ON THE HILBERT SCHEME OF PALATINI THREEFOLDS

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Abstract. In this paper we study the Hilbert scheme of Palatini threefolds X in $\mathbb{P}^5$. We prove that such a scheme has an irreducible component containing X which is birational to the Grassmannian $\mathbb{G}(3, \check{\mathbb{P}}^{14})$ and we determine the exceptional locus of the birational map.

1. INTRODUCTION

Let X be a smooth non degenerate 3-fold in $\mathbb{P}^5$ which is a scroll over a smooth surface S. There are four examples, all classical, of such scrolls in $\mathbb{P}^5$: the Segre scroll, the Bordiga scroll, the Palatini scroll and the $K3$-scroll, of degree 3, 6, 7, 9 respectively. Ottaviani in [O] proved that in $\mathbb{P}^5$ these are the only smooth 3-dimensional scrolls over a surface.

The first two scrolls are arithmetically Cohen-Macaulay threefolds, defined by the maximal minors of a 3×2, or of a 4×3 matrix of linear forms respectively and their Hilbert schemes are described by Ellingsrud ([E]).

In this paper we are interested in studying the Hilbert scheme of Palatini scrolls.

By definition, a Palatini scroll is the degeneracy locus of a general morphism $\phi : \mathcal{O}_{\mathbb{P}^5}^{\oplus 4} \longrightarrow \Omega_{\mathbb{P}^5}(2)$. This definition is the straightforward analogous in $\mathbb{P}^5$ of the classical construction performed by Guido Castelnuovo in 1891 for the projected Veronese surface S in $\mathbb{P}^4$ (see [C]). In the modern language, he showed indeed that any such surface S can be interpreted as the degeneracy locus of a morphism defined by three independent sections of $\Omega_{\mathbb{P}^4}(2)$.

This kind of morphisms are closely connected to the notion of linear complexes of lines (see [C], [BM]). In fact, it is possible to interpret the

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degeneracy locus of a general morphism $\phi : \mathcal{O}_{\mathbb{P}^5}^{\oplus 4} \longrightarrow \Omega_{\mathbb{P}^5}(2)$ as the set of centres of linear complexes in $\mathbb{P}^5$ belonging to a general linear system Δ of dimension 3 (web for short) of linear complexes in $\mathbb{P}^5$. This latter interpretation will be very important in proving our results.

In the case of Veronese surfaces, Castelnuovo proved also that the net of complexes, giving raise to the morphism $\mathcal{O}_{\mathbb{P}^4}^{\oplus 3} \longrightarrow \Omega_{\mathbb{P}^4}(2)$, can be reconstructed from S, as the span of the locus of trisecant lines to S inside the embedding space of the Grassmannian $\mathbb{G}(1,4)$. This is equivalent to the claim that the irreducible component of the Hilbert scheme of the Veronese surface is birational to $\mathbb{G}(2, H^0(\Omega_{\mathbb{P}^4}(2))$. Here we study the analogous problem for Palatini threefolds.

One of the first things we prove, see Proposition (3.1), is that the Hilbert scheme of the Palatini scroll has a distinguished reduced irreducible component $\mathcal{H}$ which is smooth at the point representing X and of dimension 44. Such dimension is equal to the dimension of $\mathbb{G}(3, \check{\mathbb{P}}^{14})$, the Grassmannian parametrizing maps $\mathcal{O}_{\mathbb{P}^5}^{\oplus 4} \longrightarrow \Omega_{\mathbb{P}^5}(2)$.

It's known that there is a natural map $\rho : \mathbb{G}(3, \check{\mathbb{P}}^{14}) - - \rightarrow \mathcal{H}$ and we study such map. The main result we obtain is the following theorem. It relies on a careful description of the variety of four-secant lines of the Palatini scroll. We prove that this variety is the union of C_4, the base locus of a general web of linear complexes, and of one more component, the one given by the lines contained in X. Hence the situation is a bit different from the one encountered by Castelnuovo in the case of a general net of linear complexes in $\mathbb{P}^4$. But nevertheless we can reconstruct C_4 from X in a geometrical way, as it was done by Castelnuovo in the case of Veronese surfaces.

THEOREM 1.1. *Let $X \subset \mathbb{P}^5$ be a smooth Palatini scroll of degree 7. Let $\mathcal{H}$ be the irreducible component of the Hilbert scheme containing X. Then the rational map*

$$\rho : \mathbb{G}(3, \check{\mathbb{P}}^{14}) - - \rightarrow \mathcal{H}$$

is birational.

Since the map ρ is birational, our next task is to determine the locus over which such map is not regular.

It is well-known that if the degeneracy locus of a bundle map from a vector bundle F to a vector bundle G has the expected dimension then it lies in the same Hilbert scheme as the degeneracy locus of a general map from F to G. So ρ is not regular over the points $\Delta \in \mathbb{G}(3, \check{\mathbb{P}}^{14})$ such that the set of centres of linear complexes in $\mathbb{P}^5$ belonging to Δ has dimension strictly bigger than 3.

It turns out that these are the 3-spaces of $\check{\mathbb{P}}^{14}$ which either are completely contained in the dual Grassmannian $\check{\mathbb{G}}(1,5)$, or intersect $\mathbb{G}(3,5)$, naturally identified with the singular locus of $\check{\mathbb{G}}(1,5)$, along a curve. In other words, we can consider the intersection of Δ with $\check{\mathbb{G}}(1,5)$: in general it is a cubic surface S which can be identified with the base of the scroll. The non-regularity of ρ at Δ means that either S is not defined or that it is singular along a curve.

For the precise statements about the locus over which the map ρ is not regular, we refer to §4, Theorem (4.3) and Theorem (4.9).

In the last section we will see how the Hilbert scheme of the Palatini scrolls fits into commutative diagrams involving the variety of 6×6 skew-symmetric matrices of linear forms, the space of cubic surfaces, the moduli space of rank two bundles E on a cubic surface, with $c_1(E) = \mathcal{O}_S(2), c_2(E) = 5$. The relations among the latter mathematical objects where considered in [B] to which we refer for all the details.

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2. Notations and Preliminaries

The following notation will be needed later on in the paper. We refer to [LB] for the details.

Let X be a smooth non degenerate 3-fold in $\mathbb{P}^5$. By $Hilb^4\mathbb{P}^5$, respectively $Hilb^4X$, we denote the Hilbert scheme of zero dimensional subschemes of $\mathbb{P}^5$, respectively of X, of length 4, i.e. 4-tuples.

Let $Hilb^4_c\mathbb{P}^5$ be the open smooth subset of $Hilb^4\mathbb{P}^5$ of the 4-tuples lying on a smooth curve, $Hilb^4_cX{:=}Hilb^4X \times_{Hilb^4P^5} Hilb^4_c\mathbb{P}^5$. Note that $\dim Hilb^4_c\mathbb{P}^5 = 20$.

Let $Al^4\mathbb{P}^5$ be the subvariety given by those elements in $Hilb^4_c\mathbb{P}^5$, which are on some line in $\mathbb{P}^5$.

Remark 2.1. (1) $Al^4\mathbb{P}^5$ is a smooth subvariety of $Hilb^4_c\mathbb{P}^5$ of dimension 12.

(2) the map $a : Al^4\mathbb{P}^5 \longrightarrow \mathbb{G}(1,5)$ which sends an element of $Al^4\mathbb{P}^5$ to the line on which it lies is a fibration of fibre type $Hilb^4\mathbb{P}^1 \cong \mathbb{P}^4$.

We recall the definition of the embedded 4-secant variety of X.

Let $Al^4X{:=}Al^4\mathbb{P}^5 \times_{Hilb^4\mathbb{P}^5} Hilb^4_cX$. We denote by $\Sigma_4(X)$ the closure of $a(Al^4X)$ in $\mathbb{G}(1,5)$. Let

$$\mathcal{F} := \{(x, L) \in \mathbb{P}^5 \times \mathbb{G}(1,5) | x \in L\}$$

be the flag manifold and let $p_1 : \mathcal{F} \longrightarrow \mathbb{P}^5$, $p_2 : \mathcal{F} \longrightarrow \mathbb{G}(1,5)$ be the two projections.

Definition 2.2. $S_4(X) := p_1(p_2^{-1}(\Sigma_4(X))) \subset \mathbb{P}^5$ is the embedded 4-secant variety of X.

We recall a basic result on the Hilbert scheme which will be important for us.
Let Z be a smooth connected projective variety. Let X be a connected submanifold of Z with $H^1(X,N) = 0$ where N is the normal bundle of X. The following proposition holds, see ([G], [S]).

Proposition 2.3. *Let Z and X be as above. There exist irreducible projective varieties $\mathcal{Y}$ and $\mathcal{H}$ with the following properties:*

(i) *$\mathcal{Y} \subset \mathcal{H} \times Z$ and the map $p : \mathcal{Y} \longrightarrow \mathcal{H}$ induced by the product projection is a flat surjection,*
(ii) *there is a smooth point $x \in \mathcal{H}$ with p of maximal rank in a neighborhood of $p^{-1}(x)$,*
(iii) *q identifies $p^{-1}(x)$ with X where $q : \mathcal{Y} \longrightarrow Z$ is the map induced by the product projection, and*
(iv) *$H^0(N)$ is naturally identified with $T_{\mathcal{H},x}$ where $T_{\mathcal{H},x}$ is the Zariski tangent space of $\mathcal{H}$ at x.*

3. Webs of linear complexes in $\mathbb{P}^5$

In this section we will study the maps $\mathcal{O}_{\mathbb{P}^5}^{\oplus 4} \longrightarrow \Omega_{\mathbb{P}^5}(2)$ and their degeneracy loci.

Note that a general morphism $\phi : \mathcal{O}_{\mathbb{P}^5}^{\oplus 4} \longrightarrow \Omega_{\mathbb{P}^5}(2)$ is assigned by given 4 general global sections of $\Omega_{\mathbb{P}^5}(2)$.

We recall first two interpretations of global sections of $\Omega_{\mathbb{P}^5}(2)$, that explain the link with the classical construction of Castelnuovo ([C]).

- Geometric interpretation of ϕ (see [O]).

Let $\mathbb{P}^5 = \mathbb{P}(V)$. Consider the twisted dual Euler sequence on $\mathbb{P}^5$

$$0 \to \Omega_{\mathbb{P}^5}(2) \to \overset{6}{\oplus}\mathcal{O}_{\mathbb{P}^5}(1) \to \mathcal{O}_{\mathbb{P}^5}(2) \to 0$$

Being $V^* \cong H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1))$, we have the natural identification $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^5}, \Omega_{\mathbb{P}^5}(2)) \cong \wedge^2 V^*$, where a morphism $\mathcal{O}_{\mathbb{P}^5} \to \Omega_{\mathbb{P}^5}(2)$ is given (after choosing a basis in V and its dual basis in V^*) by a skew-symmetric 6×6 matrix $A = (a_{ij})$, $a_{ij} \in \mathbb{C}$ and corresponds to the morphism $V \to V^*$ given by

$$(x_0, ..., x_5) \to (\sum a_{0i}x_i, ..., \sum a_{5i}x_i)$$

The morphism $\phi : \mathcal{O}^4_{\mathbb{P}^5} \longrightarrow \Omega_{\mathbb{P}^5}(2)$ is given by four generic 6×6 skew-symmetric matrices A, B, C, D.

The equations of the degeneracy locus X of ϕ are the 4×4 minors of the 4×6 matrix

$$F = \begin{pmatrix} \sum a_{0i}x_i & . & . & \sum a_{5i}x_i \\ \sum b_{0i}x_i & . & . & \sum b_{5i}x_i \\ \sum c_{0i}x_i & . & . & \sum c_{5i}x_i \\ \sum d_{0i}x_i & . & . & \sum d_{5i}x_i \end{pmatrix} \tag{1}$$

If $P = (x_0, \dots, x_5) \in X$ then there exists $(x, y, z, t) \neq (0, 0, 0, 0)$ such that

$$(xA + yB + zC + tD)P = 0 \tag{2}$$

or equivalently

$$\sum_{i=0}^{5} (xa_{ji} + ya_{ji} + za_{ji} + ta_{ji})x_i = 0, \qquad j = 0, ..., 5 \tag{3}$$

Hence the matrix $xA + yB + zC + tD$ has to be a degenerate skew-symmetric matrix and its pfaffian has to vanish. Let (x, y, z, t) be homogeneous coordinates in $\mathbb{P}^3$, then the vanishing of the pfaffian of $xA + yB + zC + tD$ defines a hypersurface S of degree 3 in $\mathbb{P}^3$.

For ϕ general and a fixed $(x, y, z, t) \in S$, the matrix $xA+yB+zC+tD$ has rank four, so we find in X a line of solutions of the equation (3) and thus X is a scroll over S. Such X is a Palatini scroll of degree 7, see ([O], [BM]).

Let $f : X \to S$ denote the scroll map. If we fix a point $P \in X$, then $f(P)$ is the unique solution of the equation (2), interpreted as an equation in (x, y, z, t). The unicity of the solution is equivalent to the fact that the 4×6 matrix F in (1) has rank 3.

The morphism $f : X \to S$ is associated to the line bundle $K_X + 2H$, ([O], 3.3), where H is the hyperplane divisor. Moreover $X \cong \mathbb{P}_S(E)$ where $E := f_*H$ is a rank two vector bundle on S with $c_1(E) = \mathcal{O}_S(2), c_2(E) = 5$.

- Global sections of $\Omega_{\mathbb{P}^5}(2)$ as linear complexes.

Consider $\mathbb{G}(1, 5)$, the Grassmannian of lines in $\mathbb{P}^5$, embedded in $\mathbb{P}(\wedge^2 V) \cong \mathbb{P}^{14}$ via the Plücker map. The dual space $\mathbb{P}(\wedge^2 V^*) \cong \check{\mathbb{P}}^{14}$ parametrizes hyperplane sections of $\mathbb{G}(1, 5)$ or, in the old terminology, linear complexes in $\mathbb{P}^5$. A linear complex Γ in $\mathbb{P}^5$ is represented by a linear equation: $\sum_{0 \leq i < j \leq 5} a_{ij} p_{ij}$ in the Plücker coordinates p_{ij}. We associate to Γ the skew-symmetric matrix $A = (a_{ij})$ of order 6. A point

$P \in \mathbb{P}^5$ is called a centre of Γ if all lines through P belong to Γ. The space $\mathbb{P}(Ker(A))$ is the set of centres of Γ and it is called the singular space of Γ. Since we are in $\mathbb{P}^5$ a general linear complex Γ does not have any centre. In fact let A be the skew-symmetric matrix associated to Γ. Being Γ general it follows that $rkA = 6$. Γ is said to be special if its singular space is at least a line. The special complexes can be of first type or of second type depending on whether they have a line or a $\mathbb{P}^3$ as a singular space.

Note that a special Γ corresponds to a tangent hyperplane section of $\mathbb{G}(1,5)$, or equivalently to a point of $\check{\mathbb{P}}^{14}$ lying in $\check{\mathbb{G}}(1,5)$, the dual variety of $\mathbb{G}(1,5)$.

From this it follows that it is possible to interpret the degeneracy locus X of a general morphism $\phi : \mathcal{O}_{\mathbb{P}^5}^{\oplus 4} \longrightarrow \Omega_{\mathbb{P}^5}(2)$ as the set of centres of linear complexes belonging to a general linear system Δ of dimension 3 (web for short) of linear complexes in $\mathbb{P}^5$. Since Δ is general, it does not contain any special complex of the second type. Moreover such Δ is spanned by four linearly independent complexes $\Gamma_1, ..., \Gamma_4$ and it corresponds to a linear subspace $\mathbb{P}^3$ in $\check{\mathbb{P}}^{14}$: it intersects $\check{\mathbb{G}}(1,5)$ (which is a hypersurface of degree three) along a cubic surface which is disjoint from $\mathbb{G}(3,5)$. Its points represent the special complexes of Δ and so the surface can be identified with S (see [BM]).

Hence given a general web of linear complexes in $\mathbb{P}^5$ its set of centres is the degeneracy locus X of a general morphism $\phi : \mathcal{O}_{\mathbb{P}^5}^{\oplus 4} \longrightarrow \Omega_{\mathbb{P}^5}(2)$ which is a Palatini scroll of degree 7.

We see next that the Hilbert scheme of the Palatini scroll has a distinguished reduced irreducible component of dimension equal to 44.

Proposition 3.1. *Let $X \subset \mathbb{P}^5$ be the Palatini scroll of degree 7. Then the Hilbert scheme of X has an irreducible component, $\mathcal{H}$, which is smooth at the point representing X and of dimension 44.*

Proof. We have seen that $X \cong \mathbb{P}_S(E)$ where E is a rank two vector bundle on a smooth cubic surface S with $c_1(E) = \mathcal{O}_S(2), c_2(E) = 5$. Note that E is stable with respect to $\mathcal{O}_S(1)$ and therefore simple: in fact $f_*H = E$, where f is the adjunction mapping of X, hence $H^0(S, E(-1)) = H^0(X, -(K+H)) = 0$.

Let N denote the normal bundle of X in $\mathbb{P}^5$. We will show that $H^1(X, N) = 0$. Let

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)^{\oplus 6} \longrightarrow T_{\mathbb{P}^5|X} \longrightarrow 0 \tag{4}$$

be the Euler sequence on $\mathbb{P}^5$ restricted to X. From the cohomology sequence associated to (4) we have $h^i(X, T_{\mathbb{P}^5|X}) = 0$ for $i \geq 1$. Using the following exact sequence

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^5|X} \longrightarrow N \longrightarrow 0 \tag{5}$$

and the fact that $h^i(X, T_{\mathbb{P}^5|X}) = 0$ for $i \geq 1$ we get that

$$h^i(X, N) = h^{i+1}(X, T_X) \qquad \text{for} \quad i \geq 1. \tag{6}$$

Hence in order to compute $h^1(X, N)$ will be enough to compute $h^2(X, T_X)$.

Let $f : \mathbb{P}(E) \longrightarrow S$ be the scroll map, where E is a rank two vector bundle on S as above. We have the following sequences:

$$0 \longrightarrow \Omega_{X|S} \longrightarrow (f^*E)(-1) \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{7}$$

$$0 \longrightarrow T_{X|S} \longrightarrow T_X \longrightarrow f^*T_S \longrightarrow 0 \tag{8}$$

Dualizing (7) we get

$$0 \longrightarrow \mathcal{O}_X \longrightarrow (f^*E^*)(1) \longrightarrow T_{X|S} \longrightarrow 0 \tag{9}$$

By the projection formula and the fact that $R^1 f_* \mathcal{O}_X(1) = 0$ it follows that the 1-st direct image $R^1 f_*((f^*E^*)(1)) = E^* \otimes R^1 f_* \mathcal{O}_X(1) = 0$. Hence by the Leray spectral sequence it follows that

$$H^i(X, (f^*E^*)(1)) \cong H^i(S, f_*(f^*E^*)(1)) \cong H^i(S, E^* \otimes E)$$

for $i \geq 0$. In particular we have that $h^3(X, (f^*E^*)(1)) = 0$ being $h^3(S, E^* \otimes E) = 0$. On the other hand $h^i(X, \mathcal{O}_X) = 0$ for $i > 0$. Hence the cohomology sequence associated to (9) gives that $h^3(X, T_{X|S}) = 0$. We also know that $H^i(X, f^*T_S) \cong H^i(S, T_S)$, $i \geq 0$ and that $h^3(S, T_S) = 0$. Thus $h^3(X, f^*T_S) = 0$ and by (8) it follows that $h^3(X, T_X) = 0$. This latter fact along with (6) gives $h^2(X, N) = 0$.

We now compute $H^i(S, T_S)$. Since S is a smooth cubic surface in $\mathbb{P}^3$, the Euler sequence on $\mathbb{P}^3$ restricted to S along with

$$0 \longrightarrow T_S \longrightarrow T_{\mathbb{P}^3|S} \longrightarrow \mathcal{O}_S(3) \longrightarrow 0$$

and the fact that $h^i(S, \mathcal{O}_S(3)) = 0$ for $i > 0$ give that $h^2(S, T_S) = 0$ and thus $h^2(X, f^*T_S) = 0$. It remains to prove that $h^2(X, T_{X|S}) = 0$. From the cohomology sequence associated to (9) it follows that $h^2(X, T_{X|S}) = h^2(S, E^* \otimes E)$. Note that

$$h^2(S, E^* \otimes E) = h^2(S, \mathcal{E}\backslash\lceil(E)) = h^2(S, \mathcal{O}_S) + h^2(S, \mathcal{E}\backslash\lceil_0(E))$$

The last equality follows from the fact that $\mathcal{E}\backslash\lceil(E) \cong \mathcal{O}_S \oplus \mathcal{E}\backslash\lceil_0(E)$, where $\mathcal{E}\backslash\lceil_0(E)$ is the bundle of the traceless endomorphisms of E, see ([?], pg. 121). Since S is a smooth cubic surface in $\mathbb{P}^3$ it follows that $h^2(S, \mathcal{O}_S) = 0$. Moreover $h^2(S, \mathcal{E}\backslash\lceil_0(E)) = 0$, see ([B], proof of

Lemma 7.7). Hence the cohomology sequence associated to (8) gives that $h^2(X, T_X) = 0$ and thus, by (6), $h^1(X, N) = 0$. In order to compute the dimension of $\mathcal{H}$, by ((2.3), (iv)) we need to compute $h^0(X, N)$. But $h^0(N) = \chi(N)$ and by the Hirzebruch-Riemann-Roch theorem we know that

$$\chi(N) = \frac{1}{6}(n_1^3 - 3n_1n_2 + 3n_3) + \frac{1}{4}c_1(n_1^2 - 2n_2) + \frac{1}{12}(c_1^2 + c_2)n_1 + r\chi(\mathcal{O}_X) \tag{10}$$

where $n_i = c_i(N), c_i = c_i(X)$ and $r = rk(N) = 2$.

Note that $n_3 = 0$ since $rk(N) = 2$. We compute the remaining Chern classes of N from the exact sequence (5) and we get:

$$n_1 = K + 6H; \quad n_2 = 15H^2 + 6HK + K^2 - c_2$$

The numerical invariants of the Palatini scroll and its Hilbert polynomial have been computed in [O] and they are:

$$KH^2 = -8; K^2H = 7; K^3 = -2; -Kc_2 = 24$$

$$\chi(\mathcal{O}_X(t)) = \frac{7}{6}t^3 + 2t^2 + \frac{11}{6}t + 1$$

From these we get that $c_2H = 15$. Plugging these in (10) we get that $\chi(N) = 44$ and thus $h^0(N) = \chi(N) = 44$. □

Let Δ be a web of linear complexes in $\mathbb{P}^5$. Δ is spanned by 4 independent linear complexes in $\mathbb{P}^5$ and hence it corresponds to a linear $\mathbb{P}^3$ in $\mathbb{P}(\wedge^2 V^*) \cong \check{\mathbb{P}}^{14}$. Thus the webs of linear complexes in $\mathbb{P}^5$ are parametrized by $\mathbb{G}(3, \check{\mathbb{P}}^{14})$, the Grassmannian of $\mathbb{P}^3$'s in $\check{\mathbb{P}}^{14}$. Therefore we can define a natural rational map

$$\rho : \mathbb{G}(3, \check{\mathbb{P}}^{14}) - - \to \mathcal{H}ilb(X).$$

The map ρ sends a general web Δ to its singular set X. By Propositions (2.3) and (3.1), the Hilbert scheme of X, $\mathcal{H}ilb(X)$, has an irreducible component $\mathcal{H}$ which is smooth at the point representing X and of dimension 44. It is well known that the image of ρ is dense in $\mathcal{H}$. We let ρ denote also the map $\rho : \mathbb{G}(3, \check{\mathbb{P}}^{14}) - - \to \mathcal{H}$. Since domain and codomain of ρ have the same dimension 44 and ρ is dominant, a general fibre of ρ is finite. So it is natural to ask for the degree of such fibre.

An answer to this question is one of the main results of this paper.

THEOREM 3.2. *Let $X \subset \mathbb{P}^5$ be a smooth Palatini scroll of degree 7. Let $\mathcal{H}$ be the irreducible component of the Hilbert scheme containing*

X. Then the rational map

$$\rho : \mathbb{G}(3, \overset{\vee}{\mathbb{P}}{}^{14}) - - \rightarrow \mathcal{H}$$

is birational.

The proof of Theorem (3.2) will follow at once after we have proved the following Claims and Proposition.

Let us point out that to prove this theorem we have tried to adapt Castelnuovo's result, ([C], §7, 8) to our case. Unfortunately his whole argument does not extend and the difficulty lies in the fact that the locus of the 4-secants of X in $\mathbb{G}(1,5)$ contains besides C_4, also other lines. By C_4 we denote the base locus of a general web Δ of linear complexes in $\mathbb{P}^5$ of which X is the singular set. In fact we prove, see Proposition (3.6), that there is only one more component, the one given by the lines of the ruling of X.

Let Δ be a general web of linear complexes. Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be four linear complexes of $\mathbb{P}^5$ generating Δ. Let $C_4 \subset \mathbb{G}(1,5)$ denote the base of the web Δ, that is, C_4 is the family of lines in $\mathbb{P}^5$ which are common to $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$: it is irreducible and 4-dimensional, being the intersection of the Grassmannian with a general linear space of codimension 4. Let X be the set of centres of complexes belonging to Δ. The following claim is the analogous of the Castelnuovo's result about trisecants of the Veronese surface in $\mathbb{P}^{\ntrianglerighteq}$.

Claim 3.3. The lines of C_4 are 4-secants of X.

Proof. (of Claim) The equations of the degeneracy locus X are the fifteen 4×4 minors F_{ij} of the 4×6 matrix F, with F as in (1). If F_{12} denotes the 4×4 minor obtained by deleting the 1st and 2nd column of F then

$$\{F_{12} = 0\}$$

is the variety of degree 4 which is made out of the lines of C_4 which intersect the 3-plane $x_0 = x_1 = 0$.

Let l be a line in C_4 which does not intersect the 3-plane $x_0 = x_1 = 0$. Let $l \cap \{F_{12} = 0\} = \{P_1, P_2, P_3, P_4\}$: then each P_i is also on a line r of C_4 which intersects the 3-plane $x_0 = x_1 = 0$. Hence there is a pencil of lines of C_4 through P_i: the pencil spanned by l and r, and so $P_i \in X$. This is true for every P_i, $i = 1, ..., 4$, hence l is a 4-secant of X. $\square$

Claim 3.4. C_4 is a congruence of lines of order one, i.e. for a generic point $x \in \mathbb{P}^5$, there is only one line of C_4 passing through x.

Proof. (of Claim) In fact, as a cycle, C_4 coincides with σ_1^4, where σ_1 is the Schubert cycle of lines of $\mathbb{P}^5$ intersecting a fixed $\mathbb{P}^3$. Using Pieri formula, it is easy to show that $\sigma_1^4 = a_0\sigma_4 + a_1\sigma_{31} + a_2\sigma_{22} = \sigma_4 + 3\sigma_{31} + 2\sigma_{22}$ (see [DP]). Since the coefficient of σ_4 is equal to the number of lines of σ_1^4 passing through a general point, we have the claim. □

Claim 3.5. C_4 is an irreducible component of $\Sigma_4(X)$, the locus of all 4-secant lines of X in the Grassmannian $\mathbb{G}(1,5)$.

Proof. (of Claim) Let $\Sigma_k(X)$ denote the locus of all k-secant lines of X in the Grassmannian $\mathbb{G}(1,5)$. Note that every irreducible component of $\Sigma_4(X)$ has dimension ≤ 5. If otherwise then, since $\Sigma_4(X) \subseteq \Sigma_2(X)$ and since $\Sigma_2(X)$ is irreducible of dimension 6, it would follow that every secant line is also a 4-secant, which is impossible.

So either C_4 is a whole irreducible component of $\Sigma_4(X)$, or it is contained in an irreducible component Σ' of dimension 5. But, in the second case, it follows from [M], Theorem 2.3, that the lines of Σ' cannot fill up $\mathbb{P}^5$, against the fact that C_4 has order one. □

Proposition 3.6. $\Sigma_4(X) = C_4 \cup \Sigma_\infty(X)$, *where* $\Sigma_\infty(X)$ *is the variety of lines contained in* X.

Proof. (of Proposition) The number $q_4(X)$ of the 4-secant lines of X passing through a general point of $\mathbb{P}^5$ is finite by [M]. Hence $q_4(X)$ can be computed using the formula given by Kwak in [Kw] and it turns out that $q_4(X) = 1$. So C_4 is the unique irreducible component of $\Sigma_4(X)$ whose lines fill up $\mathbb{P}^5$. We have to exclude the existence of another irreducible component Σ', of dimension 4 of 5, such that the union of the lines of Σ' is strictly contained in $\mathbb{P}^5$.

If such a component Σ' exists, it follows from [M] that X contains either a one-dimensional family of surfaces of $\mathbb{P}^3$ of degree at least 4, or a 2-dimensional family of plane curves of degree at least 4. In the first case, every hyperplane H containing such a surface S of $\mathbb{P}^3$ cuts X along a reducible surface: $H \cap X = S \cup S'$, where S' is a surface of degree $7 - k \leq 3$. So on X there should be also a 2-dimensional family of surfaces of degree ≤ 3. But then a general hyperplane section of X should contain a 2-dimensional family of curves of degree ≤ 3. This surface is a rational non-special surface of $\mathbb{P}^4$, which has been extensively studied (see for instance [A]): it can be easily excluded that it contains such a family of curves.

In the second case, every 3-space containing a plane curve on X, of degree $k \geq 4$, should cut X residually in a curve of degree ≤ 3. So we would get on X also a 3-dimensional family of curves of degree

≤ 3. Now, X contains in fact both a three-dimensional family of conics and of cubics, but a computation in the Picard group of a general hyperplane section of X shows that their residual curves cannot be plane.

So both possibilities are excluded and the proof of the Proposition is accomplished. $\square$

Proof. (of Theorem) Let $X = \rho(\Delta)$ be a general threefold in the image of ρ. Then C_4 is the unique irreducible component of $\Sigma_4(X)$ whose lines fill up $\mathbb{P}^5$. This implies that X comes via ρ from a unique web Δ. Therefore, by the theorem on the dimension of the fibres, there is an open subset in $\mathcal{H}$ such that the fibres of ρ over this open subset are finite and of degree one. This proves the theorem. $\square$

4. Regularity of the map ρ

We will see that there are webs Δ over which the map ρ is not regular. Our next task is to determine such webs.

It is well-known that if the degeneracy locus of a bundle map from a vector bundle F to a vector bundle G has the expected dimension then it lies in the same Hilbert scheme as the degeneracy locus of a general map from F to G. So ρ is not regular over the webs Δ such that the corresponding degeneracy locus has dimension strictly bigger than 3.

We recall the following facts about dual Grassmannians that will be used in the sequel (see for instance [Ha]):

FACTS 4.1.

- $\mathbb{G}(3,5)$ can be naturally embedded in $\check{\mathbb{G}}$, where $\check{\mathbb{G}}$ stands for the dual Grassmannian $\check{\mathbb{G}}(1,5)$. We can interpret $\mathbb{G}(3,5)$ as the set of singular complexes of the second type, because a complex of second type is determined uniquely by its singular space $\mathbb{P}^3$: it is formed by the lines intersecting that $\mathbb{P}^3$.
- $\check{\mathbb{G}} = Sec(\mathbb{G}(3,5))$, the variety of secant lines, and $\mathbb{G}(3,5) = Sing(\check{\mathbb{G}})$, hence $\check{\mathbb{G}} = Sec(Sing(\check{\mathbb{G}}))$.
- The linear spaces contained in $\mathbb{G}(3,5)$ have dimension ≤ 4. In particular, a linear $\mathbb{P}^3$ in $\mathbb{G}(3,5)$ represents the set of the 3-spaces of a fixed $\mathbb{P}^4$ passing through a fixed point.

We will use Plücker coordinates p_{ij} on $\mathbb{P}^{14}$ and the dual coordinates m_{ij} on its dual space.

Let Δ be a web of linear complexes in $\mathbb{P}^5$. Let $\Gamma_1, ..., \Gamma_4$ be four linearly independent complexes which span Δ. Hence Δ corresponds to a $\mathbb{P}^3 \subset \check{\mathbb{P}}^{14}$. The special complexes of Δ are parametrized by $\Delta \cap \check{\mathbb{G}}(1,5)$. In fact the space $\check{\mathbb{P}}^{14}$ parametrizes all linear complexes: special complexes correspond to tangent hyperplane sections of $\mathbb{G}(1,5)$, that is to points of $\check{\mathbb{G}}(1,5)$ which is a cubic hypersurface in $\check{\mathbb{P}}^{14}$. Moreover special complexes of second type can be interpreted as points in $\mathbb{G}(3,5)$ (which is also embedded in $\check{\mathbb{P}}^{14}$), because a special complex of second type is uniquely determined by its singular space $\mathbb{P}^3$: it is formed by the lines intersecting that $\mathbb{P}^3$.

Hence the following situations can occur:

α) $\Delta \subset \check{\mathbb{G}}(1,5)$, that is, all the complexes of Δ are special, or

β) $\Delta \not\subset \check{\mathbb{G}}(1,5)$ and $\Delta \cap \check{\mathbb{G}}(1,5)$ is a cubic surface S, possibly singular.

Let us consider case α): $\Delta \subset \check{\mathbb{G}}(1,5)$, that is, the case in which all the complexes of Δ are special. Let A, B, C, D be four 6×6 skew-symmetric matrices associated to the complexes $\Gamma_1, ..., \Gamma_4$, respectively. Hence these matrices span Δ. Note that in this case for all $(x, y, z, t) \in \mathbb{P}^3$ we have that $pf(xA + yB + zC + tD) \equiv 0$. Hence the equation $pf(xA + yB + zC + tD) = 0$ does not define any surface in $\mathbb{P}^3$.

Before stating our result concerning case α) we recall few facts, already introduced in [BM], which will be needed. Let

$$\psi : \check{\mathbb{G}}(1,5) - - \to \mathbb{G}(1,5)$$

be the rational surjective map which sends a special complex of first type to its singular line, see [BM]. So ψ is regular on $\check{\mathbb{G}}(1,5) \setminus \mathbb{G}(3,5)$. The fibre $\psi^{-1}(l)$ is formed by the special complexes having l as singular line. The closures of these fibres are 5-dimensional linear spaces which are denoted by $\mathbb{P}^5_l$. In fact we may think of $\psi^{-1}(l)$ as the linear system of hyperplanes in $\mathbb{P}^{14}$ containing the tangent space to $\mathbb{G}(1,5)$ at the point l: $T_{l,\mathbb{G}(1,5)} \cong \mathbb{P}^8$, see ([BM], section 3) for the details.

Remark 4.2. Let $l, m \in \mathbb{G}(1,5)$ be lines of $\mathbb{P}^5$. Then the intersection of $\mathbb{P}^5_l$ with $\mathbb{G}(3,5)$ is a smooth quadric of dimension 4. Also $\mathbb{P}^5_l \cap \mathbb{P}^5_m$ is contained in $\mathbb{G}(3,5)$ and is just one point if l, m do not intersect, or a plane if they intersect, see ([BM], Remark 1).

THEOREM 4.3. *Let Δ be a web of linear complexes in $\mathbb{P}^5$ as in case α). Then the rational map*

$$\rho : \mathbb{G}(3, \check{\mathbb{P}}^{14}) - - \to \mathcal{H}$$

is not regular at the point corresponding to Δ.

Proof. Since the web Δ is as in case α) then the following situations can occur:

$\alpha 1$) $\Delta \subset \mathbb{G}(3,5)$, or

$\alpha 2$) $\Delta \not\subset \mathbb{G}(3,5)$ and $\Delta \subset \check{\mathbb{G}}$.

Let Δ be as in case $\alpha 1$), i.e. $\Delta \subset \mathbb{G}(3,5)$. Then by (4.1) we see that Δ represents the set of the 3-dimensional linear spaces of a fixed $\mathbb{P}^4$ passing through a fixed point. Let us fix the flag: $E_0 \subset H_5$, where $E_0 = (1,0,0,0,0,0) \in \mathbb{P}^5$ and H_5 is the hyperplane in $\mathbb{P}^5$ whose equation is $x_5 = 0$.

Hence Δ represents the 3-dimensional linear spaces of $\mathbb{P}^5$ whose equations are: $x_5 = 0, xx_1 + yx_2 + zx_3 + tx_4 = 0$, with $(x,y,z,t) \in \mathbb{P}^3$. The lines intersecting such $\mathbb{P}^3$ satisfy the following equations in the Plücker coordinates p_{ij}: $xp_{15} + yp_{25} + zp_{35} + tp_{45} = 0$. Thus Δ has equations: $m_{01} = m_{02} = m_{03} = m_{04} = m_{05} = m_{12} = m_{13} = m_{14} = m_{23} = m_{24} = m_{34} = 0$ in $\check{\mathbb{P}}^{14}$.

The corresponding matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & t \\ 0 & -x & -y & -z & -t & 0 \end{pmatrix}$$

Hence a generic element of Δ can be written in the form $xA + yB + zC + tD$, where $(x,y,z,t) \in \mathbb{P}^3$ and A, B, C, D are 6×6 skew-symmetric matrices. Note that

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and similarly we can write down B, C, D. So X is the variety whose equations are the 4×4 minors F_{ij} of the 4×6 matrix

$$F = \begin{pmatrix} \sum a_{0i}x_i & . & . & \sum a_{5i}x_i \\ \sum b_{0i}x_i & . & . & \sum b_{5i}x_i \\ \sum c_{0i}x_i & . & . & \sum c_{5i}x_i \\ \sum d_{0i}x_i & . & . & \sum d_{5i}x_i \end{pmatrix} = \begin{pmatrix} 0 & x_5 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & x_5 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & x_5 & 0 & -x_3 \\ 0 & 0 & 0 & 0 & x_5 & -x_4 \end{pmatrix} \quad (11)$$

It is straightforward to see that $X = 3H_5$, where $H_5 : x_5 = 0$. Hence the map ρ is not regular in this case.

Let Δ be as in case $\alpha 2$), i.e. $\Delta \not\subset \mathbb{G}(3,5)$ and $\Delta \subset \overset{\vee}{\mathbb{G}}$. According to the position of Δ with respect to the fibre of $\psi : \overset{\vee}{\mathbb{G}}(1,5) - - \to \mathbb{G}(1,5)$, we have the following situations:

$\alpha 2.1$) there exists a unique line l such that $\Delta \subset \mathbb{P}^5_l$,
$\alpha 2.2$) $\Delta \not\subset \mathbb{P}^5_l$, for every l, but there exists a line m such that $\Delta \cap \mathbb{P}^5_m = \pi$, with π a plane,
$\alpha 2.3$) the general fibre of $\psi_{|\Delta}$ is a line,
$\alpha 2.4$) the general fibre of $\psi_{|\Delta}$ is a point.

In the case $\alpha 2.1$) since $\Delta \subset \mathbb{P}^5_l$, then $\Delta \cap \mathbb{G}(3,5) = \Delta \cap (\mathbb{P}^5_l \cap \mathbb{G}(3,5))$. Note that $\mathbb{P}^5_l \cap \mathbb{G}(3,5)$ is a smooth 4-dimensional quadric, Δ is a $\mathbb{P}^3$ and thus $\Delta \cap \mathbb{G}(3,5)$ is a 2-dimensional quadric of $\mathbb{P}^3$: its points correspond to complexes having a $\mathbb{P}^3$ containing l as a singular set, these $\mathbb{P}^3$ are contained in the degeneracy locus X and thus $\dim X > 3$.

In the case $\alpha 2.2$) since there exists a line m such that $\Delta \cap \mathbb{P}^5_m = \pi$, with π a plane, then $\Delta \cap \mathbb{G}(3,5) \supset (\Delta \cap \mathbb{P}^5_m) \cap (\mathbb{G}(3,5) \cap \mathbb{P}^5_m)$. This latter intersection is either π or a conic, since $\mathbb{G}(3,5) \cap \mathbb{P}^5_m$ is a smooth 4-dimensional quadric and $\Delta \cap \mathbb{P}^5_m = \pi$.

As in the previous case, the points in such intersection correspond to complexes having a $\mathbb{P}^3$ as a singular set which is contained in the degeneracy locus X and thus $\dim X > 3$.

In the case $\alpha 2.3$) the general fibre of $\psi_{|\Delta} : \Delta - - \to \psi_{|\Delta}(\Delta) \subset \mathbb{G}(1,5)$ is a line. Thus $T = \psi_{|\Delta}(\Delta)$ is a surface in $\mathbb{G}(1,5)$. The intersection $(\Delta \cap \mathbb{P}^5_l) \cap (\mathbb{G}(3,5) \cap \mathbb{P}^5_l) \neq \emptyset$, for every $l \in T$. Hence for every $l \in T$, we find a $\mathbb{P}^3$ contained in X and thus $\dim X > 3$.

In the case $\alpha 2.4$) the general fibre of $\psi_{|\Delta} : \Delta - - \to \psi_{|\Delta}(\Delta) \subset \mathbb{G}(1,5)$ is a point. This means that $T = \psi_{|\Delta}(\Delta)$ is a 3-fold in $\mathbb{G}(1,5)$ and that a general complex in Δ has a line as singular set. Thus there is a 3-dimensional family of such lines and the variety X is their union.

It could a priori happen that $\dim X = 3$, but then infinitely many lines of the family pass through a general point of X. Hence the matrix

F (see (1)) should have rank < 3 at every point of X. This means that the four hyperplanes, whose coordinates are the rows of F, belong to a pencil, whose support is a 3-space Λ_P: it is the union of the lines of C_4 (the base of the web Δ) passing through P. Assume that the first two rows of F are linearly independent, then the Plücker coordinates of Λ_P are the order two minors of the first two rows of F. If F_{0123} is the minor of the last two columns, then its vanishing at P represents the condition that Λ_P meets the line $x_0 = x_1 = x_2 = x_3 = 0$ (as in Claim 3.4). If Λ_P is disjoint from this line, let Q_P be the quadric surface $\Lambda_P \cap \{F_{0123} = 0\}$: we claim that $Q_P \subset X$. Indeed, if $z \in Q_P$, then both the line of C_4 through P and the 3-plane Λ_z contain z. Let P' be the point $\Lambda_z \cap \{x_0 = x_1 = x_2 = x_3 = 0\}$: the line $\overline{zP'}$ is in C_4 but not in Λ_P, so $z \in X$, and the claim is proved.

Let now P, P' be two distinct points in X: if $\Lambda_P = \Lambda'_P$, then $\Lambda_P \subset X$: in fact if r is a line of Λ_P through P, then $r \in C_4$, so if $R \in r$, then $R \in \Lambda'_P$ so the line $RP' \in C_4$ and $R \in X$. Therefore we can deduce that, if $\dim X = 3$, to each point $P \in X$ we can associate a $\mathbb{P}^3$ Λ_P and different points give different 3-spaces. So also the quadrics Q_P are two by two distinct and X contains a family of dimension 3 of quadric surfaces: a contradiction. Hence we conclude that $\dim X > 3$. $\square$

Before considering case β) we make the following remark about equations of cubic surfaces in $\mathbb{P}^3$ which will be used later on in the paper.

Remark 4.4. Let S be a cubic surface in $\mathbb{P}^3$. If S is smooth then it can be defined by an equation $pfM = 0$, where M is a 6×6 skew-symmetric matrix of linear forms ([B], Prop. 7.6).

Let S be a singular element of $|\mathcal{O}_{P^3}(3)|$. If S has a finite number of lines then by ([BL]) it follows that its equation can be expressed as the determinant of a 3×3 matrix N of linear forms, except if S is the surface whose class of projective equivalence is called T_1. Note that if S is defined by $detN = 0$, then S is also defined by $pfM = 0$, where $M = \begin{pmatrix} 0 & N \\ -{}^tN & 0 \end{pmatrix}$, since $pfM = detN$. Up to automorphisms the equation of the surface in the class T_1 is the following: $x_0x_1^2 + x_1x_3^2 + x_2^3 = 0$. It is easy to check that it is the pfaffian of the following matrix

$$\begin{pmatrix} 0 & -x_0 & 0 & 0 & x_2 & x_1 \\ x_0 & 0 & -x_0 & x_3 & 0 & x_2 \\ 0 & x_0 & 0 & x_2 & x_3 & 0 \\ 0 & -x_3 & -x_2 & 0 & x_1 & 0 \\ -x_2 & 0 & -x_3 & -x_1 & 0 & 0 \\ -x_1 & -x_2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If S has an infinite number of lines then by a well known result (see for instance [fC]) S is either a reducible cubic surface, or an irreducible cone, or an irreducible cubic surface with a double line (i.e a general ruled cubic surface). But in all of these cases S can be defined by an equation $detN$, where N is a 3×3 matrix of linear forms, and hence also by $pfM = 0$, with M as above.

Hence we can conclude that every cubic surface can be expressed as a pfaffian.

Let us consider now case β). Let S be the cubic surface $\Delta \cap \check{\mathbb{G}}(1,5)$. One of the following may happen:

$\beta 1)$ $\Delta \cap \mathbb{G}(3,5) = \emptyset$, or

$\beta 2)$ $\Delta \cap \mathbb{G}(3,5) \neq \emptyset$, but $S \not\subset \mathbb{G}(3,5) = Sing(\check{\mathbb{G}})$, or

$\beta 3)$ $\Delta \cap \mathbb{G}(3,5) \neq \emptyset$ and $S \subset \mathbb{G}(3,5)$, that is, $S \subset Sing(\check{\mathbb{G}})$.

Let Δ be as in $\beta 1$). Then either S is smooth, or it is singular and its singularities correspond to tangency points of Δ to $\check{\mathbb{G}}(1,5)$. The case of S smooth was considered in Theorem (3.2). We give next an explicit example of a surface $S = \Delta \cap \check{\mathbb{G}}(1,5)$ with $\Delta \cap \mathbb{G}(3,5) = \emptyset$ and S singular.

Example 4.5. Let S be the cubic surface whose equation is $x^2y - x^2z - xy^2 + xz^2 + y^3 - y^2t + yzt = 0$. The class of projective equivalence of this surface is called T_4 in [BL]. Its equation can be written as $\det N = 0$, where N is the following matrix: $\begin{pmatrix} t & x & y \\ y+z & -y & 2x+t \\ y & 0 & x+y-z \end{pmatrix}$ or as $pfM = 0$, where M is obtained from N as in (4.4).

S has only one singularity at the point $(0,0,0,1)$, which corresponds to the only matrix of rank less than 6 obtained from M, for a particular choice of x, y, z, t. It is easy to see that its rank is 4.

Note that if Δ is as in $\beta 1$), for all $(x,y,z,t) \in S$, the matrix $xA + yB + zC + tD$ has rank four, so it determines always a line of solutions of the equation $(xA+yB+zC+tD)^t(x_0\dots x_5)=0$ on the degeneracy locus X. Thus $\dim X=3$.

Remark 4.6. It is possible that, for certain surfaces S, the matrices A,B,C,D, appearing in a pfaffian $pf(xA+yB+zC+tD)$ giving the equation of S, are linearly dependent, so they don't generate a 3-space Δ in $\check{\mathbb{P}}^{14}$, but only a plane.

For example, let S be the union of 3 planes, then S is defined by $det(M)=0$, where M is a 3×3 diagonal matrix whose non zero entries are the linear forms defining the three planes: $F=ax+by+cz+td, G=a'x+b'y+c'z+t'd, H=a''x+b''y+c''z+t''d$.

Note that S is also defined by $pf(xA+yB+zC+tD)=0$, where

$$A=\begin{pmatrix} 0 & 0 & 0 & a & 0 & 0\\ 0 & 0 & 0 & 0 & a' & 0\\ 0 & 0 & 0 & 0 & 0 & a''\\ -a & 0 & 0 & 0 & 0 & 0\\ 0 & -a' & 0 & 0 & 0 & 0\\ 0 & 0 & -a'' & 0 & 0 & 0 \end{pmatrix}$$

The remaining matrices B,C,D are of the same type where the entries a,a',a'' are replaced by b,b',b'', c,c',c'', d,d',d'', respectively. Thus the matrices A,B,C,D are linearly dependent.

Another example is that of cones: if S is a cone of vertex $(0,0,0,1)$ over a smooth plane elliptic curve, then its equation can be put in Weierstrass normal form $y^2z=x(x-z)(x-cz)$: this can be interpreted as the determinant of a 3×3 matrix whose entries are linear forms in the variables x,y,z only.

Nevertheless, for both examples, it is possible to find also another pfaffian expression of the equation of S, with a matrix which is a linear combination of four independent matrices. In the first case, for example, the equation of S is the pfaffian of the matrix M:

$$\begin{pmatrix} 0 & 0 & 0 & F & 0 & 0\\ 0 & 0 & 0 & 0 & G & x\\ 0 & 0 & 0 & 0 & 0 & H\\ -F & 0 & 0 & 0 & 0 & 0\\ 0 & -G & 0 & 0 & 0 & 0\\ 0 & -x & -H & 0 & 0 & 0 \end{pmatrix}$$

In the second case, the equation of S is the pfaffian of

$$\begin{pmatrix} 0 & x & t & y & 0 & 0 \\ -x & 0 & 0 & 0 & y & y \\ -t & 0 & 0 & x & -z & 0 \\ -y & 0 & -x & 0 & -l & -l \\ 0 & -y & z & l & 0 & x \\ 0 & -y & 0 & l & -x & 0 \end{pmatrix}$$

where $l = (c+1)x - cz$.

We prove next that the case $\beta 3)$ does not occur.

Proposition 4.7. *The case $\beta 3)$ does not occur.*

Proof. Let S be as in $\beta 3)$. Let $P, Q \in S$ and let L denote the line through P and Q. Then $L \subset \Delta$. Moreover $L \subset \check{\mathbb{G}}$ since $\check{\mathbb{G}} = Sec(\mathbb{G}(3,5))$. Thus $L \subset S$ and hence $Sec(S) = S$. This latter fact implies that (the support of) S is linear, hence it is a triple plane. So Δ should be tangent to $\mathbb{G}(3,5)$ along a plane. But this is impossible, because every tangent space to $\mathbb{G}(3,5)$ is tangent at one point only, and the tangent spaces intersect two by two along a plane. $\square$

Let us consider now the case $\beta 2)$, that is: $\Delta \cap \mathbb{G}(3,5) \neq \emptyset$ and $S \cap \mathbb{G}(3,5) = S \cap Sing(\check{\mathbb{G}}) \neq \emptyset$, where S is the cubic surface $S = \Delta \cap \check{\mathbb{G}}$, which in this case is singular.

For all points (x,y,z,t) in $\Delta \cap \mathbb{G}(3,5)$, the equation

$$(xA + yB + zC + tD)\ {}^t(x_0 \dots x_5) = 0$$

is satisfied by the points of a $\mathbb{P}^3$, which enters in the degeneracy locus X. If the intersection $\Delta \cap \mathbb{G}(3,5)$ is a finite set of points, say d points, then X has dimension 3 and contains d 3-spaces as irreducible components. This number d is at most 4 (see [BL]). If the intersection is infinite, then it contains at least a curve C, and over every point of C there is a 3-space contained in X, therefore $\dim X \geq 4$.

The latter situation can appear only if S is irreducible with a double line, or reducible in the union of a plane with a quadric (possibly reducible). We will give examples of both these situations.

Example 4.8. (i) Let S be a cubic surface having the line r of equation $x = y = 0$ as double line, and containing also the lines $y = z = 0$ and $x = t = 0$. Then its equation takes the form $F(x,y,z,t) = 0$, where

$$F(x,y,z,t) = ax^2y + bx^2z + cxy^2 + dxyz + exyt + fy^2t$$

Note that $F = det\ N$, where $N = \begin{pmatrix} ex+fy & bx+dy & ax+cy \\ 0 & -y & z \\ -x & 0 & t \end{pmatrix}$, or $F = pfM$ where M is the corresponding skew-symmetric matrix $xA+yB+zC+tD$ (as in (4.4)). It is easy to check that the rank of M is less than 4 precisely at the points of r. Therefore, if Δ is the 3-space generated by A, B, C, D, r is the intersection $\Delta \cap \mathbb{G}(3,5)$.

(ii) Let now S be the union of the quadric $Q : xz - yt = 0$ with the plane $\pi : L(x,y,z,t) = 0$. Then the determinant $\begin{vmatrix} x & y & 0 \\ t & z & 0 \\ 0 & 0 & L \end{vmatrix}$ clearly vanishes on S. Hence the rank of the corresponding skew-symmetric matrix M is 2 along the conic $Q \cap \pi$. If we replace Q with a quadric cone Q', with similar computations we get rank 2 on the conic $Q' \cap \pi$ and moreover in the vertex of Q'.

We conclude this section collecting the results so far obtained about case β) in the following Theorem:

THEOREM 4.9. *Let Δ be a web of linear line complexes in $\mathbb{P}^5$ as in case β). Then the rational map*

$$\rho : \mathbb{G}(3, \overset{\vee}{\mathbb{P}}{}^{14}) - - \to \mathcal{H}$$

is not regular at the point corresponding to Δ if and only if the intersection of Δ with $\mathbb{G}(3,5)$ contains a line or a conic.

5. The pfaffian map

Let $\mathcal{S}_3$ be the variety of the 6×6 skew-symmetric matrices of linear forms on $\mathbb{P}^3$. Let $pf : \mathcal{S}_3 - - \to |\mathcal{O}_{\mathbb{P}^3}(3)|$ be the rational map which sends $M \in \mathcal{S}_3$ to the cubic surface S defined by $pfM = 0$. It is regular on the open set of matrices whose pfaffian is not identically zero. Note that the equation $pfM = 0$ is equivalent to $pf({}^tPMP) = 0$, with $P \in GL(6)$. In fact $pf({}^tPMP) = (detP)(pfM)$ and thus S can also be defined by $pf({}^tPMP) = 0$.

Let $GL(6)$ act on $\mathcal{S}_3$ by congruence, that is the action is given by ${}^tPMP$ where $M \in \mathcal{S}_3$ and $P \in GL(6)$. As noted in [B] the group $GL(6)$ acts freely and properly on $\mathcal{S}_3$ and the map pf factors through $\mathcal{S}_3/GL(6)$. Hence we have the following commutative diagram:

$$\begin{tikzcd} & \mathcal{S}_3 \arrow[dl, dashed, "\pi"'] \arrow[d, dashed, "pf"] \\ \mathcal{S}_3/GL(6) \arrow[r, dashed, "\overline{pf}"'] & |\mathcal{O}_{\mathbb{P}^3}(3)| \end{tikzcd}$$

It's easy to see that $\dim \mathcal{S}_3/GL(6) = 60 - 36 = 24$. By Remark (4.4) it follows that the pfaffian map, and by consequence also $\overline{pf}$ is surjective.

If S is a generic cubic surface defined by an equation $pfM = 0$, Beauville in [B] has also proved that expressing S as a linear pfaffian is equivalent to produce a rank 2 vector bundle E_M on S which is so defined

$$E_M := coker(\mathcal{O}_{\mathbb{P}^3}(-1)^6 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^3}^6).$$

Let $\mathcal{S}_3^s$ be the scheme of matrices $M \in \mathcal{S}_3$ such that pfM is smooth: it follows that the map M is injective and E_M is a stable bundle. Then $\mathcal{S}_3^s$ is an open non-empty set of $\mathcal{S}_3$. Going through the proof of Lemma 7.7 in [B], one gets that for the generic S the fibre $\overline{pf}^{-1}(S)$ can be identified with an open subset of the moduli space of simple rank 2 vector bundles E on S with $c_1(E) = \mathcal{O}_S(2), c_2(E) = 5$. These are precisely the bundles on S such that $\mathbb{P}(E)$ is a Palatini scroll over S ([O]).

Moreover the quotient $\mathcal{S}_3^s/GL(6)$ can be canonically identified with the set of pairs (S, E), where S is a smooth cubic surface and E is a rank two bundle on S, of the previous type.

There is a natural rational map $\Phi : \mathcal{H} - - \rightarrow |\mathcal{O}_{\mathbb{P}^3}(3)|$, where $\mathcal{H}$ is the component of the Hilbert scheme studied in Section 3. The map Φ sends a scroll X to the base S of the scroll. If X is smooth then S is its image via the adjunction map. Moreover we have the following commutative diagram of rational maps:

$$\begin{tikzcd} & \mathcal{H} \arrow[d, dashed, "\Phi"] \\ \mathcal{S}_3 \arrow[ur, dashed, "\Psi"] \arrow[r, dashed, "pf"'] & |\mathcal{O}_{\mathbb{P}^3}(3)| \end{tikzcd}$$

where Ψ is the map which sends the matrix $M = Ax + By + Cz + Dt$ to the degeneracy locus X as in Section 3.

Let X be a Palatini scroll and let Δ be the web associated to it. The fibre of Ψ over X is 16-dimensional: its elements correspond to the different choices of a base of Δ. Note that Ψ does not induce any map from the quotient $\mathcal{S}_3/GL(6)$ to $\mathcal{H}$.

As a consequence of the interpretation of $\mathcal{S}_3/GL(6)$ seen above, $\overline{pf}$ can be interpreted as a forgetful map and we get a factorization of Φ through $\mathcal{S}_3/GL(6)$, as $\Phi = \overline{pf} \circ \overline{\Phi}$.

The new map $\overline{\Phi} : \mathcal{H} - - \to \mathcal{S}_3/GL(6)$ sends a Palatini scroll $X = \mathbb{P}(E)$ to the corresponding pair (S, E). Since $\dim \mathcal{H} = 44$ and $\dim \mathcal{S}_3/GL(6) = 24$, the fibres of $\overline{\Phi}$ are 20-dimensional.

To explain this number, let us observe that the points of the fibre $\overline{\Phi}^{-1}(S, E)$ are nothing more than the different embeddings of the projective bundle $\mathbb{P}(E)$ on S in $\mathbb{P}^5$, i.e. the automorphisms of $\mathbb{P}^5$ preserving X.

We claim that if an automorphism of $\mathbb{P}^5$ preserves X, then it belongs to the subgroup $\overline{P}$ of $PGL(5)$ of automorphisms inducing the identity on $\mathbb{P}^3 = \mathbb{P}(H^0(\mathcal{O}_S(1)))$.

Indeed, if G is an automorphism of $\mathbb{P}^5$ which preserves X, it preserves also $\mathbb{P}^3$, because

$$\mathbb{P}^3 = \mathbb{P}(H^0(\mathcal{O}_X(K + 2H))) \simeq \mathbb{P}(H^4(\mathcal{I}_X(-2))),$$

so it induces on $\mathbb{P}^3$ an action which preserves S. But every automorphism of S is trivial, being S general, so the restriction of G to $\mathbb{P}^3$ is the identity (see [K]). This shows that the fibre of $\overline{\Phi}$ is isomorphic to the subgroup $\overline{P}$, whose dimension is 20.

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