Studia Mathematica
Vol. 151, No. 2 (2002), 109–140

# CURVED THIN DOMAINS AND PARABOLIC EQUATIONS

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Abstract. Consider the family

$$(E_\epsilon)\qquad \begin{aligned} &u_t = \Delta u + G(u), && t > 0,\ x \in \Omega_\epsilon,\\ &\partial_{\nu_\epsilon} u = 0, && t > 0,\ x \in \partial\Omega_\epsilon \end{aligned}$$

of semilinear Neumann boundary value problems, where, for $\epsilon > 0$ small, the set $\Omega_\epsilon$ is a thin domain in $\mathbb{R}^\ell$, possibly with holes, which collapses, as $\epsilon \to 0^+$, onto a (curved) $k$-dimensional submanifold of $\mathbb{R}^\ell$. If $G$ is dissipative, then equation $(E_\epsilon)$ has a global attractor $\mathcal{A}_\epsilon$.

We identify a 'limit' equation for the family $(E_\epsilon)$, prove convergence of trajectories and establish an upper semicontinuity result for the family $\mathcal{A}_\epsilon$ for $\epsilon \to 0^+$.

## 1. Introduction

Let $\omega$ be a bounded domain in $\mathbb{R}^\ell$ with Lipschitz boundary. Consider the Neumann boundary value problem

$$(1)\qquad \begin{aligned} &u_t = \Delta u + G(u), && t > 0,\ x \in \omega,\\ &\partial_\nu u = 0, && t > 0,\ x \in \partial\omega \end{aligned}$$

on $\omega$. Here, $\nu$ is the exterior normal vector field on $\partial\omega$ and $G\colon \mathbb{R} \to \mathbb{R}$ is a given nonlinearity satisfying appropriate growth and regularity conditions.

Now suppose that $\omega := \Omega_\epsilon$ where $\Omega_\epsilon$ is a domain which depends on a small parameter $\epsilon > 0$. Intuitively, the set $\Omega_\epsilon$ (or a part of it) is 'thin' of order $\epsilon$ and, as $\epsilon \to 0^+$, the domain $\Omega_\epsilon$ 'degenerates' to some limit set, which may no longer be a domain in $\mathbb{R}^\ell$.

One may now naturally ask what happens to solutions of equation (1) on $\omega := \Omega_\epsilon$ as $\epsilon \to 0^+$. Is there a limit equation and, if so, do some solutions of the limit equation persist for small $\epsilon > 0$?

One of the earliest papers devoted to such questions is the seminal work [16] by Hale and Vegas. In that paper a bifurcation mechanism is described which produces stable spatially nonhomogeneous stationary solutions of equation (1). The basic idea is simple: start with a (disconnected) set $\Omega_0 \subset \mathbb{R}^2$ consisting of two disjoint disk-like open sets $B_1$ and $B_2$ and connect $B_1$ with $B_2$ by a small channel of thickness $\epsilon > 0$ to produce the two-dimensional dumbbell domain $\Omega_\epsilon$. Suppose $a$ and $b \in \mathbb{R}$ are two different zeros of the function $G$. Then the function $u\colon \Omega \to \mathbb{R}$ defined by $u(x) \equiv a$ on $B_1$ and $u(x) \equiv b$ on $B_2$ is an equilibrium of equation (1) on $\omega := \Omega_0$. Under appropriate hypotheses on $G$ and the thin channel in $\Omega_\epsilon$ it is shown in [16] that the equilibrium $u$ can be continued to a family of stable equilibria of equation (1) on $\omega := \Omega_\epsilon$, for $\epsilon > 0$ small.

The paper [16] gave rise to a number of other important articles on dumbbell type domains, see [2], [3], [18], [20] and the reference section in the survey [25] by Raugel.

Now suppose that $\omega = \Omega_\epsilon$ is everywhere 'thin' of order $\epsilon > 0$. Then the domain $\Omega_\epsilon$ collapses, as $\epsilon \to 0^+$, to a lower-dimensional set. Suppose also that the nonlinearity $G$ is dissipative so equation (1) has a global attractor $\mathcal{A}_\epsilon$ on an appropriate phase space. In addition to the questions mentioned above one may now ask what happens to the family of attractors $\mathcal{A}_\epsilon$ as $\epsilon \to 0^+$.

This problem was first considered in [13] by J. Hale and G. Raugel for domains $\Omega_\epsilon$ having the special form

$$\Omega_\epsilon = \{\, (x_1, x_2) \in \mathbb{R}^{\ell-1} \times \mathbb{R} = \mathbb{R}^\ell \mid x_1 \in \tilde{\omega} \text{ and } 0 < x_2 < \epsilon g(x_1) \,\},$$

where $g$ is a smooth positive function defined on an $(\ell-1)$-dimensional domain $\tilde{\omega}$. For dissipative nonlinearities they prove that the limit equation is the $(\ell-1)$-dimensional boundary value problem

$$\begin{aligned} &u_t = (1/g)\operatorname{div}(g\nabla u) + G(u), && t > 0,\, x \in \tilde{\omega}, \\ &\partial_\nu u = 0, && t > 0,\, x \in \partial\tilde{\omega}. \end{aligned} \tag{2}$$

They compare the semiflows generated by these equations and establish an important upper-semicontinuity result for the corresponding family of attractors. For $\ell = 2$, Hale and Raugel also prove existence of inertial manifolds containing these attractors. In a subsequent paper [14] the same authors established an upper semicontinuity result in the context of damped wave equations. See also [4], [11] and [25] for some other upper semicontinuity results for problems on thin domains.

In [21], the first and the third author of the present paper considered equation (1) on a class of thin domains which are much more general than those considered in [13], including, e.g., domains with holes. In this context they developed an abstract framework for the analysis of the questions mentioned above, based on a property of strong spectral convergence satisfied by the linear part of the equation.

Let us describe in some detail the results of [21]. Let $k \in \mathbb{N}$ with $k < \ell$ be arbitrary and $\Omega$ be an arbitrary nonempty bounded domain in $\mathbb{R}^\ell = \mathbb{R}^k \times \mathbb{R}^{\ell-k}$ with Lipschitz boundary and $\epsilon > 0$ be arbitrary. Write $x = (x_1, x_2)$ for a generic point of $\mathbb{R}^k \times \mathbb{R}^{\ell-k}$. Given $\epsilon > 0$ squeeze $\Omega$ by the factor $\epsilon$ in the $y$-direction to obtain the *flatly squeezed domain* $\Omega_\epsilon$. More precisely, let $T_\epsilon \colon \mathbb{R}^k \times \mathbb{R}^{\ell-k} \to \mathbb{R}^k \times \mathbb{R}^{\ell-k}$ be the *flat squeezing* transformation

$$(x_1, x_2) \mapsto (x_1, \epsilon x_2)$$

and set

$$\Omega_\epsilon := T_\epsilon(\Omega).$$

Note that the domains considered by Hale and Raugel arise from the flat squeezing of the domain

$$\Omega := \{\, (x_1, x_2) \mid x_1 \in \tilde{\omega} \text{ and } 0 < x_2 < g(x_1).\,\}$$

Under appropriate conditions on $G$, equation (1) on the varying domain $\omega := \Omega_\epsilon$ can then equivalently be described, in abstract terms, by the equation

$$\dot{u} + \tilde{A}_\epsilon u = \hat{G}(u) \tag{3}$$

on $H^1(\Omega_\epsilon)$. Here $\tilde{A}_\epsilon$ is the self-adjoint operator defined by the bilinear form

$$\tilde{a}_\epsilon(u,v) = \int_{\Omega_\epsilon} \nabla u \cdot \nabla v \; \mathrm{d}x$$

on $H^1(\Omega_\epsilon)$ and $\hat{G}$ is the Nemitski operator defined by the function $G$. Via the change of variables $u(x) \to u(\tilde{x})$, where $\tilde{x} = T_\epsilon(x)$, we can transform equation (3) to the equivalent equation

$$\dot{u} + A_\epsilon u = \hat{G}(u) \tag{4}$$

on the fixed space $H^1(\Omega)$. Here, $A_\epsilon$ is the operator defined by the formula

$$A_\epsilon(u \circ T_\epsilon) = \tilde{A}_\epsilon(u) \circ T_\epsilon.$$

Equation (4) defines a (local) semiflow $\pi_\epsilon$ on $H^1(\Omega)$ which, for dissipative nonlinearities $G$, possesses a global attractor $\mathcal{A}_\epsilon$.

Note that the operator $A_\epsilon$ is the linear operator induced by the bilinear form

$$a_\epsilon(u,v) := \int_\Omega \left( \nabla_{x_1} u \cdot \nabla_{x_1} v + \frac{1}{\epsilon^2} \nabla_{x_2} u \cdot \nabla_{x_2} v \right) \; \mathrm{d}x$$

on $H^1(\Omega)$. Observe that for $u \in H^1(\Omega)$

$$\lim_{\epsilon \to 0^+} a_\epsilon(u,u) = \begin{cases} \int_\Omega |\nabla_{x_1} u|^2 \; \mathrm{d}x, & \text{if } \nabla_{x_2} u = 0 \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus the family $a_\epsilon(u,u)$, $\epsilon > 0$, of real numbers has a finite limit (as $\epsilon \to 0^+$) if and only if $u \in H^1_s(\Omega)$, where we define

$$H^1_s(\Omega) := \{\, u \in H^1(\Omega) \mid \nabla_{x_2} u = 0 \,\}.$$

This is a closed linear subspace of $H^1(\Omega)$. The corresponding limit bilinear form is given by the formula:

$$a_0(u,v) := \int_\Omega \nabla_{x_1} u \cdot \nabla_{x_1} v \, \mathrm{d}x, \quad u, v \in H^1_s(\Omega).$$

The form $a_0$ uniquely determines a densely defined selfadjoint linear operator

$$A_0 \colon D(A_0) \subset H^1_s(\Omega) \to L^2_s(\Omega)$$

by the usual formula

$$a_0(u,v) = \langle A_0 u, v \rangle_{L^2(\Omega)}, \qquad \text{for } u \in D(A_0) \text{ and } v \in H^1_s(\Omega).$$

Here, $L^2_s(\Omega)$ is the closure of $H^1_s(\Omega)$ in the $L^2$-norm, so $L^2_s(\Omega)$ is a closed linear subspace of $L^2(\Omega)$. It turns out that, as $\epsilon \to 0^+$, the operators $A_\epsilon$ converge to the operator $A_0$ in some spectral sense and the linear semigroups $e^{-tA_\epsilon}$ 'singularly' converge to the semigroup $e^{-tA_0}$ in some strong sense.

We can now consider the abstract parabolic equation

$$\dot{u} = -A_0 u + \hat{G}(u). \tag{5}$$

on the space $H^1_s(\Omega)$. This equation defines a (local) semiflow $\pi_0$ on $H^1_s(\Omega)$ which, for dissipative $G$, possesses a global attractor $\mathcal{A}_0$. It turns out that, as $\epsilon \to 0^+$, the family $\pi_\epsilon$ of semiflows converges, in some strong singular sense, to the semiflow $\pi_0$. As a consequence we obtain, for dissipative nonlinearities $G$, an upper semicontinuity result for the resulting family $(\mathcal{A}_\epsilon)_{\epsilon \geq 0}$ of attractors, which extends the corresponding result of [13] to the present case. As it is proved in [22], the inertial manifold result from [13] also holds in this more general situation.

The analogues of the results of [21] for damped wave equations are proved in the recent paper [7], which, in particular, contains an extension of the upper semicontinuity result from [14]. Finally, some applications of Conley index to reaction-diffusion equations and damped wave equations on squeezed domains appear in [6] and [8].

Note that the above papers all deal with a rather special *flat* squeezing of $\Omega$ onto a lower dimensional subspace of $\mathbb{R}^\ell$, cf Figure 1.

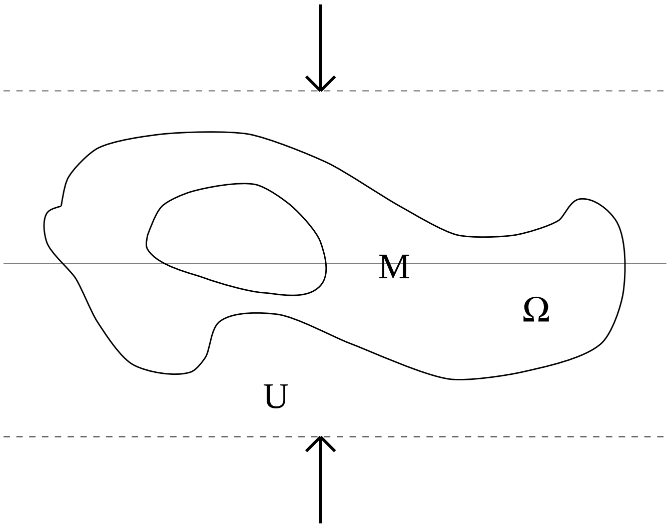


Figure 1: flat squeezing

It is geometrically much more appealing and also more realistic from the point of view of applications to consider general squeezing of the domain $\Omega$ onto a *curved* lower dimensional submanifold $\mathcal{M}$ of $\mathbb{R}^\ell$ and to study the effect of such squeezing upon the behavior of the solutions of reaction-diffusion equations. This is the purpose of the present paper.

Let us briefly describe the geometry of the problem considered here. Let $\ell$, $k$ and $r$ be positive integers with $r \geq 2$, $\ell \geq 2$ and $k < \ell$. Let $\mathcal{M} \subset \mathbb{R}^\ell$ be an arbitrary imbedded $k$-dimensional submanifold of $\mathbb{R}^\ell$ of class $C^r$. Note that, in the general case considered here, the manifold is *global*, i.e. $\mathcal{M}$ need not be included in a single coordinate chart. Let us also remark that we do *not* assume $\mathcal{M}$ to be orientable.

By the Tubular neighborhood theorem there exists an open set $\mathcal{U}$ in $\mathbb{R}^\ell$ and a map $\phi\colon \mathcal{U} \to \mathcal{M}$ of class $C^{r-1}$ such that whenever $x \in \mathcal{U}$ and $p \in \mathcal{M}$ then $\phi(x) = p$ if and only if the vector $x - p$ is orthogonal to $T_p\mathcal{M}$; moreover, $\epsilon x + (1-\epsilon)\phi(x) \in \mathcal{U}$ for all $x \in \mathcal{U}$ and all $\epsilon \in [0,1]$.

For $\epsilon \in [0,1]$ let us define the *curved squeezing* transformation

$$\Phi_\epsilon\colon \mathcal{U} \to \mathbb{R}^\ell,$$

$$\Phi_\epsilon(x) := \epsilon x + (1-\epsilon)\phi(x) = \phi(x) + \epsilon(x - \phi(x)). \tag{6}$$

Now let $\Omega$ be an arbitrary nonempty bounded domain in $\mathbb{R}^\ell$ with Lipschitz boundary and such that $\mathrm{Cl}\,\Omega \subset \mathcal{U}$. For $\epsilon \in ]0,1]$, define the *curved squeezed domain*

$$\Omega_\epsilon := \Phi_\epsilon(\Omega).$$

(Cf Figure 2.)

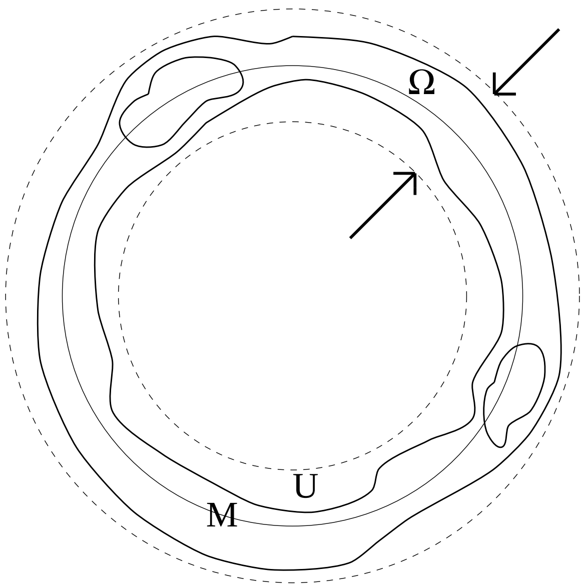


Figure 2: curvilinear squeezing

Note that setting, in particular, $\mathcal{M} := \mathbb{R}^k \times \{0\}$, $\mathcal{U} := \mathbb{R}^k \times \mathbb{R}^{\ell-k}$ and letting $\phi$ be the orthogonal projection $(x_1, x_2) \mapsto (x_1, 0)$ of $\mathbb{R}^k \times \mathbb{R}^{\ell-k}$ onto $\mathcal{M}$ we are reduced to the flat squeezing case described above.

Consider equation (1) on the curved squeezed domain $\omega := \Omega_\epsilon$. This equation can again be described in abstract terms as the equation

$$\dot{u} + \tilde{A}_\epsilon u = \hat{G}(u) \tag{7}$$

on $H^1(\Omega_\epsilon)$. Here, the operator $\tilde{A}_\epsilon$ is induced by the bilinear form

$$\tilde{a}_\epsilon(u, v) = \int_{\Omega_\epsilon} \nabla u \cdot \nabla v \; \mathrm{d}x.$$

We can now use the change of variables $u(x) \mapsto u(\tilde{x})$, where $\tilde{x} = \Phi_\epsilon(x)$, to transform equation (7) to the equivalent problem

$$\dot{u} + A_\epsilon u = \hat{G}(u) \tag{8}$$

on $H^1(\Omega)$. Here, the operator $A_\epsilon$ is defined by the formula

$$A_\epsilon(u \circ \Phi_\epsilon) = (\tilde{A}_\epsilon u) \circ \Phi_\epsilon.$$

Equation (8) defines a (local) semiflow $\pi_\epsilon$ on $H^1(\Omega)$, which, for dissipative $G$, possesses a global attractor $\mathcal{A}_\epsilon$.

For $x \in \mathcal{U}$ denote by $Q(x)\colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ the orthogonal projection of $\mathbb{R}^\ell \cong T_p\mathbb{R}^\ell$ onto $T_p\mathcal{M}$, where $p := \phi(x)$. Let $P(x) = I - Q(x)$. Note that $P(x)$ is the the orthogonal projection of $\mathbb{R}^\ell \cong T_p\mathbb{R}^\ell$ onto the orthogonal complement of $T_p\mathcal{M}$ in $T_p\mathbb{R}^\ell \cong \mathbb{R}^\ell$.

Now let us define the space

$$H^1_s(\Omega) := \{\, u \in H^1(\Omega) \mid P(x)\nabla u(x) = 0 \ \text{ a.e.}\,\}. \tag{9}$$

Note that $H^1_s(\Omega)$ is a closed linear subspace of the Hilbert space $H^1(\Omega)$. Let $L^2_s(\Omega)$ be the closure in $L^2(\Omega)$ of $H^1_s(\Omega)$.

It is one of the main contributions of this paper to show that the family of operators $(A_\epsilon)_{\epsilon\in]0,1]}$ converges in a strong spectral sense to a densely defined selfadjoint operator $A_0$ in $L^2_s(\Omega)$ (cf sections 2 and 3 below).

Both the space $H^1_s(\Omega)$ defined above and the limit operator $A_0$ defined in section 3 below strongly depend on the geometry of $\mathcal{M}$ and especially on its curvature. The presence of curvature, reflected in the nonlinearity of the mapping $\Phi_\epsilon$, renders the proof of the existence of $A_0$ much more involved than the corresponding proof in the flat squeezing case.

We can now consider the abstract parabolic equation

$$\dot u + A_0 u = \hat G(u). \tag{10}$$

on the space $H^1_s(\Omega)$, where $H^1_s(\Omega)$ is defined in (9). Equation (10) defines a (local semiflow) $\pi_0$ on $H^1_s(\Omega)$, which, for dissipative $G$, possesses a global attractor $\mathcal{A}_0$.

We prove in section 4 that, as $\epsilon \to 0^+$, the linear semigroups $e^{-tA_\epsilon}$ singularly converge to the semigroup $e^{-tA_0}$ and the semiflows $\pi_\epsilon$ singularly converge to $\pi_0$. In the dissipative case, we also obtain an upper semicontinuity result for the family $(\mathcal{A}_\epsilon)_{\epsilon\in[0,1]}$ of attractors.

We thus obtain a far reaching generalization of the results proved in [21] for the flat squeezing case. Further developments of this research will appear in the paper [24], where some of the results given here are used to prove existence of inertial manifolds on some genuinely high dimensional thin domains.

Lastly, it is worth mentioning that various thin domain problems have also been studied in the context of elasticity theory. This is a vast and fascinating subject for which we refer the reader to the monograph [10] by Ciarlet.

## 2. Properties of curved squeezing transformations

In this section we define curved squeezing transformations and discuss their fundamental properties. The results of this section are crucial for the rest of this paper.

We assume throughout that $\ell$, $k$ and $r$ are positive integers with $r \ge 2$, $\ell \ge 2$ and $k < \ell$. By $\langle\cdot,\cdot\rangle$ we denote the standard inner product in $\mathbb{R}^\ell$.

Let $\mathcal{M} \subset \mathbb{R}^\ell$ be an imbedded $k$-dimensional submanifold of $\mathbb{R}^\ell$ of class $C^r$. For $p \in \mathcal{M}$ we denote by $T_p\mathcal{M}$ the tangent space to $\mathcal{M}$ at the point $p$. We will identify $T_p\mathcal{M}$ with a subspace of $\mathbb{R}^\ell$.

**Definition 2.1.** *An open set $\mathcal{U}$ in $\mathbb{R}^\ell$ is called a* normal neighborhood of $\mathcal{M}$ *if there is a map $\phi\colon \mathcal{U} \to \mathcal{M}$ of class $C^{r-1}$ such that:*

(1) *whenever* $x \in \mathcal{U}$ *and* $p \in \mathcal{M}$ *then* $\phi(x) = p$ *if and only if the vector* $x - p$ *is orthogonal to* $T_p\mathcal{M}$*;*
(2) $\epsilon x + (1-\epsilon)\phi(x) \in \mathcal{U}$ *for all* $x \in \mathcal{U}$ *and all* $\epsilon \in [0,1]$.

The following properties easily follow from the above definition:

**Proposition 2.2.** *Let* $\mathcal{U}$ *be a normal neighborhood of* $\mathcal{M}$*. Then the map* $\phi$ *of Definition 2.1 is uniquely determined by item 1 of that definition. Moreover,*

(1) $\phi(\mathcal{U}) = \mathcal{M}$ *and* $\phi(x) = x$ *if and only if* $x \in \mathcal{M}$*;*
(2) $D\phi(x)\nu = 0$ *for all* $x \in \mathcal{U}$ *and all vectors* $\nu$ *orthogonal to* $T_p\mathcal{M}$*, where* $p := \phi(x)$. □

Note that, by the Tubular neighborhood theorem (see, e.g. [5]), a normal neighborhood of $\mathcal{M}$ always exists.

In the sequel we consider a fixed normal neighborhood $\mathcal{U}$ of $\mathcal{M}$ and let the map $\phi$ be as in Definition 2.1. For $x \in \mathcal{U}$ we denote by $Q(x)\colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ the orthogonal projection of $\mathbb{R}^\ell \cong T_p\mathbb{R}^\ell$ onto $T_p\mathcal{M}$, where $p := \phi(x)$. Let $P(x) = I - Q(x)$. Note that $P(x)$ is the the orthogonal projection of $\mathbb{R}^\ell \cong T_p\mathbb{R}^\ell$ onto the orthogonal complement of $T_p\mathcal{M}$ in $T_p\mathbb{R}^\ell \cong \mathbb{R}^\ell$.

For $\epsilon \in [0,1]$ let us define the *curved squeezing* transformation

$$\Phi_\epsilon\colon \mathcal{U} \to \mathbb{R}^\ell,$$

$$\Phi_\epsilon(x) := \epsilon x + (1-\epsilon)\phi(x) = \phi(x) + \epsilon(x - \phi(x)). \tag{11}$$

The following properties are an immediate consequence of the definition:

**Proposition 2.3.** *The map* $[0,1] \times \mathcal{U} \to \mathbb{R}^\ell$*,* $(\epsilon, x) \mapsto \Phi_\epsilon(x)$*, is continuous.*

*Let* $\epsilon \in \,]0,1]$ *be arbitrary. Then*

(1) $\Phi_\epsilon(\mathcal{U}) = \{\, y \in \mathcal{U} \mid \phi(y) + (1/\epsilon)(y - \phi(y)) \in \mathcal{U} \,\}$, $\Phi_\epsilon(\mathcal{U})$ *is open in* $\mathbb{R}^\ell$ *and* $\Phi_\epsilon\colon \mathcal{U} \to \Phi_\epsilon(\mathcal{U})$ *is a diffeomorphism of class* $C^{r-1}$ *with*
$$\Phi_\epsilon^{-1}(y) = \phi(y) + (1/\epsilon)(y - \phi(y)), \quad y \in \Phi_\epsilon(\mathcal{U});$$
(2) $\phi(\Phi_\epsilon(x)) = \phi(x)$ *for* $x \in \mathcal{U}$.

The following result is of crucial importance for the whole paper:

**Theorem 2.4.** *For* $x \in \mathcal{U}$ *and* $\epsilon \in [0,1]$ *define*

$$J_\epsilon(x) := \begin{cases} \epsilon^{-(\ell-k)/2}|\det D\Phi_\epsilon(x)| & \text{if } \epsilon > 0, \\ |\det(D\phi(x)|_{T_{\phi(x)}\mathcal{M}})| & \text{otherwise.} \end{cases}$$

*Then*

$$J_\epsilon(x) > 0 \quad \text{for all } \epsilon \in [0,1] \text{ and } x \in \mathcal{U}. \tag{12}$$

*Moreover, the function* $[0,1] \times \mathcal{U} \to \mathbb{R}$*,* $(\epsilon, x) \mapsto J_\epsilon(x)$*, is continuous.*

*For every* $\epsilon \in [0,1]$ *and for every* $x \in \mathcal{U}$ *there exists a linear map* $S_\epsilon(x)\colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ *such that, for* $\epsilon \in \,]0,1]$,

$$D\Phi_\epsilon^{-1}(\Phi_\epsilon(x)) = S_\epsilon(x) + (1/\epsilon)P(x) \quad \text{for all } x \in \mathcal{U}.$$

*Accordingly,*

$$(D\Phi_\epsilon^{-1}(\Phi_\epsilon(x)))^T = S_\epsilon(x)^T + (1/\epsilon)P(x) \quad \text{for all } x \in \mathcal{U}.$$

*The following properties are satisfied:*

(1) *The maps* $[0,1]\times\mathcal{U}\to\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)$,
$$(\epsilon,x)\mapsto S_\epsilon(x)\quad\text{and } (\epsilon,x)\mapsto S_\epsilon(x)^T$$
*are continuous;*

(2) *for every* $\epsilon\in[0,1]$, *for every* $x\in\mathcal{U}$ *and for every* $\nu$ *orthogonal to* $T_{\phi(x)}\mathcal{M}$
$$S_\epsilon(x)\nu=S_\epsilon(x)^T\nu=0;$$

(3) *for every* $\epsilon\in[0,1]$ *and for every* $x\in\mathcal{U}$ *the maps*
$$S_\epsilon(x)|_{T_{\phi(x)}\mathcal{M}}\colon T_{\phi(x)}\mathcal{M}\to T_{\phi(x)}\mathcal{M},$$
$$S_\epsilon(x)^T|_{T_{\phi(x)}\mathcal{M}}\colon T_{\phi(x)}\mathcal{M}\to T_{\phi(x)}\mathcal{M}$$
*are well-defined and bijective. Furthermore,*
$$(S_0(x)|_{T_{\phi(x)}\mathcal{M}})^{-1}=D\phi(x)|_{T_{\phi(x)}\mathcal{M}}$$
*and*
$$(S_0(x)^T|_{T_{\phi(x)}\mathcal{M}})^{-1}=D\phi(x)^T|_{T_{\phi(x)}\mathcal{M}}.$$

*Finally,* $\phi\colon\mathcal{U}\to\mathcal{M}$ *is an open map.*

*Proof.* The rest of this section will be devoted to the proof of Theorem 2.4.

We begin by an obvious local result:

**Proposition 2.5.** *For every* $p\in\mathcal{M}$ *there is an open set* $V_p$ *in* $\mathcal{M}$ *and* $C^{r-1}$*-maps* $h_i=h_{p,i}\colon V_p\to\mathbb{R}^\ell$, $i=1,\ldots,k$, $\nu_j=\nu_{p,j}\colon V_p\to\mathbb{R}^\ell$ *and* $\alpha_j=\alpha_{p,j}\colon\phi^{-1}(V_p)\to\mathbb{R}$, $j=1,\ldots,\ell-k$ *such that for every* $q\in V_p$ *the vectors* $h_i(q)$, $i=1,\ldots,k$, *form an orthonormal basis of* $T_q\mathcal{M}$, *and the vectors* $\nu_j(q)$, $j=1,\ldots,\ell-k$, *form an orthonormal basis of the orthogonal complement of* $T_q\mathcal{M}$ *in* $T_q\mathbb{R}^\ell\cong\mathbb{R}^\ell$. *Moreover, for* $x\in\phi^{-1}(V_p)$, $y\in\Phi_\epsilon(\phi^{-1}(V_p))$ *and* $h\in\mathbb{R}^\ell$

$$Q(x)h=\sum_{i=1}^{k}\langle h,h_i(\phi(x))\rangle h_i(\phi(x)),\tag{13}$$

$$P(x)h=\sum_{j=1}^{\ell-k}\langle h,\nu_j(\phi(x))\rangle \nu_j(\phi(x)),\tag{14}$$

$$x=\phi(x)+\sum_{j=1}^{\ell-k}\alpha_j(x)\nu_j(\phi(x)),\tag{15}$$

$$\Phi_\epsilon(x)=\phi(x)+\epsilon\sum_{j=1}^{\ell-k}\alpha_j(x)\nu_j(\phi(x))\tag{16}$$

*and*

$$\Phi_\epsilon^{-1}(y)=\phi(y)+(1/\epsilon)\sum_{j=1}^{\ell-k}\alpha_j(y)\nu_j(\phi(y)).\tag{17}$$

□

Since $\phi\colon \mathcal{U} \to \mathcal{M}$, we have of course that $D\phi(x)h \in T_{\phi(x)}\mathcal{M}$ for all $x \in \mathcal{U}$ and for all $h \in \mathbb{R}^\ell$. In particular, we see that

$$D\phi(x)|_{T_{\phi(x)}\mathcal{M}}\colon T_{\phi(x)}\mathcal{M} \to T_{\phi(x)}\mathcal{M}. \tag{18}$$

Now let $p \in \mathcal{M}$ be arbitrary and let the set $V_p$ and the maps $h_i = h_{p,i}$, $i = 1, \ldots, k$, $\nu_j = \nu_{p,j}$, and $\alpha_j = \alpha_{p,j}$, $j = 1, \ldots, \ell - k$, be as in Proposition 2.5. Since $\nu_j(q)$ has norm one for all $q \in V_p$ and $j = 1, \ldots, \ell - k$, differentiating the equality

$$\langle \nu_j(\phi(x)), \nu_j(\phi(x))\rangle = 1, \quad x \in \phi^{-1}(V_p),$$

we obtain that

$$\langle D\nu_j(\phi(x))D\phi(x)h, \nu_j(\phi(x))\rangle = 0$$

for all $x \in \phi^{-1}(V_p)$, $h \in \mathbb{R}^\ell$ and for $j = 1, \ldots, \ell - k$.

Differentiating the identity

$$x = \phi(x) + \sum_{j=1}^{\ell-k} \alpha_j(x)\nu_j(\phi(x)), \quad x \in \phi^{-1}(V_p),$$

we get

$$h = D\phi(x)h + \sum_{j=1}^{\ell-k}(D\alpha_j(x)h)\nu_j(\phi(x)) + \sum_{j=1}^{\ell-k} \alpha_j(x)D\nu_j(\phi(x))D\phi(x)h$$

for all $x \in \phi^{-1}(V_p)$ and all $h \in \mathbb{R}^\ell$.

For $x \in \phi^{-1}(V_p)$ and $h \in T_{\phi(x)}\mathcal{M}$ we thus obtain

$$h = D\phi(x)h + \sum_{j=1}^{\ell-k} \alpha_j(x)Q(x)D\nu_j(\phi(x))D\phi(x)h. \tag{19}$$

For $x \in \phi^{-1}(V_p)$, let us set

$$S_0(x)h := Q(x)h + \sum_{j=1}^{\ell-k} \alpha_j(x)Q(x)D\nu_j(\phi(x))Q(x)h, \quad h \in \mathbb{R}^\ell. \tag{20}$$

Then we have

$$S_0(x)|_{T_{\phi(x)}\mathcal{M}}\colon T_{\phi(x)}\mathcal{M} \to T_{\phi(x)}\mathcal{M}$$

and, using (20), (19) and (18) we obtain

$$(S_0(x)|_{T_{\phi(x)}\mathcal{M}}) \circ (D\phi(x)|_{T_{\phi(x)}\mathcal{M}}) = I_{T_{\phi(x)}\mathcal{M}}$$

for all $x \in \phi^{-1}(V_p)$. Since $T_{\phi(x)}\mathcal{M}$ is finite-dimensional, this implies that $D\phi(x)|_{T_{\phi(x)}\mathcal{M}}$ is bijective and

$$(D\phi(x)|_{T_{\phi(x)}\mathcal{M}})^{-1} = S_0(x)|_{T_{\phi(x)}\mathcal{M}}, \quad x \in \phi^{-1}(V_p). \tag{21}$$

Since $p \in \mathcal{M}$ is arbitrary, we obtain that,

$$J_0(x) > 0, \quad x \in \mathcal{U}. \tag{22}$$

We also conclude that $D\phi(x)\colon \mathbb{R}^\ell \to T_{\phi(x)}(\mathcal{M})$ is surjective for all $x \in \mathcal{U}$. By the surjective mapping theorem (cf [1, Theorem 3.5.2]) we conclude that $\phi\colon \mathcal{U} \to \mathcal{M}$ is an open map.

Now let $\epsilon \in ]0,1]$. Formula (11) and Proposition 2.3 imply, for all $x \in \mathcal{U}$, $y \in \Phi_\epsilon(\mathcal{U})$ and every $h \in \mathbb{R}^\ell$,

$$D\Phi_\epsilon(x)h = D\phi(x)h + \epsilon(h - D\phi(x)h) \tag{23}$$

and

$$D\Phi_\epsilon^{-1}(y)h = D\phi(y)h + (1/\epsilon)(h - D\phi(y)h). \tag{24}$$

This immediately implies that

$$D\Phi_\epsilon(x)|_{T_{\phi(x)}\mathcal{M}} \colon T_{\phi(x)}\mathcal{M} \to T_{\phi(x)}\mathcal{M} \tag{25}$$

and

$$D\Phi_\epsilon^{-1}(y)|_{T_{\phi(y)}\mathcal{M}} \colon T_{\phi(y)}\mathcal{M} \to T_{\phi(y)}\mathcal{M}. \tag{26}$$

Thus, using Proposition 2.5 we obtain for $x \in \phi^{-1}(V_p)$ and $h \in T_{\phi(x)}\mathcal{M}$

$$D\Phi_\epsilon(x)h = D\phi(x)h + \epsilon \sum_{j=1}^{\ell-k} \alpha_j(x)Q(x)D\nu_j(\phi(x))D\phi(x)h. \tag{27}$$

For $y \in \Phi_\epsilon(\phi^{-1}(V_p))$ and $h \in T_{\phi(y)}\mathcal{M}$ we get

$$D\Phi_\epsilon^{-1}(y)h = D\phi(y)h + (1/\epsilon) \sum_{j=1}^{\ell-k} \alpha_j(y)Q(y)D\nu_j(\phi(y))D\phi(y)h. \tag{28}$$

Moreover, for $x \in \phi^{-1}(V_p)$ and $y \in \Phi_\epsilon(\phi^{-1}(V_p))$ we also obtain

$$D\Phi_\epsilon(x)\nu_j(\phi(x)) = \epsilon\nu_j(\phi(x)) \tag{29}$$

and

$$D\Phi_\epsilon^{-1}(y)\nu_j(\phi(y)) = (1/\epsilon)\nu_j(\phi(y)) \tag{30}$$

for $j = 1, \ldots, \ell - k$.

For $x \in \phi^{-1}(V_p)$, we also have

$$|\det D\Phi_\epsilon(x)| = (\det G_\epsilon(x))^{1/2},$$

where $G_\epsilon(x)$ is the symmetric $\ell \times \ell$ matrix whose entries are

$$G_\epsilon(x)_{ij} = \begin{cases} \langle D\Phi_\epsilon(x)h_i(\phi(x)), D\Phi_\epsilon(x)h_j(\phi(x))\rangle, & 1 \le i,j \le k \\ \langle D\Phi_\epsilon(x)\nu_{i-k}(\phi(x)), D\Phi_\epsilon(x)\nu_{j-k}(\phi(x))\rangle, & k+1 \le i,j \le \ell \\ \langle D\Phi_\epsilon(x)\nu_{i-k}(\phi(x)), D\Phi_\epsilon(x)h_j(\phi(x))\rangle, & k+1 \le i \le \ell,\, 1 \le j \le k. \end{cases}$$

Hence, by (29) and (25), we get

$$|\det D\Phi_\epsilon(x)| = \epsilon^{(\ell-k)/2}(\det \tilde{G}_\epsilon(x))^{1/2} > 0,$$

where $\tilde{G}_\epsilon(x)$ is the $k \times k$ matrix whose entries are

$$\tilde{G}_\epsilon(x)_{ij} = \langle D\Phi_\epsilon(x)h_i(\phi(x)), D\Phi_\epsilon(x)h_j(\phi(x))\rangle.$$

Thus, for every $x \in \phi^{-1}(V_p)$

$$J_\epsilon(x) = \begin{cases} (\det(\langle D\Phi_\epsilon(x)h_i(\phi(x)), D\Phi_\epsilon(x)h_j(\phi(x))\rangle)_{ij})^{1/2} > 0, & \epsilon \in ]0,1], \\ (\det(\langle D\phi(x)h_i(\phi(x)), D\phi(x)h_j(\phi(x))\rangle)_{ij})^{1/2} > 0, & \epsilon = 0. \end{cases}$$

Since $p \in \mathcal{M}$ is arbitrary we therefore conclude all the statements of Theorem 2.4 concerning $J_\epsilon$.

Next we analyze the linear operator

$$L_\epsilon(x) := D\Phi_\epsilon^{-1}(\Phi_\epsilon(x)). \tag{31}$$

for $\epsilon \in ]0,1]$. Formula (30) implies

$$L_\epsilon(x)\nu_j(\phi(x)) = (1/\epsilon)\nu_j(\phi(x)) \quad \text{for } x \in \phi^{-1}(V_p) \text{ and } j = 1, \ldots, \ell - k.$$

Moreover, Formula (28) and Proposition 2.5 imply, for $x \in \phi^{-1}(V_p)$ and $h \in T_{\phi(x)}\mathcal{M}$,

$$\begin{aligned} L_\epsilon(x)h &= D\phi(\Phi_\epsilon(x))h + (1/\epsilon)\sum_{j=1}^{\ell-k}\alpha_j(\Phi_\epsilon(x))Q(\Phi_\epsilon(x))D\nu_j(\phi(\Phi_\epsilon(x)))D\phi(\Phi_\epsilon(x))h \\ &= D\phi(\Phi_\epsilon(x))h + \sum_{j=1}^{\ell-k}\alpha_j(x)Q(x)D\nu_j(\phi(x))D\phi(\Phi_\epsilon(x))h. \end{aligned} \tag{32}$$

Set

$$B_\epsilon(x) := D\phi(\Phi_\epsilon(x)), \quad N_j(x) := D\nu_j(\phi(x)).$$

Then we can write

$$L_\epsilon(x)h = B_\epsilon(x)h + \sum_{j=1}^{\ell-k}\alpha_j(x)Q(x)N_j(x)B_\epsilon(x)h \quad \text{for } x \in \phi^{-1}(V_p) \text{ and } h \in T_{\phi(x)}\mathcal{M}.$$

Thus, for $x \in \phi^{-1}(V_p)$ and $h \in \mathbb{R}^\ell$, we get

$$L_\epsilon(x)h = Q(x)B_\epsilon(x)Q(x)h + \sum_{j=1}^{\ell-k}\alpha_j(x)Q(x)N_j(x)B_\epsilon(x)Q(x)h + (1/\epsilon)P(x)h. \tag{33}$$

(Here we used the fact that $B_\epsilon(x) = D\phi(\Phi_\epsilon(x))$ maps $\mathbb{R}^\ell$ onto $T_{\phi(x)}\mathcal{M}$, so $B_\epsilon(x) = Q(x)B_\epsilon(x)$ for $x \in \mathcal{U}$.)

Now observe that, since $\Phi_\epsilon(x) \to \phi(x)$ as $\epsilon \to 0$ in $\mathbb{R}^\ell$, uniformly on compact subsets of $\mathcal{U}$, we also have that

$$B_\epsilon(x) = D\phi(\Phi_\epsilon(x)) \to D\phi(\phi(x)) \quad \text{as } \epsilon \to 0$$

in $\mathcal{L}(\mathbb{R}^\ell, \mathbb{R}^\ell)$, uniformly on compact subsets of $\mathcal{U}$. Since $\phi(x) \equiv x$ on $\mathcal{M}$, we have that $D\phi(\phi(x))h = h$ for all $x \in \mathcal{U}$ and all $h \in T_{\phi(x)}\mathcal{M}$. Moreover, whenever $\nu$ is orthogonal to $T_{\phi(x)}\mathcal{M}$ then $D\phi(\phi(x))\nu = 0$. It follows that

$$D\phi(\phi(x)) = Q(x) \quad \text{for } x \in \mathcal{U}, \tag{34}$$

and hence

$$B_\epsilon(x) \to Q(x) \quad \text{as } \epsilon \to 0 \tag{35}$$

in $\mathcal{L}(\mathbb{R}^\ell, \mathbb{R}^\ell)$, uniformly on compact subsets of $\mathcal{U}$. For $x \in \mathcal{U}$ set

$$S_\epsilon(x) := L_\epsilon(x) - (1/\epsilon)P(x).$$

It follows that

$$S_\epsilon(x)|_{T_{\phi(x)}\mathcal{M}} \colon T_{\phi(x)}\mathcal{M} \to T_{\phi(x)}\mathcal{M}$$

and

$$S_\epsilon(x)|_{T_{\phi(x)}\mathcal{M}} = L_\epsilon(x)|_{T_{\phi(x)}\mathcal{M}}.$$

This implies that

$$S_\epsilon(x)|_{T_{\phi(x)}\mathcal{M}} \colon T_{\phi(x)}\mathcal{M} \to T_{\phi(x)}\mathcal{M}$$

is bijective. Moreover, using (33), we see that for $x \in \phi^{-1}(V_p)$

$$S_\epsilon(x) := Q(x)B_\epsilon(x)Q(x) + \sum_{j=1}^{\ell-k} \alpha_j(x)Q(x)N_j(x)B_\epsilon(x)Q(x).$$

Therefore

$$|S_\epsilon(x) - S_0(x_0)|_{\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)} \to 0 \quad \text{as } \epsilon \to 0 \text{ and } x \to x_0 \text{ in } \phi^{-1}(V_p) \tag{36}$$

where

$$S_0(x) = Q(x) + \sum_{j=1}^{\ell-k} \alpha_j(x)Q(x)N_j(x)Q(x)$$

is the linear operator defined in (20). We recall that, by (21),

$$S_0(x)|_{T_{\phi(x)}\mathcal{M}} \colon T_{\phi(x)}\mathcal{M} \to T_{\phi(x)}\mathcal{M}$$

is bijective and

$$(S_0(x)^T|_{T_{\phi(x)}\mathcal{M}})^{-1} = D\phi(x)^T|_{T_{\phi(x)}\mathcal{M}}$$

We now have to compute $L_\epsilon(x)^T$. Since $P(x)$ and $Q(x)$ are orthogonal projections in $\mathbb{R}^\ell$, we have that $P(x)^T = P(x)$ and $Q(x)^T = Q(x)$ for all $x \in \mathcal{U}$. For $x \in \phi^{-1}(V_p)$ we have

$$S_\epsilon(x)^T = Q(x)B_\epsilon(x)^T Q(x) + \sum_{j=1}^{\ell-k} \alpha_j(x)Q(x)B_\epsilon(x)^T N_j(x)^T Q(x)$$

and

$$S_0(x)^T = Q(x) + \sum_{j=1}^{\ell-k} \alpha_j(x)Q(x)N_j(x)^T Q(x).$$

It follows that $S_\epsilon(x)^T\nu_j(\phi(x)) = 0$ and $S_0(x)^T\nu_j(\phi(x)) = 0$ for $j = 1, \ldots, \ell - k$. We also have

$$\langle S_\epsilon(x)^T h, \nu_j(\phi(x))\rangle = \langle h, S_\epsilon(x)\nu_j(\phi(x))\rangle = 0$$

and

$$\langle S_0(x)^T h, \nu_j(\phi(x))\rangle = \langle h, S_0(x)\nu_j(\phi(x))\rangle = 0$$

for all $x \in \phi^{-1}(V_p)$, $h \in \mathbb{R}^\ell$ and $j = 1, \ldots, \ell - k$. It follows that

$$S_\epsilon(x)^T|_{T_{\phi(x)}\mathcal{M}} \colon T_{\phi(x)}\mathcal{M} \to T_{\phi(x)}\mathcal{M}$$

and

$$S_\epsilon(x)^T|_{T_{\phi(x)}\mathcal{M}} = L_\epsilon(x)^T|_{T_{\phi(x)}\mathcal{M}}.$$

This implies that

$$S_\epsilon(x)^T|_{T_{\phi(x)}\mathcal{M}} \colon T_{\phi(x)}\mathcal{M} \to T_{\phi(x)}\mathcal{M}$$

is bijective. Moreover,

$$S_0(x)^T|_{T_{\phi(x)}\mathcal{M}} \colon T_{\phi(x)}\mathcal{M} \to T_{\phi(x)}\mathcal{M}$$

and

$$|S_\epsilon(x)^T - S_0(x_0)^T|_{\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)} \to 0 \quad \text{as } \epsilon \to 0 \text{ and } x \to x_0 \text{ in } \phi^{-1}(V_p). \tag{37}$$

Now given $x \in \mathcal{U}$, $h \in T_{\phi(x)}\mathcal{M}$ and $\nu$ orthogonal to $T_{\phi(x)}\mathcal{M}$ we have

$$\langle D\phi(x)^T h, \nu\rangle = \langle h, D\phi(x)\nu\rangle = 0,$$

so

$$D\phi(x)^T|_{T_{\phi(x)}\mathcal{M}} \colon T_{\phi(x)}\mathcal{M} \to T_{\phi(x)}\mathcal{M}.$$

Finally, observe that, by (21), for an arbitrary $h$ and $h' \in T_{\phi(x)}\mathcal{M}$ we have

$$\langle D\phi(x)^T S_0(x)^T h, h'\rangle = \langle h, S_0(x) D\phi(x) h'\rangle = \langle h, h'\rangle.$$

It follows that

$$(D\phi(x)^T|_{T_{\phi(x)}\mathcal{M}}) \circ (S_0(x)^T|_{T_{\phi(x)}\mathcal{M}}) = I|_{T_{\phi(x)}\mathcal{M}}.$$

This implies that $S_0(x)^T|_{T_{\phi(x)}\mathcal{M}}$ is bijective and

$$(S_0(x)^T|_{T_{\phi(x)}\mathcal{M}})^{-1} = D\phi(x)^T|_{T_{\phi(x)}\mathcal{M}}. \tag{38}$$

The proof of Theorem 2.4 is complete. □

## 3. Spectral convergence

For the rest of this paper let $\Omega$ be a nonempty open bounded set in $\mathbb{R}^\ell$ with Lipschitz boundary. Suppose that $\mathrm{Cl}\,\Omega \subset \mathcal{U}$. For $\epsilon \in ]0,1]$ let $\Omega_\epsilon := \Phi_\epsilon(\Omega)$ be the *squeezed domain.* In this section we will study the bilinear forms on $H^1(\Omega_\epsilon)$ stemming from the Laplacian on $\Omega_\epsilon$ with Neumann boundary conditions. We will show that, for $\epsilon \to 0^+$ these bilinear forms tend in a strong spectral sense to a limit form defined on a *subspace* of $H^1(\Omega_\epsilon)$. This result is the basis of all the applications presented in the next section.

We begin with some useful definitions:

**Definition 3.1.** *Let $H$ be a vector space and $V$ be a linear subspace of $H$. Let $a\colon V\times V \to \mathbb{R}$ be a bilinear form on $V$ and $b\colon H\times H \to \mathbb{R}$ be a bilinear form on $H$. If $\lambda \in \mathbb{R}$, $u \in V\setminus\{0\}$ satisfy*

$$a(u,v) = \lambda b(u,v) \qquad \textit{for all } v \in V$$

*then we say that $\lambda$ is* an eigenvalue of the pair $(a,b)$ *and $u$ is* an eigenvector of the pair $(a,b)$, corresponding to $\lambda$. *The dimension of the span of all eigenvectors of $(a,b)$ corresponding to $\lambda$ is called* the multiplicity of $\lambda$. *If the set of eigenvalues of $(a,b)$ is countably infinite, contains a smallest element and if each eigenvalue has finite multiplicity then* the repeated sequence of the eigenvalues of $(a,b)$ *is the uniquely determined nondecreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ which contains exactly the eigenvalues of $(a,b)$ and the number of occurrences of each eigenvalue in this sequence is equal to its multiplicity.*

*Given $a$ and $b$ as above define $R = R(a,b)$ to be the set of all pairs $(u,w) \in V\times H$ such that $a(u,v) = b(w,v)$ for all $v \in V$. We call $R$ the* operator relation generated by the pair $(a,b)$. *If $R$ is the graph of a mapping $A\colon D(A) \to H$, then this map is called the* operator generated by the pair $(a,b)$.

The following properties are obvious:

**Proposition 3.2.** *Let $H$, $V$, $a$, $b$ and $R$ be as in Definition 3.1 Then $R$ is a linear subspace of $V\times H$. Moreover, $(\lambda,u)$ is an eigenvalue-eigenvector pair of $(a,b)$ if and only if $\lambda \in \mathbb{R}$, $u \in V$, $u \neq 0$ and $(u,\lambda u) \in R$. Thus if $R$ is the graph of a map $A$, then $A$ is linear and $(\lambda,u)$ is an eigenvalue-eigenvector pair of $(a,b)$ if and only if $(\lambda,u)$ is an eigenvalue-eigenvector pair of $A$.* □

The following proposition is well-known:

**Proposition 3.3.** *Let $V$, $H$ be two infinite dimensional Hilbert spaces. Suppose $V \subset H$ with compact inclusion, and $V$ is dense in $H$. Let $\|\cdot\|$ and $|\cdot|$ denote the norms of $V$ and $H$ respectively, and $b$ be the inner product of $H$. Let $a\colon V \times V \to \mathbb{R}$ be a symmetric bilinear form on $V$. Assume that there are constants $d$, $C$, $\alpha \in \mathbb{R}$, $\alpha > 0$, such that, for all $u$, $v \in V$,*

$$|a(u,v)| \le C\|u\|\|v\|$$
$$a(u,u) \ge \alpha\|u\|^2 - d|u|^2.$$

*Then the set of eigenvalues of $(a,b)$ is countably infinite, it has a smallest element and each eigenvalue has finite multiplicity. Moreover, the operator relation generated by $(a,b)$ is the graph of a linear selfadjoint operator $A$ on $(H,\langle\cdot,\cdot\rangle)$ with compact resolvent.* □

For $\epsilon \in ]0,1]$ define the following bilinear forms:

$$\tilde{a}_\epsilon\colon H^1(\Omega_\epsilon) \times H^1(\Omega_\epsilon) \to \mathbb{R},$$

$$\tilde{a}_\epsilon(u,v) := \int_{\Omega_\epsilon} \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x; \tag{39}$$

$$a_\epsilon\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R},$$

$$a_\epsilon(u,v) := \int_\Omega J_\epsilon(x)\langle S_\epsilon(x)^T\nabla u(x), S_\epsilon(x)^T\nabla v(x)\rangle\,\mathrm{d}x + \frac{1}{\epsilon^2}\int_\Omega J_\epsilon(x)\langle P(x)\nabla u(x), P(x)\nabla v(x)\rangle\,\mathrm{d}x. \tag{40}$$

Moreover, for $\epsilon \in [0,1]$ define the following bilinear forms:

$$\tilde{b}_\epsilon\colon L^2(\Omega_\epsilon) \times L^2(\Omega_\epsilon) \to \mathbb{R}$$

$$\tilde{b}_\epsilon(u,v) := \int_{\Omega_\epsilon} u(x)v(x)\,\mathrm{d}x; \tag{41}$$

$$b_\epsilon\colon L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$$

$$b_\epsilon(u,v) := \int_\Omega J_\epsilon(x)u(x)v(x)\,\mathrm{d}x \tag{42}$$

Note that, by Theorem 2.4 there are constants $C$, $c \in ]0,\infty[$ such that

$$cb_\epsilon(u,u) \le |u|^2_{L^2(\Omega)} \le Cb_\epsilon(u,u), \quad \text{for } \epsilon \in [0,1] \text{ and } u \in L^2(\Omega). \tag{43}$$

It is clear that, for $\epsilon \in ]0,1]$, the assignment

$$u \mapsto u \circ \Phi_\epsilon$$

restricts to linear isomorphisms $L^2(\Omega_\epsilon) \to L^2(\Omega)$ and $H^1(\Omega_\epsilon) \to H^1(\Omega)$. Using the change-of-variables formula and Theorem 2.4 we see that, for $\epsilon \in ]0,1]$,

$$a_\epsilon(u \circ \Phi_\epsilon, v \circ \Phi_\epsilon) = \epsilon^{-(\ell-k)/2}\tilde{a}_\epsilon(u,v)$$

for all $u, v \in H^1(\Omega_\epsilon)$. Moreover,

$$b_\epsilon(u \circ \Phi_\epsilon, v \circ \Phi_\epsilon) = \epsilon^{-(\ell-k)/2}\tilde{b}_\epsilon(u,v)$$

for all $u, v \in L^2(\Omega_\epsilon)$.

We thus obtain the following

**Proposition 3.4.** *The (linear) operators $\tilde{A}_\epsilon$ (resp. $A_\epsilon$) defined by $(\tilde{a}_\epsilon, \tilde{b}_\epsilon)$ (resp. $(a_\epsilon, b_\epsilon)$) satisfy the following properties:*

(1) *$u \in \tilde{D}(A_\epsilon)$ if and only if $u \circ \Phi_\epsilon \in D(A_\epsilon)$;*
(2) *$A_\epsilon(u \circ \Phi_\epsilon) = (\tilde{A}_\epsilon u) \circ \Phi_\epsilon$ for $u \in D(\tilde{A}_\epsilon)$.*

Notice that, if $u \in H^1(\Omega)$,

$$a_\epsilon(u,u) \to \begin{cases} \int_\Omega J_0(x)\langle S_0(x)^T \nabla u(x), S_0(x)^T \nabla v(x)\rangle \,\mathrm{d}x & \text{if } P(x)\nabla u(x) = 0 \text{ a.e.,} \\ +\infty & \text{otherwise.} \end{cases} \tag{44}$$

Let us define the space

$$H^1_s(\Omega) := \{\, u \in H^1(\Omega) \mid P(x)\nabla u(x) = 0 \ \text{ a.e.}\,\}.$$

Note that $H^1_s(\Omega)$ is a closed linear subspace of the Hilbert space $H^1(\Omega)$.

We have the following

**Proposition 3.5.** *The space $H^1_s(\Omega)$ is infinite dimensional.*

*Proof.* Since $\Omega$ is open and nonempty and $\phi\colon \mathcal{U} \to \mathcal{M}$ is open by Theorem 2.4, it follows that $V := \phi(\mathcal{U})$ is open and nonempty in $\mathcal{M}$. Let $C^1_c(V)$ be the linear space of all real-valued $C^1$-functions on $\mathcal{M}$ with compact support contained in $V$. For every $u \in C^1_c(V)$ the function $u\circ\phi$ is bounded and has continuous and bounded derivatives. Thus $u\circ\phi \in H^1(\Omega)$. Moreover, for $x \in \Omega$ the vector $\nu = \nu(x) := P(x)\nabla(u \circ \phi)(x)$ is orthogonal to $T_{\phi(x)}(\mathcal{M})$. Thus

$$\langle \nu, \nu\rangle = \langle P(x)\nabla(u\circ\phi)(x), \nu\rangle = \langle \nabla(u\circ\phi)(x), \nu\rangle = D(u\circ\phi)(x)\nu = 0$$

since $D\phi(x)\nu = 0$. Hence $\nu(x) = 0$ on $\Omega$, i.e. $u \circ \phi \in H^1_s(\Omega)$. Define the map $\Gamma\colon C^1_c(V) \to H^1_s(\Omega)$ by $u \mapsto u \circ \phi$. Clearly, $\Gamma$ is injective. Since $C^1_c(V)$ has infinite dimension, so does $H^1_s(\Omega)$. □

Now define the 'limit' bilinear form

$$a_0\colon H^1_s(\Omega) \times H^1_s(\Omega) \to \mathbb{R}$$

$$a_0(u,v) := \int_\Omega J_0(x)\langle S_0(x)^T \nabla u(x), S_0(x)^T \nabla v(x)\rangle \,\mathrm{d}x. \tag{45}$$

Finally, let $L^2_s(\Omega)$ be the closure of $H^1_s(\Omega)$ in $L^2(\Omega)$. We will denote by $A_0$ the operator generated by the pair $(a_0, b_0|_{L^2_s(\Omega)\times L^2_s(\Omega)})$.

For $\epsilon \in [0,1]$ and $u \in L^2(\Omega)$ set

$$|u|_\epsilon := b_\epsilon(u,u)^{1/2}.$$

For $\epsilon \in ]0,1]$ for $u \in H^1(\Omega)$ set

$$\|u\|_\epsilon := (a_\epsilon(u,u) + b_\epsilon(u,u))^{1/2}.$$

Finally, for $\epsilon = 0$ and $u \in H^1_s(\Omega)$ set

$$\|u\|_0 := (a_0(u,u) + b_0(u,u))^{1/2}.$$

We require the following propositions.

**Proposition 3.6.** *For every compact set $K \subset \mathcal{U}$ and for every $\delta \in ]0,1[$ there exists an $\overline{\epsilon} \in ]0,1]$ such that, for all $\epsilon \in ]0,\overline{\epsilon}]$, for all $x \in K$ and for all $h \in T_{\phi(x)}\mathcal{M}$ the following inequality is satisfied:*

$$(1-\delta)\langle S_0(x)^T h, S_0(x)^T h\rangle \leq_{(1)} \langle S_\epsilon(x)^T h, S_\epsilon(x)^T h\rangle \leq_{(2)} (1+\delta)\langle S_0(x)^T h, S_0(x)^T h\rangle. \tag{46}$$

*Proof.* Assume by contradiction that $\leq_{(1)}$ is false. Then there exists a $\delta \in ]0,1[$, and for all $m \in \mathbb{N}$ there exist an $x_m \in K$, an $\epsilon_m \in \mathbb{R}$, $0 < \epsilon_m < (1/m)$ and an $h_m \in T_{\phi(x_m)}\mathcal{M}$, such that

$$(1-\delta)\langle S_0(x_m)^T h_m, S_0(x_m)^T h_m\rangle \geq \langle S_{\epsilon_m}(x_m)^T h_m, S_{\epsilon_m}(x_m)^T h_m\rangle. \tag{47}$$

We can assume without loss of generality that $|h_m| = 1$ for all $m$. Then, up to a subsequence, we have that $x_m \to x$ and $h_m \to h$ as $m \to \infty$, where $x \in K$ and $|h| = 1$. It is not difficult to see that $h \in T_{\phi(x)}\mathcal{M}$. Then, by letting $m \to \infty$ in (47) and applying Theorem 2.4 we see that

$$(1-\delta)\langle S_0(x)^T h, S_0(x)^T h\rangle \geq \langle S_0(x)^T h, S_0(x)^T h\rangle.$$

It follows that $S_0(x)^T h = 0$, so $h = 0$ by Theorem 2.4, a contradiction. The proof of $\leq_{(2)}$ is analogous. □

**Proposition 3.7.** *For every compact set $K \subset \mathcal{U}$ there is a constant $\gamma = \gamma(K) \in ]0,\infty[$ such that, for all $x \in K$, $\epsilon \in [0,1]$ and $h \in T_{\phi(x)}\mathcal{M}$, the following estimate holds:*

$$\langle S_\epsilon(x)^T h, S_\epsilon(x)^T h\rangle \geq \gamma |h|^2. \tag{48}$$

*Proof.* Assume by contradiction that the proposition is false. Then for all $m \in \mathbb{N}$ there exist $x_m \in K$, $\epsilon_m \in [0,1]$ and $h_m \in T_{\phi(x_m)}\mathcal{M}$, such that

$$\frac{1}{m}|h_m|^2 \geq \langle S_{\epsilon_m}(x_m)^T h_m, S_{\epsilon_m}(x_m)^T h_m\rangle. \tag{49}$$

Without loss of generality, we can assume that $|h_m| = 1$ for all $m$. So, up to a subsequence, we have that $x_m \to x$, $\epsilon_m \to \epsilon$ and $h_m \to h$ as $m \to \infty$, where $x \in K$, $\epsilon \in [0,1]$ and $|h| = 1$. It is not difficult to see that $h \in T_{\phi(x)}\mathcal{M}$. Then, by letting $m \to \infty$ in (49) and applying Theorem 2.4 we obtain

$$\langle S_\epsilon(x)^T h, S_\epsilon(x)^T h\rangle = 0.$$

It follows that $S_0(x)^T h = 0$, so $h = 0$ by Theorem 2.4, a contradiction.

□

As a consequence of Propositions 3.6 and 3.7, we have:

**Proposition 3.8.** *For every $\delta \in ]0,1[$ there exists an $\overline{\epsilon} \in ]0,1]$ such that*

$$(1-\delta)b_0(u,u) \leq b_\epsilon(u,u) \leq (1+\delta)b_0(u,u) \tag{50}$$

*for all $u \in L^2$ and $\epsilon \in ]0,\overline{\epsilon}]$, and*

$$(1-\delta)a_0(u,u) \leq a_\epsilon(u,u) \leq (1+\delta)a_0(u,u) \tag{51}$$

*for all $u \in H^1_s(\Omega)$ and $\epsilon \in ]0,\overline{\epsilon}[$.*

*Besides, whenever $u$ and $v \in L^2$, then*

$$b_\epsilon(u,v) \to b_0(u,v) \quad \text{as } \epsilon \to 0. \tag{52}$$

*Moreover, on $H_s^1(\Omega)$ the norms $\|\cdot\|_\epsilon$ and $\|\cdot\|_0$ are equivalent, with equivalence constants independent of $\epsilon \in \,]0,1]$ and*

$$a_\epsilon(u,u) \to a_0(u,u) \quad \text{as } \epsilon \to 0, \tag{53}$$

*for all $u \in H_s^1(\Omega)$. Finally, there exists a $\gamma > 0$ such that*

$$\gamma |u|_{H^1} \leq \|u\|_\epsilon \quad \text{for all } \epsilon \in \,]0,1] \text{ and } u \in H^1. \tag{54}$$

In the rest of this section we will present a basic spectral convergence result, which generalizes [21, Theorem 3.3] and has far-reaching implications concerning the dynamics of both reaction-diffusion equations and damped wave equations on thin domains.

Since we are working on a fixed domain $\Omega$, we can write for short $L^2 := L^2(\Omega)$, $H^1 := H^1(\Omega)$, $L_s^2 := L_s^2(\Omega)$ and $H_s^1 := H_s^1(\Omega)$. By (43) the norms $|\cdot|_\epsilon$, $\epsilon \in [0,1]$, are all equivalent to the usual norm on $L^2$, with equivalence constants independent of $\epsilon$.

For $\epsilon \in \,]0,1]$ the norm $\|\cdot\|_\epsilon$ is equivalent to the usual norm on $H^1$ and so $(H^1, \|\cdot\|_\epsilon)$ is densely and compactly imbedded in $(L^2, |\cdot|_\epsilon)$.

Consequently well known results (cf [21, Proposition 2.2]) imply that, for all $\epsilon \in \,]0,1]$, there exists a sequence $(\lambda_j^\epsilon, w_j^\epsilon)_{j\in\mathbb{N}}$ of eigenvalue-eigenvector pairs of $(a_\epsilon, b_\epsilon)$ such that

$$\lambda_1^\epsilon \leq \lambda_2^\epsilon \leq \lambda_3^\epsilon \leq \dots$$

and $(w_j^\epsilon)_{j\in\mathbb{N}}$ is a complete $b_\epsilon$-orthonormal system in $L^2$.

Similarly, the space $H_s^1$ is densely and compactly imbedded in $L_s^2$. Since $H_s^1$ has infinite dimension by Proposition 3.5, [21, Proposition 2.2] again implies that there exists a sequence $(\lambda_j^0, w_j^0)_{j\in\mathbb{N}}$ of eigenvalue-eigenvector pairs of $(a_0, b_0)$ such that

$$\lambda_1^0 \leq \lambda_2^0 \leq \lambda_3^0 \leq \dots$$

and $(w_j^0)_{j\in\mathbb{N}}$ is a complete $b_0$-orthonormal system in $L_s^2$.

We need the following

**Lemma 3.9.** *Let $(\epsilon_m)_{m\in\mathbb{N}}$ be a sequence in $]0,1]$ converging to zero. Let $(u_m)_{m\in\mathbb{N}}$ be a sequence in $H^1$ and assume there exists $u \in H_s^1$ such that $\|u_m - u\|_{\epsilon_m} \to 0$ as $m \to \infty$. Then*

$$|a_{\epsilon_m}(u_m,u_m) - a_0(u,u)| \to 0 \quad \text{as } m \to \infty. \tag{55}$$

*Proof.* We begin by computing

$$\begin{aligned} &\big|a_{\epsilon_m}(u_m,u_m) - a_0(u,u)\big| \\ &\leq \big|a_{\epsilon_m}(u_m,u_m) + b_{\epsilon_m}(u_m,u_m) - a_{\epsilon_m}(u,u) - b_{\epsilon_m}(u,u)\big| \\ &+ \big|a_{\epsilon_m}(u,u) - a_0(u,u)\big| + \big|b_{\epsilon_m}(u_m,u_m) - b_{\epsilon_m}(u,u)\big| \\ &= \big|\|u_m\|_{\epsilon_m}^2 - \|u\|_{\epsilon_m}^2\big| + \big|a_{\epsilon_m}(u,u) - a_0(u,u)\big| + \big|b_{\epsilon_m}(u_m,u_m) - b_{\epsilon_m}(u,u)\big|. \end{aligned} \tag{56}$$

Since $\|\cdot\|_{\epsilon_m}$ and $|\cdot|_{\epsilon_m}$ are norms we have

$$\big|\|u_m\|_{\epsilon_m} - \|u\|_{\epsilon_m}\big| \leq \|u_m - u\|_{\epsilon_m} \to 0 \quad \text{as } m \to \infty$$

and

$$\big||u_m|_{\epsilon_m} - |u|_{\epsilon_m}\big| \leq |u_m - u|_{\epsilon_m} \to 0 \quad \text{as } m \to \infty.$$

However,

$$\|u\|_{\epsilon_m} \to \|u\|_0 \quad \text{and} \quad |u|_{\epsilon_m} \to |u|_0$$

as $m \to \infty$ by (52) and (53). Thus also $\|u_m\|_{\epsilon_m} \to \|u\|_0$ and hence

$$\left| \|u_m\|_{\epsilon_m}^2 - \|u\|_{\epsilon_m}^2 \right| \to 0. \tag{57}$$

In the same way, also $|u_m|_{\epsilon_m} \to |u|_0$ and hence

$$\left| b_{\epsilon_m}(u_m, u_m) - b_{\epsilon_m}(u, u) \right| = \left| |u_m|_{\epsilon_m}^2 - |u|_{\epsilon_m}^2 \right| \to 0. \tag{58}$$

Finally, (57), (58) and (53) imply that (55) is satisfied. ☐

With these preliminaries, we have the following

**Theorem 3.10.** *For $\epsilon \in ]0,1]$ let*

$$\lambda_1^\epsilon \le \lambda_2^\epsilon \le \lambda_3^\epsilon \le \dots$$

*be the repeated sequence of eigenvalues of the pair $(a_\epsilon, b_\epsilon)$ and $w_1^\epsilon, w_2^\epsilon, w_3^\epsilon, \dots$ be a corresponding complete $(L^2, b_\epsilon)$-orthonormal sequence of eigenvectors. Moreover, let*

$$\lambda_1^0 \le \lambda_2^0 \le \lambda_3^0 \le \dots$$

*be the repeated sequence of eigenvalues of $(a_0, b_0)$. Then the following properties hold:*

1. *For every $j \in \mathbb{N}$,*
$$\lambda_j^0 = \lim_{\epsilon \to 0^+} \lambda_j^\epsilon;$$
2. *let $(\epsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers converging to $0$. Then there is a subsequence of $(\epsilon_n)_{n\in\mathbb{N}}$, again denoted by $(\epsilon_n)_{n\in\mathbb{N}}$, and there exists a complete $(L_s^2, b_0)$-orthonormal system $(w_j^0)_{j\in\mathbb{N}}$ of eigenvectors of $(a_0, b_0)$ corresponding to $(\lambda_j^0)_{j\in\mathbb{N}}$ such that, for every $j \in \mathbb{N}$,*
$$\|w_j^{\epsilon_n} - w_j^0\|_{\epsilon_n} \to 0$$
*as $n \to \infty$.*

*Proof.* Let $j \in \mathbb{N}$ be arbitrary. By [21, Proposition 2.2],

$$\lambda_j^\epsilon = \min_{E \in \mathcal{F}_j} \max_{u \in E\setminus\{0\}} \frac{a_\epsilon(u,u)}{b_\epsilon(u,u)},$$

where $\mathcal{F}_j$ is the set of all $j$-dimensional linear subspaces of $H^1$. Since $H_s^1 \subset H^1$, by (50) and (51) we have that, for every $\delta \in ]0,1[$ there exists an $\bar\epsilon(\delta) \in ]0,1]$ such that, for all $\epsilon \in ]0, \bar\epsilon]$,

$$0 \le \lambda_j^\epsilon = \min_{E \in \mathcal{F}_j} \max_{u \in E\setminus\{0\}} \frac{a_\epsilon(u,u)}{b_\epsilon(u,u)} \le \frac{1+\delta}{1-\delta} \min_{E \in \mathcal{F}_{sj}} \max_{u \in E\setminus\{0\}} \frac{a_0(u,u)}{b_0(u,u)} = \frac{1+\delta}{1-\delta}\lambda_j^0$$

where $\mathcal{F}_{sj}$ is the set of all $j$-dimensional linear subspaces of $H_s^1$. Thus the set $\{\, \lambda_j^\epsilon \mid \epsilon \in ]0,1[\,\}$ is bounded in $\mathbb{R}$. Now let $(\epsilon_m)_{m\in\mathbb{N}}$ be a sequence of positive numbers converging to zero. It follows that there is a subsequence of $(\epsilon_m)_{m\in\mathbb{N}}$ (again denoted by $(\epsilon_m)_{m\in\mathbb{N}}$) and a number $\mu_j$ such that

$$\mu_j = \lim_{m\to\infty} \lambda_j^{\epsilon_m}.$$

Let $(\delta_m)_{m\in\mathbb{N}}$ be any sequence of positive numbers converging to zero. Then we can assume without loss of generality that $\epsilon_m \le \bar\epsilon(\delta_m)$ for all $m$. Note that $\mu_j \le \lambda_j^0$.

Now, for $j$ fixed and for all $m \in \mathbb{N}$, we have:

$$\begin{aligned} a_{\epsilon_m}(w_j^{\epsilon_m}, w_j^{\epsilon_m}) &= \lambda_j^{\epsilon_m}, \\ a_{\epsilon_m}(w_j^{\epsilon_m}, w) &= \lambda_j^{\epsilon_m} b_{\epsilon_m}(w_j^{\epsilon_m}, w) \quad \text{for all } w \in H^1. \end{aligned} \tag{59}$$

Hence

$$\begin{aligned} \gamma |w_j^{\epsilon_m}|_{H^1}^2 &\le a_{\epsilon_m}(w_j^{\epsilon_m}, w_j^{\epsilon_m}) + b_{\epsilon_m}(w_j^{\epsilon_m}, w_j^{\epsilon_m}) \\ &= \lambda_j^{\epsilon_m} + 1 \le \frac{1+\delta_m}{1-\delta_m} \lambda_j^0 + 1. \end{aligned} \tag{60}$$

It follows that there exists a subsequence of $(\epsilon_m)_{m\in\mathbb{N}}$ (again denoted by $(\epsilon_m)_{m\in\mathbb{N}}$) and a function $w_j \in H^1(\Omega)$ such that

$$w_j^{\epsilon_m} \rightharpoonup w_j \quad \text{in } H^1(\Omega)$$

as $m \to \infty$. Since $\Omega$ has Lipschitz boundary, the space $H^1(\Omega)$ is compactly imbedded into $L^2(\Omega)$, so

$$w_j^{\epsilon_m} \to w_j \quad \text{in } L^2(\Omega).$$

For all $m$ we have

$$(1-\delta_m) b_0(w_j^{\epsilon_m}, w_j^{\epsilon_m}) \le b_{\epsilon_m}(w_j^{\epsilon_m}, w_j^{\epsilon_m}) = 1 \le (1+\delta_m) b_0(w_j^{\epsilon_m}, w_j^{\epsilon_m}),$$

so, passing to the limit, we obtain that $b_0(w_j, w_j) = 1$. Next we show that $w_j \in H_s^1$.

Observe that

$$\begin{aligned} \frac{1}{\epsilon_m^2} \int_\Omega J_\epsilon(x) \langle P(x)\nabla w_j^{\epsilon_m}(x), P(x)\nabla w_j^{\epsilon_m}(x)\rangle \,\mathrm{d}x& \\ \le a_{\epsilon_m}(w_j^{\epsilon_m}(x), w_j^{\epsilon_m}(x)) = \lambda_j^{\epsilon_m} &\le \frac{1+\delta_m}{1-\delta_m} \lambda_j^0, \end{aligned} \tag{61}$$

which implies that

$$\int_\Omega J_{\epsilon_m}(x) \langle P(x)\nabla w_j^{\epsilon_m}(x), P(x)\nabla w_j^{\epsilon_m}(x)\rangle \,\mathrm{d}x \to 0 \quad \text{as } m \to \infty,$$

so $P(\cdot)\nabla w_j^{\epsilon_m} \to 0$ in $L^2(\Omega, \mathbb{R}^\ell)$. However, since $\nabla w_j^{\epsilon_m} \rightharpoonup \nabla w_j$ in $L^2(\Omega, \mathbb{R}^\ell)$ and the map $x \mapsto P(x)$ is continuous from $\overline{\Omega}$ to $\mathcal{L}(\mathbb{R}^\ell, \mathbb{R}^\ell)$, we have that $P(\cdot)\nabla w_j^{\epsilon_m} \rightharpoonup P(\cdot)\nabla w_j$ in $L^2(\Omega, \mathbb{R}^\ell)$ as $m \to \infty$. It follows that $P(x)\nabla w_j(x) = 0$ almost everywhere in $\Omega$, i.e. $w_j \in H_s^1$.

Now we prove that $(\mu_j, w_j)$ is an eigenvalue-eigenvector pair of $(a_0, b_0)$. Let $w \in H_s^1$ be arbitrary. Then, using the fact that $P(x)\nabla w(x) = 0$ almost everywhere, we have

$$\begin{aligned} \int_\Omega J_{\epsilon_m}(x) \langle S_{\epsilon_m}(x)^T \nabla w_j^{\epsilon_m}(x), S_{\epsilon_m}(x)^T \nabla w(x)\rangle \,\mathrm{d}x& \\ = a_{\epsilon_m}(w_j^{\epsilon_m}, w) = \lambda_j^{\epsilon_m} b_{\epsilon_m}(w_j^{\epsilon_m}, w) &= \lambda_j^{\epsilon_m} \int_\Omega J_{\epsilon_m}(x) w_j^{\epsilon_m}(x) w(x) \,\mathrm{d}x. \end{aligned} \tag{62}$$

Since $w_j^{\epsilon_m} \rightharpoonup w_j$ in $H^1(\Omega)$, $S_{\epsilon_m}(x)^T \to S_0(x)^T$ in $\mathcal{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)$ uniformly on $\overline{\Omega}$, $J_{\epsilon_m}(x) \to J_0(x)$ in $\mathbb{R}$ uniformly on $\overline{\Omega}$, $\lambda_j^{\epsilon_m} \to \mu_j$ and $w_j^{\epsilon_m} \to w_j$ in $L^2(\Omega)$ as $m \to \infty$, we obtain that

$$a_0(w_j, w) = \int_\Omega J_0(x)\langle S_0(x)^T \nabla w_j(x), S_0(x)^T \nabla w(x)\rangle \,\mathrm{d}x = \mu_j b_0(w_j, w),$$

i.e. $(\mu_j, w_j)$ is an eigenvalue-eigenvector pair of $(a_0, b_0)$.

Now we prove that $\|w_j^{\epsilon_m} - w_j\|_{\epsilon_m} \to 0$ as $m \to \infty$. We have

$$\begin{aligned}
&\|w_j^{\epsilon_m} - w_j\|^2_{\epsilon_m} \\
&= a_{\epsilon_m}(w_j^{\epsilon_m} - w_j, w_j^{\epsilon_m} - w_j) + b_{\epsilon_m}(w_j^{\epsilon_m} - w_j, w_j^{\epsilon_m} - w_j) \\
&= \lambda_j^{\epsilon_m} - 2a_{\epsilon_m}(w_j^{\epsilon_m}, w_j) + a_{\epsilon_m}(w_j, w_j) + b_{\epsilon_m}(w_j^{\epsilon_m} - w_j, w_j^{\epsilon_m} - w_j) \\
&= \lambda_j^{\epsilon_m} - 2\lambda_j^{\epsilon_m} b_{\epsilon_m}(w_j^{\epsilon_m}, w_j) + a_{\epsilon_m}(w_j, w_j) + b_{\epsilon_m}(w_j^{\epsilon_m} - w_j, w_j^{\epsilon_m} - w_j).
\end{aligned} \tag{63}$$

But now observe that $b_{\epsilon_m}(w_j^{\epsilon_m}, w_j) \to b_0(w_j, w_j) = 1$, $\lambda_j^{\epsilon_n} \to \mu_j$ and $a_{\epsilon_m}(w_j, w_j) \to a_0(w_j, w_j) = \mu_j$ as $m \to \infty$. Moreover,

$$b_{\epsilon_m}(w_j^{\epsilon_m} - w_j, w_j^{\epsilon_m} - w_j) \le (1 + \delta_m) b_0(w_j^{\epsilon_m} - w_j, w_j^{\epsilon_m} - w_j) \to 0$$

as $m \to \infty$. These properties together imply that $\|w_j^{\epsilon_m} - w_j\|_{\epsilon_m} \to 0$ as $m \to \infty$. In particular, by (54), we have that $w_j^{\epsilon_m} \to w_j$ in $H^1$ as $m \to \infty$.

By the Cantor diagonal procedure, given a sequence $(\epsilon_m)_{m\in\mathbb{N}}$ of positive numbers converging to zero, we can find a subsequence of $(\epsilon_m)_{m\in\mathbb{N}}$ (again denoted by $(\epsilon_m)_{m\in\mathbb{N}}$) and a family $(w_j)_{j\in\mathbb{N}}$ of functions in $H^1_s$ with the following properties:

(1) for every $j \in \mathbb{N}$, $(\mu_j, w_j)$ is an eigenvalue-eigenvector pair of $(a_0, b_0)$;
(2) $b_0(w_j, w_j) = 1$;
(3) $\|w_j^{\epsilon_m} - w_j\|_{\epsilon_m} \to 0$ as $m \to \infty$.

In order to complete the proof, we have to show the following:

(1) the family $(w_j)_{j\in\mathbb{N}}$ is a complete $b_0$-orthonormal system in $L^2_s$;
(2) $\mu_j = \lambda_j^0$ for all $j \in \mathbb{N}$.

First we show that $(w_j)_{j\in\mathbb{N}}$ is a $b_0$-orthonormal system in $L^2_s$. Let $j_1, j_2 \in \mathbb{N}$, $j_1 \neq j_2$. Then

$$\int_\Omega J_{\epsilon_m}(x) w_{j_1}^{\epsilon_m}(x) w_{j_2}^{\epsilon_m}(x) \,\mathrm{d}x = 0 \quad \text{for all } m \in \mathbb{N}.$$

Since $w_j^{\epsilon_m} \to w_j$ in $L^2$ as $m \to \infty$ for all $j \in \mathbb{N}$ and $J_{\epsilon_m} \to J_0$ uniformly on $\overline{\Omega}$, it follows that

$$\int_\Omega J_0(x) w_{j_1}(x) w_{j_2}(x) \,\mathrm{d}x = 0,$$

that is $w_{j_1}$ and $w_{j_2}$ are $b_0$-orthogonal. By [21, Proposition 2.2], the proof will be complete if we show that, for every $j \in \mathbb{N}$,

$$\begin{aligned}
\mu_j &= a_0(w_j, w_j) \\
&= \min\{\, a_0(w, w) \mid w \in H^1_s(\Omega), b_0(w, w) = 1, b_0(w, w_h) = 0 \text{ for } h = 1, \ldots, j-1 \,\}.
\end{aligned} \tag{64}$$

Fix $j \in \mathbb{N}$, and let $w \in H^1_s$ be arbitrary with $b_0(w,w) = 1$, such that $b_0(w, w_h) = 0$ for $h = 1,\dots, j-1$. For $m \in \mathbb{N}$, define

$$v_m := w - \sum_{h=1}^{j-1} b_{\epsilon_m}(w, w_h^{\epsilon_m}) w_h^{\epsilon_m}.$$

Then obviously $b_{\epsilon_m}(v_m, w_h^{\epsilon_m}) = 0$ for $h = 1,\dots, j-1$. We claim that $\|v_m - w\|_{\epsilon_m} \to 0$ as $m \to \infty$. In fact,

$$\begin{aligned}
\|v_n - w\|^2_{\epsilon_m} &= \left\| \sum_{h=1}^{j-1} b_{\epsilon_m}(w, w_h^{\epsilon_m}) w_h^{\epsilon_m} \right\|^2_{\epsilon_m} \\
&= a_{\epsilon_m}\left( \sum_{h=1}^{j-1} b_{\epsilon_m}(w, w_h^{\epsilon_m}) w_h^{\epsilon_m}, \sum_{h=1}^{j-1} b(w, w_h^{\epsilon_m}) w_h^{\epsilon_m} \right) + \left| \sum_{h=1}^{j-1} b_{\epsilon_m}(w, w_h^{\epsilon_n}) w_h^{\epsilon_n} \right|^2_{\epsilon_m} \\
&= \sum_{h,h'=1}^{j-1} b_{\epsilon_m}(w, w_h^{\epsilon_m}) b_{\epsilon_m}(w, w_{h'}^{\epsilon_m}) a_{\epsilon_m}(w_h^{\epsilon_m}, w_{h'}^{\epsilon_m}) + \sum_{h=1}^{j-1} |b_{\epsilon_m}(w, w_h^{\epsilon_m})|^2 \\
&= \sum_{h,h'=1}^{j-1} b_{\epsilon_m}(w, w_h^{\epsilon_m}) b_{\epsilon_m}(w, w_{h'}^{\epsilon_m}) \lambda_h^{\epsilon_m} b_{\epsilon_m}(w_h^{\epsilon_m}, w_{h'}^{\epsilon_m}) + \sum_{h=1}^{j-1} |b_{\epsilon_m}(w, w_h^{\epsilon_m})|^2.
\end{aligned} \tag{65}$$

Since $w_h^{\epsilon_m} \to w_h$ in $L^2$ as $m \to \infty$ for every $h = 1,\dots, j-1$, since $J_{\epsilon_m} \to J_0$ uniformly on $\overline{\Omega}$ and since $b_0(w, w_h) = 0$ for $h = 1,\dots, j-1$, it follows that $\|v_m - w\|_{\epsilon_m} \to 0$ as $m \to \infty$ and the claim is proved. In view of (54), it follows that $v_m \to w$ in $L^2$, so that $b_{\epsilon_m}(v_m, v_m) = |v_m|_{\epsilon_m} \to 1$ as $m \to \infty$. We can therefore assume that $b_{\epsilon_m}(v_m, v_m) \neq 0$ for all $m \in \mathbb{N}$. Define

$$w_m := |v_m|^{-1}_{\epsilon_m} v_m.$$

We have $|w_m|_{\epsilon_m} = 1$ and $b_{\epsilon_m}(w_m, w_h^{\epsilon_m}) = 0$ for $h = 1,\dots, j-1$. Moreover,

$$\begin{aligned}
&\|w_n - w\|_{\epsilon_m} \\
&= \left\| |v_m|^{-1}_{\epsilon_m} v_m - w \right\|_{\epsilon_m} \\
&= |v_n|^{-1}_{\epsilon_m} \|v_m - |v_m|_{\epsilon_m} w\|_{\epsilon_m} \\
&= |v_m|^{-1}_{\epsilon_m} \left( \|v_m - w\|_{\epsilon_m} + \|w - |v_m|_{\epsilon_m} w\|_{\epsilon_m} \right) \\
&= |v_m|^{-1}_{\epsilon_m} \left( \|v_m - w\|_{\epsilon_m} + (1 - |v_m|_{\epsilon_m}) \|w\|_{\epsilon_m} \right).
\end{aligned} \tag{66}$$

Since $|v_m|_{\epsilon_m} \to 1$ and $\|w\|_{\epsilon_m} \to \|w\|_0$ (as $w \in H^1_s$), it follows that

$$\|w_m - w\|_{\epsilon_m} \to 0 \quad \text{as } m \to \infty.$$

Using Lemma 3.9 we obtain

$$a_{\epsilon_m}(w_m, w_m) \to a_0(w, w) \quad \text{as } m \to \infty.$$

We already know that

$$a_{\epsilon_m}(w_j^{\epsilon_m}, w_j^{\epsilon_m}) = \lambda_j^{\epsilon_m} \to \mu_j = a_0(w_j, w_j) \quad \text{as } m \to \infty.$$

Moreover, for every $m$ we have

$$a_{\epsilon_m}(w_j^{\epsilon_m}, w_j^{\epsilon_m}) \le a_{\epsilon_m}(w_m, w_m),$$

as $(w_j^{\epsilon_m})_{j\in\mathbb{N}}$ is a complete $b_{\epsilon_m}$-orthonormal system of eigenvectors for $(a_{\epsilon_m}, b_{\epsilon_m})$. By letting $m \to \infty$, we finally obtain

$$a_0(w_j, w_j) \le a_0(w, w).$$

The theorem is proved. □

## 4. Applications to reaction-diffusion equations

In this section we will apply our preceding results to reaction-diffusion equations on the squeezed domains $\Omega_\epsilon = \Phi_\epsilon(\Omega)$. We will, in particular, show that, as $\epsilon \to 0^+$, these equations converge, in some singular sense, to a limit equation. Under some dissipativity condition we will also establish an upper semicontinuity result for the resulting family of global attractors.

Let us first recall the concept of a semiflow:

Let $X$ be a topological space, let $D$ be an open subset of $[0,\infty[ \times X$ and $\pi\colon D \to X$ be a continuous map. We write $x\pi t := \pi(t,x)$ for $(t,x) \in D$. The map $\pi$ is called a *local semiflow on $X$* if the following properties are satisfied:

(1) For every $x \in X$ there is a number $\omega_x = \omega_x^\pi \in ]0,\infty]$ such that $(t,x) \in D$ if and only if $0 \le t < \omega_x$.
(2) $x\pi 0 = x$ for all $x \in X$.
(3) If $(t,x) \in D$ and $(s, x\pi t) \in D$ then $(t+s, x) \in D$ and

$$x\pi(t+s) = (x\pi t)\pi s.$$

If $\omega_x = \infty$ for every $x \in X$, then $\pi$ is called a *global semiflow on $X$*.

Let $J$ be an arbitrary interval in $\mathbb{R}$. A map $\sigma\colon J \to X$ is called a *solution of $\pi$* if for all $t \in J$ and $s \in [0,\infty[$ for which $t+s \in J$ , it follows that $\sigma(t)\pi s$ is defined and $\sigma(t)\pi s = \sigma(t+s)$. If $0 \in J$ and $\sigma(0) = x$, we say that $\sigma$ is a *solution through $x$*. The image $\sigma(J)$ of a solution is called an *orbit of $\pi$*. If $J = \mathbb{R}$ then $\sigma$ is called a *full solution* relative to $\pi$ and its image is called a *full orbit of $\pi$*.

**Example 4.1.** *Let $X$ be a Banach space and $A$ be a sectorial operator in $X$ generating the family $X^\beta$, $\beta \ge 0$, of fractional power spaces. Fix an $\alpha \in [0,1[$ and suppose $f\colon X^\alpha \to X$ is a locally Lipschitzian map. The equation*

$$\dot u = -Au + f(u)$$

*defines, in the usual way, a local semiflow $\pi_{A,f}$ on $X^\alpha$. (see* [17] *or* [26]*). If $f$ is globally Lipschitzian on $X^\alpha$, then $\pi_{A,f}$ is a global semiflow.*

Let $U$ be a nonempty bounded open subset of $\mathbb{R}^\ell$ with Lipschitz boundary. Let $A_U$ be the operator defined by the pair $(a,b)$, where $a\colon H^1(U) \times H^1(U) \to \mathbb{R}$, is defined by

$$a(u,v) := \int_U \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x. \tag{67}$$

Then $A_U$ is selfadjoint on $X = L^2(U)$. In particular, $A_U$ is sectorial on $X$ and $X^{1/2} = H^1(U)$ with equivalent norms. We interpret $A_U$ as the *weak Laplacian on $U$ with Neumann boundary condition.* More generally, consider the following reaction-diffusion equation on

$U$:

$$\begin{aligned} u_t &= \Delta u + f(u), && t > 0,\ x \in U, \\ \partial_\nu u &= 0, && t > 0,\ x \in \partial U, \end{aligned} \tag{68}$$

where $\nu$ is the exterior normal vector field on $\partial U$ and $f\colon H^1(U) \to L^2(U)$ is a locally Lipschitzian map. We interpret equation (68) as being *equivalent* to the abstract parabolic equation

$$\dot{u} + A_U u = f(u) \tag{69}$$

on $H^1(U)$.

In particular, using the notation of Section 3 we see that, for $\epsilon \in ]0,1]$, $\tilde{A}_\epsilon$ is the weak Laplacian on $\Omega_\epsilon$ with Neumann boundary condition. Moreover, the operator $A_\epsilon$ is sectorial on $X = L^2(\Omega)$ and the corresponding fractional power space $X^\alpha$ with $\alpha = 1/2$ satisfies $X^\alpha = H^1(\Omega)$. If $f_\epsilon\colon H^1(\Omega) \to L^2(\Omega)$ (resp. $\tilde{f}_\epsilon\colon H^1(\Omega_\epsilon) \to L^2(\Omega_\epsilon)$) is a locally Lipschitzian map we thus obtain the corresponding local semiflow $\pi_{\epsilon,f_\epsilon} := \pi_{A_\epsilon,f_\epsilon}$ (resp. $\tilde{\pi}_{\epsilon,\tilde{f}_\epsilon} := \pi_{\tilde{A}_\epsilon,\tilde{f}_\epsilon}$) on $H^1(\Omega)$ (resp. on $H^1(\Omega_\epsilon)$). Note that, given a locally Lipschitzian map $\tilde{f}_\epsilon\colon H^1(\Omega_\epsilon) \to L^2(\Omega_\epsilon)$, the linear transformation $\Phi_\epsilon^*(u) := u \circ \Phi_\epsilon$ *conjugates* the semiflow $\tilde{\pi}_{\epsilon,\tilde{f}_\epsilon}$ with the semiflow $\pi_{\epsilon,f_\epsilon}$ where $f_\epsilon := \Phi_\epsilon^* \circ \tilde{f}_\epsilon \circ (\Phi_\epsilon^*)^{-1}$.

In particular, let $g_\epsilon\colon \Omega_\epsilon \times \mathbb{R} \to \mathbb{R}$ be a given function and $\tilde{f}_\epsilon := \hat{g}_\epsilon$ be the Nemitski operator generated by $g_\epsilon$, i.e. for $u\colon \Omega_\epsilon \to \mathbb{R}$ set

$$\tilde{f}_\epsilon(u)(x) := g_\epsilon(x, u(x)), \text{ for } x \in \Omega_\epsilon.$$

Suppose $\tilde{f}_\epsilon$ restricts to a locally Lipschitzian map from $H^1(\Omega_\epsilon)$ to $L^2(\Omega_\epsilon)$. Then the map $f_\epsilon := \Phi_\epsilon^* \circ \tilde{f}_\epsilon \circ (\Phi_\epsilon^*)^{-1}$ is clearly given by

$$f_\epsilon(u)(x) := g_\epsilon(\Phi_\epsilon(x), u(x)), \text{ for } u \in H^1(\Omega) \text{ and } x \in \Omega. \tag{70}$$

Now note that the "limit" operator $A_0$ is sectorial on $X = L_s^2(\Omega)$ and the corresponding fractional power space $X^\alpha$ with $\alpha = 1/2$ satisfies $X^\alpha = H_s^1(\Omega)$. If $f_0\colon H_s^1(\Omega) \to L_s^2(\Omega)$ is a locally Lipschitzian map we thus obtain the corresponding local semiflow $\pi_{0,f_0} := \pi_{A_0,f_0}$ on $H_s^1(\Omega)$. Again, if $f_0$ is globally Lipschitzian, then $\pi_{0,f_0}$ is a global semiflow.

Finally, note that, for $\epsilon \in ]0,1]$, the operator $A_\epsilon$ generates a $C^0$-semigroup $e^{-tA_\epsilon}$, $t \in [0,\infty[$, of linear operators on $L^2(\Omega)$, while the operator $A_0$ generates a $C^0$-semigroup $e^{-tA_0}$, $t \in [0,\infty[$, of linear operators on $L_s^2(\Omega)$.

We can now state the following linear singular convergence result, extending, to curved squeezing, [21, Theorem 4.1]:

**Theorem 4.2.** *Let $(\epsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers converging to zero. Moreover, assume that $u \in L_s^2(\Omega)$ and $(u_n)_{n\in\mathbb{N}}$ is a sequence in $L^2(\Omega)$ such that*

$$|u_n - u|_{L^2} \to 0 \quad \text{as } n \to \infty.$$

*Then for all $\beta$ and $\gamma \in ]0,\infty[$ with $\beta < \gamma$*

$$\sup_{t\in[\beta,\gamma]} \left\| e^{-tA_{\epsilon_n}} u_n - e^{-tA_0} u \right\|_{\epsilon_n} \to 0 \quad \text{as } n \to \infty.$$

*Proof.* Let $\delta > 0$ be arbitrary. Then there is an $s_0 = s_0(\delta, \beta) > 0$ such that $(s+1)e^{-st} < \delta$ for $s \geq s_0$ and $t \geq \beta$. There is a $j_0 = j_0(\delta, \beta)$ such that $\lambda_{j_0} > s_0$. Thus there is an $n_0 = n_0(\delta, \beta)$ such that $\lambda_{j_0}^{\epsilon_n} > s_0$ for $n \geq n_0$. Therefore we obtain

$$\lambda_j^{\epsilon_n} \geq s_0(\delta, \beta) \quad \text{for } n \geq n_0(\delta, \beta) \text{ and } j \geq j_0(\delta, \beta). \tag{71}$$

Let $P_n \colon L^2(\Omega) \to L^2(\Omega)$ be the $b_{\epsilon_n}$-orthogonal projection of $L^2(\Omega)$ onto the span of $\{w_1^{\epsilon_n}, \dots, w_{j_0-1}^{\epsilon_n}\}$. Let $P \colon L^2_s(\Omega) \to L^2_s(\Omega)$ be the $b_0$-orthogonal projection of $L^2_s(\Omega)$ onto the span of $\{w_1, \dots, w_{j_0-1}\}$. Let $t \in [\beta, \gamma]$ be arbitrary. Then, for all $n \in \mathbb{N}$ and $u \in L^2(\Omega)$

$$P_n e^{-tA_{\epsilon_n}} u = e^{-tA_{\epsilon_n}} P_n u \in D(A_{\epsilon_n}) \subset H^1(\Omega), \tag{72}$$

$$(I - P_n) e^{-tA_{\epsilon_n}} u = e^{-tA_{\epsilon_n}} (I - P_n) u \in D(A_{\epsilon_n}) \subset H^1(\Omega). \tag{73}$$

Analogously, for all $u \in L^2_s(\Omega)$,

$$P e^{-tA_0} u = e^{-tA_0} P u \in D(A_0) \subset H^1_s(\Omega), \tag{74}$$

$$(I - P) e^{-tA_0} u = e^{-tA_0} (I - P) u \in D(A_0) \subset H^1_s(\Omega). \tag{75}$$

It follows that

$$\begin{aligned} \|e^{-tA_{\epsilon_n}} u_n - e^{-tA} u\|_{\epsilon_n} &\leq \|P_n e^{-tA_{\epsilon_n}} u_n - P e^{-tA} u\|_{\epsilon_n} \\ &\quad + \|(I - P_n) e^{-tA_{\epsilon_n}} u_n\|_{\epsilon_n} + \|(I - P) e^{-tA_0} u\|_{\epsilon_n}. \end{aligned} \tag{76}$$

For every $\epsilon \in [0, 1]$ let $(\lambda_j^\epsilon)_{j \in \mathbb{N}}$ be the repeated sequence of eigenvalues of $A_\epsilon$ and $(w_j^\epsilon)_{j \in \mathbb{N}}$ be a corresponding $L^2$-orthonormal system of eigenvectors. Write $\lambda_j$ and $w_j$ for $\lambda_j^0$ and $w_j^0$, respectively. Then

$$\begin{aligned} \|P_n e^{-tA_{\epsilon_n}} u_n - P e^{-tA_0} u\|_{\epsilon_n} &\leq \sum_{k=1}^{j_0-1} \|e^{-t\lambda_k^{\epsilon_n}} b_{\epsilon_n}(u_n, w_k^{\epsilon_n}) w_k^{\epsilon_n} - e^{-t\lambda_k} b_0(u, w_k) w_k\|_{\epsilon_n} \\ \leq \sum_{k=1}^{j_0-1} (|e^{-t\lambda_k^{\epsilon_n}} b_{\epsilon_n}(u_n, w_k^{\epsilon_n})| \|w_k^{\epsilon_n} - w_k\|_{\epsilon_n} &+ |e^{-t\lambda_k^{\epsilon_n}} b_{\epsilon_n}(u_n, w_k^{\epsilon_n}) - e^{-t\lambda_k} b_0(u, w_k)| \|w_k\|_{\epsilon_n}). \end{aligned} \tag{77}$$

Since $w_k^{\epsilon_n} \to w_k$ and $u_n \to u$ in $L^2(\Omega)$ it follows that $b_{\epsilon_n}(u_n, w_k^{\epsilon_n}) \to b_0(u, w_k)$ as $n \to \infty$, for $k = 1, \dots, j_0 - 1$. Moreover since $\lambda_k^{\epsilon_n} \to \lambda_k$ as $n \to \infty$ we obtain that

$$\sup_{t \in [\beta, \gamma]} |e^{-t\lambda_k^{\epsilon_n}} - e^{-t\lambda_k}| \to 0 \tag{78}$$

as $n \to \infty$, for $k = 1, \dots, j_0 - 1$. Furthermore $\|w_k^{\epsilon_n} - w_k\|_{\epsilon_n} \to 0$ as $n \to \infty$. Finally, $\|w_k\|^2_{\epsilon_n} \leq C' \|w_k\|^2_0$ for some constant $C' \in ]0, \infty[$ independent of $n \in \mathbb{N}$. All this, together with (77), implies that

$$\sup_{t \in [\beta, \gamma]} \|P_n e^{-tA_{\epsilon_n}} u_n - P e^{-tA_0} u\|_{\epsilon_n} \to 0 \quad \text{as } n \to \infty. \tag{79}$$

Now

$$\begin{aligned}(80)\quad \|(I-P_n)e^{-tA_{\epsilon_n}}u_n\|_{\epsilon_n}^2 &= \sum_{j=1}^{\infty}(\lambda_j^{\epsilon_n}+1)|b_{\epsilon_n}((I-P_n)e^{-tA_{\epsilon_n}}u_n, w_j^{\epsilon_n})|^2 \\ = \sum_{j=j_0}^{\infty}(\lambda_j^{\epsilon_n}+1)|b_{\epsilon_n}(e^{-tA_{\epsilon_n}}u_n, w_j^{\epsilon_n})|^2 &= \sum_{j=j_0}^{\infty}(\lambda_j^{\epsilon_n}+1)(e^{-t\lambda_j^{\epsilon_n}})^2|b_{\epsilon_n}(u_n, w_j^{\epsilon_n})|^2 \\ &\le \delta b_{\epsilon_n}(u_n,u_n) \le \delta c^{-1}C'' < \infty.\end{aligned}$$

Here $C'' := \sup_{n\in\mathbb{N}}|u_n|_{L^2}^2 < \infty$. We have used (43) and (71) above. Finally, since $(I-P)e^{-tA_0}u \in H_s^1(\Omega)$, it follows that

$$\begin{aligned}(81)\quad \|(I-P)e^{-tA_0}u\|_{\epsilon_n}^2 \le C'\|(I-P)e^{-tA_0}u\|_0^2 &= C'\sum_{j=1}^{\infty}(\lambda_j+1)|b_0((I-P)e^{-tA_0}u, w_j)|^2 \\ &\le C'\delta|u|_{L^2}^2 \le \delta C'C'',\end{aligned}$$

by the same argument as in (80). Since $\delta$ is arbitrary, we conclude from (71), (79), (80) and (81) that

$$\sup_{t\in[\beta,\gamma]}\left\|e^{-tA_{\epsilon_n}}u_n - e^{-tA_0}u\right\|_{\epsilon_n} \to 0 \quad \text{as } n\to\infty.$$

The theorem is proved. □

The following concept, introduced in [6], plays a crucial role in the nonlinear singular convergence result established below:

**Definition 4.3.** *Given $\epsilon_0$ with $0 < \epsilon_0 \le 1$ we say that the family $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ of maps* satisfies hypothesis $(A1)$ *if the following properties are satisfied:*

1. *$f_\epsilon\colon H^1(\Omega)\to L^2(\Omega)$ for every $\epsilon\in\,]0,\epsilon_0]$ and $f_0\colon H_s^1(\Omega)\to L_s^2(\Omega)$.*
2. *$\lim_{\epsilon\to 0^+}|f_\epsilon(u)-f_0(u)|_{L^2} = 0$ for every $u\in H_s^1(\Omega)$.*
3. *For every $M\in[0,\infty[$ there is an $L = L_M\in[0,\infty[$ such that*

$$|f_\epsilon(u)-f_\epsilon(v)|_{L^2} \le L\|u-v\|_\epsilon$$

*for $\epsilon\in\,]0,\epsilon_0]$ and $u$, $v\in H^1(\Omega)$ satisfying $\|u\|_\epsilon$, $\|v\|_\epsilon\le M$. Moreover,*

$$|f_0(u)-f_0(v)|_{L^2} \le L\|u-v\|_0$$

*for $u$, $v\in H_s^1(\Omega)$ satisfying $\|u\|_0$, $\|v\|_0\le M$.*

The following simple extension of [6, Proposition 2.6] shows how we can obtain, in applications, families of maps satisfying hypothesis $(A1)$:

**Proposition 4.4.** *Let $G\colon\mathbb{R}\times\mathbb{R}^\ell\times\mathbb{R}\to\mathbb{R}$, $(\epsilon,x,\xi)\mapsto G(\epsilon,x,\xi)$, be a $C^1$-function for which there are constants $\beta$, $\delta$ and $C\in[0,\infty[$ such that for all $(\epsilon,x,\xi)\in\mathbb{R}\times\mathbb{R}^\ell\times\mathbb{R}$ the following estimates are satisfied:*

1. *$|\partial_\epsilon G(\epsilon,x,\xi)| \le C(1+|\xi|^\beta)$;*
2. *$|\nabla_x G(\epsilon,x,\xi)| \le C(1+|\xi|^\beta)$;*
3. *$|\partial_\xi G(\epsilon,x,\xi)| \le C(1+|\xi|^\delta)$.*

*If $\ell > 2$ then we also assume that $\beta \le 2^*/2$ and $\delta \le (2^*/2) - 1$, where $2^* := 2\ell/(\ell-2)$.*

*For $\epsilon > 0$ define the map $f_\epsilon$ by*

$$f_\epsilon(u)(x) := G(\epsilon, \Phi_\epsilon(x), u(x)),\ \textit{for } u \in H^1(\Omega) \textit{ and } x \in \Omega.$$

*Furthermore, for $u \in H^1_s(\Omega)$ define $f_0(u)\colon \Omega \to \mathbb{R}$ by*

$$f_0(u)(x) := G(0, \phi(x), u(x)) \textit{ for } x \in \Omega.$$

*Then $f_0(u) \in L^2_s(\Omega)$ and the family $(f_\epsilon)_{\epsilon\in[0,1]}$ satisfies hypothesis $(A1)$.*

*Proof.* That $f_0$ maps $H^1_s(\Omega)$ into $L^2_s(\Omega)$ follows by a modification of the proof of [21, Theorem 5.3]. (Cf also the proof of Proposition 3.5 in Section 3.) All the other assertions follow by an application of the Sobolew imbedding theorems, the Hölder inequality and the mean value theorem. The details are left to the reader. □

**Remark 4.5.** *Proposition 4.4 can be generalized to functions $G$ taking values in $\mathbb{R}^p$ and having the argument $\xi$ lying in $\mathbb{R}^p$. This allows applications to* systems *of reaction-diffusion equations.*

We can also state the following nonlinear singular convergence theorem, generalizing [21, Theorem 5.1] and [6, Theorem 2.13]:

**Theorem 4.6.** *Let $(f_\epsilon)_{\epsilon\in[0,1]}$ satisfy hypothesis $(A1)$ and $(\epsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers convergent to zero. Write $\pi_n := \pi_{A_{\epsilon_n}, f_{\epsilon_n}}$ and $\pi := \pi_{A_0,f_0}$. Assume that $u \in H^1_s(\Omega)$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $H^1(\Omega)$ such that*

$$|u_n - u|_{L^2} \to 0 \quad \textit{as } n \to \infty.$$

*Let $b \in ]0,\infty[$ and suppose that $u_n\pi_n s$ and $u\pi s$ are defined for all $s \in [0,b]$, all $n \in \mathbb{N}$ and*

$$\sup_{n\in\mathbb{N}} \sup_{s\in[0,b]} \|u_n\pi s\|_{\epsilon_n} \le M \textit{ and } \sup_{n\in\mathbb{N}} \sup_{s\in[0,b]} \|u\pi s\|_{\epsilon_n} \le M$$

*for some constant $M \in [0,\infty[$.*

*Finally let $t_0 \in ]0,b]$ and $(t_n)_{n\in\mathbb{N}}$ is a sequence in $]0,b]$ converging to $t_0$.*

*Under these assumptions*

$$\left\|u_n\pi_n t_n - u\pi t_0\right\|_{\epsilon_n} \to 0 \quad \textit{as } n \to \infty.$$

*Proof.* Let us first note that there is a positive real constant $C_1$ such that for all $\epsilon \in ]0,1]$, $r \in ]0,\infty[$ and $v \in L^2(\Omega)$

$$\|e^{-A_\epsilon r}v\|_\epsilon \le C_1(r^{-1/2}+1)|v|_{L^2} \tag{82}$$

and, for all $v \in L^2_s(\Omega)$

$$\|e^{-A_0 r}v\|_\epsilon \le C_1(r^{-1/2}+1)|v|_{L^2}. \tag{83}$$

Let $L = L_M$ be as in Condition $(A1)$. For every $t \in [0,b]$ we have, by the variation-of-constants formula,

$$u_n\pi_n t - u\pi t = e^{-A_{\epsilon_n}t}u_n - e^{-A_0 t}u +$$
$$\int_0^t e^{-A_{\epsilon_n}(t-s)}\big(f_{\epsilon_n}(u_n\pi_n s) - f_{\epsilon_n}(u\pi s)\big)\,\mathrm{d}s + \int_0^t \big(e^{-A_{\epsilon_n}(t-s)}f_{\epsilon_n}(u\pi s) - e^{-A_0(t-s)}f_0(u\pi s)\big)\,\mathrm{d}s.$$

Define the function $g_n\colon [0,b]\times[0,b] \to \mathbb{R}$ as follows: if $0 < s < t$ then set

$$g_n(t,s) = \|e^{-A_{\epsilon_n}(t-s)}f_{\epsilon_n}(u\pi s) - e^{-A_0(t-s)}f_0(u\pi s)\|_{\epsilon_n}$$

and set $g_n(t,s)=0$ otherwise. The function $g_n$ restricted to the set of $(s,t)$ with $0<s<t$ is continuous by Condition $(A1)$ and Theorem 4.2. Thus $g_n$ is measurable on $[0,b]\times[0,b]$. By Fubini's theorem the function

$$c_n(t) := \int_0^b g_n(t,s)\ \mathrm{d}s = \int_0^t g_n(t,s)\ \mathrm{d}s$$

is a.e. defined and measurable on $[0,b]$. Set

$$a_n(t) := \|e^{-A_{\epsilon_n}t}u_n - e^{-A_0 t}u\|_{\epsilon_n} + c_n(t) \quad \text{for } t\in\,]0,b]$$

and $a_n(0):=0$. It follows that $a_n$ is measurable on $[0,b]$. Using the estimates (82) and (83) we obtain

$$|g_n(t,s)| \le 2C_2C_1\big((t-s)^{-1/2}+1\big) \quad \text{whenever } 0<s<t\le b$$

where

$$C_2 := \max\{\sup_{s\in[0,b]}\sup_{n\in\mathbb{N}}|f_{\epsilon_n}(u\pi s)|_{L^2},\ \sup_{s\in[0,b]}|f_0(u\pi s)|_{L^2}\}.$$

Let $L=L_M$ be as in Condition $(A1)$. Note that, by hypothesis $(A1)$

$$|f_{\epsilon_n}(u\pi s)|_{L^2} \le |f_{\epsilon_n}(u\pi s)-f_{\epsilon_n}(u)|_{L^2}+|f_{\epsilon_n}(u)|_{L^2} \le L|u\pi s-u|_{\epsilon_n}+|f_{\epsilon_n}(u)|_{L^2}\le M'$$

for some constant $M'<\infty$, independent of $n\in\mathbb{N}$ and $s\in[0,b]$. Similarly, we may assume that

$$|f_0(u\pi s)|_{L^2}\le M',\ s\in[0,b].$$

This shows that $C_2<\infty$.

If $0<s<t_0$ then for some $n_0$ and some $\beta>0$, $t_n-s>\beta$ for $n\ge n_0$. By Theorem 4.2 $g_n(t_n,s)\to 0$. If $0<t<s$ then for some $n_0$, $t_n<s$ and so $g_n(t_n,s)=0$ for all $n\ge n_0$. Again $g_n(t_n,s)\to 0$. It follows from the dominated convergence theorem that

$$c_n(t_n)\to 0.$$

Thus, again using Theorem 4.2 we obtain

$$a_n(t_n)\to 0.$$

In particular,

$$a_n(t)\to 0 \quad \text{for all } t\in\,]0,b]. \tag{84}$$

Furthermore, the definition of $a_n$ clearly implies that

$$a_n(t)\le C_3(t^{-1/2}+1) \quad \text{for } t\in\,]0,b].$$

An application of Henry's Inequality ([17, Lemma 7.1.1]) implies that

$$\|u_n\pi_n t-u\pi t\|_{\epsilon_n}\le a_n(t)+\int_0^t \rho(t-s)a_n(s)\ \mathrm{d}s \text{ for } t\in\,]0,b],$$

where

$$\rho(x) := \sum_{n=1}^{\infty}\frac{(L\Gamma(\beta))^n}{\Gamma(n\beta)}x^{n\beta-1}$$

with $\beta:=1/2$.

The function $\rho\colon\ ]0,\infty[\,\to\,]0,\infty[$ is well defined and continuous on $]0,\infty[$ and it satisfies the estimate

$$\rho(x)\le C_4x^{-1/2}+C_4 \quad \text{for } x\in\,]0,b].$$

Fix a $\delta_0$ with $0 < \delta_0 < t$ and let $\delta > 0$ with $2\delta < \delta_0$ be arbitrary. There is an $n_0 = n_0(\delta)$ such that $|t_n - t| < \delta$ for $n \geq n_0$. Therefore for all such $n$ and all $s \in [0, t - 2\delta]$ it follows that $t_n - s > \delta$ so $\rho(t_n - s) \leq C_4\delta^{-1/2} + C_4$. Thus

$$\rho(t_n - s)a_n(s) \leq C_5(s^{-1/2} + 1) \quad \text{for } s \in \,]0, t - 2\delta].$$

Therefore (84) and the dominated convergence theorem show that

$$\int_0^{t-2\delta} \rho(t_n - s)a_n(s) \ \mathrm{d}s \to 0.$$

On the other hand, for $s \in [t - 2\delta, t_n]$ we have that $s \geq t - \delta_0 > 0$ so $a_n(s) \leq C_6$. Therefore

$$\int_{t-2\delta}^{t_n} \rho(t_n - s)a_n(s) \ \mathrm{d}s \leq C_7(\delta^{1/2} + \delta).$$

Since $\delta < \delta_0$ is arbitrary, it follows that

$$\int_0^{t_n} \rho(t_n - s)a_n(s) \ \mathrm{d}s \to 0.$$

Consequently,

$$\|u_n\pi_n t_n - u\pi t_n\|_{\epsilon_n} \to 0.$$

Since, for some constant $C' \in \,]0, \infty[$ independent of $n \in \mathbb{N}$,

$$\|u\pi t_n - u\pi t_0\|_{\epsilon_n} \leq C'\|u\pi t_n - u\pi t_0\|_0 \to 0$$

we obtain that

$$\|u_n\pi_n t_n - u\pi t_0\|_{\epsilon_n} \to 0.$$

The theorem is proved. □

**Corollary 4.7.** *Let $(f_\epsilon)_{\epsilon\in[0,1]}$ satisfy hypothesis (A1) and $(\epsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers convergent to zero. Write $\pi_n := \pi_{A_{\epsilon_n}, f_{\epsilon_n}}$ and $\pi := \pi_{A_0, f_0}$. Let $C \in [0, \infty[$ and for every $n \in \mathbb{N}$ let $\sigma_n \colon \mathbb{R} \to H^1(\Omega)$ be a solution of $\pi_n$ such that*

$$\sup_{t\in\mathbb{R}} \|\sigma_n(t)\|_{\epsilon_n} \leq C.$$

*Assume also that $\pi$ is a global semiflow.*

*Under these hypotheses, there is a subsequence of $(\sigma_n)_n$, denoted by the same symbol $(\sigma_n)_n$, and there is a solution $\sigma \colon \mathbb{R} \to H^1_s$ of $\pi$ such that*

$$\|\sigma_n(t) - \sigma(t)\|_\epsilon \to 0 \quad \textit{for every } t \in \mathbb{R}.$$

*Proof.* Let $(u_n)_n$ be an arbitrary sequence in $H^1(\Omega)$ such that

$$\sup_{n\in\mathbb{N}} \|u_n\|_{\epsilon_n} \leq C. \tag{85}$$

Then there is a subsequence of $(u_n)_n$, again denoted by $(u_n)_n$, such that $(u_n)_n$ converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to some $u \in H^1(\Omega)$. In view of (85) we have $P(\cdot)\nabla u_n \to 0$ in $L^2(\Omega)$, which easily implies that $P(\cdot)\nabla u = 0$, i.e. $u \in H^1_s(\Omega)$. Applying this to the sequences $(\sigma_n(-k))_n$ for every $k \in \mathbb{N}_0$ and using Cantor's diagonal procedure we easily obtain the existence of a subsequence of $(\sigma_n)_n$, denoted by the same symbol $(\sigma_n)_n$, and a sequence $v(-k) \in H^1_s(\Omega)$, $k \in \mathbb{N}_0$, such that for every $k \in \mathbb{N}_0$ the subsequence $(\sigma_n(-k))$ converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to $v(-k)$. If follows that

$$\tau_k(t) := v(-k)\pi(t + k)$$

is well-defined for all $t \in [-k, \infty[$. For every $k \in \mathbb{N}_0$ and every $t \in ]-k, \infty[$

$$\sigma_n(t) = \sigma_n(-k)\pi_n(t+k)$$

so by Theorem 4.6

$$\|\sigma_n(t) - \tau_k(t)\|_{\epsilon_n} \to 0.$$

In particular, if $l$ and $k \in \mathbb{N}_0$ with $l > k$, this implies that $\tau_l(t) = \tau_k(t)$ for $t \in [-k, \infty[$. It follows that there exists a unique function $\sigma\colon \mathbb{R} \to H^1_s(\Omega)$ such that $\sigma(t) = \tau_k(t)$ for $t \in [-k, \infty[$. Consequently, $\sigma$ is a solution of $\pi$ satisfying

$$\|\sigma_n(t) - \sigma(t)\|_{\epsilon_n} \to 0 \quad \text{for every } t \in \mathbb{R}.$$

The corollary is proved. □

**Remark 4.8.** *Note that Theorem 4.6 and its corollary remain valid, with obvious changes and the same proof, if the operators $A_\epsilon$, $\epsilon \in [0,1]$, are replaced by 'decoupled' systems of operators $B_\epsilon\colon (D(A_\epsilon))^p \to (L^2(\Omega))^p$, $\epsilon \in ]0,1]$, and $B_0\colon (D(A_0))^p \to (L^2_s(\Omega))^p$, with $B_\epsilon(u_1, \dots, u_p) := (d_1 A_\epsilon, \dots, d_p A_\epsilon)$, where $(d_1, \dots, d_p)$ is a fixed vector in $\mathbb{R}^p$ with positive components. Thus Theorem 4.6 and its corollary are applicable to systems of reaction-diffusion equations on squeezed domains. Cf also Remark 4.5.*

We will now show that, under the usual dissipativeness assumption, every semiflow $\pi_\epsilon$, $\epsilon \in [0,1]$, possesses a global attractor $\mathcal{A}_\epsilon$ and that this family of attractors is uppersemicontinuous at $\epsilon = 0$. This generalizes, to curved squeezing, the corresponding results from [21]. We assume that the reader is familiar with some basic theory of attractors for evolution equations, as expounded in the monographs [12], [19] or [9].

**Theorem 4.9.** *Assume that $G\colon \mathbb{R} \to \mathbb{R}$ is a $C^1$-function such that $|G'(s)| \le C(|s|^\beta + 1)$ for $s \in \mathbb{R}$, where $C$ and $\beta \in [0, \infty[$ are arbitrary real constants. If $\ell > 2$ then assume in addition that $\beta \le (2^*/2) - 1$, where $2^* = 2\ell/(\ell - 2) > 2$. Moreover, suppose that $G$ is* dissipative *in the sense that*

$$\limsup_{|s| \to \infty} G(s)/s \le -\zeta \text{ for some } \zeta > 0.$$

*For $\epsilon \in ]0,1]$ define the map $f_\epsilon\colon H^1(\Omega) \to L^2(\Omega)$ by $f_\epsilon(u) = G \circ u$. Furthermore, define the map $f_0\colon H^1_s(\Omega) \to L^2_s(\Omega)$ by $f_0(u) = G \circ u$. For $\epsilon \in [0,1]$ let $\pi_\epsilon := \pi_{A_\epsilon, f_\epsilon}$, and let $\mathcal{A}_\epsilon$ be the union of all full bounded orbits of $\pi_\epsilon$.*

*Then, for all $\epsilon \in [0,1]$, $\pi_\epsilon$ is a global semiflow and the set $\mathcal{A}_\epsilon$ is nonempty, compact, connected in $H^1(\Omega)$. Furthermore, the set $\mathcal{A}_\epsilon$* attracts *every set $B$ which is bounded in $H^1(\Omega)$ for $\epsilon \in ]0,1]$ and in $H^1_s(\Omega)$ for $\epsilon = 0$. In other words, for every such $B$*

$$\lim_{t \to \infty} \sup_{u \in B} \inf_{v \in \mathcal{A}_\epsilon} \|u\pi_\epsilon t - v\|_\epsilon = 0.$$

*The family $(\mathcal{A}_\epsilon)_{\epsilon \in [0, \epsilon_0]}$ is upper-semicontinuous at $\epsilon = 0$ with respect to the family $\|\cdot\|_\epsilon$ of norms, i.e.*

$$\lim_{\epsilon \to 0^+} \sup_{u \in \mathcal{A}_\epsilon} \inf_{v \in \mathcal{A}_0} \|u - v\|_\epsilon = 0.$$

*Proof.* Proposition 4.4 implies that the maps $f_\epsilon$ and the local semiflows $\pi_\epsilon$, $\epsilon \in [0,1]$, are well-defined. All the other assertions follow exactly like those of [21, Theorems 5.8 and 5.10]. In particular, the upper-semicontinuity of the attractor family follows by an application of Corollary 4.7. We omit the easy details. □

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