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A REMARK ON WELL-POSEDNESS FOR HYPERBOLIC EQUATIONS WITH SINGULAR COEFFICIENTS

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ABSTRACT. We prove some C^∞ and Gevrey well-posedness results for hyperbolic equations with singular coefficients.

1. INTRODUCTION

This work is devoted to the study of the well-posedness of the Cauchy problem for a linear hyperbolic operator whose coefficients depend only on time.

We consider the equation

$$u_{tt} - \sum_{i,j=1}^{n} a_{ij}(t) u_{x_i x_j} = 0 \tag{1.1}$$

in $[0,T] \times \mathbb{R}^n$, with initial data

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{1.2}$$

in $\mathbb{R}^n$. The matrix (a_{ij}) is supposed to be real and symmetric. Setting

$$a(t,\xi) := \sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j/|\xi|^2, \quad (t,\xi) \in [0,T] \times (\mathbb{R}^n \setminus \{0\}), \tag{1.3}$$

we assume that $a(\cdot,\xi) \in L^1(0,T)$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

We suppose that the equation (1.1) is hyperbolic i.e.

$$a(t,\xi) \geq \lambda_0 \geq 0 \tag{1.4}$$

for all $(t,\xi) \in [0,T] \times (\mathbb{R}^n \setminus \{0\})$.

In the strictly hyperbolic case (i.e. $\lambda_0 > 0$) it is well known that if the coefficients a_{ij} are Lipschitz-continuous then the Cauchy problem (1.1), (1.2) is well-posed in Sobolev spaces. In the same case if the a_{ij}'s are Log-Lipschitz-continuous or Hölder-continuous of index α, (1.1), (1.2) is well-posed in C^∞ or in the Gevrey space $\gamma^{(s)}$ for $s < \frac{1}{1-\alpha}$ respectively (see [1]). In the weakly hyperbolic case (i.e. $\lambda_0 = 0$) if the coefficients are $C^{k,\alpha}$ then

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the problem (1.1), (1.2) is $\gamma^{(s)}$-well-posed for $s < 1 + \frac{k+\alpha}{2}$ (see [4]). Some counter examples show that all these results are sharp (see also [5]).

Recently Colombini, Del Santo and Kinoshita have considered the same problem for operators having coefficients which are C^1 on $[0,T]\setminus\{t_0\}$ with a singularity concentrated at t_0. In this situation, under the main assumptions that

$$
\begin{aligned}
&|t_0 - \cdot|^p a'(\cdot,\xi) = \beta(\cdot,\xi) \in L^\infty(0,T) \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\} \\
&|t_0 - \cdot|^r a(\cdot,\xi) = \alpha(\cdot,\xi) \in L^\infty(0,T) \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}
\end{aligned}
\tag{1.5}
$$

it is possible to show that the Cauchy problem (1.1), (1.2) is $\gamma^{(s)}$-well-posed, the value of s depending on p and r (see [2] and[3]) (here and in the following "$'$" denotes the differentiation with respect to t).

The aim of the present work is to improve the results of [2] and [3] allowing the function β in (1.5) to be in a L^q space and removing the growth assumption on a. We make the following assumptions: let $1 \le q \le +\infty$ and $p \ge 0$ and let $t_0 \in [0,T]$; suppose that

(H1) $a(\cdot,\xi) \in \cap_{\varepsilon>0} W^{1,1}(]0,t_0-\varepsilon[\cup]t_0+\varepsilon,T[)$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$;

(H2) $|t_0 - \cdot|^p a'(\cdot,\xi) = \beta(\cdot,\xi) \in L^q(0,T)$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

In the weakly hyperbolic case the results are the following.

Theorem 1. *Assume that $3 \le (p+1/q)$. Then the Cauchy problem (1.1), (1.2) is $\gamma^{(\sigma)}$-well-posed for $1 \le \sigma < \frac{(p+1/q)-\frac{3}{2}}{(p+1/q)-2}$. If moreover*

$$|t_0 - \cdot|^r a(\cdot,\xi) = \alpha(\cdot,\xi) \in L^s(0,T) \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \tag{1.6}$$

with $r \ge 0$, $1 \le s \le +\infty$ and $(r+1/s) \le 1$, then the Cauchy problem (1.1), (1.2) is $\gamma^{(\sigma)}$-well-posed for $1 \le \sigma < \frac{(p+1/q)-\frac{3}{2}(r+1/s)}{(p+1/q)-(r+1/s)-1}$.

Theorem 2. *Assume that $(p+1/q) < 3$. Then the Cauchy problem (1.1), (1.2) is $\gamma^{(\sigma)}$-well-posed for all $1 \le \sigma < \frac{3}{2}$.*

The result concerning the strictly hyperbolic case are contained in the following theorems.

Theorem 3. *Assume that $1 < (p+1/q) < 3$. Moreover, assume that $\lambda_0 > 0$. Then the Cauchy problem (1.1), (1.2) is $\gamma^{(\sigma)}$-well-posed for all $1 \le \sigma < \frac{(p+1/q)}{(p+1/q)-1}$.*

Theorem 4. *Assume that $(p+1/q) \le 1$. Moreover, assume that $\lambda_0 > 0$. Then the Cauchy problem (1.1), (1.2) is C^∞-well-posed.*

Remark 1. Adapting to the present situation some counter examples contained in [4], [2], and [3] it is possible to see that the results of Theorems 1–4 are optimal. Let us show this in some detail in the case of Theorem 1. Suppose $p_0+1/q_0 = 3$. In this case $\frac{(p_0+1/q_0)-\frac{3}{2}}{(p_0+1/q_0)-2} = \frac{(p_0+1/q_0)-\frac{3}{2}(r_0+1/s_0)}{(p_0+1/q_0)-(r_0+1/s_0)-1} = \frac{3}{2}$; consequently Theorem 2 in [4] shows that this value of the Gevrey index cannot be improved. Consider next the case that $p_0+1/q_0 > 3$ and $(r_0+1/s_0) \le 1$.

Let $\bar\sigma > \sigma_0 = \frac{(p_0+1/q_0)-\frac{3}{2}(r_0+1/s_0)}{(p_0+1/q_0)-(r_0+1/s_0)-1}$. We fix $q_1 > q_0$ and $s_1 > s_0$ in such a way that $p_0 + 1/q_1 > 3$, $r_0 + 1/s_1 < 1$ and $\sigma_0 < \sigma_1 := \frac{(p_0+1/q_1)-\frac{3}{2}(r_0+1/s_1)}{(p_0+1/q_1)-(r_0+1/s_1)-1} < \bar\sigma$. From Theorem 4 in [3] we have that there exists a function $a : [0,1[\to [1/2,+\infty[$ such that $a \in C^\infty([0,1[)$ and

$$(1-t)^{p_0+1/q_1}a'(t) \in L^\infty, \quad (1-t)^{r_0+1/s_1}a(t) \in L^\infty,$$

and there exist u_0, $u_1 \in \gamma^{(\sigma)}$ for all $\sigma > \sigma_1$ such that the Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0, \qquad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{1.7}$$

has no solution in $W^{2,1}([0,1],\mathcal{D}'^{(\sigma)}(\mathbb{R}))$ for all $\sigma > \sigma_1$. Consequently

$$(1-t)^{p_0}a'(t) \in L^{q_0}, \quad (1-t)^{r_0}a(t) \in L^{s_0},$$

u_0, $u_1 \in \gamma^{(\bar\sigma)}$ and the Cauchy problem (1.7) does not have a solution in $W^{2,1}([0,1],\mathcal{D}'^{(\bar\sigma)}(\mathbb{R}))$.

Remark 2. Let us remark that Theorem 1 is a nontrivial improvement of Theorem 2 in [3] also in the case of $q = \infty$. In fact the growth condition on a is removed and the result is sharp (see [3, Th. 4]).

2. Proof of Theorems 1–4

As a preliminary step, let us observe that, since the coefficients a_{ij} are real integrable functions, the Cauchy problem (1.1), (1.2) is well posed in $\mathcal{A}'(\mathbb{R}^n)$, the space of real analytic functionals. Moreover, if the initial data vanish in a ball, then the solution vanishes in a cone, whose slope depends on the coefficients a_{ij}. Therefore it will be sufficient to show that, under the hypotheses of each theorem, if u_0 and u_1 have compact support then the corresponding solution u is not only in $W^{2,1}([0,T],\mathcal{A}'(\mathbb{R}^n))$, but it belongs to a Gevrey space in the x variable. Our main tools in doing this will be the Paley-Wiener theorem (in the version of [1, p. 517], to which we refer here and throughout) and some energy estimates.

Denoting by v the Fourier transform of u with respect to x, equation (1.1) reads

$$v''(t,\xi) + a(t,\xi)|\xi|^2 v(t,\xi) = 0. \tag{2.1}$$

Let ϵ be a positive parameter and for each ϵ let $a_\varepsilon\colon [0,T]\times(\mathbb{R}^n\setminus\{0\}) \to \mathbb{R}$ be a strictly positive real function such that $a_\varepsilon(\cdot,\xi) \in W^{1,1}(0,T)$ for all $\xi \in \mathbb{R}^n\setminus\{0\}$. We define the *approximate energy* of v by

$$E_\varepsilon(t,\xi) := a_\varepsilon(t,\xi)|\xi|^2|v(t,\xi)|^2 + |v'(t,\xi)|^2, \quad (t,\xi) \in [0,T]\times(\mathbb{R}^n\setminus\{0\}). \tag{2.2}$$

Differentiating E_ε with respect to t and using (2.1) we get

$$\begin{aligned} E'_\varepsilon(t,\xi) = a'_\varepsilon(t,\xi)|\xi|^2|v(t,\xi)|^2 &+ 2a_\varepsilon(t,\xi)|\xi|^2\mathrm{Re}(v'(t,\xi)\bar v(t,\xi)) \\ &+ 2\mathrm{Re}(v''(t,\xi)\bar v'(t,\xi)) \\ &\le \left(\frac{|a'_\varepsilon(t,\xi|}{a_\varepsilon(t,\xi)} + \frac{|a_\varepsilon(t,\xi) - a(t,\xi)|}{a_\varepsilon(t,\xi)^{1/2}}|\xi|\right) E_\varepsilon(t,\xi). \end{aligned}$$

By Gronwall's lemma we obtain
(2.3)

$$E_\varepsilon(t,\xi) \le E_\varepsilon(0,\xi) \exp\left(\int_0^T \frac{|a'_\varepsilon(t,\xi|}{a_\varepsilon(t,\xi)}\, dt + |\xi| \int_0^T \frac{|a_\varepsilon(t,\xi) - a(t,\xi)|}{a_\varepsilon(t,\xi)^{1/2}}\, dt \right)$$

for all $t \in [0,T]$ and for all $\xi \in \mathbb{R}^n$, $|\xi| \ge 1$.

Now we are able to give the

Proof of Theorem 1. First of all, observe that condition (1.6) is always satisfied at least with $r = 0$ and $s = 1$ (recall that $a_{ij} \in L^1(0,T)$).

Since u_0, $u_1 \in \gamma^{(\sigma)} \cap C_0^\infty$, the Paley-Wiener theorem ensures that there exist $M, \delta > 0$ such that

$$|v(0,\xi)|^2 + |v'(0,\xi)|^2 \le M \exp(-\delta |\xi|^{1/\sigma}) \tag{2.4}$$

for all $\xi \in \mathbb{R}^n$, $|\xi| \ge 1$. To verify that $u \in W^{2,1}([0,T], \gamma^{(\sigma)})$ it is sufficient to show that there exist $M', \delta' > 0$ such that

$$|v(t,\xi)|^2 + |v'(t,\xi)|^2 \le M' \exp(-\delta' |\xi|^{1/\sigma}) \tag{2.5}$$

for all $t \in [0,T]$ and for all $\xi \in \mathbb{R}^n$, $|\xi| \ge 1$. We consider first the case $t_0 = T$. For $\varepsilon \in]0,T]$, we set

$$a_\varepsilon(t,\xi) := \begin{cases} a(t,\xi) + \varepsilon^{2-(r+1/s)}(T-t)^{-2} & \text{for } 0 \le t \le T-\varepsilon \\ \varepsilon^{-(z+r)} a(t,\xi)(T-t)^{z+r} + \varepsilon^{-(r+1/s)} & \text{for } T-\varepsilon \le t \le T \end{cases} \tag{2.6}$$

where z is any positive number such that

$$z > \max\{1/s, (p+1/q) - r - 1\}. \tag{2.7}$$

Then
(2.8)

$$a_\varepsilon(t,\xi) = \begin{cases} \alpha(t,\xi)(T-t)^{-r} + \varepsilon^{2-(r+1/s)}(T-t)^{-2} & \text{for } 0 \le t \le T-\varepsilon \\ \varepsilon^{-(z+r)} \alpha(t,\xi)(T-t)^{z} + \varepsilon^{-(r+1/s)} & \text{for } T-\varepsilon \le t \le T \end{cases}$$

and
(2.9)

$$a'_\varepsilon(t,\xi) = \begin{cases} \beta(t,\xi)(T-t)^{-p} - 2\varepsilon^{2-(r+1/s)}(T-t)^{-3} & \text{for } 0 \le t \le T-\varepsilon \\ \varepsilon^{-(z+r)} \left(\beta(t,\xi)(T-t)^{z+r-p} \right. & \\ \qquad\qquad \left. -(z+r)\alpha(t,\xi)(T-t)^{z-1}\right) & \text{for } T-\varepsilon \le t \le T \end{cases}$$

Our choice of z implies that $a_\epsilon(\cdot,\xi) \in W^{1,1}(0,T)$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. By (2.8) and (2.9) we get

$$\begin{aligned}\int_0^T \frac{|a'_\varepsilon(t,\xi|}{a_\varepsilon(t,\xi)}\,dt \le& \int_0^{T-\varepsilon} \frac{|\beta(t,\xi)|(T-t)^{-p}}{\varepsilon^{2-(r+1/s)}(T-t)^{-2}}\,dt \\ &+ \int_0^{T-\varepsilon} \frac{2\varepsilon^{2-(r+1/s)}(T-t)^{-3}}{\varepsilon^{2-(r+1/s)}(T-t)^{-2}}\,dt \\ &+ \int_{T-\varepsilon}^{T} \frac{\varepsilon^{-(z+r)}|\beta(t,\xi)|(T-t)^{z+r-p}}{\varepsilon^{-(r+1/s)}}\,dt \\ &+ \int_{T-\varepsilon}^{T} \frac{\varepsilon^{-(z+r)}(z+r)|\alpha(t,\xi)|(T-t)^{z-1}}{\varepsilon^{-(r+1/s)}}\,dt\end{aligned}$$

The choice of z allows us to use Hölder inequality; an easy computation shows that

$$\int_0^T \frac{|a'_\varepsilon(t,\xi|}{a_\varepsilon(t,\xi)}\,dt \le C'(1+|\log\varepsilon|)\varepsilon^{-(p+1/q)+(r+1/s)+1}, \tag{2.10}$$

where C' is a constant depending only on C, r, s, p, q and z. On the other hand,

$$\begin{aligned}\int_0^T \frac{|a_\varepsilon(t,\xi)-a(t,\xi)|}{a_\varepsilon(t,\xi)^{1/2}}\,dt =& \int_0^{T-\varepsilon} \frac{\varepsilon^{2-(r+1/s)}(T-t)^{-2}}{\varepsilon^{1-(1/2)(r+1/s)}(T-t)^{-1}}\,dt \\ &+ \int_{T-\varepsilon}^{T} \frac{\varepsilon^{-(z+r)}\alpha(t,\xi)(T-t)^{z}}{\varepsilon^{-(1/2)(r+1/s)}}\,dt \\ &+ \int_{T-\varepsilon}^{T} \frac{\varepsilon^{-(r+1/s)}}{\varepsilon^{-(1/2)(r+1/s)}}\,dt + \int_{T-\varepsilon}^{T} \frac{\alpha(t,\xi)(T-t)^{-r}}{\varepsilon^{-(1/2)(r+1/s)}}\,dt.\end{aligned}$$

The first three summands on the right hand side can be estimated again by using Hölder inequality. In order to estimate the fourth summand, we shall distinguish the case $(r+1/s)<1$ and $(r+1/s)=1$. In the first case, we use once more Hölder inequality; in the second case, we use the fact that $\alpha(t,\xi)(T-t)^{-r} = a(t,\xi) \in L^1(0,T)$. At the end, we get

$$\int_0^T \frac{|a_\varepsilon(t,\xi)-a(t,\xi)|}{a_\varepsilon(t,\xi)^{1/2}}\,dt \le C''(1+|\log\varepsilon|)\varepsilon^{-(1/2)(r+1/s)+1}, \tag{2.11}$$

where C'' is a constant depending only on C, r, s, p, q and z. By (2.3), (2.10) and (2.11) we obtain

$$\begin{aligned}&E(t,\xi) \\ &\quad\le E(0,\xi)\exp\left(\tilde{C}(1+|\log\varepsilon|)(\varepsilon^{-(p+1/q)+(r+1/s)+1} + |\xi|\varepsilon^{-(1/2)(r+1/s)+1})\right)\end{aligned} \tag{2.12}$$

for all $t \in [0,T]$ and for all $\xi \in \mathbb{R}^n$, $\xi \ge 1$, where $\tilde{C}$ is a positive constant depending only on C, r, s, p, q and z.

Now, by (2.2) and (2.6), we have

$$E_\varepsilon(0,\xi) \le \left(a(0,\xi) + T^{-(r+1/s)}\right)|\xi|^2|v(0,\xi)|^2 + |v'(0,\xi)|^2 \tag{2.13}$$

and

$$E_\varepsilon(t,\xi) \geq T^{-2}\varepsilon^{2-(r+1/s)}|\xi|^2|v(t,\xi)|^2 + |v'(t,\xi)|^2. \tag{2.14}$$

Then choosing $\varepsilon := |\xi|^{-[(p+1/q)-\frac{3}{2}(r+1/s)]^{-1}}$ we deduce

$$\begin{aligned} &T^{-2}|\xi|^{2-\frac{2-(r+1/s)}{(p+1/q)-\frac{3}{2}(r+1/s)}}|v(t,\xi)|^2 + |v'(t,\xi)|^2 \\ &\quad \leq (\tilde{K}|\xi|^2|v(0,\xi)|^2 + |v'(0,\xi)|^2)\exp\big(\tilde{C}(1+|\log|\xi||)|\xi|^{\frac{(p+1/q)-(r+1/s)-1}{(p+1/q)-\frac{3}{2}(r+1/s)}}\big). \end{aligned}$$

Using the Paley-Wiener theorem, the well-posedness follows for all $1 \leq \sigma < \frac{(p+1/q)-\frac{3}{2}(r+1/s)}{(p+1/q)-(r+1/s)-1}$.

If $t_0 = 0$, for $\varepsilon \in]0,T]$ we set

$$a_\varepsilon(t,\xi) := \begin{cases} \varepsilon^{-(z+r)}a(t,\xi)t^{z+r} + \varepsilon^{-(r+1/s)} & \text{for } 0 \leq t \leq \varepsilon \\ a(t,\xi) + \varepsilon^{2-(r+1/s)}t^{-2} & \text{for } \varepsilon \leq t \leq T \end{cases} \tag{2.15}$$

where z satisfies (2.7). Our choice of z implies that $a_\epsilon(\cdot,\xi) \in W^{1,1}(0,T)$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. So, in particular, $a_\epsilon(\cdot,\xi)$ is continuous on $[0,T]$. Arguing as before, we obtain (2.12). An easy computation shows that $|a(t,\xi)| \leq \tilde{K}t^{1-(p+1/q)}$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. It follows that

$$\begin{aligned} a_\varepsilon(0,\xi) = \lim_{\tau\to 0} a_\varepsilon(\tau,\xi) &= \lim_{\tau\to 0}(\varepsilon^{-(z+r)}a(\tau,\xi)\tau^{z+r} + \varepsilon^{-(r+1/s)}) \\ &\leq \tilde{K}\limsup_{\tau\to 0}(\varepsilon^{-(z+r)}\tau^{z+r+1-(p+1/q)} + \varepsilon^{-(r+1/s)}). \end{aligned}$$

By (2.7) we deduce that $a_\varepsilon(0,\xi) \leq \tilde{K}\varepsilon^{-(r+1/s)}$. It follows that

$$E_\varepsilon(0,\xi) \leq \tilde{K}\varepsilon^{-(r+1/s)}|\xi|^2|v(0,\xi)|^2 + |v'(0,\xi|^2. \tag{2.16}$$

Moreover, we have also

$$E_\varepsilon(t,\xi) \geq T^{-2}\varepsilon^{2-(r+1/s)}|\xi|^2|v(t,\xi)|^2 + |v'(t,\xi)|^2. \tag{2.17}$$

Then, choosing again $\varepsilon := |\xi|^{-[(p+1/q)-\frac{3}{2}(r+1/s)]^{-1}}$, we deduce

$$\begin{aligned} |\xi|^{2-\frac{2-(r+1/s)}{(p+1/q)-\frac{3}{2}(r+1/s)}}|v(t,\xi)|^2 + |v'(t,\xi)|^2 &\leq (\tilde{K}|\xi|^{2+\frac{(r+1/s)}{(p+1/q)-\frac{3}{2}(r+1/s)}}|v(0,\xi)|^2 \\ &\quad + |v'(0,\xi)|^2)\exp\big(\tilde{C}(1+|\log|\xi||)|\xi|^{\frac{(p+1/q)-(r+1/s)-1}{(p+1/q)-\frac{3}{2}(r+1/s)}}\big). \end{aligned}$$

Using the Paley-Wiener theorem, the well-posedness follows again for all $1 \leq \sigma < \frac{(p+1/q)-\frac{3}{2}(r+1/s)}{(p+1/q)-(r+1/s)-1}$.

Finally, if $t_0 \in]0,T[$, it will be sufficient to solve first the Cauchy problem in $[0,t_0]$, then to solve the problem in $[t_0,T]$ with the initial data obtained from the previous one and finally to glue together the two solutions. $\square$

In order to prove Theorem 2, we proceed exactly like in the proof of Theorem 1. In this case the role of condition (1.6) is played by the estimate

$$a(t,\xi) \leq C'|t-t_0|^{-(p+1/q)+1} \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \tag{2.18}$$

which is a direct consequence of condition (H2). The function $a_\varepsilon(\cdot,\xi)$ is defined by

$$(2.19)\qquad a_\varepsilon(t,\xi):=\begin{cases} a(t,\xi)+\varepsilon^{3-(p+1/q)}(T-t)^{-2} & \text{for } 0\le t\le T-\varepsilon\\ a(T-\varepsilon,\xi)+\varepsilon^{1-(p+1/q)} & \text{for } T-\varepsilon\le t\le T\end{cases}$$

if $t_0=T$ and by

$$(2.20)\qquad a_\varepsilon(t,\xi):=\begin{cases} a(\varepsilon,\xi)+\varepsilon^{1-(p+1/q)} & \text{for } 0\le t\le \varepsilon\\ a(t,\xi)+\varepsilon^{3-(p+1/q)}t^{-2} & \text{for } \varepsilon\le t\le T\end{cases}$$

if $t_0=0$. Arguing like in the proof of Theorem 1, we get

$$(2.21)\qquad \int_0^T \frac{|a_\varepsilon'(t,\xi|}{a_\varepsilon(t,\xi)}\,dt \le C''(1+|\log\varepsilon|)\varepsilon^{(p+1/q)-3},$$

and

$$(2.22)\qquad \int_0^T \frac{|a_\varepsilon(t,\xi)-a(t,\xi)|}{a_\varepsilon(t,\xi)^{1/2}}\,dt \le C''(1+|\log\varepsilon|)\varepsilon^{-(1/2)(p+1/q)+3/2}$$

and the conclusion follows by choosing $\varepsilon:=|\xi|^{-(2/3)[3-(p+1/q)]^{-1}}$.

Theorem 3 is the strictly hyperbolic version of Theorem 2. We define again a_ε by (2.19) and (2.20), but in this case the positive lower bound for $a(t,\xi)$ allows us to obtain better estimates for $\int_0^T \frac{|a_\varepsilon'(t,\xi|}{a_\varepsilon(t,\xi)}\,dt$. Let us consider, for example, the case $t_0=T$. First observe that, by rescaling the x variable if necessary, we can always assume that $\lambda_0=1$. Then we can minorize $a_\varepsilon(t,\xi)$ by the constant 1 on $[0,T-\varepsilon^{(1/2)[3-(p+1/q)]}]$ and by $\varepsilon^{3-(p+1/q)}(T-t)^{-2}$ on $[T-\varepsilon^{(1/2)[3-(p+1/q)]},T-\varepsilon]$. So we obtain that

$$(2.23)\qquad \int_0^T \frac{|a_\varepsilon'(t,\xi|}{a_\varepsilon(t,\xi)}\,dt \le C''(1+|\log\varepsilon|)\varepsilon^{(1/2)((p+1/q)-1)((p+1/q)-3)}.$$

The conclusion follows by choosing $\varepsilon:=|\xi|^{-2[p+1/q]^{-1}[3-(p+1/q)]^{-1}}$.

Finally, we give the

Proof of Theorem 4. Since $u_0,\ u_1\in C_0^\infty$, the Paley-Wiener theorem ensures that for all $\zeta>0$ there exists $M_\zeta>0$ such that

$$(2.24)\qquad |v(0,\xi)|^2+|v'(0,\xi)|^2\le M_\zeta|\xi|^{-\zeta}$$

for all $\xi\in\mathbb{R}^n$, $|\xi|\ge 1$. To verify that $u\in W^{2,1}([0,T],C_0^\infty)$ it is sufficient to show that for all $\eta>0$ there exists $M_\eta>0$ such that

$$(2.25)\qquad |v(t,\xi)|^2+|v'(t,\xi)|^2\le M_\eta|\xi|^{-\eta}$$

for all $t\in[0,T]$ and for all $\xi\in\mathbb{R}^n$, $|\xi|\ge 1$. We give the details only in the case $t_0=T$. If $q=1$, then necessarily $p=0$. This means that $a(\cdot,\xi)\in W^{1,1}(0,T)$ and it is well known that this is enough to detect C^∞-well-posedness of the Cauchy problem (1.1), (1.2). If $q>1$, for $\varepsilon\in]0,T]$, we set

$$a_\varepsilon(t,\xi) := \begin{cases} a(t,\xi) & \text{for } 0 \le t \le T-\varepsilon \\ a(T-\varepsilon,\xi) & \text{for } T-\varepsilon \le t \le T \end{cases} \tag{2.26}$$

Now observe that

$$|a(t,\xi)| \le |a(0,\xi)| + \int_0^t |a'(\tau,\xi)|\,d\tau \le |a(0,\xi)| + \int_0^t \beta(\tau,\xi)(T-\tau)^{-p}\,d\tau$$

$$\le |a(0,\xi)| + \|\beta(\cdot,\xi)\|_{L^q}\left(\int_0^t (T-\tau)^{-pq'}\,d\tau\right)^{1/q'} \le C(1+|\log(T-t)|^{1/q'})$$

An easy computation shows that

$$\int_0^T |a'_\varepsilon(t,\xi)|\,dt \le C'|\log\varepsilon|^{1/q'} \tag{2.27}$$

and

$$\int_0^T |a_\varepsilon(t,\xi) - a(t,\xi)|\,dt \le C'\varepsilon|\log\varepsilon|^{1/q'}. \tag{2.28}$$

Then we deduce by (2.3) that

$$\begin{aligned} &|\xi|^2|v(t,\xi)|^2 + |v'(t,\xi)|^2 \\ &\quad\le (a(0,\xi)|\xi|^2|v(0,\xi)|^2 + v'(0,\xi)|^2)\exp(C'|\log\varepsilon|^{1/q'} + C'|\xi|\varepsilon|\log\varepsilon|^{1/q'}). \end{aligned} \tag{2.29}$$

Here, for simplicity, we have assumed that $\lambda_0 = 1$. Choosing $\varepsilon := |\xi|^{-1}$, we obtain

$$\begin{aligned} &|\xi|^2|v(t,\xi)|^2 + |v'(t,\xi)|^2 \\ &\qquad\qquad \le (a(0,\xi)|\xi|^2|v(0,\xi)|^2 + v'(0,\xi)|^2)\exp(C'|\log|\xi||^{1/q'}). \end{aligned} \tag{2.30}$$

Now, for $|\xi| \ge e$, we have $|\log|\xi||^{1/q'} \le |\log|\xi||$, and hence

$$|\xi|^2|v(t,\xi)|^2 + |v'(t,\xi)|^2 \le (a(0,\xi)|\xi|^2|v(0,\xi)|^2 + v'(0,\xi)|^2)|\xi|^{C'}. \tag{2.31}$$

By the Paley-Wiener theorem, the well-posedness in C_0^∞ follows. □

References

[1] F. Colombini, E. De Giorgi and S. Spagnolo, *Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temp*, Ann. Sc. Norm. Sup. Pisa, **6** (1979), 511-559.

[2] F. Colombini, D. Del Santo and T. Kinoshita, *Well-posedness of the Cauchy problem for a hyperbolic equation with non-Lipschitz coefficients*, Ann. Sc. Norm. Sup. Pisa, to appear.

[3] F. Colombini, D. Del Santo and T. Kinoshita, *Gevrey-well-posedness for weakly hyperbolic operators with non-regular coefficients*, J. Math. Pures Appl., to appear.

[4] F. Colombini, E. Jannelli and S. Spagnolo, *Well-posedness in the Gevrey classes of the Cauchy problem for a non-strictly hyperbolic equation with coefficients depending on time*, Ann. Sc. Norm. Sup. Pisa, **10** (1983), 291-312.

[5] F. Colombini and N. Lerner, *Hyperbolic operators with non-Lipschitz coefficients*, Duke Math. J., **77** (1995), 657-698.

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