On the Construction of Some Buchsbaum Varieties and the Hilbert Scheme of Elliptic Scrolls in $\mathbb{P}^5$

DOLORES BAZAN (bazan@mathsun1.univ.trieste.it)
Via degli Orti 10, Gradisca (GO), Italy

EMILIA MEZZETTI (mezzette@univ.trieste.it)*
Università degli Studi di Trieste, Dipartimento di Scienze Matematiche, Via Valerio 12/1, 34127 Trieste, Italy

Abstract. We study the degeneracy loci of general bundle morphisms of the form $\mathcal{O}_{\mathbb{P}^n}^{\oplus m} \longrightarrow \Omega_{\mathbb{P}^n}(2)$, also from the point of view of the classical geometrical interpretation of the sections of $\Omega_{\mathbb{P}^n}(2)$ as linear line complexes in $\mathbb{P}^n$. We consider in particular the case of $\mathbb{P}^5$ with $m = 2, 3$. For $n = 5$ and $m = 3$ we give an explicit description of the Hilbert scheme $\mathcal{H}$ of elliptic normal scrolls in $\mathbb{P}^5$, by defining a natural rational map $\rho : \mathbb{G}(2, 14) \dashrightarrow \mathcal{H}$, which results to be dominant with general fibre of degree four.

Keywords: Cotangent sheaf, linear complex, elliptic scroll, Hilbert scheme

Mathematics Subject Classification (1991): 14F05, 14H52, 14J10

1. Introduction

Degeneracy loci of general bundle morphisms of the form

$$\mathcal{O}_{\mathbb{P}^n}^{\oplus m} \longrightarrow \Omega_{\mathbb{P}^n}(2)$$

have been studied by several authors, for example M.C. Chang ((Chang, 1988)) and G. Ottaviani ((Ottaviani, 1992)). We have performed here a study of these loci in the general case, computing in particular a locally free resolution of their ideals, the cohomology groups of their ideal sheaf and their degree. They result to be all arithmetically Buchsbaum varieties, with interesting geometrical properties. This can be seen by the classical geometrical interpretation of the sections of $\Omega_{\mathbb{P}^n}(2)$ as linear line complexes in $\mathbb{P}^n$. For example, if n is odd, they are scrolls, while, if n is even, they are unirational varieties.

We have considered then the particular case of $\mathbb{P}^5$ with $m = 2, 3$. If $m = 2$, the degeneracy locus X is the union of three lines. We have analyzed the possible configurations of these lines, obtaining the result that X has the expected dimension 1 if and only if X is exactly

* supported by MURST and GNSAGA, member of EAGER

the union of three skew lines generating $\mathbb{P}^5$, i.e. without any common secant line.

This has been applied to the case $m = 3$. Here, if the degeneracy locus X has the expected dimension 2, X is an elliptic normal scroll of degree six. It is known that the Hilbert scheme $\mathcal{H}$ of elliptic normal scrolls in $\mathbb{P}^5$ has dimension 36 (see (Ionescu, 1984)), which is equal to the dimension of $\mathbb{G}(2, 14)$, the Grassmannian parametrizing maps $\mathcal{O}^3_{\mathbb{P}^5} \longrightarrow \Omega_{\mathbb{P}^5}(2)$. Moreover there is an open subset in $\mathcal{H}$ corresponding to elliptic scrolls with only two distinct unisecant cubics. There is a natural rational map $\rho : \mathbb{G}(2, 14) \dashrightarrow \mathcal{H}$: we have studied this map, in particular to understand if it is dominant and what are its fibres. This problem had been tackled classically by G. Fano in (Fano, 1930). Our result, which was already known to Fano, is Theorem 2, saying that if X is an elliptic normal scroll in $\mathbb{P}^5$ with only two distinct unisecant cubics, then there are exactly four general nets of linear line complexes having X as singular surface. So the map ρ is dominant with general fibre of degree four.

If X is an elliptic normal scroll with only one unisecant cubic or with a 1-dimensional family of unisecant cubics, this result remains still valid, but *one* of the four nets having X as singular surface is not general, i.e. it contains some special complexes of second type. In this situation, the singular locus of such a net does not have the expected codimension 3, but 2: indeed it is the union of X with the singular 3-space of the special complexes of second type contained in the net.

We will denote by $\mathbb{P}^n$ the projective space of dimension n over an algebraically closed field K of characteristic 0. If $\mathcal{F}$ is a sheaf on $\mathbb{P}^n$, the direct sum of m copies of $\mathcal{F}$ will be denoted by $\mathcal{F}^{\oplus m}$ or else by $m\mathcal{F}$.

2. Degeneracy locus of a morphism $\varphi : \mathcal{O}^{\oplus m}_{\mathbb{P}^n} \longrightarrow \Omega_{\mathbb{P}^n}(2)$

Let $\mathcal{O}_{\mathbb{P}^n}$ be the sheaf of regular functions on $\mathbb{P}^n$ and $\Omega_{\mathbb{P}^n}$ be the cotangent sheaf. $\Omega_{\mathbb{P}^n}$ is locally free of rank n. We recall that the cohomology groups $H^0(\Omega_{\mathbb{P}^n}(k))$ are zero if $k < 2$, while $\Omega_{\mathbb{P}^n}(2)$ is generated by its global sections and $\dim H^0(\Omega_{\mathbb{P}^n}(2)) = \binom{n+1}{2}$.

A morphism $\varphi : \mathcal{O}^{\oplus m}_{\mathbb{P}^n} \longrightarrow \Omega_{\mathbb{P}^n}(2)$ is assigned by giving m global sections of $\Omega_{\mathbb{P}^n}(2)$. If $m \leqslant n$, then m general sections of $\Omega_{\mathbb{P}^n}(2)$ are linearly independent, so a *general* morphism φ is generically injective.

We want to study such a morphism φ for $m \leqslant n$ and, in particular, its degeneracy locus X. From Bertini type theorems we have that, if non-empty, X has the expected codimension $n-m+1$, that is $\dim X =$

$m-1$. Furthermore the singular locus of X, $Sing\ X$, has dimension at most $2m-n-4$. In particular X is smooth if $m < \frac{n+4}{2}$.

Eagon-Northcott's theorem (see for instance (Gruson et al., 1982)) gives a locally free resolution of $\mathcal{O}_X$. Indeed, φ is general, therefore the Eagon-Northcott's complexes associated to the dual morphism φ^* are exact and have length equal to the codimension of X. The first of such complexes can be written as follows:

$$\begin{array}{lll} 0 & \to \wedge^n(\mathcal{T}_{\mathbb{P}^n}(-2)) \otimes S^{n-m}(\mathcal{O}_{\mathbb{P}^n}^{\oplus m}) & \to \wedge^{n-1}(\mathcal{T}_{\mathbb{P}^n}(-2)) \otimes S^{n-m-1}(\mathcal{O}_{\mathbb{P}^n}^{\oplus m}) \to \\ \cdots & \to \wedge^m(\mathcal{T}_{\mathbb{P}^n}(-2)) & \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0 \end{array} \tag{1}$$

with natural maps induced by φ.

Furthermore the class of X in the Chow ring of $\mathbb{P}^n$ is the top Chern class $c_{n-m+1}(\text{coker } \varphi)$; in particular we can compute the degree of X, which is:

$$\deg X = \sum_{i=0}^{n-m+1} (-1)^i \binom{n-i}{m-1}.$$

We recall the following isomorphisms:

$$S^k(\mathcal{O}_{\mathbb{P}^n}^{\oplus m}) \simeq \binom{m+k-1}{k} \mathcal{O}_{\mathbb{P}^n}$$

$$\wedge^j \mathcal{T}_{\mathbb{P}^n} \simeq (\wedge^{n-j} \mathcal{T}_{\mathbb{P}^n}) \otimes \wedge^n \mathcal{T}_{\mathbb{P}^n} \simeq \Omega_{\mathbb{P}^n}^{n-j}(n+1)$$

Substituting in (1) and twisting by $n-1$, we obtain at last :

$$\begin{array}{ll} 0 & \to \binom{n-1}{m-1}\mathcal{O}_{\mathbb{P}^n} \to \binom{n-2}{m-1}\Omega_{\mathbb{P}^n}(2) \to \binom{n-3}{m-1}\Omega^2_{\mathbb{P}^n}(4) \to \cdots \\ \cdots \to & \binom{n-j-1}{m-1}\Omega^j_{\mathbb{P}^n}(2j) \to \cdots \to \Omega^{n-m}_{\mathbb{P}^n}(2n-2m) \to \mathcal{I}_X(n-1) \to 0 \end{array} \tag{2}$$

where $\mathcal{I}_X$ is the ideal sheaf of X. From this resolution, we can compute the dimensions of the cohomology groups of $\mathcal{I}_X$.

PROPOSITION 1. *The cohomology groups $H^i(\mathcal{I}_X(p))$ for $i > 0$ are all zero, except the following ones:*
$H^1(\mathcal{I}_X(m-2)), H^3(\mathcal{I}_X(m-4)), \ldots, H^{n-m}(\mathcal{I}_X(2m-n-1))$ *if $n-m$ is odd,* $H^2(\mathcal{I}_X(m-3)), H^4(\mathcal{I}_X(m-5)), \ldots, H^{n-m}(\mathcal{I}_X(2m-n-1))$ *if $n-m$ is even.*
In particular, X is arithmetically Buchsbaum.

Proof. The dimensions of the groups $H^i(\mathcal{I}_X(p))$ can be computed from (2). In particular, the multiplication maps

$$H^i(\mathcal{I}_X(p)) \to H^i(\mathcal{I}_X(p+1))$$

are all zero for $i > 0$ and the following condition of Stückrad-Vogel ((Stückrad et al., 1986), see also (Chang, 1988)), which ensures that X is arithmetically Buchsbaum, is fulfilled: if $H^i(\mathcal{I}_X(p)) \neq 0$ and $H^j(\mathcal{I}_X(q)) \neq 0$ with $i < j$, then $j - i \neq p - q - 1$. $\square$

We recall now a very nice geometrical interpretation of a morphism φ as above (see (Ottaviani, 1992)).

From the Euler sequence for $\mathbb{P}^n = \mathbb{P}(V)$ twisted by two:

$$0 \to \Omega_{\mathbb{P}^n}(2) \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n}(2) \to 0$$

taking global sections and noting that $H^0(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)}) \simeq V^* \times V^*$ and $H^0(\mathcal{O}_{\mathbb{P}^n}(2)) \simeq S^2V^*$, we obtain $H^0(\Omega_{\mathbb{P}^n}(2)) \simeq (\wedge^2 V)^*$.

This allows to interpret a global section of $\Omega_{\mathbb{P}^n}(2)$ as a bilinear alternating form on V, or as a skew-symmetric matrix of type $(n+1) \times (n+1)$ with entries in the base field. So a general morphism φ is assigned by giving m general skew-symmetric matrices $A_1, \ldots, A_m$ and the corresponding degeneracy locus X in $\mathbb{P}^n$ is the set of the points P such that:

$$(\lambda_1 A_1 + \cdots + \lambda_m A_m)[P] = 0 \qquad (3)$$

for some $(\lambda_1, \ldots, \lambda_m) \neq (0, \ldots, 0)$, where $[P]$ denotes the column matrix of the coordinates of P.

Since a skew-symmetric matrix always is of even rank, to study equation (3) we have to distinguish two cases:

1. if n is even, for every m-tuple $\lambda_1, \ldots, \lambda_m$ equation (3) has at least one solution $P \in X$; in this case X is an unirational variety parametrized by $\mathbb{P}^{m-1}$;

2. if n is odd, then the vanishing of the Pfaffian of the matrix $\lambda_1 A_1 + \cdots + \lambda_m A_m$ defines a hypersurface Z of degree $\frac{n+1}{2}$ in $\mathbb{P}^{m-1}$, where $\lambda_1, \ldots, \lambda_m$ are homogeneus coordinates. Furthermore, if φ is *general*, for a fixed point $[\lambda] \in Z$, we find a line of solutions of (3) in X, so that X is a scroll over Z.

In particular, from these observations we can conclude that X is always non-empty.

To see another interpretation of global sections of $\Omega_{\mathbb{P}^n}(2)$, let us consider $\mathbb{G}(1, n) \subseteq \mathbb{P}(\wedge^2 V)$, the Grassmannian of lines in $\mathbb{P}^n$, embedded in $\mathbb{P}(\wedge^2 V)$ via the Plücker map. The dual space $\mathbb{P}(\wedge^2 V^*)$ parametrizes hyperplane sections of $\mathbb{G}(1,n)$ or, in the old terminology, the *linear line complexes* in $\mathbb{P}^n$. Such a linear complex Γ is represented by a linear equation in the Plücker coordinates p_{ij}: $\sum_{0\leqslant i<j\leqslant n} a_{ij}p_{ij}=0$. We can associate to it a skew-symmetric matrix $A=(a_{ij})$ of order $n+1$. A point $P\in\mathbb{P}^n$ is called a *centre* of Γ if all lines through P belong to Γ. The space $\mathbb{P}(\ker(A))$ results to be the set of centres of Γ: it is called the *singular space* of Γ.

Once again we have to distinguish the following two cases:

1. if n is even, then every linear complex Γ possesses at least one centre. Γ is said to be *special* if its singular space is at least a plane. A special Γ corresponds to a hyperplane section of $\mathbb{G}(1,n)$ with a tangent hyperplane or, equivalently, to a point of $\check{\mathbb{G}}(1,n)$, the dual variety of $\mathbb{G}(1,n)$. It is known that, n being even, the dual Grassmannian $\check{\mathbb{G}}(1,n)$ has codimension three in $\mathbb{P}(\wedge^2 V^*)$.

2. If n is odd, then a general linear complex Γ does not have any centre, whereas it is said to be *special* if its singular space is at least a line. As above, if Γ is special, it corresponds to a tangent hyperplane section of $\mathbb{G}(1,n)$ or, which is the same, to a point of $\check{\mathbb{G}}(1,n)$ which in this case is a hypersurface in $\mathbb{P}(\wedge^2 V^*)$ of degree $\frac{n+1}{2}$.

From this discussion, it follows that it is possible to interpret the degeneracy locus X of a general morphism $\varphi:\mathcal{O}_{\mathbb{P}^n}^{\oplus m}\longrightarrow\Omega_{\mathbb{P}^n}(2)$ as the set of centres of complexes belonging to a general linear system Δ of dimension $m-1$ of complexes in $\mathbb{P}^n$. Such a Δ is generated by m independent complexes and corresponds to a linear subspace $\mathbb{P}^{m-1}$ in $\mathbb{P}(\wedge^2 V^*)$. The special complexes in Δ are parametrized by the intersection $\mathbb{P}^{m-1}\cap\check{\mathbb{G}}(1,n)$.

We conclude this section by giving some examples.

- $n=3, m=2$: X is the union of two skew lines;
- $n=3, m=3$: X is a smooth quadric, it can be seen as a scroll of degree two over a conic;
- $n=4, m=2$: X is an irreducible conic;
- $n=4, m=3$: X is a smooth projected Veronese surface;
- $n=4, m=4$: X is a so-called Segre cubic 3-fold, which has been extensively studied. It is singular with singular locus formed by ten distinct points;

- $n=5, m=2$: X is the union of three lines in $\mathbb{P}^5$; we shall study the possible configurations of X in §3;
- $n=5, m=3$: X is a scroll over a plane cubic, of degree six in $\mathbb{P}^5$, i.e. an elliptic normal scroll in $\mathbb{P}^5$; we shall study this situation in §4;
- $n=5, m=4$: X is a 3-fold of degree seven, which is scroll over a cubic surface in $\mathbb{P}^3$. It is also known as Palatini scroll ((Ottaviani, 1992)).

3. Pencils of linear complexes in $\mathbb{P}^5$

In this section we will study in detail the pencils of linear complexes in $\mathbb{P}^5$, or, in other words, maps $\mathcal{O}_{\mathbb{P}^5}^{\oplus 2} \longrightarrow \Omega_{\mathbb{P}^5}(2)$ and their degeneracy loci.

We will denote by the same symbol a line both as a subset of $\mathbb{P}^5$ and as a point of $\mathbb{G}(1,5)$. Let us recall that the Grassmannian $\mathbb{G}(1,5)$ is an 8-dimensional variety in $\mathbb{P}^{14}$. A general linear complex of lines in $\mathbb{P}^5$ does not have any centre, whereas the special complexes can be of *first or second type*, if they have a line or a $\mathbb{P}^3$ as singular space, respectively. The dual space $\check{\mathbb{P}}^{14}$ parametrizes linear line complexes: special complexes correspond to points of the cubic hypersurface $\check{\mathbb{G}}(1,5) \subseteq \check{\mathbb{P}}^{14}$. Furthermore special complexes of second type can be interpreted as points of the Grassmannian $\mathbb{G}(3,5)$ (which is also embedded in $\check{\mathbb{P}}^{14}$), because a special complex of second type is determined uniquely by its singular space $\mathbb{P}^3$: it is formed by the lines intersecting that $\mathbb{P}^3$.

We introduce a rational surjective map:

$$\psi : \check{\mathbb{G}}(1,5) \dashrightarrow \mathbb{G}(1,5)$$

which maps a special complex (of first type) to its singular line. So ψ is regular on $\check{\mathbb{G}}(1,5) \setminus \mathbb{G}(3,5)$.

The closure of the fibre $\psi^{-1}(l)$ over a line l is formed by all special complexes having l as singular line. These fibres are linear spaces of dimension five; in fact we may interpret $\psi^{-1}(l)$ as the linear system of hyperplanes in $\mathbb{P}^{14}$ containing the tangent space to $\mathbb{G}(1,5)$ at the point l: $T_{l,\mathbb{G}} \simeq \mathbb{P}^8$. We will use the notation $\mathbb{P}^5_l$ for $\psi^{-1}(l)$.

REMARK 1. *Let $l, m \in \mathbb{G}(1,5)$ be lines of $\mathbb{P}^5$. Then the intersection of $\mathbb{P}^5_l$ with $\mathbb{G}(3,5)$ is a smooth quadric of dimension 4. In fact, it is formed by the $\mathbb{P}^3$'s containing l: this is a Schubert subvariety of the Grassmannian and precisely a quadric.*

Also the intersection of the fibres $I\!P^5_l \cap I\!P^5_m$ *is contained in* $\mathbb{G}(3,5)$ *and is just one point, if* l *and* m *do not intersect, or a plane, if they intersect. Indeed note that* $I\!P^5_l \cap I\!P^5_m$ *is the set of the special complexes of second type, whose singular space contains both* l *and* m*. If* $l \cap m = \emptyset$*, then the linear span* $\langle l, m\rangle$ *is a* $I\!P^3$*, so* $I\!P^5_l \cap I\!P^5_m$ *is the point corresponding to the special complex of second type whose singular space is* $\langle l, m\rangle$*. Whereas, if* $l \cap m \neq \emptyset$*, then* $\langle l, m\rangle$ *is a plane* π *and* $I\!P^5_l \cap I\!P^5_m$ *is the plane corresponding to the linear system of* $I\!P^3$*'s containing* π*.*

Let L be a line in $\check{I\!P}^{14}$: L represents a pencil of linear complexes. In the general case, L meets $\check{\mathbb{G}}(1,5)$ at three points, which do not belong to $\mathbb{G}(3,5)$. Therefore the most general pencil of linear complexes in $I\!P^5$ contains three special complexes of first type. Let l_1, l_2, l_3 be the singular lines of these three complexes. We distinguish four possibilities for the reciprocal positions of the three lines.

Case 1. The lines l_i are two by two skew and generate the whole $I\!P^5$. This is the most general situation. In this case the three lines do not have any trisecant line.

Case 2. The lines l_i are two by two skew and generate a $I\!P^4$. Equivalently, the three lines have one and only one trisecant line.

Case 3. The lines l_i are two by two skew and generate a $I\!P^3$.

Case 4. The lines l_i are not two by two skew.

Let us analyze separately the four cases. We will see that only in the first case the pencil of linear complexes is general.

Case 1. Let $I\!P^5_i := \psi^{-1}(l_i)$ be the space that parametrizes special complexes having l_i as singular line and let H_{ij} be the linear span in $I\!P^5$ of l_i and l_j for $1 \leqslant i < j \leqslant 3$. Note that, by the assumption on the reciprocal position of the lines, $H_{ij} \cap H_{ik} = l_i$. We will use the same notation for H_{ij} as a 3-space in $I\!P^5$ and for H_{ij} as a point of $\mathbb{G}(3,5)$. So, by Remark 1, we may write $I\!P^5_i \cap I\!P^5_j = H_{ij}$.

We can prove the following:

PROPOSITION 2. *In the situation of case 1., let* $L \subseteq \check{I\!P}^{14}$ *be a line representing a pencil of linear complexes having* l_1, l_2, l_3 *as singular lines. Then* L *belongs to one of the following families:*

a) lines of the plane $\sigma := \langle H_{ij}, 1 \leqslant i < j \leqslant 3\rangle \subseteq \check{I\!P}^{14}$*;*

b) lines through H_{ij} *intersecting* $I\!P^5_k$*, for some* $1 \leqslant i < j \leqslant 3$*,* $k \neq i, j$*.*

In particular, lines L *representing general pencils (i. e. pencils not containing any special complex of the second type) correspond to lines of* σ *not passing through any* H_{ij}*.*

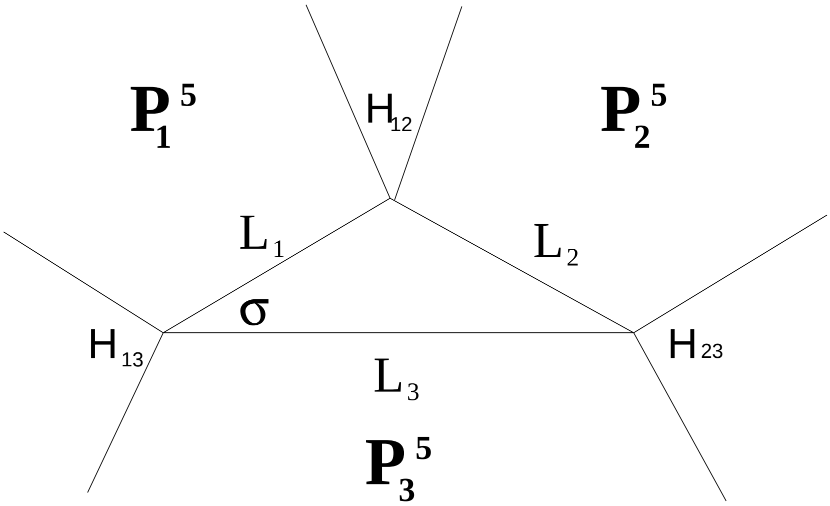

Figure 1.

Proof. We observe first that our assumption on L is equivalent to the fact that L intersects all the three spaces $\mathbb{P}^5_i$. Moreover $\mathbb{P}^5_i \cap \sigma$ is the line L_i through H_{ij} and H_{ik} (see Figure 1.). So every line in σ meets L_1, L_2 and L_3 and therefore represents a pencil having l_1, l_2, l_3 as singular lines.

Now let $L \nsubseteq \sigma$ be any line intersecting the three spaces $\mathbb{P}^5_i$: we want to show that L is a line of type b). We define $R_i := L \cap \mathbb{P}^5_i$, for $i = 1, 2, 3$. Since $L \nsubseteq \sigma$, we can suppose that $R_2, R_3 \notin \sigma$. If $R_2 = R_3$, then this point is H_{23} and L is of type b). So we assume $R_2 \neq R_3$: then L is contained in $\langle \mathbb{P}^5_2, \mathbb{P}^5_3 \rangle$, which is a $\mathbb{P}^{10}$ by Grassmann relation, and $R_1 = L \cap \mathbb{P}^5_1$ belongs to $\langle \mathbb{P}^5_2, \mathbb{P}^5_3 \rangle \cap \mathbb{P}^5_1 = L_1$. Now, note that L is contained both in $\langle \mathbb{P}^5_2, L_1 \rangle$ and in $\langle \mathbb{P}^5_3, L_1 \rangle$ and therefore in their intersection, which is the plane σ (again by Grassmann relation). So we have a contradiction that makes us conclude that $R_2 = R_3 = H_{12}$.

From Remark 1, it follows that each line L_i meets $\mathbb{G}(3, 5)$ only at the two points H_{ij} and H_{ik}, otherwise it would be contained in $\mathbb{G}(3, 5)$, which is excluded by our assumptions. So lines of σ not passing through any H_{ij} represent general pencils. $\square$

Case 2. Assume that l_1, l_2, l_3 are two by two skew but generate a hyperplane H in $\mathbb{P}^5$. Then they have exactly one trisecant line r: it can be constructed as follows. Let P be the intersection $\langle l_1, l_2 \rangle \cap l_3$: r is the only line through P meeting both l_1 and l_2.

In this case, we can consider the spaces H_{ij}, the lines L_i and the plane σ as in case 1., but σ is now contained in $\mathbb{G}(3, 5)$. Indeed, the H_{ij}'s intersect two by two along a plane in H, precisely $H_{ij} \cap H_{ik} = \langle l_i, r \rangle$, for all i, j, k. This means that the lines L_1, L_2, L_3, joining the corresponding points in the Plücker embedding, are completely contained in $\mathbb{G}(3,5)$, which is a smooth quadric, and this implies that also $\sigma \subseteq \mathbb{G}(3,5)$.

The pencils of line complexes having l_1, l_2, l_3 as singular lines can be obtained exactly as in case 1., but *all of them contain at least one special complex of second type.* In particular, those corresponding to the lines in σ are pencils of complexes all special of second type.

Case 3. If l_1, l_2, l_3 are two by two skew but generate a $\mathbb{P}^3$, then $H_{12} = H_{13} = H_{23}$ and σ "collapses" to a point. It is easy to see that a line meeting $\mathbb{P}^5_i$, for all i, necessarily passes through this point, so the corresponding pencil contains at least one special complex of second type.

Case 4. When at least two among l_1, l_2, l_3 intersect each other, several configurations are possible. It is clear from the previous discussion that, in all cases, we get pencils containing special complexes of the second type.

We can conclude with the following:

THEOREM 1. *If the degeneracy locus X of a map $\varphi : \mathcal{O}_{\mathbb{P}^5}^{\oplus 2} \longrightarrow \Omega_{\mathbb{P}^5}(2)$ has the expected dimension one, then X is the union of three skew lines generating $\mathbb{P}^5$.*

Proof. Indeed, if the pencil of linear complexes corresponding to φ contains a special complex Γ of the second type, then the singular $\mathbb{P}^3$ of Γ is contained in X. So the theorem follows from the previous analysis of cases 1. – 4. $\square$

REMARK 2. *Let $\mathbb{G}(1,5)^{(3)}$ denote the third symmetric power of the Grassmannian of lines in $\mathbb{P}^5$, which is a projective variety of dimension 24. $\mathbb{G}(1, \check{\mathbb{P}}^{14})$, of dimension 26, parametrizes the lines in $\check{\mathbb{P}}^{14}$, corresponding bijectively to pencils of linear line complexes in $\mathbb{P}^5$. There is a natural rational map:*

$$\alpha : \mathbb{G}(1, \check{\mathbb{P}}^{14}) \dashrightarrow \mathbb{G}(1,5)^{(3)}.$$

α maps a general pencil of linear line complexes of $\mathbb{P}^5$ into the triple of its singular lines. The results of this section show that α is dominant and Proposition 2 describes its general fibre.

4. Nets of linear complexes in $\mathbb{P}^5$

We will study now the nets of linear complexes in $\mathbb{P}^5$, i.e. the maps $\mathcal{O}_{\mathbb{P}^5}^{\oplus 3} \longrightarrow \Omega_{\mathbb{P}^5}(2)$ and their degeneracy loci.

As we have seen in §2, the singular surface X of a general net Δ of linear complexes in $\mathbb{P}^5$ is an elliptic normal scroll surface, of degree six. Since Δ is general, it does not contain any special complex of second type, so there is a natural surjective regular map, whose fibres are lines: $\varphi : X \longrightarrow C$, where C is the smooth plane cubic, defined by the vanishing of the Pfaffian of the skew-symmetric matrix associated to the net. The fibres of φ are just the singular lines of the special complexes of Δ.

We can interpret Δ as a general plane in $\check{\mathbb{P}}^{14}$: it intersects $\check{\mathbb{G}}(1,5)$ along a cubic curve which is disjoint from $\mathbb{G}(3,5)$. Its points represent the special complexes of Δ, so the curve can be identified with C.

Let us briefly recall the well-known classification of elliptic normal scrolls in $\mathbb{P}^5$ (see (Hartshorne, 1977)). Every such scroll X is isomorphic to $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank two normalized locally free sheaf over the base curve C. The invariant e is necessarily zero in this case, so $\deg \mathcal{E} = 0$. $\mathbb{P}(\mathcal{E})$ is embedded in $\mathbb{P}^5$ by the very ample linear system $|C_0 + 3f|$, where C_0 is a minimal unisecant curve with $C_0^2 = 0$ and f is a fibre.

There are three cases, according to the Atiyah classification of vector bundles over elliptic curves ((Atiyah, 1957)):

1. $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$, with $\mathcal{L}$ a non-trivial invertible sheaf of degree 0. There is a one-dimensional family of such sheaves up to isomorphism, parametrized by C. In this case, X has two unisecant cubics, γ and γ'. This is the most general case;

2. $\mathcal{E}$ is indecomposable and is unique up to isomorphism. X has only one unisecant cubic;

3. $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{O}_C$. $X \simeq C \times \mathbb{P}^1$ has a 1-dimensional family of unisecant cubics.

The Hilbert scheme $\mathcal{H}$ of elliptic normal scrolls in $\mathbb{P}^5$ has dimension 36 and is smooth at points representing smooth surfaces (see (Ionescu, 1984)). From the above classification, it follows that the surfaces of the first type form a dense subset in $\mathcal{H}$, whereas those of the second type form a subvariety of dimension 35 and those of the third type a subvariety of dimension 33.

The nets of linear complexes in $\mathbb{P}^5$ are parametrized by $\mathbb{G}(2, \check{\mathbb{P}}^{14})$, the Grassmannian of planes in $\check{\mathbb{P}}^{14}$, therefore we can define a natural rational map:

$$\rho : \mathbb{G}(2, \check{\mathbb{P}}^{14}) \dashrightarrow \mathcal{H}.$$

ρ maps a general net Δ to its singular surface X. Since domain and codomain of ρ have the same dimension 36, the first natural question is if ρ is dominant or, in other words, if an elliptic scroll X of the first type is always the degeneracy locus of a suitable bundle map $\mathcal{O}_{\mathbb{P}^5}^{\oplus 3} \longrightarrow \Omega_{\mathbb{P}^5}(2)$. In the affirmative case, a general fibre of ρ is finite. So the second natural question is what is the degree of that fibre.

An answer to this questions is the main result of this paper. It has been inspired by an article of Gino Fano (Fano, 1930).

THEOREM 2. *Let X be an elliptic normal scroll in $\mathbb{P}^5$ whose associated sheaf is $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$, with $\mathcal{L}$ non-trivial. Then there are exactly four general nets of linear line complexes, whose singular surface is X.*

The proof of Theorem 2 will follow after several Lemmas and Remarks. We will study first the relations between the properties of a net Δ having X as singular surface and the geometry of X. In particular we will find some interesting linear series, associated to Δ, on the elliptic curve C_X in the Grassmannian $\mathbb{G}(1,5)$ representing the lines of X. This will allow us to explicitly construct all nets having a fixed X as singular surface.

Let X be an elliptic normal scroll of type 1. with unisecant cubics γ and γ', generating the planes π and π' respectively.

LEMMA 1. *If Δ is a general net of linear line complexes in $\mathbb{P}^5$ having X as singular surface, then every line contained in π or in π' belongs to all complex of Δ.*

Proof. Let r be a general line in π. Put $r \cap \gamma = \{P_1, P_2, P_3\}$. Let l_i be the line of X through P_i: l_i is the singular line of a complex Γ_i of Δ and r belongs to Γ_i for i =1,2,3. The complexes $\Gamma_1, \Gamma_2, \Gamma_3$ do not belong to the same pencil: indeed the lines l_i have the trisecant r, so a pencil Φ containing $\Gamma_1, \Gamma_2, \Gamma_3$ should contain at least one special complex of second type (by Theorem 2), but this contradicts the assumption that Δ is general. Therefore r belongs to three independent complexes of Δ, hence to the base locus of Δ. The same conclusion holds true if $r \subseteq \pi'$. □

LEMMA 2. *Let k be a line of X. The special complexes having k as singular line and containing all lines of π and π' form a 3-dimensional linear system. None of these complexes contains all the lines of X.*

Proof. Let $\mathbb{P}^5_k \subseteq \check{\mathbb{P}}^{14}$ be the space which parametrizes the special complexes having k as singular line (notation as in §3). The condition that the ruled planes π and $\pi^{'}$ are contained in a complex of $\mathbb{P}^5_k$ is expressed by two linear conditions, so that these complexes form a $\mathbb{P}^3_k \subseteq \mathbb{P}^5_k$. Indeed, every point of k is a centre of every complex $\Gamma \in \mathbb{P}^5_k$, so it suffices to impose that one line of π (resp. $\pi^{'}$), not passing through $k \cap \pi$ (resp. $k \cap \pi^{'}$), belongs to Γ.

We assume now by contradiction that a complex Γ of $\mathbb{P}^3_k$ contains all lines of X. Let us consider the projection p_k from k to a complementar space $\mathbb{P}^3$. Let $X' := p_k(X)$: it is a ruled surface of degree 4. Observe that $r := p_k(\pi)$ and $r' := p_k(\pi^{'})$ are skew lines. Every line in X, different from k, is projected to a line meeting r and $r^{'}$. Every point $P \in r$ (resp. $r^{'}$) comes from two points of γ (resp. $\gamma^{'}$), which are collinear with $k \cap \gamma$ (resp. $k \cap \gamma^{'}$). Therefore in X' there are two lines through P intersecting $r^{'}$ (resp. r).

The projection of Γ in $\mathbb{P}^3$ via p_k is a linear complex Γ' of lines of $\mathbb{P}^3$ containing r, r' and all lines of X'. If Γ' is a singular complex, then it is formed by the lines meeting a fixed line, which is impossible because X' is elliptic. If Γ' is non-singular, the lines of Γ' passing through a point P of r should belong to a pencil, containing r and the two lines of X' intersecting r': also this is impossible because r and r' are skew. □

Let now X be the singular surface of a net Δ. We will consider the linear series of divisors on the elliptic curve C_X, of degree six in $\mathbb{G}(1,5)$, associated to X. The hyperplane linear series on C_X is a g^5_6: $|\, H_X \,|$. If k is a line on X, we will denote by k also the corresponding point on C_X and by Γ_k the line complex of Δ having k as singular line. We can interpret it as a hyperplane in $\mathbb{P}^{14}$, tangent to $\mathbb{G}(1,5)$ at the point corresponding to k. Since Γ_k is also tangent to C_X at k, the intersection of C_X and Γ_k, residually to k, is a divisor D_k of degree four on C_X, such that $D_k + 2k$ belongs to the hyperplane series $|\, H_X \,|$. If k varies in C_X, the divisors D_k describe a non-linear series of degree 4 and dimension 1, a γ^1_4. On the other hand, letting Γ vary in $\mathbb{P}^3_k$, we obtain a complete linear series $|\, D_k \,|$, depending on k, of dimension three and degree four, i.e. a g^3_4. The following lemma is classical.

LEMMA 3. *Let g^{2n-1}_{2n} be a complete linear series on an elliptic curve E. Then there exist exactly four distinct linear series $g^{n-1}_n = |\, G_i \,|$, i=1,…,4, such that $|\, 2G_i \,| = g^{2n-1}_{2n}$.*

In particular, for all line k there exist four divisors in $|\, D_k \,|$ of the form $2E^{(k)}_i$, where $|\, E_i \,|$ is a g^1_2.

Similarly there are exactly four linear series such that $|\,H_X\,| = |\,2F_i\,|$, and $|\,F_i\,|$ is a g^2_3. But $k + E_i^{(k)}$ has degree three and $2(k + E_i^{(k)}) \in |\,2k + D_k\,| = |\,H_X\,|$. Therefore $|\,k + E_i^{(k)}\,|$ is one of the four $|\,F_i\,|$.

We interpret now Δ as a plane in $\check{\mathbb{P}}^{14}$ and its special complexes as the points of $\Delta \cap \check{\mathbb{G}}(1,5)$, which is a plane cubic C. C and C_X are isomorphic via the map ψ (see §3) which associates to each special complex its singular line.

The hyperplane series $|\,H_C\,|$ on C is a g^2_3, representing on X triples of lines which are singular loci of pencils contained in Δ.

Claim. $\psi^*(H_X) \in |\,2H_C\,|$. Then, up to the isomorphism ψ, $|\,H_C\,|$ is one of the four series $|\,F_i\,|$.

Proof of the claim. We consider $\psi^{-1}(C_X \cap H)$, where $H \in \Delta$ is a hyperplane in $\mathbb{P}^{14}$. Note that a line k_Γ, singular for a complex Γ, belongs to H if and only if H belongs to $\check{k}_\Gamma$ in $\check{\mathbb{P}}^{14}$ and $\check{k}_\Gamma$ is tangent to $\check{\mathbb{G}}(1,5)$ at all points representing special complexes having k_Γ as singular line, in particular at Γ. So $\psi^{-1}(C_X \cap H)$ is formed by special complexes Γ of Δ such that $k_\Gamma \in H$, or equivalently that $H \in T_{\Gamma,\check{\mathbb{G}}}$. This again is equivalent to the condition that the line $\overline{\Gamma H}$ be tangent to $C = \Delta \cap \check{\mathbb{G}}$ at Γ. So the points of $\psi^{-1}(C_X \cap H)$ are in bijection with the tangent lines to C passing through H. We can conclude that the linear system on C defining ψ contains the system cut on C by the polar conics of the points of its plane Δ, which proves the claim.

We have now a rather complete picture of linear series on C_X, if Δ is given a priori. In particular a g^5_6 and a g^2_3 with $g^5_6 = 2g^2_3$ are fixed.

We start now from X, so the curve C_X with hyperplane divisor H_X and the four divisors F_i are fixed. We choose one of them: F. Every $k \in C_X$ has a residual g^1_2 with respect to F. By Lemma 3, this g^1_2 has four double points $P_1, \dots, P_4$.

Claim. There exists a well determined linear complex Γ_k in $\mathbb{P}^3_k$ containing the four lines in X, which correspond to $P_1, \dots, P_4$.

Proof of the claim. Indeed, if we embed C_X in the plane as a plane cubic using $|F|$, then Γ_k corresponds to the conic section cut out by the polar conic of k with respect to C_X: it is tangent to C_X at k and it gets through the four contact points of the tangent lines to C_X passing through k. So we have the Claim.

Letting k vary on C_X, we obtain an elliptic system S of linear complexes $\{\Gamma_k\}$. To prove Theorem 2, it remains to verify that the minimum linear system of complexes containing S is a net. Since in $\check{\mathbb{P}}^{14}$ S is an elliptic curve, it suffices to prove that S is a cubic: the plane of S is then the sought net Δ. To compute the degree of S, we intersect S with $\check{k}$ where $k \in C_X$ and $\check{k}$ is the hyperplane in $\check{\mathbb{P}}^{14}$ which parametrizes the complexes containing k.

There are only two complexes of S containing k, i.e. Γ_k, whose singular line is k, and Γ', that corresponds to the g_2^1 contained in $|F|$, having k as double element. Γ_k and Γ' are the unique points of $\check{k} \cap S$.

The intersection multiplicity of $\check{k}$ and S at Γ_k is two, because Γ_k corresponds to a tangent hyperplane to $\mathbb{G}(1,5)$ at k. We prove finally that the intersection of $\check{k}$ and S at Γ' is transversal.

Let $\check{\Gamma}'$ be the hyperplane, through k, in $\mathbb{P}^{14}$, which parametrizes the hyperplanes in $\check{\mathbb{P}}^{14}$ through Γ'. Observe that $\check{\Gamma}' \cap \mathbb{G}(1,5)$ is nothing but the set of the lines of Γ'. If, by contradiction, the intersection of $\check{k}$ and S at Γ' is not transversal, then $\check{k}$ is a tangent hyperplane to S at Γ', so $\check{\Gamma}'$ is tangent to C_X at k. This is impossible, because k is not the singular line of Γ'. We can conclude that S is just a cubic.

Therefore, we have found a general net of linear complexes in $\mathbb{P}^5$ with X as singular surface. There is such a net for each of the four g_3^2 on C_X above described. This completes the proof of Theorem 2.

We conclude the treatment of this general case with the following Remark, which we will use afterwards.

REMARK 3. *Let X be an elliptic normal scroll of type 1. with unisecant cubics γ and γ', generating the planes π and π' respectively. The hyperplane series $|H_\gamma|$ on γ and $|H_{\gamma'}|$ on γ' are two g_3^2 such that the sum series $|H_\gamma + H_{\gamma'}|$ is the hyperplane series $|H_X|$ on C_X, up to the Plücker embedding.*

Proof. Observe that a divisor of $|H_\gamma + H_{\gamma'}|$ corresponds to the union of $r \cap \gamma$ and $r' \cap \gamma'$, where $r \subseteq \pi$ and $r' \subseteq \pi'$ are two skew lines. The 3 -space $\langle r, r' \rangle$ determines uniquely the special complex Γ of second type, formed by all and only lines of $\mathbb{P}^5$ meeting $\langle r, r' \rangle$. Γ corresponds to a hyperplane section of C_X, and precisely to a divisor of $|H_X|$. This proves the Remark. $\square$

We may use the same arguments, with suitable changes, to study also elliptic scrolls X of type 2. and 3., but in both cases the situation results to be different.

Let X be an elliptic scroll of type 2., i.e. with unisecant cubic γ generating the plane π. Such a scroll has the property that the 4-space, generated by three lines of X meeting γ at collinear points, always contains the plane π. This follows immediately from the fact that X is embedded in $\mathbb{P}^5$ by the very ample linear system $|C_0 + 3f|$, where C_0 is the unique unisecant cubic and f is a fibre, as recalled at the beginning of this section.

LEMMA 4. *If Δ is a general net of linear line complexes in $\mathbb{P}^5$ having X as singular surface, then every line contained in π belongs to all complex of Δ.*

Proof. See the proof of Lemma 1. □

LEMMA 5. *Let k be a line of X. The special complexes having k as singular line and containing all lines of π form a 4-dimensional linear system. Only one of this complexes contains all the lines of X.*

Proof. For the first assertion see the proof of Lemma 2. Let us consider the projection p_k from a line k of X to a complementar space $\mathbb{P}^3$. Let $X' := p_k(X)$: it is a ruled surface of degree 4 with only one unisecant line $r := p_k(\pi)$. Observe that every line in X, different from k, is projected to a line meeting r, and every point $P \in r$ comes from two points of γ, which are collinear with $k \cap \gamma$. Therefore in X' there are two lines through P intersecting r.

Observe that the lines of X' are contained in the special complex Γ' of lines of $\mathbb{P}^3$ with r as singular line. Let $\Gamma := p_k^{-1}(\Gamma')$: it is a linear complex of $\mathbb{P}^5$, containing all the lines of X and, moreover, having k as singular line. Indeed, Γ' is formed by the union of the (closure of the) fibres $p_k^{-1}(l)$, when l varies among the lines meeting r. Note that $p_k^{-1}(l)$ is the set of the lines contained in the 3-space $\langle k, l \rangle$. □

REMARK 4. *If X is an elliptic scroll of type 2., then, for all line k in X, there is a unique linear special complex H_k in $\mathbb{P}^5$, having k as singular line and containing all the lines of X. So we obtain a 1-dimensional elliptic system $\{H_k\}_{k \subseteq X}$ of special complexes containing X.*

Now, let X be of type 3., i.e. $X \simeq \gamma \times \mathbb{P}^1$, where γ is a cubic generating the plane π.

LEMMA 6. *If Δ is a general net of linear line complexes in $\mathbb{P}^5$ having X as singular surface, then the base locus of Δ contains all the ∞^1 ruled planes of the cubics in X.*

Proof. See the proof of Lemma 1. □

LEMMA 7. *Let k be a line of X. Among the special complexes having k as singular line, there is a linear system of dimension 2 of such complexes containing all the lines of X.*

Proof. Let us consider the projection p_k from a line k of X to a complementar space $\mathbb{P}^3$. $X' := p_k(X)$ is a quadric surface and p_k is a $2:1$ map.

Observe that there is a linear system of dimension 2 of complexes of lines of $\mathbb{P}^3$ containing the lines of the rulings, projection via p_k of the lines of X on X'. These complexes can be lifted to special complexes of $\mathbb{P}^5$ containing X and having k as singular line. To prove this, we may use the same arguments of the proof of Lemma 5. □

If we start from X of type 2. or 3., i.e. the curve C_X with hyperplane divisor H_X and the four series $|F_i|$, such that $|2F_i| = |H_X|$, for $i = 1, \ldots, 4$, are fixed, we find that *one* of the four nets of complexes, having X as singular surface, contains some special complexes of second type. Indeed we have the following:

REMARK 5. *Let X be an elliptic normal scroll of type 2. or 3. The two series g^2_3 defined in Remark 3 coincide, because of the geometry of X. If $|H_\gamma|$ denotes this g^2_3, then $|2H_\gamma| = |H_X|$ and so $|H_\gamma|$ is one of the four F_i.*

The net Δ of complexes having X as singular surface, that can be constructed starting from the series $|H_\gamma|$, contains complexes of second type. Indeed, for every divisor of $|H_\gamma|$, we obtain three lines l_1, l_2, l_3 of X having a common trisecant, so that among the complexes of the pencil contained in Δ and having l_1, l_2, l_3 as singular lines, there are certainly special complexes of second type (see §3).

From this discussion we conclude that the singular locus of such a net Δ has codimension 2 instead of the expected codimension 3. Indeed, it is the union of X with the singular 3-space of the special complexes of second type of Δ.

References

Atiyah, M.F.: 1957 Vector Bundles over an Elliptic Curve, *Proc. London Math. Soc.* **(3) 7**, pp. 414–452

Chang, M.C.: 1988 Buchsbaum subvarieties of codimension 2 in $\mathbb{P}^n$, *Bull. (New Series) Amer. Math. Soc.* **Vol. 19**, pp. 269–272

Fano, G.: 1930 Reti di complessi lineari dello spazio S_5 aventi una rigata assegnata di rette-centri *Rend. Reale Accad. Naz. Lincei* **Vol. XI**, pp. 227–232

Gruson L., Peskine C.: 1982 Courbes de l'espace projectif: variétés de sécantes, in *Enumerative and Classical Algebraic Geometry, Proc. Conf. Nice, 1981 (P. le Barz, Y. Hervier eds.)*, Progr. Math. **24**, Birkhäuser, Boston-Basel-Stuttgart 1982, pp. 1–31

Hartshorne R., 1977 *Algebraic Geometry*, Springer-Verlag, New York-Heidelberg-Berlin

Ionescu P.: 1984 Embedded projective varieties of small invariants, in *Proceedings of the week of Algebraic Geometry, Bucharest 1982*, Lecture Notes in Mathematics **1056**, Springer-Verlag, New York-Heidelberg-Berlin 1984, pp. 142–186

Ottaviani G.: 1992 On 3-Folds in $\mathbb{P}^5$ which are Scrolls, *Ann. Scuola Norm. Sup. Pisa, cl. Sci.***(4) 19**, pp. 451–471

Stückrad J., Vogel W., 1986 *Buchsbaum Rings and Applications*, Springer- Verlag, New York-Heidelberg-Berlin