Delta_{\mathbb{S}^{n-1}(r)} u - \langle (1/\mu)\nabla\mu, \nabla u\rangle. \end{aligned}$$

Let $(\lambda_j)_{j\in\mathbb{N}}$ be the repeated sequence of the eigenvalues of the operator $-\Delta_{\mathbb{S}^{n-1}(r)}$. Moreover, for $\nu \in \mathbb{N}_0$, let $\overline{\lambda}_\nu$ denote the ν-th distinct eigenvalue of $-\Delta_{\mathbb{S}^{n-1}(r)}$. It is well known (see e.g. [4]) that

$$\overline{\lambda}_\nu = r^{-2}\nu(\nu + n - 2), \quad \text{for } \nu \in \mathbb{N}.$$

Therefore we can find arbitrarily large gaps in the spectrum of $\Delta_{\mathbb{S}^{n-1}(r)}$. More precisely, we have that

$$\limsup_{j\to\infty} \frac{\lambda_{j+1} - \lambda_j}{\lambda_j^{1/2}} = \lim_{\nu\to\infty} \frac{\overline{\lambda}_{\nu+1} - \overline{\lambda}_\nu}{\overline{\lambda}_\nu^{1/2}} = \frac{2}{r}. \tag{4.2}$$

Now we observe that A_μ is a relatively bounded perturbation of $-\Delta_{\mathbb{S}^{n-1}(r)}$. More precisely, set $A := -\Delta_{\mathbb{S}^{n-1}(r)}$ and, for $u \in H^1(\mathbb{S}^{n-1}(r))$, set

$$B_\mu u := -(1/\mu)\langle \nabla\mu, \nabla u(p)\rangle,$$

so $A_\mu = A + B_\mu$. For $u \in H^2(\mathbb{S}^{n-1}(r))$, we have that

$$\begin{aligned} |B_\mu u|^2_{L^2} &= \int_{\mathbb{S}^{n-1}(r)} |\langle \mu^{-1}\nabla\mu, \nabla u\rangle|^2 \, \mathrm{d}\mathcal{H}^{n-1} \leq C_\mu^2 \int_{\mathbb{S}^{n-1}(r)} \langle \nabla u, \nabla u\rangle \, \mathrm{d}\mathcal{H}^{n-1} \\ &= C_\mu^2 \int_{\mathbb{S}^{n-1}(r)} u \, Au \, \mathrm{d}\mathcal{H}^{n-1} \leq C_\mu^2 |u|_{L^2} |Au|_{L^2}. \end{aligned}$$

It follows that, whenever $\delta > 0$, we have

$$|B_\mu u|_{L^2} \leq \delta |Au|_{L^2} + \frac{C_\mu^2}{4\delta}|u|_{L^2} \quad \text{for all } u \in D(A). \tag{4.3}$$

Now let $\lambda > 0$ and let $d(\lambda)$ be the distance of λ from the spectrum of A. Assume that $\lambda I - A$ is invertible. Write $L^2 := L^2(\mathbb{S}^{n-1}(r))$. It is well known (see e.g. Theorem 3.17 in [13]) that a sufficient condition for $\lambda I - (A + B_\mu)$ being invertible is

$$|B_\mu(\lambda I - A)^{-1}|_{\mathcal{L}(L^2,L^2)} < 1.$$

In view of (4.3), for every $\delta > 0$ we have

$$|B_\mu(\lambda I - A)^{-1}|_{\mathcal{L}(L^2,L^2)} \leq \delta |A(\lambda I - A)^{-1}|_{\mathcal{L}(L^2,L^2)} + \frac{C_\mu^2}{4\delta}|(\lambda I - A)^{-1}|_{\mathcal{L}(L^2,L^2)}.$$

Observe that, since A is self-adjoint,

$$|(\lambda I - A)^{-1}|_{\mathcal{L}(L^2,L^2)} = \sup_{\nu\in\mathbb{N}} |\lambda - \overline{\lambda}_\nu|^{-1} \leq d(\lambda)^{-1}$$

and

$$\begin{aligned}|A(\lambda I - A)^{-1}|_{\mathcal{L}(L^2,L^2)} = \sup_{\nu\in\mathbb{N}} |\overline{\lambda}_\nu||\lambda - \overline{\lambda}_\nu|^{-1} &\leq \sup_{\nu\in\mathbb{N}}(1 + \lambda|\lambda - \overline{\lambda}_\nu|^{-1}) \\ &\leq 1 + \lambda d(\lambda)^{-1}.\end{aligned}$$

It follows that

$$|B_\mu(\lambda I - A)^{-1}|_{\mathcal{L}(L^2,L^2)} \leq \delta(1 + \lambda d(\lambda)^{-1}) + \frac{C_\mu^2}{4\delta}d(\lambda)^{-1}.$$

So a sufficient condition for $\lambda I - (A + B_\mu)$ being invertible is

$$\delta(d(\lambda) + \lambda) + \frac{C_\mu^2}{4\delta} < d(\lambda)$$

or equivalently

$$\delta\lambda + \frac{C_\mu^2}{4\delta} < (1 - \delta)d(\lambda) \quad \text{for some } \delta,\ 0 < \delta < 1. \tag{4.4}$$

Using our assumption on C_μ and choosing $\delta := (8r)^{-1}\lambda^{1/2}$ we see that (4.4) is satisfied (and so $\lambda I - A_\mu$ is invertible) whenever

$$\lambda > 1/(4r)^2 \quad \text{and} \quad d(\lambda) > \frac{1}{2r}\lambda^{1/2}. \tag{4.5}$$

Now let $\nu > 1$ be fixed. Then $\nu(\nu + n - 2) > 1/4$, so $\overline{\lambda}_\nu > 1/(4r)^2$. If $\lambda \in]\overline{\lambda}_\nu, \overline{\lambda}_{\nu+1}[$, then, in view of (4.5), $\lambda I - A_\mu$ is invertible provided

$$\lambda - \overline{\lambda}_\nu > \frac{1}{2r}\lambda^{1/2} \quad \text{and} \quad \overline{\lambda}_{\nu+1} - \lambda > \frac{1}{2r}\lambda^{1/2}.$$

Setting

$$\xi_\nu := \frac{1}{8r^2} + \left(\frac{1}{64r^4} + \frac{\overline{\lambda}_\nu}{4r^2}\right)^{1/2}$$

and

$$\eta_{\nu+1} := -\frac{1}{8r^2} + \left(\frac{1}{64r^4} + \frac{\overline{\lambda}_{\nu+1}}{4r^2}\right)^{1/2},$$

we see that, if $\overline{\lambda}_\nu + \xi_\nu < \overline{\lambda}_{\nu+1} - \eta_{\nu+1}$, then the interval

$$I_\nu :=]\overline{\lambda}_\nu + \xi_\nu, \overline{\lambda}_{\nu+1} - \eta_{\nu+1}[$$

is contained in the resolvent set of A_μ. An explicit computation shows that there is a $\nu_0 \in \mathbb{N}$ such that for all $\nu \geq \nu_0$,

$$|I_\nu| = (\overline{\lambda}_{\nu+1} - \eta_{\nu+1}) - (\overline{\lambda}_\nu + \xi_\nu) \geq \frac{1}{3}(\overline{\lambda}_{\nu+1} - \overline{\lambda}_\nu). \tag{4.6}$$

Now all eigenvalues of A_μ larger than $\overline{\lambda}_{\nu_0+1}$ are contained in

$$\bigcup_{\nu \geq \nu_0+1} J_\nu, \quad J_\nu :=]\overline{\lambda}_\nu - \eta_\nu, \overline{\lambda}_\nu + \xi_\nu[\,.$$

The conclusion follows from (4.2) and by noticing that

$$\lim_{\nu\to\infty} \frac{\overline{\lambda}_\nu^{1/2}}{(\overline{\lambda}_\nu + \xi_\nu)^{1/2}} = 1.$$

The details can be found in [21]. □

References

1. M. Berger and B. Gostiaux, *Differential geometry: manifolds, curves and surfaces*, Springer-Verlag, New York, 1988.
2. Haim Brezis, *Analyse Fonctionelle*, Masson, Paris, 1992.
3. P. Brunovský and I. Tereščák, *Regularity of invariant manifolds*, J. Dyn. Differ. Equations **3** (1991), 313 – 337.
4. I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, Orlando, Florida, 1984.
5. S. N. Chow and K. Lu, *Invariant manifolds for flows in Banach spaces*, J. Differ. Equations **74** (1988), 285 – 317.
6. I.S. Ciuperca, *Spectral properties of Schrödinger operators on domains with varying order of thinness*, J. Dyn. Differ. Equations **10** (1998), 73 – 108.
7. Jack K. Hale, *Asymptotic Behavior of Dissipative Systems*, Math. Surveys Monographs 25, AMS, Providence, 1988.
8. J. Hale and G. Raugel, *Reaction-diffusion equations on thin domains*, J. Math. Pures Appl. **IX Ser.71** (1992), 33 – 95.
9. J. Hale and G. Raugel, *A damped hyperbolic equation on thin domains*, J. Trans. Am. Math. Soc. **329** (1992), 185 – 219.
10. J. Hale and G. Raugel, *A reaction-diffusion equation on a thin L-shaped domain*, Proc. Roy. Soc. Edinb., Sect A **125** (1995), 283 – 327.
11. E. Hebey, *Sobolev spaces on Riemannian manifolds*, Lecture notes in mathematics, Vol 1635, Springer-Verlag, Heidelberg, 1996.
12. D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture notes in mathematics, Vol 840, Springer-Verlag, NY, 1981.

13. T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin Heidelberg, 1995.
14. O. Ladyzhenskaya, *Attractors for semigroups and evolution equations*, Cambridge University Press, Cambridge, 1991.
15. O. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, Cambridge University Press, Cambridge, 1991.
16. J. Mallet-Paret and G.R. Sell, *Inertial manifolds for reaction-diffusion equations in higher space dimensions*, Jour. Am. Math. Society **1** (1988), 805 – 866.
17. M. Prizzi, M. Rinaldi and K. P. Rybakowski, *Curvilinear domain squeezing and parabolic equations*, Studia Math., to appear.
18. M. Prizzi and K. P. Rybakowski, *The effect of domain squeezing upon the dynamics of reaction-diffusion equations*, JDE **173** (2001), 271 – 320.
19. M. Prizzi and K. P. Rybakowski, *Some recent results on thin domain problems*, Topol. Methods in Nonl. Anal. **14** (1999), 239 – 255.
20. M. Prizzi and K. P. Rybakowski, *Inertial manifolds on squeezed domains*, preprint.
21. M. Prizzi and K. P. Rybakowski, *On inertial manifolds for reaction-diffusion equations on genuinely high-dimensional thin domains*, preprint.
22. G. Raugel, *Dynamics of partial differential equations on thin domains*, R. Johnson (ed.), Dynamical systems. Lectures given at the 2nd session of the Centro Internazionale Matematico Estivo (CIME) held in Montecatini Terme, Italy, June 13 – 22, 1994. Lecture notes in mathematics, Vol. 1609, Springer-Verlag, Berlin, 1995, pp. 208 – 315.
23. K. P. Rybakowski, *An abstract approach to smoothness of invariant manifolds*, Appl. Analysis **49** (1993), 119 – 150.
24. G. Schwartz, *Hodge decomposition – A method for solving boundary value problems*, Lecture notes in mathematics, Vol 1607, Springer-Verlag, Heidelberg, 1995.

Martino Prizzi, Università degli Studi di Trieste, Dipartimento di Scienze Matematiche, Via Valerio 12/b, 34127 Trieste, Italy
E-mail address: prizzi@mathsun1.univ.trieste.it

Krzysztof P. Rybakowski, Universität Rostock, Fachbereich Mathematik, Universitätsplatz 1, 18055 Rostock, Germany
E-mail address: krzysztof.rybakowski@mathematik.uni-rostock.de