A REMARK ON REACTION-DIFFUSION EQUATIONS IN UNBOUNDED DOMAINS

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Abstract. We prove the existence of a compact $(L^2 - H^1)$ attractor for a reaction-diffusion equation in $\mathbb{R}^N$. This improves a previous result of B. Wang concerning the existence of a compact $(L^2 - L^2)$ attractor for the same equation.

1. Introduction

In this note we consider the reaction-diffusion equation

$$u_t = \Delta u - \lambda u + f(u) + g(x) \quad \text{in }]0, +\infty[\times\mathbb{R}^N. \tag{1.1}$$

We make the following assumptions:

$$\lambda > 0, \quad g \in L^2(\mathbb{R}^N); \tag{1.2}$$

$$f(0) = 0; \quad f(s)s \leq 0, \quad f'(s) \leq C \quad \text{for all } s \in \mathbb{R}; \tag{1.3}$$

$$|f'(s)| \leq C(1 + |s|^\beta) \quad \text{for all } s \in \mathbb{R}, \tag{1.4}$$

where C is some positive constant and

$$0 \leq \beta \quad \text{if } N \leq 2; \quad 0 \leq \beta \leq \min\{(2^*/2) - 1, 4/N\} \quad \text{if } N \geq 3. \tag{1.5}$$

In the sequel, we write L^2 and H^1 instead of $L^2(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$. Moreover, we denote by $\langle \cdot, \cdot \rangle$ the scalar product of L^2, by $\| \cdot \|$ the norm of L^2 and by $\| \cdot \|_E$ the norm of any other space E. It is well known (see e.g. [2]) that the Cauchy problem for equation (1.1) is well posed for initial data

$$u(0, x) = u_0(x) \tag{1.6}$$

in the space L^2. Actually, equation (1.1) generates a global semiflow π in L^2 and in H^1. The following lemma summarises some well known a-priori estimates for the semiflow π (see [9], Lemmas 1 and 2):

Lemma 1.1. *Assume conditions (1.2)-(1.5) are satisfied. Then*

(1) *there exists a constant $K > 0$ and for every $R > 0$ there exists $T(R) > 0$ such that, if $u \in L^2$ and $\|u\| \leq R$, then $\|\pi(t, u)\|_{H^1} \leq K$ for all $t \geq T(R)$;*
(2) *For every $R > 0$ there exists a constant $K(R)$ such that, if $u \in H^1$ and $\|u\|_{H^1} \leq R$, then $\|\pi(t, u)\|_{H^1} \leq K(R)$ for all $t \geq 0$.*

1991 *Mathematics Subject Classification.* 35K57, 35B40, 35B41.

Key words and phrases. Reaction-diffusion equation, asymptotic compactness, attractor.

It is a classical result that, if $\mathbb{R}^N$ is replaced by a bounded domain $\Omega \subset \mathbb{R}^N$, equation (1.1) possesses a compact ($L^2 - H^1$) attractor (see e.g. [1], [4] and [5]). The proof of this fact relies essentially on the compactness of the Sobolev embedding $H^1(\Omega) \subset L^2(\Omega)$. If Ω is unbounded, the Sobolev embedding may be no longer compact and new techniques are needed. In [2] Babin and Vishik proved the existence of a compact attractor for equation (1.1) in an unbounded domain, but they needed to introduce weighted spaces in order to overcome the difficulties arising from the lack of compactness. The choice of weighted spaces, however, imposes some severe conditions on the forcing term g and on the initial data. In [9], B. Wang obtained for the first time the existence of a compact attractor for equation (1.1) in the 'natural' space L^2. The crucial step in his proof is the following

Theorem 1.2 (Wang '98). *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in L^2 and let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers, $t_n \to \infty$ as $n \to \infty$. Then there exists a strictly increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ and a function $u \in L^2$ such that $\pi(t_{n_k}, u_{n_k}) \to u$ in L^2 as $k \to \infty$. In other words, π is asymptotically compact in the strong L^2 topology.*

Combining Lemma 1.1 and Theorem 1.2 with the abstract results of [4], [6] and [8], one easily obtains

Theorem 1.3 (Wang '98). *The semiflow π possesses a compact ($L^2 - L^2$) attractor.*

On the other hand, the 'natural' energy associated to equation (1.1) is given by

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} |u|^2\,dx - \int_{\mathbb{R}^N} F(u)\,dx - \int_{R^N} gu\,dx, \tag{1.7}$$

where $F'(s) = f(s)$, $s \in \mathbb{R}$. The gradient structure of equation (1.1) implies that, along any given nonconstant solution of (1.1), the energy $\mathcal{E}$ decays and asymptotically approaches some limit level. The question then arises, whether the ω-limit of a given orbit lies at the energy level asymptotically approached by the orbit itself. Since $\mathcal{E}$ is continuous on H^1, the problem becomes that of proving the asymptotic compactness of π in the strong H^1 topology. However, it seems definitely not trivial to obtain such a result by means of estimates involving the energy $\mathcal{E}$.

Very recently Efendiev and Zelik in [3] considered the more general problem

$$u_t = \Delta u - \lambda u + f(u, \nabla u) + g(x) \tag{1.8}$$

in *three* spatial dimensions. Using energy estimates and comparison arguments they proved, among other things, the existence of a compact ($H^\alpha - H^\alpha$) attractor for (1.8), where H^α is some fractional space between H^2 and H^1. Their technique, however, exploits the Sobolev imbedding $H^2 \subset L^\infty$, which is no longer true in higher space dimensions.

We have found more convenient to follow a different way: in this note we show that the asymptotic compactness in H^1 can be recovered from the asymptotic compactness in L^2 by a simple continuity argument (Theorem

2.3). As a consequence, we deduce that the $(L^2 - L^2)$ attractor, whose existence was established by B. Wang, is actually an $(L^2 - H^1)$ attractor, like in the case of bounded domains (Theorem 2.4). The proof is very simple and is based on Henry's theory of abstract parabolic equations (see [5]).

Finally, we would like to mention the recent results obtained by Zelik in [10], concerning the existence and the entropy of *locally compact* attractors for equations like (1.1). See also [11], where similar results have been obtained for a dampded wave equation in $\mathbb{R}^N$.

2. The proof

Define the bilinear form

$$a(u,v) := \int_{\mathbb{R}^N} \nabla u(x) \cdot \nabla v(x) dx, \quad u, v \in H^1. \tag{2.1}$$

The form $a(\cdot,\cdot)$ uniquely determines a self-adjoint operator $A\colon D(A) \subset L^2 \to L^2$, defined by the relations

$$\begin{cases} D(A) := \{u \in H^1 \mid \exists\, w \in L^2 \text{ such that } a(u,v) = \langle w, v\rangle \text{ for all } v \in H^1\} \\ Au := w \quad \text{for } u \in D(A) \end{cases} \tag{2.2}$$

The self-adjoint operator $-A$ is the generator of an analytic semigroup of linear operators e^{-At}, $t \geq 0$, satisfying the following estimates:

$$\|e^{-At}u\| \leq M e^{at}\|u\| \quad \text{for all } u \in L^2 \text{ and all } t \geq 0, \tag{2.3}$$

$$\|e^{-At}u\|_{H^1} \leq M e^{at} t^{-1/2}\|u\| \quad \text{for all } u \in L^2 \text{ and all } t > 0, \tag{2.4}$$

where M and a are two positive constants. We need also the following

Lemma 2.1. *The assignement $u \mapsto f \circ u$ defines a map $\hat{f}\colon H^1 \to L^2$, which is Lipschitz continuous on every bounded set in H^1.*

Proof. By (1.4) we have

$$|f(s)| \leq C(|s| + |s|^{\beta+1}).$$

Let $u \in H^1$. Then

$$\begin{aligned} \int_{\mathbb{R}^N} |f(u(x)|^2 dx &\leq 2C \left(\int_{\mathbb{R}^N} |u(x)|^2 dx + \int_{\mathbb{R}^N} |u(x)|^{2(\beta+1)} dx \right) \\ &= 2C \left(\|u\|^2 + \|u\|^{2(\beta+1)}_{L^{2(\beta+1)}} \right). \end{aligned}$$

Since $2 \leq 2(\beta+1) \leq 2^*$, H^1 is continuously imbedded in $L^{2(\beta+1)}$. Then there exists a constant C_1 such that

$$\int_{\mathbb{R}^N} |f(u(x)|^2 dx \leq 2C \left(\|u\|^2 + C_1 \|u\|^{2(\beta+1)}_{H^1} \right)$$

and hence

$$\|\hat{f}(u)\| \leq 2C \left(\|u\| + C_1 \|u\|^{(\beta+1)}_{H^1} \right).$$

This shows that $\hat{f}\colon H^1 \to L^2$ is well defined and maps bounded sets of H^1 to bounded sets of L^2.

Now let $s_1, s_2 \in \mathbb{R}$. By the mean value theorem and by (1.4) we have

$$\begin{aligned}|f(s_1) - f(s_2)| &= |f'(\theta s_1 + (1-\theta)s_2)(s_1 - s_2)| \\ &\le C(1 + |\theta s_1 + (1-\theta)s_2|^\beta)|s_1 - s_2| \le C_1(1 + |s_1|^\beta + |s_2|^\beta)|s_1 - s_2|,\end{aligned}$$

where $\theta = \theta(s_1, s_2)$, $0 \le \theta \le 1$, and C_1 is a positive constant. Let $u_1, u_2 \in H^1$, $\|u_1\|_{H^1}, \|u_2\|_{H^1} \le R$. Then we have

$$\begin{aligned}&\int_{\mathbb{R}^N} |f(u_1(x)) - f(u_2(x))|^2 dx \\ &\qquad \le 2C_1 \int_{\mathbb{R}^N} (1 + |u_1(x)|^{2\beta} + |u_2(x)|^{2\beta})|u_1(x) - u_2(x)|^2 dx.\end{aligned}$$

Let $p := 2^*/(2\beta) > 1$, $q := p/(p-1) = 2^*/(2^* - 2\beta)$. Then, by Hölder inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^N} |f(u_1(x)) - f(u_2(x))|^2 dx &\le 2C_1 \int_{\mathbb{R}^N} |u_1(x) - u_2(x)|^2 dx \\ &\quad + 2C_1 \left(\|u_1\|_{L^{2^*}}^{2\beta} + \|u_2\|_{L^{2^*}}^{2\beta} \right) \|u_1 - u_2\|_{L^{2q}}^2.\end{aligned}$$

Since $2 \le 2q \le 2^*$, H^1 is continuously imbedded in L^{2q}. Then there exists a constant C_2 such that

$$\int_{\mathbb{R}^N} |f(u_1(x)) - f(u_2(x))|^2 dx \le C_2(1 + \|u_1\|_{H^1}^{2\beta} + \|u_2\|_{H^1}^{2\beta})\|u_1 - u_2\|_{H^1}^2$$

and hence

$$\|\hat{f}(u_1) - \hat{f}(u_2)\| \le C_2^{1/2}(1 + 2R^{2\beta})^{1/2}\|u_1 - u_2\|_{H^1}.$$

The proof is complete. □

Following Henry [5], the Cauchy problem (1.1)-(1.6), for initial data $u_0 \in H^1$, can be formulated as an abstract parabolic initial value problem

$$\begin{cases} \dot{u} + Au = -\lambda u + \hat{f}(u) + g \\ u(0) = u_0 \end{cases} \tag{2.5}$$

It is well known that equation (2.5) is equivalent to the integral equation

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}(-\lambda u + \hat{f}(u(s)) + g)ds. \tag{2.6}$$

We have the following crucial

Lemma 2.2. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in H^1, let $u \in H^1$, and assume that $u_n \rightharpoonup u$ in H^1, $u_n \to u$ in L^2. Then $\pi(t, u_n) \to \pi(t, u)$ in H^1, uniformly on $[t_0, t_1]$, for all $t_1 > t_0 > 0$.*

Proof. Let $t_1 > 0$ be fixed. Since the set $\{u_n \mid n \in \mathbb{N}\} \cup \{u\}$ is bounded in H^1, by Lemma 1.1 there exists $R > 0$ such that $\|\pi(t, u_n)\|_{H^1} \leq R$ and $\|\pi(t, u)\|_{H_1} \leq R$, for all $t \in [0, t_1]$ and for all $n \in \mathbb{N}$. Let L be a Lipschitz constant for $\hat{f}$ on the ball of radius R in H^1. Write $u_n(t) := \pi(t, u_n)$ and $u(t) := \pi(t, u)$. For $t \in [0, t_1]$, we have

$$u_n(t) = e^{-At}u_n + \int_0^t e^{-A(t-s)}(-\lambda u_n(s) + \hat{f}(u_n(s)) + g)ds$$

and

$$u(t) = e^{-At}u + \int_0^t e^{-A(t-s)}(-\lambda u + \hat{f}(u(s)) + g)ds.$$

It follows that, for $t \in [0, t_1]$,

$$\begin{aligned}\|u_n(t) - u(t)\|_{H^1} &\leq Me^{at_1}t^{-1/2}\|u_n - u\| \\ &\quad + M(\lambda + L)e^{at_1}\int_0^t (t-s)^{-1/2}\|u_n(s) - u(s)\|_{H^1}ds.\end{aligned}$$

By the singular Gronwall's Lemma (see [5]), there is a constant $C_1 = C_1(\lambda, L, M, a, t_1)$ such that, for $t \in [0, t_1]$,

$$\begin{aligned}\|u_n(t) - u(t)\|_{H^1} &\leq Me^{at_1}t^{-1/2}\|u_n - u\| \\ &\quad + C_1\int_0^t (1 + (t-s)^{-1/2})Me^{at_1}s^{-1/2}\|u_n - u\|ds.\end{aligned}$$

Finally, a simple computation shows that there exists a constant $C_2 = C_2(\lambda, L, M, a, t_1)$ such that, for $t \in [0, t_1]$,

$$\|u_n(t) - u(t)\|_{H^1} \leq C_2(1 + t^{-1/2})\|u_n - u\|.$$

This completes the proof. □

Remark. It seems that such a result cannot be obtained as a consequence of a-priori estimates like the ones in [9]. In a different context, a similar observation was made also in [7].

The next theorem shows that, thanks to Lemma 2.2, the asymptotic compactness of π in L^2 implies the asymptotic compactness of π in H^1.

Theorem 2.3. *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in H^1 and let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers, $t_n \to \infty$ as $n \to \infty$. Then there exists a strictly increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ and a function $u \in H^1$ such that $\pi(t_{n_k}, u_{n_k}) \to u$ in H^1 as $k \to \infty$. In other words, π is asymptotically compact in the strong H^1 topology.*

Proof. Fix any positive T. Since $t_n \to \infty$ as $n \to \infty$, we have $t_n > T$ for all sufficiently large n, say $n \geq n_0$. Since the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in H^1, by Lemma 1.1 also the sequence $\pi(t_n - T, u_n)_{n\geq n_0}$ is bounded in H^1. Then there exists a strictly increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$, $n_k \geq n_0$ for all $k \in \mathbb{N}$, and a function $\bar{u} \in H^1$ such that $\pi(t_{n_k} - T, u_{n_k}) \rightharpoonup \bar{u}$

in H^1 as $k \to \infty$. On the other hand, by Theorem 1.2, we can choose the sequence $(n_k)_{k\in\mathbb{N}}$ in such a way that $\pi(t_{n_k} - T, u_{n_k}) \to \bar{u}$ in L^2 as $k \to \infty$. Then, by Lemma 2.2, we have

$$\pi(t_{n_k}, u_{n_k}) = \pi(T, \pi(t_{n_k} - T, u_{n_k})) \to \pi(T, \bar{u}) \quad \text{in } H^1 \text{ as } k \to \infty.$$

The proof is complete. □

Finally, we can state and prove

Theorem 2.4. *The semiflow π possesses a compact $(L^2 - H^1)$ attractor.*

Proof. By Lemma 1.1 there exists a bounded set $\mathcal{B}$ in H^1 and for any bounded set B in L^2 there exists $T(B) > 0$ such that $\pi(t, B) \subset \mathcal{B}$ for all $t \geq T(B)$. On the other hand, by Theorem 2.3 π is asymptotically compact in H^1. The conclusion follows from the abstract results of [4], [6] and [8]. □

Remark. Of course, by Lemma 1.1, the $(L^2 - L^2)$ attractor and the $(L^2 - H^1)$ attractor coincide. Moreover, by the continuity result of Lemma 2.2, the Hausdorff (or the fractal) dimension of the attractor is the same in L^2 and in H^1.

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