ON THE INTEGRABILITY OF FINITE–DIMENSIONAL SYSTEMS RELATED TO SOLITON EQUATIONS

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Abstract. A bi–Hamiltonian formulation for stationary flows of the KdV hierarchy is derived in an extended phase space. A map between stationary flows and restricted flows is constructed: in a case it connects the Henon–Heiles and the Garnier system. Moreover a new integrability scheme for Hamiltonian systems in their standard phase space is proposed.

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Contents

1. Introduction

In the last years there has been an increasing interest for the construction of finite–dimensional dynamical systems from soliton equations, through the so–called methods of *stationary flows* and *restricted flows* (see [Dic], [AW1] and references therein). Indeed the discovery of suitable sets of coordinates has allowed one to write the reduced systems as physically interesting Hamiltonian systems. In the case of the KdV hierarchy, the q–representation for stationary flows has given rise to the Henon–Heiles system [For, Wo2] and the square eigenfunctions representation for restricted flows has furnished the Neumann and the Garnier systems [Cao1, AW1]. However the relation between dynamical systems which are obtained through different reduction techniques from the same soliton hierarchy is not clear; moreover a systematic way to find the second Hamiltonian formulation for stationary flows of any order, without the use of a Miura map, is still lacking.

The aim of this paper is to give a contribution in these directions. The main results are:

i) We derive in a systematic way a bi–Hamiltonian formulation for stationary flows of the KdV hierarchy in a suitably extended phase space. As an example the bi–Hamiltonian structure of Henon–Heiles–type systems with two and four degrees of freedom is constructed.
ii) We obtain an explicit map between stationary and restricted flows of the KdV hierarchy, based on the generating function of the Gelfand–Dickey (GD) polynomials. As an application, a map between a generalized Henon–Heiles system and the Garnier system with two degrees of freedom is constructed.
iii) We propose an integrability structure, which can be applied to both stationary and restricted flows. It generalizes the structure introduced in [CRG] for the particular case of the Henon–Heiles system. Though weaker than the bi–Hamiltonian formulation, it does not require the extension of the phase spaces.

The paper is organized as follows. In Subsect. 2.1–2.3 we construct both the KdV hierarchy and the associated stationary flows through the kernel of the KdV Poisson pencil. The difference between the two cases is given by the fact that the gradients of the integrals of motion for the KdV hierarchy are the coefficients in a Laurent series, whereas in the second case they are the coefficients of a polynomial. Thus, using the generating function of GD polynomials, we give a bi–Lagrangian and a bi–Hamiltonian formulation of the Lax–Novikov stationary equations of any order; as applications, we exhibit a generalized Henon–Heiles system with two and four degrees of freedom (Subsect.s 2.2 and 4.1).

In Subsect. 2.4 we formulate the method of restricted flows in terms of the kernels of some Poisson structures extracted from the Poisson pencil and of the generating function of the GD polynomials, without the explicit use of the spectral problem as in [Cao1, AW1]. This formulation allows us to connect, in a quite natural way, restricted flows with stationary flows: this is obtained by means of an appropriate extension of the corresponding phase spaces with some free parameters, as we show in Subsect. 2.5. In the final subsection 2.7 we specialize the previous map to the Henon–Heiles and the Garnier systems.

In Sect. 3 we show that it is not possible to reduce from the extended phase space to the standard phase space the entire bi–Hamiltonian hierarchy of Henon–Heiles and Garnier systems, i.e. both the Poisson structures and the associated vector fields. For this reason, in Subsect. 3.2 we propose an integrability criterion holding for a generic finite–dimensional Hamiltonian system. It generalizes the criterion introduced in [CRG] for the particular case of the Henon–Heiles system. Though weaker than the bi–Hamiltonian scheme, it will be shown to assure Liouville–integrability of a Hamiltonian system [Arn] in its standard phase space, i.e. without the introduction of supplementary coordinates. In Subsect. 3.3 we apply this criterion to the Henon–Heiles system and the Garnier system with two degrees of freedom and then to the four–dimensional Henon–Heiles system in Subsect. 4.2.

Now we give some preliminaries to be used in the following: the aim is mainly to fix notations and terminology. Let M be a n–dimensional manifold. At any point $u \in M$, the tangent and cotangent spaces are denoted by T_uM and T^*_uM, the pairing between the two spaces by $<,>: T^*_uM \times T_uM \to \mathbb{R}$. For each smooth function $f \in C^\infty(M)$, df denotes the differential of f. M is said to be a Poisson manifold if it is endowed with a Poisson bracket $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$; the associated Poisson tensor P is defined by $\{f, g\}(u) :=< df(u), P_u\, dg(u) >$. So, at each point u, P_u is a linear map $P_u : T^*_uM \to T_uM$, skew–symmetric and with vanishing Schouten bracket [LM]. A function $h \in C^\infty(M)$ with a non trivial differential $df \in KerP$ is called a Casimir of P: $P_u\, df(u) = 0$. A Poisson morphism is a differentiable map which leaves invariant the Poisson bracket. Namely, $\Phi : M \to M$ is a Poisson morphism if $\{f, g\} \circ \Phi = \{f \circ \Phi, g \circ \Phi\}$, for each $f,\ g \in C^\infty(M)$; Φ leaves invariant the Poisson tensor P: $P_{\Phi(u)} = \Phi_*\, P_u\, \Phi^*$, where Φ_* and Φ^* denote, respectively, the tangent and the cotangent maps associated to Φ. In particular, if the Poisson bracket is non degenerate, i.e. P is invertible, and the Poisson morphism is a diffeomorphism, Φ defines a symplectic (canonical) transformation. M is said to be a bi–Hamiltonian manifold if it is endowed with two Poisson tensors P_0 and P_1 such that the associated pencil $P^\lambda := P_1 - \lambda P_0$ be itself a Poisson tensor for any $\lambda \in \mathbb{C}$ [Ma1].

2. Finite–dimensional bi–Hamiltonian systems from soliton equations

In order to reduce a bi–Hamiltonian hierarchy of soliton equations on invariant submanifolds we improve a method described in [Ton], where it was applied to *stationary flows* of the KdV hierarchy. This method adopts the unifying point of view of searching for the kernel of:

i) a given Poisson pencil in the case of *stationary flows*;
ii) some Poisson tensors extracted from the Poisson pencil in the case of *restricted flows*.

In this framework, a bi–Hamiltonian structure for the reduced vector fields can be obtained algorithmically by a systematic use of the Gelfand–Dickey (GD) polynomials. Furthermore, the use of the GD polynomials allows us to construct a map between the *stationary flows* and the *restricted flows*.

2.1. Bi–Hamiltonian hierarchies and Gelfand–Dickey polynomials. In this subsection we summarize the main facts about the bi–Hamiltonian theory, following

[CMP]; furthermore, we recall the GD polynomials and their realizations in the KdV hierarchy [Dic], showing explicitly some of their properties to be used below.
Let M be a bi–Hamiltonian manifold: if the associated Poisson pencil $P^\lambda := P_1 - \lambda P_0$ admits as a Casimir a formal Laurent series $h(\lambda)$

$$h(\lambda) := \sum_{j \geq 0} h_j \, \lambda^{-j} \ , \tag{2.1}$$

then h_0 is a Casimir of P_0 and the coefficients h_j ($j \geq 1$) are the Hamiltonian functions of a hierarchy of vector fields X_j, which are Hamiltonian with respect to both P_0 and P_1:

$$X_j = P_1 dh_j = P_0 dh_{j+1} \qquad\qquad (j \geq 0) \ . \tag{2.2}$$

At any point $u \in M$, the equations of the bi–Hamiltonian flows are given by $du/dt_j = X_j(u)$, t_j being the evolution parameter of the jth flow. The vector fields (2.2) are Hamiltonian also with respect to the Poisson pencil P^λ. In fact let us consider the polynomials $h^{(j)}(\lambda) := (\lambda^j h(\lambda))_+ = \sum_{k=0}^{j} h_k \, \lambda^{j-k}$, where the index $+$ means the projection of a Laurent series onto the purely polynomial part. Then the recursion relation (2.2) can be written as

$$X_j = P^\lambda \, dh^{(j)}(\lambda) \ . \tag{2.3}$$

Summarizing, the construction of a bi–Hamiltonian hierarchy with respect to a given Poisson pencil amounts to search the elements of its kernel which are exact 1–forms.

Remark 2.1. In the framework of stationary flows we will also consider Casimirs $\hat{h}(\lambda)$ of the Poisson pencil which are finite sums

$$\hat{h}(\lambda) = \sum_{j=0}^{n} \hat{h}_j \lambda^{-j} \ . \tag{2.4}$$

If such Casimirs exist, one has a finite hierarchy starting from a Casimir $\hat{h}_0$ of P_0 and ending with a Casimir $\hat{h}_n$ of P_1. □

Now let M be the algebra of polynomials in $u, u_x, u_{xx}, \ldots$ ($u = u(x)$ is a C^∞ function of x and the subscript x means the derivative with respect to x), and let P_0 and P_1 be two compatible Poisson tensors in M. Consider the associated Poisson pencil P^λ and look for the 1–forms

$$v(\lambda) := \sum_{j\geq 0} v_j \ \lambda^{-j} \ , \tag{2.5}$$

which are solutions of the equation

$$P^\lambda \, v(\lambda) = 0 \ . \tag{2.6}$$

For the moment we do not require that $v(\lambda)$ be an exact 1–form.

The solutions of Eq. (2.6) can be obtained as follows. Owing to the skew–symmetry of P_0 and P_1, two bilinear functions $\mathcal{B}_0(w_1, w_2)$ and $\mathcal{B}_1(w_1, w_2)$ exist such that

$$\frac{d}{dx}\mathcal{B}_k(w_1, w_2) = w_1 \ P_k w_2 + w_2 \ P_k w_1 \qquad\qquad (k = 0, 1 \ , \quad \forall\, w_1, w_2) \ . \tag{2.7}$$

These equations define $\mathcal{B}_0$ and $\mathcal{B}_1$up to additive constants which will be taken to be zero. The pencil of the two functions, denoted by B^λ:

$$B^\lambda := \mathcal{B}_1 - \lambda \mathcal{B}_0 \tag{2.8}$$

enjoys the relation

$$\frac{d}{dx}B^\lambda(w_1, w_2) = w_1 \ P^\lambda w_2 + w_2 \ P^\lambda w_1 \qquad\qquad (\forall\, w_1, w_2) \ . \tag{2.9}$$

So Eq. (2.6) is equivalent to:

$$B^\lambda(v(\lambda), v(\lambda)) = a(\lambda) \ , \tag{2.10}$$

where $a(\lambda) = \sum_{j\geq -1} a_j \lambda^{-j}$ and $\frac{d}{dx}a(\lambda) = 0$. As it is known from the literature [CMP], if the coefficients a_j are chosen independent of u, then each solution $v(\lambda)$ of (2.10) is an exact 1–form, $v(\lambda) = dh(\lambda)$, $h(\lambda)$ being a Casimir of the Poisson pencil. Moreover, the coefficients v_j are gradients of the Hamiltonians h_j of the hierarchy, fulfilling the bi–Hamiltonian relations (2.2). On account of this result, a_j will be chosen to be constant in the rest of the paper.

Remark 2.2. From the bilinearity of B^λ it follows that if $\bar{v}(\lambda)$ is a solution of (2.10) for $\bar{a}(\lambda)$ then $v(\lambda) = \sqrt{a(\lambda)/\bar{a}(\lambda)}\,\bar{v}(\lambda)$ is a solution of (2.10) for $a(\lambda)$. □

Eq. (2.10) can be solved developing the left hand side as a Laurent series:

$$B^\lambda(v(\lambda), v(\lambda)) = \sum_{k\geq -1} B_k \lambda^{-k} \ , \tag{2.11}$$

so that

$$B_{-1} = -\mathcal{B}_0(v_0, v_0) \ , \quad B_k = \sum_{j=0}^{k} \mathcal{B}_1(v_j, v_{k-j}) - \sum_{j=0}^{k+1} \mathcal{B}_0(v_j, v_{k+1-j}) \ , \tag{2.12}$$

and

$$B_k = a_k \qquad k = -1, 0, \dots \ . \tag{2.13}$$

Remark 2.3. The only term depending on v_{k+1} in each B_k is $-2\mathcal{B}_0(v_0, v_{k+1})$. □

Now let us introduce the GD polynomials. For any Laurent series $w(\lambda)$ of type (2.5), let us consider the functions $B^{(k)}(\lambda) := B^{\lambda}\left(w(\lambda), w^{(k)}(\lambda)\right)$, where $w^{(k)}(\lambda) := \left(\lambda^k w(\lambda)\right)_+$; it can be proved by direct computation that these functions have the form

$$B^{(k)}(\lambda) = \sum_{j=1}^{k+1} \lambda^j B_{k-j} + \sum_{j \geq 0} \lambda^{-j} p_{jk} \qquad (j, k \in \mathbb{N}_0) \ . \tag{2.14}$$

If $v(\lambda)$ is a solution of Eq. (2.10), the coefficients p_{jk} are called the GD polynomials and $B^{(k)}(\lambda)$ will be referred to as their generating functions. Indeed, if P_0 and P_1 are differential operators in d/dx, with a polynomial dependence on $u, u_x, u_{xx}, u_{xxx}, u^{(4)}, \dots$, (as in the KdV case), then the p_{jk} are polynomials in u and its x–derivatives.

The *fundamental* property of the GD polynomials, stemming from (2.14), (2.9) and (2.3), is the following relation with the gradients v_j and the bi–Hamiltonian vector fields X_k :

$$\frac{d}{dx} p_{jk} = v_j X_k \ . \tag{2.15}$$

Some other properties are contained in the following

Proposition 2.1. *For every $j, k \in \mathbb{N}_0$ and $w(\lambda)$ of type (2.5), the functions p_{jk} in Eq.(2.14) enjoy the properties*

$$\begin{aligned} & B^{\lambda}\left(w^{(k)}(\lambda), w^{(k)}(\lambda)\right) = \sum_{j=0}^{k} \lambda^j (2p_{k-j,k} - B_{2k-j}) + \sum_{j=k+1}^{2k+1} \lambda^j B_{2k-j} \ , \\ & B_k = p_{0k} - \mathcal{B}_0(w_0, w_{k+1}) \ , \qquad B_{2k} = 2p_{kk} - \mathcal{B}_1(w_k, w_k) \ . \end{aligned} \tag{2.16}$$

Proof. The first and the second equation are recovered evaluating by means of (2.14) the positive part, respectively, of $B^\lambda\Big(w^{(k)}(\lambda)-\lambda^k w(\lambda), w^{(k)}(\lambda)-\lambda^k w(\lambda)\Big)$ and of $B^\lambda\Big(w(\lambda), w^{(k)}(\lambda)-\lambda^k w(\lambda)\Big)$. The last equation is obtained extracting from the first one the coefficient of λ^0. $\square$

Remark 2.4. If $\hat{w}(\lambda)$ is a finite sum as in Eq. (2.4), i.e. $\hat{w}_l = 0$ for $l > n$, the coefficients $\hat{p}_{jn}$ in Eq. (2.14) are given by

$$\hat{p}_{jn} = \begin{cases} p_{jn|w_l=0\ l>n} & \text{if } j \leq n\ , \\ 0 & \text{if } j > n. \end{cases} \tag{2.17}$$

$\square$

Example: the KdV hierarchy. Let us apply the previous scheme to the KdV hierarchy. As it is well known [Ma1], it originates from the Poisson tensors

$$P_0 := \frac{d}{dx}\ , \qquad P_1 := \frac{d^3}{dx^3} + 4u\frac{d}{dx} + 2u_x\ , \tag{2.18}$$

which give rise to the following bilinear function

$$B^\lambda(w_1, w_2) := w_{1xx}w_2 + w_1 w_{2xx} - w_{1x}w_{2x} + 4uw_1w_2 - \lambda w_1 w_2\ . \tag{2.19}$$

We report some polynomials p_{jk} to be used in the following:

(2.20)

$$
\begin{aligned}
p_{00} &= 4u - w_1 \ , \\
p_{01} &= 8uw_1 - w_1^2 - w_2 + 2w_{1xx} \ , \\
p_{02} &= 4uw_1^2 + 8uw_2 - 2w_1w_2 - w_3 - w_{1x}^2 + 2w_1w_{1xx} + 2w_{2xx} \ , \\
p_{03} &= 8uw_1w_2 - w_2^2 + 8uw_3 - 2w_1w_3 - w_4 - 2w_{1x}w_{2x} + 2w_2w_{1xx} + 2w_1w_{2xx} + 2w_{3xx} \ , \\
p_{04} &= 4uw_2^2 + 8uw_1w_3 - 2w_2w_3 + 8uw_4 - 2w_1w_4 - w_5 + \\
&\quad - w_{2x}^2 - 2w_{1x}w_{3x} + 2w_3w_{1xx} + 2w_2w_{2xx} + 2w_1w_{3xx} + 2w_{4xx} \ , \\
p_{12} &= 8uw_1w_2 - w_2^2 + 4uw_3 - w_1w_3 - w_4 + 2w_{1x}w_{2x} + 2w_2w_{1xx} + 2w_1w_{2xx} + w_{3xx} \ , \\
p_{14} &= 8uw_2w_3 - w_3^2 + 8uw_1w_4 - 2w_2w_4 + 4uw_5 - w_1w_5 - w_6 + \\
&\quad - 2w_{2x}w_{3x} - 2w_{1x}w_{4x} + 2w_4w_{1xx} + 2w_2w_{3xx} + 2w_3w_{2xx} + 2w_1w_{4xx} + w_{5xx} \ , \\
p_{24} &= 4uw_3^2 + 8uw_2w_4 - 2w_3w_4 + 4uw_1w_5 - w_2w_5 + 4uw_6 - w_1w_6 - w_7 + \\
&\quad - w_{3x}^2 - 2w_{2x}w_{4x} - w_{1x}w_{5x} + 2w_4w_{2xx} + 2w_3w_{3xx} + 2w_2w_{4xx} + w_1w_{5xx} + w_{6xx} \ , \\
p_{34} &= 8uw_3w_4 - w_4^2 + 4uw_2w_5 - w_3w_5 + 4uw_1w_6 - w_2w_6 + 4uw_7 - w_1w_7 - w_8 - 2w_{3x}w_{4x} + \\
&\quad - w_{2x}w_{5x} - w_{1x}w_{6x} + w_6w_{1xx} + w_5w_{2xx} + 2w_4w_{3xx} + 2w_3w_{4xx} + w_2w_{5xx} + w_1w_{6xx} + w_{7xx} \ , \\
p_{kk} &= w_{kxx}w_k - \frac{w_{kx}^2}{2} + 2uw_k^2 + \frac{B_{2k}}{2} \ .
\end{aligned}
$$

Remark 2.5. The polynomials p_{kk} in the above list are computed using the last equation (2.16). They will have a relevant role in the construction of a second Hamiltonian formulation both for stationary flows and for restricted flows. □

Now we turn to the solution of Eq.(2.10), with B^λ given by (2.19). Since in this case $\mathcal{B}_0(w_1, w_2) = w_1w_2$, the second equation (2.16) can be written

$$B_k = p_{0k} - w_0w_{k+1} \ , \tag{2.21}$$

and the system (2.13) becomes

$$v_0^2 = -a_{-1} \ , \qquad p_{0k} - v_0v_{k+1} = a_k \qquad\qquad k \in \mathbb{N}_0 \ . \tag{2.22}$$

On account of Rem. 2.3 this system can be solved recursively with respect to v_{k+1}; for each k and for each $a(\lambda)$, it furnishes the coefficients of the unique solution (up to a sign) $v(\lambda)$. The solution corresponding to

$$\bar{a}(\lambda) = -\lambda \tag{2.23}$$

is the so–called basis solution $\bar{v}(\lambda)$; its first coefficients are:

(2.24)

$$\bar{v}_0 = 1 \ , \qquad \bar{v}_1 = 2u \ , \qquad \bar{v}_2 = 2(u_{xx} + 3u^2) \ , \qquad \bar{v}_3 = 2(u^{(4)} + 5u_x^2 + 10u_{xx}u + 10u^3)$$

and so on, namely the gradients of the first KdV Hamiltonians. In the following we shall consider also the 1–form $v(\lambda) = c(\lambda)\bar{v}(\lambda)$, which is solution of (2.10) for

$$a(\lambda) = -\lambda c^2(\lambda) \ , \qquad c_0 = 1 \tag{2.25}$$

(see Rem. (2.2)), where the coefficient c_j are free parameters. In this case the first 1–forms of the hierarchy are

(2.26)

$$v_0 = 1 \ , \quad v_1 = \bar{v}_1 + c_1 \ , \quad v_2 = \bar{v}_2 + c_1\bar{v}_1 + c_2 \ , \quad v_3 = \bar{v}_3 + c_1\bar{v}_2 + c_2\bar{v}_1 + c_3 \ .$$

Finally we denote by $\bar{p}_{jk}$ the GD polynomials corresponding to the fundamental solution $\bar{v}(\lambda)$. They are essentially the polynomials defined in [Dic, Prop. 12.1.12].

2.2. The method of stationary flows. The method of *stationary flows* [Nov] was developed in order to reduce the flows of the KdV hierarchy onto the set M_n defined, for every integer n, by

$$M_n := \{u \mid X_n(u, u_x, \ldots, u^{(2n+1)}) = 0\} \ . \tag{2.27}$$

This manifold, being the set of fixed points of the nth flow, is invariant with respect to each flow of the hierarchy; so the corresponding vector fields can be restricted to M_n. It is implicitly given by the solutions of a non linear ordinary differential equation (ODE) of order $2n+1$ and can be parametrized by the Cauchy initial data $u(x_0), u_x(x_0), \ldots, u^{(2n)}(x_0)$. As M_n is odd-dimensional it cannot be a symplectic manifold; nevertheless it will be shown to be a bi–Hamiltonian manifold, and it will be referred to as *extended phase space*. Moreover, M_n is naturally foliated, on account of (2.2) and (2.18), by a one–parameter family of $2n$–dimensional submanifolds S_n given by

$$S_n := \{u \mid v_{n+1}(u, u_x, \ldots, u^{(2n)}) = c\} \tag{2.28}$$

(c being a constant parameter), which are again invariant manifolds with respect to each vector field of the KdV hierarchy, due to the invariance of the 1–forms v_k. So M_n can be naturally parametrized by the gradients of the Hamiltonians $v_1, \ldots, v_{n+1}$ and by their x–derivatives $v_{1x}, \ldots, v_{nx}$. We shall use these coordinates in the following.

The original reduction of stationary flows consists in the restriction of the vector fields X_j of the hierarchy to a fixed leaf S_n, by a variational formulation and using the so–called Ostrogradski coordinates [BN]. In this way finite–dimensional integrable Hamiltonian systems, with x as evolution parameter, are obtained, the integrals of motion being just the GD polynomials p_{jn} $(j = 1, \dots, n)$ restricted to S_n. The last property follows from the fundamental property (2.15) of the GD polynomials. However these Hamiltonian systems are not natural in the Ostrogradski coordinates, so that they do not appear physically interesting.

Here we perform two different stationary reductions of the KdV flows by improving the procedure introduced in [Ton]. On one side, we choose as reduction submanifold $S_n^{(0)}$ just the leaf S_n, of the foliation (2.28), corresponding to $c = 0$. Owing to Eq. (2.22), it is a level set of the GD polynomial $\hat{p}_{0n}$. On account of Eq. (2.15), also the GD polynomials $\hat{p}_{jn}$, restricted to M_n, are invariant with respect to each flow of the hierarchy; thus we can choose as a second reduction submanifold $S_n^{(1)}$ a level set of $\hat{p}_{nn}$. The one–parameter family of the level sets of $\hat{p}_{nn}$ forms another foliation of the manifold M_n. In steps i)–iii), to be described below, we reduce the KdV flows onto the manifold $S_n^{(0)}$ and $S_n^{(1)}$ respectively; furthermore, in step iv) we construct the bi–Hamiltonian structure of the reduced flows in the extended phase space M_n.

From the computational point of view, the previous geometric reduction can be performed as follows. Due to (2.3), the manifold M_n is defined by the solutions u of the equation

$$P^\lambda \hat{v}^{(n)}(\lambda) = 0 \ , \tag{2.29}$$

where $\hat{v}(\lambda) = \sum_{j=0}^{n} \hat{v}_j \lambda^{-j}$, $\hat{v}^{(n)}(\lambda) = (\lambda^n \hat{v}(\lambda))_+ = \lambda^n \hat{v}(\lambda)$.Taking into account (2.9) and that $\hat{v}(\lambda)$ is a finite sum, this equation is equivalent to:

$$B^\lambda\Big(\hat{v}(\lambda), \hat{v}^{(n)}(\lambda)\Big) = \lambda^n \hat{a}(\lambda) \ , \tag{2.30}$$

where $\hat{a}(\lambda) = \sum_{j=-1}^{2n} a_j \lambda^{-j}$, (from now on we choose $a_{-1} = -1$ for convenience).

Remark 2.6. In particular if $\hat{a}(\lambda) = -\lambda c^2(\lambda)$, as in (2.25), M_n is given by

$$M_n = \left\{ u | \ \bar{X}_n + \sum_{j=1}^{n} c_j \bar{X}_{n-j} = 0 \right\} \ , \tag{2.31}$$

i.e. by the solutions of the Lax–Novikov equations [Lax]. □

Equating in Eq. (2.30) the coefficients of the same powers of λ and taking into account (2.14), (2.21) and Rem. 2.4 we get the following system

$$\begin{cases} \hat{v}_0^2 = 1 \\ \hat{p}_{0k} - \hat{v}_0\hat{v}_{k+1} = a_k & (k = 0, \ldots, n-1), \\ \hat{p}_{jn} = a_{n+j} & (j = 0, \ldots, n). \end{cases} \tag{2.32}$$

Remark 2.7. Since $\hat{v}^{(n)}$ is a finite sum, (2.30) is equivalent to the equation $B^\lambda\Big(\hat{v}^{(n)}(\lambda), \hat{v}^{(n)}(\lambda)\Big) = \lambda^{2n}\hat{a}(\lambda)$, used by Alber [Alb]. Due to (2.16), taking the coefficients of λ^j $(j = 0 \ldots, n)$ in this equation one obtains

$$2\hat{p}_{jn} - B_{n+j} = a_{n+j} \qquad (j = 0, \ldots, n) \ . \tag{2.33}$$

Comparing Eq. (2.33) with the last $n+1$ equations (2.32) one has

$$B_{n+j} = \hat{p}_{jn} \qquad (j = 0, \ldots, n) \ .\square \tag{2.34}$$

In order to obtain the dynamical equations corresponding to the reduction of the first vector field of the KdV hierarchy on the submanifolds $S_n^{(0)}$ and $S_n^{(1)}$, we make the following steps:
i) The choice of the system of $(n+1)$ equations in the variables $(u, \hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n)$ given by

$$\hat{p}_{0k} - \hat{v}_{k+1} = a_k \quad (k = 0, \ldots, n-1) \ ; \qquad \hat{p}_{0n} = a_n \ . \tag{2.35}$$

It is equivalent to the first $n+2$ equations (2.32) putting $\hat{v}_0 = 1$. The remaining equations will furnish a set of integrals of motion, whose independence has been proved in [Dic]. As it will be shown later on, this system furnish the first Poisson structure of the reduced system and will be referred to as P_0–system. In order to obtain a second Poisson structure, we consider the following system (P_1–system)

$$\hat{p}_{0k} - \hat{v}_{k+1} = a_k \quad (k = 0, \ldots, n-1) \ ; \qquad \hat{p}_{nn} = a_{2n} \ , \tag{2.36}$$

which differs from the previous one only for the last equation; $\hat{p}_{nn}$ is computed from the list (2.20) taking into account (2.34).
ii) The previous systems are decoupled by using the first equation to eliminate u: $u = \hat{v}_1/2 + a_0/4$. This procedure gives rise to the reduced systems of second order ODE's in the n variables $\hat{v}_j$ $(j = 1, \ldots, n)$:

$$\hat{p}_{0k} - \hat{v}_{k+1} = a_k \quad (k = 1, \ldots, n-1) \ ; \qquad \hat{p}_{0n} = a_n \ , \tag{2.37}$$

$$\hat{p}_{0k} - \hat{v}_{k+1} = a_k \quad (k = 1, \dots, n-1) \; ; \qquad \hat{p}_{nn} = a_{2n} \; . \tag{2.38}$$

iii) The system (2.37) can be written in Lagrangian form. To this purpose, we use the so-called Newton or r–representation introduced in [Wo2]. Namely, we choose as new coordinates in $S_n^{(0)}$ the first n coefficients r_j of the formal series $r(\lambda) := \sqrt{\hat{v}(\lambda)}$, i.e.,

$$r_k = \Delta_{-k}(\sqrt{\hat{v}(\lambda)} \qquad (k = 1, \dots, n) \; , \tag{2.39}$$

where Δ_k means the coefficient of λ^k in a Laurent series. Taking into account Eq.(2.30), with the substitution $\hat{v}^{(n)}(\lambda) = r^2(\lambda)^{(n)}$ and observing that $2r_{n+1} = -\sum_{j=1}^n r_j r_{n-j+1}$, Eqs. (2.37) are equivalent to

$$\Big(\lambda^n \Big(r_{xx} + (r_1 + \frac{a_0 - \lambda}{4})r - \frac{\hat{a}}{4r^3}\Big)\Big)_+ = 0 \; . \tag{2.40}$$

This system is Lagrangian, with Lagrangian function

$$L_n^{(0)} = \Delta_{-(n+1)}\Big(\mathcal{L}(\lambda; r(\lambda)\Big) \; , \tag{2.41}$$

where $\mathcal{L}\Big(\lambda; w(\lambda)\Big)$ is given, for each Laurent series $w(\lambda)$, by

$$\mathcal{L}(\lambda; w(\lambda) := \frac{1}{2}\Big(w_x(\lambda)\Big)^2 - \frac{1}{2}(w_1 + \frac{a_0 - \lambda}{4})w^2(\lambda) - \frac{\hat{a}(\lambda)}{8w^2(\lambda)} \; . \tag{2.42}$$

The Lagrangian gradients $\frac{\delta}{\delta r_k} := \frac{\partial}{\partial r_k} - \frac{d}{dx}\frac{\partial}{\partial r_{kx}}$ of $L_n^{(0)}$ are

$$\frac{\delta L_n^{(0)}}{\delta r_k} = \Delta_{k-1}\Big(\lambda^n\Big(- r_{xx} - (r_1 + \frac{a_0 - \lambda}{4})r + \frac{\hat{a}}{4r^3}\Big)\Big)_+ \quad (k = 1, \dots, n) \; . \tag{2.43}$$

Our new result is that it is possible to put also the P_1–system (2.38) in Lagrangian form with the following choice of coordinates in $S_n^{(1)}$

$$q_k = \Delta_{-k}\Big(\sqrt{\hat{v}(\lambda)}\Big) \qquad (k = 1 \dots, n-1); \qquad q_n = -\sqrt{v_n} \; . \tag{2.44}$$

In other words we take $q_k = r_k$ $(k = 1, \dots, n-1)$ and $q_n = \sqrt{-v_n}$. The choice of the last variable q_n is motivated by the form (2.20) of p_{nn} (see also Rem. 2.5). Indeed, putting $v_n = -q_n^2$, we can write the equation $p_{nn} = a_{2n}$ as a Newton equation without q_{nx}. Furthermore, defining $q(\lambda) := \sqrt{\hat{v}(\lambda)}$, we observe that the system (2.38) is equivalent to

$$\begin{aligned} &\left(\lambda^{n-1}(q_{xx} + (q_1 + \frac{a_0 - \lambda}{4})q - \frac{\hat{a}}{4q^3})\right)_+ + \frac{1}{2}q_n^2 = 0 \ , \\ &q_{nxx} + (q_1 + \frac{a_0}{4})q_n - \frac{a_{2n}}{4q_n^3} = 0 \ . \end{aligned} \tag{2.45}$$

This is a Lagrangian system with Lagrangian

$$L_n^{(1)} = \Delta_{-n}\Big(\mathcal{L}(\lambda; q(\lambda)\Big) + \frac{1}{2}q_{nx}^2 - \frac{1}{2}(q_1 + \frac{a_0}{4})q_n^2 - \frac{a_{2n}}{8q_n^2} \ . \tag{2.46}$$

Indeed it can be verified that the Lagrangian gradients of $L_n^{(1)}$ are

(2.47)
$$\begin{aligned} \frac{\delta L_n^{(1)}}{\delta q_1} &= \Delta_0 \left(\lambda^{n-1}\Big(- q_{xx} - (q_1 + \frac{a_0 - \lambda}{4})q + \frac{\hat{a}}{4q^3}\Big)\right)_+ - \frac{1}{2}q_n^2 \\ \frac{\delta L_n^{(1)}}{\delta q_k} &= \Delta_{k-1} \left(\lambda^{n-1}(-q_{xx} - \Big(q_1 + \frac{a_0 - \lambda}{4})q + \frac{\hat{a}}{4q^3}\Big)\right)_+ \quad (k = 2, \dots, n-1) \\ \frac{\delta L_n^{(1)}}{\delta q_n} &= -q_{nxx} - (q_1 + \frac{a_0}{4})q_n + \frac{a_{2n}}{4q_n^3} \end{aligned}$$

The two Lagrangian systems can be put in Hamiltonian form. For the P_0-system the canonical momenta are

$$s_{n+1-k} = r_{kx} \qquad (k = 1, \dots, n) \tag{2.48}$$

and the Hamiltonian function

$$H_n^{(0)} = \Delta_{-(n+1)}\Big(\mathcal{H}(\lambda; r(\lambda), s(\lambda))\Big) \tag{2.49}$$

where $\mathcal{H}(\lambda; w(\lambda), z(\lambda))$ is given by

$$\mathcal{H}(\lambda; w(\lambda), z(\lambda)) = \frac{1}{2}z^2(\lambda) + \frac{1}{2}\Big(w_1 + \frac{a_0 - \lambda}{4}\Big)w^2(\lambda) + \frac{\hat{a}(\lambda)}{8w^2(\lambda)} \ , \tag{2.50}$$

and $s(\lambda) = \sum_{j=1}^n s_j \lambda^{-j}$. For the P_1–system the canonical momenta are

$$p_n = q_{nx} \ , \qquad p_{n-k} = q_{kx} \qquad (k = 1, \dots, n) \ , \tag{2.51}$$

and the Hamiltonian function is

$$H_n^{(1)} = \Delta_{-n}\Big(\mathcal{H}(\lambda; q(\lambda), p(\lambda))\Big) + \frac{1}{2}p_n^2 + \frac{1}{2}(q_1 + \frac{a_0}{4})q_n^2 + \frac{a_{2n}}{8q_n^2} \ , \tag{2.52}$$

with $p(\lambda) = \sum_{j=1}^{n} p_j \lambda^{-j}$.

The two Hamiltonian functions depend on the two sets of coordinates and momenta (r_k, s_k), (q_k, p_k) respectively and on the two sets of free parameters $(a_0, \dots, a_{n-1}, a_n)$ and $(a_0, \dots, a_{n-1}, a_{2n})$.

iv) Now let us consider the manifold M_n (2.27), which can be parametrized by (r_k, s_k, a_n), or by (q_k, p_k, a_{2n}), with the parameters a_n and a_{2n} regarded as additional dynamical variables in M_n. On this manifold one can extend trivially the canonical Poisson structures, the Hamiltonians and the vector fields associated with each one of the two systems, following a method introduced in [AFW]. In particular the vector fields can be extended in such a way that they are tangent to the foliations $S^{(0)}_{a_n}$ and $S^{(1)}_{a_{2n}}$. We denote by a tilda the extended tensor fields. Taking into account, on one side, the relation between the two sets of coordinates through the original variables (v_k, v_{kx}), on the other side the relation between the two integrals of motion a_n and a_{2n} through the GD polynomials p_{0n} and p_{nn}, a map $\Phi : M_n \to M_n, (r_k, s_k, a_n) \mapsto (q_k, p_k, a_{2n})$ can be systematically constructed. It relates the Hamiltonians and the vector fields of one system with the corresponding ones of the other system. Since this map is not a Poisson morphism, the extended canonical Poisson structures associated with one chart is mapped into a Poisson structure different from the extended canonical structure associated with the other chart. If this second Poisson tensor is compatible with the extended canonical one, a bi–Hamiltonian formulation of the two system is obtained.

In conclusion the previous steps can be summarized as follows:

Proposition 2.2. *The systems* (2.37) *and* (2.38), *written respectively in the coordinates* (2.39) *and* (2.44), *are natural Lagrangian systems. The corresponding canonical Hamiltonian systems*

$$r_{kx} = \frac{\partial H_n^{(0)}}{\partial s_k} \ , \qquad s_{kx} = -\frac{\partial H_n^{(0)}}{\partial r_k} \ , \tag{2.53}$$

$$q_{kx} = \frac{\partial H_n^{(1)}}{\partial p_k} \ , \qquad p_{kx} = -\frac{\partial H_n^{(1)}}{\partial q_k} \ , \tag{2.54}$$

have n integrals of motion given by

(2.55)
$$K_j \equiv -\frac{1}{8}\hat{p}_{jn_{|Y}} = a_{n+j} \ (j = 1, \dots , n) \ , \quad H_j \equiv -\frac{1}{8}\hat{p}_{jn_{|X}} = a_{n+j} \ (j = 0, \dots , n-1) \ .$$

Moreover, the map $\Phi : M_n \to M_n$ in the extended phase space generates a second Poisson structure.

Remark 2.8. The symbols $|Y$ and $|X$ in (2.55) mean that, in the GD polynomials $\hat{p}_{jk}$, the coordinates (v_k, v_{kx}) must be replaced by the canonical coordinates (r_k, s_k) and (q_k, p_k) respectively and that the first order x–derivatives of momenta must be eliminated by means of the Hamiltonian dynamical equations (2.53), (2.54). □

In the next Subsection we shall give some applications of the results stated in this proposition.

2.3. Example I: the bi–Hamiltonian structure of a Henon–Heiles system. Here we present the bi–Hamiltonian structure of a generalized Henon–Heiles system with two degrees of freedom. Its Hamiltonian is

$$H_0 = \frac{1}{2}\left(p_1^2 + p_2^2\right) + q_1^3 + \frac{1}{2}q_1 q_2^2 + \frac{a_4}{8q_2^2} + \frac{a_0}{2}\left(q_1^2 + \frac{1}{4}q_2^2\right) - \frac{a_1}{4}q_1 \tag{2.56}$$

where q_1, q_2, p_1, p_2 are the canonical coordinates and momenta and a_0, a_1, a_4 are free constant parameters.

Remark 2.9. The previous Hamiltonian encompasses the two cases $a_0 = a_4 = 0$ and $a_0 = a_1 = 0$ introduced in [BW]. Moreover H_0 is related with the Hamiltonian

$$H_H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{1}{2}\left(Aq_1'^2 + Bq_2'^2\right) + q_1'^3 + \frac{1}{2}q_1' q_2'^2 + \frac{a_4}{8q_2'^2} \ , \tag{2.57}$$

through the map

(2.58)
$$q_1 = q_1' + \frac{A}{2} - 2B \ , \quad q_2 = q_2' \ , \qquad a_0 = -2A + 12B \ , \ a_1 = -A^2 + 16AB - 48B^2 \ .$$

H_H is the Hamiltonian of the classical integrable Henon–Heiles system [Tab] with the additional term $a_4/8q_2'^2$. □

The Hamiltonian (2.56) can be recovered, together with a second independent integral of motion by applying the method discussed in Subsect. 2.2. On account of Rem. 2.6 it corresponds to the reduction of the first vector field of the KdV hierarchy on the stationary manifold

(2.59)
$$M_2 = \left\{ u | u^{(5)} + 10u_{xxx}u + 20u_{xx}u_x + 30u_x u^2 + c_1(u_{xxx} + 6u_x u) + c_2 u_x = 0 \right\} \ ,$$

where $c_1 = -a_0/2$, $c_2 = -a_1/2 + a_0^2/4$.

One proceeds as follows:

i) The P_0–system and the P_1–system of ODE's corresponding to Eqs. (2.35) and (2.36) become respectively

$$\begin{aligned} & 2u - \hat{v}_1 = \frac{a_0}{2} \ , \qquad\qquad 8u\hat{v}_1 - \hat{v}_1^2 + 2\hat{v}_{1xx} - 2\hat{v}_2 = a_1 \ , \\ & 4u\hat{v}_1^2 + 8u\hat{v}_2 - 2\hat{v}_1\hat{v}_2 - \hat{v}_{1x}^2 + 2\hat{v}_1\hat{v}_{1xx} + 2\hat{v}_{2xx} = a_2 \ , \end{aligned} \tag{2.60}$$

$$\begin{aligned} & 2u - \hat{v}_1 = \frac{a_0}{2} \ , \qquad\qquad 8u\hat{v}_1 - \hat{v}_1^2 + 2\hat{v}_{1xx} - 2\hat{v}_2 = a_1 \ , \\ & 2\hat{v}_{2xx}\hat{v}_2 - \hat{v}_{2x}^2 + 4u\hat{v}_2^2 = a_4 \ . \end{aligned} \tag{2.61}$$

ii) The reduced systems corresponding to (2.37) and (2.38) are obtained by eliminating u by means of the first equation, so that

$$\begin{aligned} \hat{v}_{1xx} &= -\frac{3}{2}\hat{v}_1^2 + \hat{v}_2 - a_0\hat{v}_1 + \frac{a_1}{2} \\ \hat{v}_{2xx} &= \frac{1}{2}\hat{v}_1^3 - 2\hat{v}_1\hat{v}_2 + \frac{1}{2}\hat{v}_{1x}^2 + \frac{a_0}{2}\hat{v}_1^2 - a_0\hat{v}_2 - \frac{a_1}{2}\hat{v}_1 + \frac{a_2}{2} \ , \end{aligned} \tag{2.62}$$

$$\hat{v}_{1xx} = -\frac{3}{2}\hat{v}_1^2 + \hat{v}_2 - a_0\hat{v}_1 + \frac{a_1}{2} \ , \qquad \hat{v}_{2xx} = \frac{\hat{v}_{2x}^2}{2\hat{v}_2} - \hat{v}_1\hat{v}_2 + \frac{a_4}{2\hat{v}_2} - \frac{a_0}{2}\hat{v}_2 \ . \tag{2.63}$$

iii) Introduce in (2.62) the coordinates (2.39) $r_1 = \hat{v}_1/2, r_2 = \hat{v}_2/2 - \hat{v}_1^2/4$ and in (2.63) the coordinates (2.44) $q_1 = \hat{v}_1/2, q_2 = \sqrt{-\hat{v}_2}$; the two systems take the Lagrangian form

$$\begin{aligned} r_{1xx} &= -\frac{5}{2}r_1^2 + r_2 - a_0 r_1 + \frac{a_1}{4} \\ r_{2xx} &= \frac{5}{2}r_1^3 - 5r_1 r_2 + \frac{3}{2}a_0 r_1^2 - \frac{3}{4}a_1 r_1 - a_0 r_2 + \frac{a_2}{4} \ , \end{aligned} \tag{2.64}$$

$$q_{1xx} = -3q_1^2 - \frac{1}{2}q_2^2 - a_0 q_1 + \frac{a_1}{4} \quad , \quad q_{2xx} = -q_1 q_2 + \frac{a_4}{4q_2^3} - \frac{a_0}{4}q_2 \ , \tag{2.65}$$

with Lagrangian functions

$$
(2.66)\quad L_2^{(0)} = r_{1x} r_{2x} - V_2^{(0)} \ ,
$$

$$
V_2^{(0)} = -\frac{5}{8} r_1^4 + \frac{5}{2} r_1^2 r_2 - \frac{1}{2} r_2^2 - \frac{a_0}{2} r_1^3 + \frac{3}{8} a_1 r_1^2 + a_0 r_1 r_2 - \frac{a_2}{4} r_1 - \frac{a_1}{4} r_2 \ ,
$$

(2.67)

$$
L_2^{(1)} = \frac{1}{2}\left(q_{1x}^2 + q_{2x}^2\right) - V_2^{(1)} \ , \quad V_2^{(1)} = q_1^3 + \frac{1}{2} q_1 q_2^2 + \frac{a_4}{8 q_2^2} + \frac{a_0}{2} q_1^2 + \frac{a_0}{8} q_2^2 - \frac{a_1}{4} q_1 \ .
$$

These system can be put in Hamiltonian form, taking $s_1 = r_{2x}$, $s_2 = r_{1x}$ and $p_1 = q_{1x}$, $p_2 = q_{2x}$ as canonical momenta (see (2.48) and (2.51)). The integrals of motion which are obtained by the reduction of the GD polynomials are

$$
K_0 \equiv -\frac{1}{8} \hat{p}_{02|Y} = -\frac{a_2}{8} \ , \tag{2.68}
$$

$$
K_1 \equiv -\frac{1}{8} \hat{p}_{12|Y} = s_1 s_2 + V_2^{(0)} \ , \tag{2.69}
$$

$$
\begin{aligned}
K_2 \equiv -\frac{1}{8} \hat{p}_{22|Y} &= -s_2^2 r_2 + s_1 s_2 r_1 + \frac{1}{2} s_1^2 - \frac{1}{2} r_1^5 + 2 r_1 r_2^2 - \frac{3}{8} a_0 r_1^4 + \\
&+ \frac{a_1}{4} r_1^3 - \frac{a_0}{2} r_1^2 r_2 + \frac{a_1}{2} r_1 r_2 + \frac{a_0}{2} r_2^2 - \frac{a_2}{8} r_1^2 - \frac{a_2}{4} r_2 \ ,
\end{aligned} \tag{2.70}
$$

and

$$
H_0 \equiv -\frac{1}{8} \hat{p}_{02|X} = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V_2^{(1)} \ , \tag{2.71}
$$

$$
H_1 \equiv -\frac{1}{8} \hat{p}_{12|X} = p_2^2 q_1 - p_1 p_2 q_2 - \frac{1}{2} q_1^2 q_2^2 - \frac{1}{8} q_2^4 + \frac{a_4 q_1}{4 q_2^2} - \frac{a_0}{4} q_1 q_2^2 + \frac{a_1}{8} q_2^2 \ , \tag{2.72}
$$

$$
H_2 \equiv -\frac{1}{8} \hat{p}_{22|X} = -\frac{a_4}{8} \ , \tag{2.73}
$$

The corresponding Hamiltonian vector fields will be respectively denoted by $Y_j := P_0\, dK_j$ and $X_{j+1} := P_1\, dH_j$ $(j = 0, 1, 2)$; P_0 and P_1 are represented in the corresponding coordinates by the canonical Poisson matrix $E = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$, where $\mathbf{0}$ and $\mathbf{1}$ are 2×2 zero and identity matrices respectively.

iv) Let M_2 be the 5–dimensional extended phase space parametrized by $(r_1, r_2, s_1, s_2; a_2)$ or $(q_1, q_2, p_1, p_2; a_4)$. It is convenient to make use of block notations. So, for example, we denote with $m = (r, s; a)$ the 5–tuple $(r_1, r_2, s_1, s_2; a_2)$, with $\tilde{X} = [\tilde{X}^r, \tilde{X}^s; \tilde{X}^a]^T$

the generic vector field and with $d\tilde{K} = [\partial\tilde{K}/\partial r, \partial\tilde{K}/\partial s; \partial\tilde{K}/\partial a]^T$ the generic gradient of a function $\tilde{K}$ (the superscript T means transposition). In this notation a vector field $\tilde{X} = \tilde{P}\, d\tilde{K}$ with Hamiltonian function $\tilde{K}$ with respect to a Poisson tensor $\tilde{P}$ will be written

$$\begin{bmatrix} \tilde{X}^r \\ \tilde{X}^s \\ \tilde{X}^a \end{bmatrix} = \begin{bmatrix} P^{rr} & P^{rs} & P^{ra} \\ P^{sr} & P^{ss} & P^{sa} \\ P^{ar} & P^{as} & P^{aa} \end{bmatrix} \begin{bmatrix} \frac{\partial\tilde{K}}{\partial r} \\ \frac{\partial\tilde{K}}{\partial s} \\ \frac{\partial\tilde{K}}{\partial a} \end{bmatrix} , \tag{2.74}$$

where $P^{sr} = -(P^{rs})^T$ etc. ..., for the skew–symmetry of P. From the definition of r_1, r_2 and q_1, q_2 in terms of v_1 and v_2, and from (2.73) and (2.70) one obtains the following map $\Phi : M_2 \to M_2, (r, s; a_2) \mapsto (q, p; a_4)$

$$\begin{aligned} & q_1 = r_1 \ , \qquad q_2 = (-2r_2 - r_1^2)^{1/2} \\ & p_1 = s_2 \ , \qquad p_2 = -\frac{s_1 + r_1 s_2}{(-2r_2 - r_1^2)^{1/2}} \ , \qquad a_4 = -8K_2 \end{aligned} \tag{2.75}$$

with K_2 given by Eq. (2.70). In these two charts let us consider the extended Hamiltonians $\tilde{H}_j$, $\tilde{K}_j$, the vector fields $\tilde{X}_j$, with components $\tilde{X}_j^r = X_j^r$, $\tilde{X}_j^s = X_j^s$, $\tilde{X}_j^a = 0$ and $\tilde{Y}_j$, with components $\tilde{Y}_j^r = Y_j^r$, $\tilde{Y}_j^s = Y_j^s$, $\tilde{Y}_j^a = 0$, and the extension of the canonical Poisson structure, $\tilde{E} := \left(\begin{smallmatrix} \mathbf{0} & \mathbf{1} & 0 \\ -\mathbf{1} & \mathbf{0} & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. The following proposition holds

Proposition 2.3. *The action of the map $\Phi : M_2 \to M_2$ defined by (2.75) on the Hamiltonians $\tilde{H}_j$, the vector fields $\tilde{Y}_j$ and the Poisson tensor $\tilde{P}_0' := \tilde{E}$ is given by*

$$\Phi^*(\tilde{H}_j) = \tilde{K}_j \ , \qquad \Phi_*(\tilde{Y}_j) = \tilde{X}_j \tag{2.76}$$

$$\tilde{P}_0 := \Phi_* \tilde{P}_0' \Phi^* = \begin{bmatrix} 0 & A & -8\tilde{X}_2^q \\ -A^T & B & -8\tilde{X}_2^p \\ 8(\tilde{X}_2^q)^T & 8(\tilde{X}_2^p)^T & 0 \ , \end{bmatrix} \tag{2.77}$$

where

$$A = \begin{bmatrix} 0 & -\frac{1}{q_2} \\ -\frac{1}{q_2} & \frac{2q_1}{q_2^2} \end{bmatrix} , \qquad B = \begin{bmatrix} 0 & -\frac{p_2}{q_2^2} \\ \frac{p_2}{q_2^2} & 0 \end{bmatrix} . \tag{2.78}$$

So, the map Φ is not a Poisson morphism.

Thus we have recovered in the extended phase space M_2 a second Poisson tensor $\tilde{P}_0$. We can check that $\tilde{P}_0$ is compatible with $\tilde{P}_1 = \tilde{E}$. Furthermore $\tilde{P}_0$ and $\tilde{P}_1$ give rise to the following bi–Hamiltonian hierarchy

$$\tilde{X}_{j+1} := \tilde{P}_1 \, d\tilde{H}_j = \tilde{P}_0 \, d\tilde{H}_{j+1} \qquad (j = 0, 1) \ , \tag{2.79}$$

the Hamiltonians $\tilde{H}_0$ and $\tilde{H}_2$ being Casimirs of $\tilde{P}_0$ and $\tilde{P}_1$ respectively.

2.4. The method of restricted flows. The method of restricted flows was introduced in [Mos] as a *non linearization* of the KdV spectral problem and was generalized in [Cao1, AW1]. Our formulation of this method puts the emphasis on the role played by the GD polynomials and their generating function; by their use, a map connecting restricted and stationary flows will be algorithmically constructed in the next subsection. In this subsection, we apply the method to the KdV hierarchy and we recover the Garnier system.

Let us consider the following system

$$\hat{p}_{00} - \hat{v}_1 = a_0, \quad P_0\Big(\hat{v}_1 - \sum_{j=1}^{n} \beta_j\Big) = 0, \quad P^{\lambda_k}\beta_k = 0 \quad (k = 1, \ldots, n) \tag{2.80}$$

where: $\lambda_1, \ldots, \lambda_n$ are distinct fixed parameters, $P^{\lambda_k} := P_1 - \lambda_k P_0$ (P_0 and P_1 being the two KdV Poisson tensors (2.18)). This is a system of $(n + 2)$ equations in $u, \hat{v}_1, \beta_1, \ldots, \beta_n$. The second equation will be referred to as the P_0*–restriction* of the first KdV flow $X_0 = P_0 \, \hat{v}_1 = \hat{v}_{1x}$, and the last n equations define the kernel of n Poisson tensors extracted from the Poisson pencil. Written in terms of

$$\psi_k = \sqrt{\beta_k} \ , \tag{2.81}$$

the last n equations are called *square eigenfunction relations* (SER) and the ψ_k are referred to as Bargmann coordinates. On account of (2.20), (2.18) and (2.9), the previous system is equivalent to the following one

$$u = \frac{\hat{v}_1}{2} + \frac{a_0}{4} \ , \qquad \hat{v}_1 = \sum_{j=1}^{n} \beta_j + c \ , \qquad B^{\lambda_k}(\beta_k, \beta_k) = f_k \ , \tag{2.82}$$

where B^λ is the bilinear function (2.19), c and f_k are free parameters. This system is interesting for it has a close relationship with the GD polynomials, $B^\lambda\left(\hat{v}(\lambda), \hat{v}^{(k)}(\lambda)\right)$ being just their generating functions (2.14).

By using the first two equations in order to eliminate u and $\hat{v}_1$ from the last n equations, one gets a system of n ODE's of second order for $\beta_1, \ldots, \beta_n$: .

$$2\beta_{kxx}\beta_k - \beta_{kx}^2 + 2\beta_k^2(\sum_{j=1}^n \beta_j + d) - \lambda_k\beta_k^2 = f_k \quad (k = 1, \ldots, n) \ , \tag{2.83}$$

where $d := c + a_0/2$. On account of (2.81) this system takes the Lagrangian form

$$\psi_{kxx} = -\frac{1}{2}\psi_k \left(\sum_{j=1}^n \psi_j^2\right) + \frac{1}{4}(\lambda_k - 2d)\psi_k + \frac{1}{4}\frac{f_k}{\psi_k^3} \ , \tag{2.84}$$

with Lagrangian function

$$\mathcal{L}^{(0)} = \frac{1}{2}\sum_{j=1}^n \psi_{jxx}^2 - \frac{1}{8}\left[\left(\sum_{k=1}^n \psi_j^2\right)^2 - \sum_{j=1}^n(\lambda_j - 2d)\psi_j^2 + \sum_{j=1}^n \frac{f_j}{\psi_j^2}\right] \ . \tag{2.85}$$

The corresponding Hamiltonian is

$$\mathcal{K}_G = \frac{1}{2}\sum_{j=1}^n \chi_j^2 + \frac{1}{8}\left[\left(\sum_{k=1}^n \psi_j^2\right)^2 - \sum_{j=1}^n(\lambda_j - 2d)\psi_j^2 + \sum_{j=1}^n \frac{f_j}{\psi_j^2}\right] \ , \tag{2.86}$$

where $\chi_j = \psi_{jx}$ are canonical momenta. The Hamiltonian vector field $\mathcal{Y}_G = \mathcal{E}\, d\mathcal{K}_G$ (where $\mathcal{E} := \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix}$, $\mathbf{0}_n$ and $\mathbf{1}_n$ being the $n \times n$ zero and identity matrices respectively) gives the Garnier system with n degrees of freedom [AW1]

$$\psi_{jx} = \frac{\partial \mathcal{K}_G}{\partial \chi_j} \ , \qquad \chi_{jx} = -\frac{\partial \mathcal{K}_G}{\partial \psi_j} \qquad (j = 1, \ldots, n) \ . \tag{2.87}$$

2.5. A map between stationary flows and restricted flows. Now we shall construct a map between the nth stationary flow and the previous restricted flow of the KdV hierarchy. To this end we extend the corresponding phase spaces, regarding some free parameters in the Hamiltonian functions as additional dynamical variables. Let us consider the P_1–formulation of the stationary flow (2.54) and let us extend its phase space to a $(3n+1)$–dimensional space, $\tilde{M}_n$, with coordinates $(q_k, p_k; a_0, \ldots, a_{n-1}, a_{2n})$. Analogously let us consider the P_0–formulation of the first restricted flow (2.87) in the extended space $\tilde{\mathcal{M}}_n$ with coordinates $(\psi_k, \chi_k; f_1, \ldots, f_n, d)$.

Let us recall that if q_k, p_k are solutions of the dynamical equations (2.54) then the pair $(u, \hat{v}^{(n)}(\lambda))$ given by

$$u = \frac{\hat{v}_1}{2} + \frac{a_0}{4} \ , \qquad \hat{v}^{(n)}(\lambda) = \lambda \left(q^2(\lambda)\right)^{(n-1)} - q_n^2 \ , \tag{2.88}$$

with $q(\lambda)=1+\sum_{j=1}^n q_j\lambda^{-j}$, satisfies (2.30) and consequently the following equation

$$B^\lambda\Big(\hat{v}^{(n)}(\lambda),\hat{v}^{(n)}(\lambda)\Big)=\lambda^{2n}\hat{a}(\lambda)\ , \tag{2.89}$$

(see Rem. 2.7). So, for each n–tuple of distinct complex parameters λ_j, any solution $(u,\hat{v}^{(n)}(\lambda))$ of Eq. (2.89) fulfills the following system

$$u=\frac{\hat{v}_1}{2}+\frac{a_0}{4}\ ,\quad B^{\lambda_k}\Big(\hat{v}^{(n)}(\lambda_k),\hat{v}^{(n)}(\lambda_k)\Big)=\lambda_k^{2n}\hat{a}(\lambda_k)\ \ (k=1,\dots,n)\ , \tag{2.90}$$

where $\hat{v}^{(n)}(\lambda_k):=\hat{v}^{(n)}(\lambda)_{|\lambda=\lambda_k}$. In order to have a solution of this system satisfying also a constraint condition as the second equation (2.82), the so–called Lagrange interpolation formula can be used [Alb, CW]. It allows us to represent the polynomial $\hat{v}^{(n)}(\lambda)$ by

$$\hat{v}^{(n)}(\lambda)=\hat{v}_0\,p(\lambda)+\sum_{j=1}^n\frac{p(\lambda)}{\lambda-\lambda_j}\beta_j\ , \tag{2.91}$$

where $p(\lambda)=\prod_{j=1}^n(\lambda-\lambda_j)$, and

$$\beta_k=\frac{\hat{v}^{(n)}(\lambda_k)}{p'(\lambda_k)}\qquad\qquad (k=1,\dots,n)\ , \tag{2.92}$$

($p'(\lambda)$ means the derivative of $p(\lambda)$ with respect to λ). Obviously the n functions β_k (2.92) and u are solutions of the following system

$$u=\frac{\hat{v}_1}{2}+\frac{a_0}{4}\qquad B^{\lambda_k}(\beta_k,\beta_k)=\frac{\lambda_k^{2n}\hat{a}(\lambda_k)}{(p'(\lambda_k))^2}\ . \tag{2.93}$$

Furthermore, developing both sides of (2.91) in power of λ, it follows that β_k satisfy the so–called Bargmann constraint

$$\hat{v}_1=\sum_{j=1}^n\beta_j-\hat{v}_0\sum_{j=1}^n\lambda_j\ . \tag{2.94}$$

So, by eliminating u by means of the first equation (2.93) one gets

(2.95)
$$2\beta_{kxx}\beta_k-\beta_{kx}^2+2\beta_k^2\Big(\sum_{j=1}^n\beta_j+\frac{a_0}{2}-\sum_{j=1}^n\lambda_j\Big)-\lambda_k\beta_k^2=\frac{\lambda_k^{2n}\hat{a}(\lambda_k)}{(p'(\lambda_k))^2}\qquad (k=1,\dots,n)\ .$$

Then at each point of $\tilde{\mathcal{M}}_n$ the functions β_k defined by Eq. (2.92) are solutions of the system (2.83), provided that

$$f_k = \frac{\lambda_k^{2n}\hat{a}(\lambda_k)}{(p'(\lambda_k))^2}\,, \qquad d = \frac{a_0}{2} - \sum_{j=1}^{n}\lambda_j\,. \tag{2.96}$$

Taking into account Eq. (2.81), (2.92), (2.96) and (2.55), we obtain the following

Proposition 2.4. *Let* $\Psi : \tilde{M}_n \to \tilde{\mathcal{M}}_n, (q, p; a_0, \dots, a_{n-1}, a_{2n}) \mapsto (\psi, \chi; f_1, \dots, f_n, d)$ *be the map:*

(2.97)

$$\psi_k = \Big(\frac{\sum_{j=0}^{n-1}\sum_{l=0}^{j} q_l q_{j-l}\,\lambda_k^{n-j} - q_n^2}{p'(\lambda_k)}\Big)^{1/2}\,, \quad \chi_k = \frac{\sum_{j=1}^{n-1}\sum_{l=1}^{j} q_{j-l}p_{n-l}\,\lambda_k^{n-j} - q_n p_n}{\Big(p'(\lambda_k)\Big(\sum_{j=0}^{n-1}\sum_{l=0}^{j} q_l q_{j-l}\,\lambda_k^{n-j} - q_n^2\Big)\Big)^{1/2}}\,,$$

$$f_k = \frac{1}{(p'(\lambda_k))^2}\Big(a_{2n} - 8\sum_{j=0}^{n} H_{n-j}\,\lambda_k^j + \sum_{j=n+1}^{2n+1} a_{2n-j}\,\lambda_k^j\Big)\,, \quad d = \frac{a_0}{2} - \sum_{j=1}^{n}\lambda_j$$

$(k = 1, \dots, n)$*, where* H_j *are the Hamiltonian functions* (2.55)*. If* (q_k, p_k) *are solutions of the stationary flows* (2.54)*, then* (ψ_k, χ_k) *are solutions of the Garnier system* (2.87) *for* f_k *and* d *given by* (2.97)*.*

The function B^λ is also a generating function of integrals of motion for the Garnier system. Indeed evaluating the function B^λ by means of (2.81), (2.91) and eliminating the second x–derivatives of ψ_k by means of the Newton equations (2.84), one gets

$$4\sum_{j=1}^{n}\frac{I_j}{\lambda - \lambda_j} + \sum_{j=1}^{n}\frac{f_j}{(\lambda - \lambda_j)^2} + 2d - \lambda = \frac{\lambda^{2n}\hat{a}(\lambda)}{(p(\lambda))^2} \tag{2.98}$$

where I_j are given by

(2.99)

$$I_j = \chi_j^2 + \sum_{\substack{k=1\\k\neq j}}^{n}\frac{(\psi_j\chi_k - \psi_k\chi_j)^2}{\lambda_j - \lambda_k} + \frac{1}{4}\left(\psi_j^2\Big(2d - \lambda_j + \sum_{k=1}^{n}\psi_k^2\Big) + \frac{f_j}{\psi_j^2} + \sum_{\substack{k=1\\k\neq j}}^{n}\frac{1}{\lambda_j - \lambda_k}\Big(\frac{f_j\psi_k^2}{\psi_j^2} + \frac{f_k\psi_j^2}{\psi_k^2}\Big)\right)\,.$$

Taking the residues at $\lambda = \lambda_j$ it follows that the functions I_j are integrals of motion along the flow (2.87). Its Hamiltonian function $\mathcal{K}_G$ (2.86) is dependent on I_j because $\mathcal{K}_G = 1/2\sum_{j=1}^{n} I_j$. The integrals I_j were obtained in [Wo2] by means of a Lax representation. Another family of n integrals of motion (dependent on I_j) can be obtained by multiplying both members of (2.98) by $p(\lambda)^2$ and equating the coefficients of the powers of λ. A few ones are

(2.100)
$$\mathcal{K}_0 \equiv \frac{a_0}{8} = \frac{1}{4}(d + \sum_{j=1}^{n} \lambda_j) \ , \qquad \mathcal{K}_1 \equiv \frac{a_1}{8} = \mathcal{K}_G - \frac{d}{2}(\sum_{j=1}^{n} \lambda_j) - \frac{1}{8}(\sum_{j=1}^{n} \lambda_j)^2 \ ,$$
$$\mathcal{K}_n \equiv \frac{a_{2n}}{8} = \frac{d}{4}\prod_{j=1}^{n} \lambda_j^2 + \frac{1}{8}\sum_{j=1}^{n} f_j \left(\prod_{k\neq j} \lambda_k^2\right) - \frac{1}{2}\left(\prod_{j=1}^{n} \lambda_j\right)\sum_{j=1}^{n} I_j \left(\prod_{k\neq j} \lambda_k\right) \ ,$$

where $\mathcal{K}_G$ is the Hamiltonian (2.86).

2.6. Example II: the bi–Hamiltonian structure of the Garnier system. In the previous section we have recovered a map between the nth stationary flow and the Garnier system with n degrees of freedom. Now we shall use implicitly this map in order to construct a second Hamiltonian formulation for the Garnier vector field $\mathcal{Y}_G$. Let us consider the system

(2.101)
$$\hat{p}_{nn} = a_{2n} \ , \qquad P_1 \left(\hat{v}_0 - \sum_{j=1}^{n} \frac{\beta_j}{\lambda_j}\right) = 0 \ , \qquad B^{\lambda_k}(\beta_k, \beta_k) = f_k \qquad (k = 1, \dots, n) \ .$$

and let us compare it with the first Hamiltonian formulation (2.80). The first constraint equation has this motivation: it can be shown, by means of the map (2.97), that the solutions of the system (2.80) comes from solutions of Eq. (2.89). Just this equation implies the first constraint (2.101). The second equation (2.101) is due to the bi–Hamiltonian property (2.2) of the KdV hierarchy and will be referred to as the P_1–formulation of the first restricted flow. The first equation can be solved with respect to u:

$$u = -\frac{\hat{v}_{nxx}}{2\hat{v}_n} + \frac{\hat{v}_{nx}^2 + a_{2n}}{4\hat{v}_n^2} \ , \tag{2.102}$$

and the second equation is solved by

$$\hat{v}_0 - \sum_{j=1}^{n} \frac{\beta_j}{\lambda_j} = \mu \hat{v}_n \qquad \mu \in \mathbb{R} \ , \tag{2.103}$$

on account of (2.89). Thus we can eliminate u in the last n equations (2.101), which become (for $\hat{v}_0 = 1$)

$$2\beta_{kxx}\beta_k - \beta_{kx}^2 + 2\beta_k^2\Big(\frac{\sum_{j=1}^n \frac{\beta_{jxx}}{\lambda_j}}{1-\sum_{j=1}^n \frac{\beta_j}{\lambda_j}} + \frac{\Big(\sum_{j=1}^n \frac{\beta_{jxx}}{\lambda_j}\Big)^2 + \frac{a_{2n}}{\lambda_1^2\lambda_2^2}}{2\Big(1-\sum_{j=1}^n \frac{\beta_j}{\lambda_j}\Big)^2}\Big) - \lambda_k\beta_k^2 = f_k \tag{2.104}$$

for $k = 1, \ldots, n$. This system, with the substitution $\beta_k = \psi_k^2$, becomes the Lagrangian system

$$\psi_{kxx} - \frac{\lambda_k}{4}\psi_k - \frac{f_k}{4\psi_k^3} + \psi_k\Big(\frac{\langle\psi_x, \Lambda^{-1}\psi_x\rangle + \langle\psi, \Lambda^{-1}\psi_{xx}\rangle}{1-\langle\psi, \Lambda^{-1}\psi\rangle} + \frac{\langle\psi, \Lambda^{-1}\psi_x\rangle^2 + \frac{a_{2n}}{4\lambda_1^2\lambda_2^2}}{\Big(1-\langle\psi, \Lambda^{-1}\psi\rangle\Big)^2} = 0 \ , \tag{2.105}$$

with Lagrangian function

$$\mathcal{L}_a = \frac{1}{2}\langle\psi_x, \Lambda^{-1}\psi_x\rangle + \frac{\langle\psi, \Lambda^{-1}\psi_x\rangle^2}{2\Big(1-\langle\psi, \Lambda^{-1}\psi\rangle\Big)} + \frac{1}{8}\langle\psi,\psi\rangle - \frac{a_{2n}}{8\lambda_1^2\lambda_2^2\Big(1-\langle\psi, \Lambda^{-1}\psi\rangle\Big)} - \frac{1}{8}\langle F\psi^{-1}, \Lambda^{-1}\psi^{-1}\rangle \ , \tag{2.106}$$

where we have used the vector notation $\psi = [\psi_1, \ldots, \psi_n]^T$, $\psi^{-1} = [\psi_1^{-1}, \ldots, \psi_n^{-1}]^T$, $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$, $F = diag(f_1, \ldots, f_n)$ and $\langle\phi, \psi\rangle = \sum_{j=1}^n \phi_j\psi_j$. The corresponding Hamiltonian is

$$\mathcal{H}_G = \frac{1}{2}\langle\theta, \Lambda\theta\rangle - \frac{1}{2}\langle\theta, \psi\rangle^2 - \frac{1}{8}\langle\psi,\psi\rangle + \frac{a_{2n}}{8\lambda_1^2\lambda_2^2\Big(1-\langle\psi, \Lambda^{-1}\psi\rangle\Big)} + \frac{1}{8}\langle F\psi^{-1}, \Lambda^{-1}\psi^{-1}\rangle \ , \tag{2.107}$$

where the canonical momenta θ_k are given by

$$\theta_k = \frac{\psi_{kx}}{\lambda_k} + \frac{\psi_k}{\lambda_k}\,\frac{\langle\psi, \Lambda^{-1}\psi_x\rangle}{1-\langle\psi, \Lambda^{-1}\psi\rangle} \ ; \tag{2.108}$$

the corresponding Hamiltonian vector field is $\mathcal{X}_G = \mathcal{E}\, d\mathcal{H}_G$. As well as for the first Hamiltonian formulation we can use the function B^λ as a generating function of the integrals of motion for the vector field $\mathcal{X}_G$. For the sake of simplicity we report here only one of these

$$\mathcal{H}_0 = \frac{a_0}{8} = \mathcal{H}_G + \frac{1}{4}(\sum_{j=1}^n \lambda_j) \ . \tag{2.109}$$

Now let us consider in the extended phase space $\mathcal{M}_n$ with coordinates $(\psi_k, \theta_k; a)$, the extended Hamiltonian $\tilde{\mathcal{H}}_G$, the vector field $\tilde{\mathcal{X}}_G$, the Poisson structure $\tilde{\mathcal{E}} = \left(\begin{smallmatrix} \mathbf{0}_n & \mathbf{1}_n & 0 \\ -\mathbf{1}_n & \mathbf{0}_n & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and in the coordinates $(\psi_k, \chi_k; d)$ the extended Hamiltonian $\tilde{\mathcal{K}}_G$ (see (2.86)) and the vector field $\tilde{\mathcal{Y}}_G = \tilde{\mathcal{E}}\, d\tilde{\mathcal{K}}_G$. By inverting (2.108) and comparing (2.109) with the first Eq. (2.100) it is easy to prove the following result

Proposition 2.5. *Let $\Phi : \mathcal{M}_n \to \mathcal{M}_n, (\psi_k, \theta_k; a_{2n}) \mapsto (\psi_k, \chi_k; d)$ be the map:*

$$\chi_k = \lambda_k \theta_k - \psi_k \sum_{j=1}^{n} \theta_j \psi_j \ , \qquad d = 4\mathcal{H}_G \ , \tag{2.110}$$

where $\mathcal{H}_G$ is as in (2.107). Its action on the vector field $\tilde{\mathcal{X}}_G$ and on the Poisson tensor $\tilde{\mathcal{P}}'_1 := \tilde{\mathcal{E}}$ is given by

$$\Phi_*(\tilde{\mathcal{X}}_G) = \tilde{\mathcal{Y}}_G \tag{2.111}$$

$$\tilde{\mathcal{P}}_1 := \Phi_* \tilde{\mathcal{P}}'_1 \Phi^* = \begin{bmatrix} 0 & \Lambda - \psi \otimes \psi & 4\tilde{\mathcal{Y}}_G^{\psi} \\ -(\Lambda - \psi\otimes\psi)^T & \chi\otimes\psi - \psi\otimes\chi & 4\tilde{\mathcal{Y}}_G^{\chi} \\ -4(\tilde{\mathcal{Y}}_G{}^{\psi})^T & -4(\tilde{\mathcal{Y}}_G{}^{\chi})^T & 0 \end{bmatrix} \ , \tag{2.112}$$

where $\otimes$ denotes the tensor product.

Since the Poisson tensor $\tilde{\mathcal{P}}'_1$ is mapped into a Poisson structure $\tilde{\mathcal{P}}_1$, which can be verified to be compatible with $\tilde{\mathcal{P}}_0 := \tilde{\mathcal{E}}$, $\tilde{\mathcal{P}}_1$ and $\tilde{\mathcal{P}}_0$ endow the manifold $\mathcal{M}_n$ with a bi–Hamiltonian structure. It coincides with the one obtained in [AW3] for the particular case $f_k = 0\,(k = 1, \dots, n)$. Let us specialize the previous results for $n = 2$. In this case the Hamiltonian (2.86) becomes

$$\mathcal{K}_G = \frac{1}{2}(\chi_1^2 + \chi_2^2) + \frac{1}{8}[(\psi_1^2 + \psi_2^2)^2 - (\lambda_1 - 2d)\psi_1^2 - (\lambda_2 - 2d)\psi_2^2 + \frac{f_1}{\psi_1^2} + \frac{f_2}{\psi_2^2}] \ , \tag{2.113}$$

and the corresponding Hamiltonian vector field $\mathcal{Y}_G = \mathcal{E}\, d\mathcal{K}_G$ is

$$\mathcal{Y}_G = [\chi_1, \chi_2, -\frac{1}{2}(\psi_1^2 + \psi_2^2)\psi_1 + \frac{1}{4}(\lambda_1 - 2d)\psi_1 + \frac{f_1}{4\psi_1^3}, -\frac{1}{2}(\psi_1^2 + \psi_2^2)\psi_2 + \frac{1}{4}(\lambda_2 - 2d)\psi_2 + \frac{f_2}{4\psi_2^3}]^T \ . \tag{2.114}$$

In the five–dimensional extended phase space $\mathcal{M}_2$ with coordinates $(\psi_1, \psi_2, \chi_1, \chi_2; d)$ let us consider the extended Hamiltonian $\tilde{\mathcal{K}}_G$, the vector field $\tilde{\mathcal{Y}}_G$ and the Poisson tensors $\tilde{\mathcal{P}}_0$ and $\tilde{\mathcal{P}}_1$ (2.112). In this space, combining the integrals of motion I_j (2.99), one can construct a bi–Hamiltonian chain, starting and ending with a Casimir of $\tilde{\mathcal{P}}_0$ and $\tilde{\mathcal{P}}_1$:

Proposition 2.6. *The Garnier vector field $\tilde{\mathcal{Y}}_G = \tilde{\mathcal{Y}}_1$ belongs to the following bi–Hamiltonian hierarchy*

$$\tilde{\mathcal{Y}}_{j+1} = \tilde{\mathcal{P}}_1\, d\tilde{\mathcal{G}}_j = \tilde{\mathcal{P}}_0\, d\tilde{\mathcal{G}}_{j+1} \qquad (j = 0, 1)\ , \tag{2.115}$$

where the Hamiltonians $\tilde{\mathcal{G}}_j$ are given by

$$\begin{aligned} &\tilde{\mathcal{G}}_0 = \frac{d}{4}\ , \qquad \tilde{\mathcal{G}}_1 = -(\lambda_1 + \lambda_2)\frac{d}{4} + \frac{1}{2}(I_1 + I_2) \\ &\tilde{\mathcal{G}}_2 = \lambda_1\lambda_2\frac{d}{4} - \frac{1}{2}(\lambda_1 + \lambda_2)(I_1 + I_2) + \frac{1}{2}(\lambda_1 I_1 + \lambda_2 I_2)\ , \end{aligned} \tag{2.116}$$

$\tilde{\mathcal{G}}_0$ and $\tilde{\mathcal{G}}_2$ being Casimirs of $\tilde{\mathcal{P}}_0$ and $\tilde{\mathcal{P}}_1$ respectively, and I_1, I_2 being the integrals of motion (2.99).

As in the case of the Henon–Heiles system, a bi–Hamiltonian structure for the Garnier system seems to naturally exist only in its extended phase space. Nevertheless in Subsect. 3.3 a realization of the integrability structure introduced in Prop. 3.1 will be constructed in the original four dimensional phase space.

2.7. Example III: a map between the Henon–Heiles and the Garnier system. Now we specialize the map of Subsect. 2.5 to the Henon–Heiles system and the Garnier system with two degrees of freedom: we obtain the surprising result that the Henon–Heiles vector field is mapped into the Garnier vector field. Let us consider the seven–dimensional phase space of the Henon–Heiles system $\tilde{M}_2$ with coordinates $(q, p; a_0, a_1, a_4)$. Similarly, for the Garnier system let us select the parameters f_1, f_2, d (whereas λ_1,λ_2 have to be considered fixed) and we enlarge the phase space to a seven–dimensional phase space $\tilde{\mathcal{M}}_2$, with coordinates $(\psi, \chi; f_1, f_2, d)$. It is easy to prove

Proposition 2.7. *Let us consider the map $\Psi : \tilde{M}_2 \to \tilde{\mathcal{M}}_2, (q, p; a_0, a_1, a_4) \mapsto (\psi, \chi; f_1, f_2, d)$ defined by*

(2.117)

$$\psi_1 = \Big(\frac{\lambda_1^2 + 2\lambda_1 q_1 - q_2^2}{(\lambda_1-\lambda_2)}\Big)^{1/2} , \qquad \psi_2 = \Big(\frac{-\lambda_2^2 - 2\lambda_2 q_1 + q_2^2}{(\lambda_1-\lambda_2)}\Big)^{1/2} ,$$

$$\chi_1 = \frac{(\lambda_1 p_1 - q_2 p_2)}{\Big((\lambda_1-\lambda_2)\ (\lambda_1^2 + 2\lambda_1 q_1 - q_2^2)\Big)^{1/2}} , \qquad \chi_2 = \frac{(\lambda_2 p_1 - q_2 p_2)}{\Big((\lambda_1-\lambda_2)\ (-\lambda_2^2 - 2\lambda_2 q_1 + q_2^2)\Big)^{1/2}} ,$$

$$f_1 = \frac{(-\lambda_1^5 + a_0\lambda_1^4 + a_1\lambda_1^3 - 8H_0\lambda_1^2 - 8H_1\lambda_1 + a_4)}{(\lambda_1-\lambda_2)^2} ,$$

$$f_2 = \frac{(-\lambda_2^5 + a_0\lambda_2^4 + a_1\lambda_2^3 - 8H_0\lambda_2^2 - 8H_1\lambda_2 + a_4)}{(\lambda_1-\lambda_2)^2} , \quad d = \frac{a_0}{2} - (\lambda_1+\lambda_2) .$$

The tangent map Φ_* maps the extended Henon-Heiles vector fields $\tilde{X}_1, \tilde{X}_2$ (2.79) into the extended Garnier vector fields $\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2$ (2.115):

$$\Phi_*\left(\tilde{X}_1\right) = \tilde{\mathcal{Y}}_1 \qquad \Phi_*\left(\tilde{X}_2\right) = \tilde{\mathcal{Y}}_2 \quad . \tag{2.118}$$

Moreover the pull–backs of the Garnier integrals of motion $\mathcal{G}_1$ and $\mathcal{G}_2$ are

$$\begin{aligned} \Phi^*(\mathcal{G}_1) &= -\frac{1}{8}(\lambda_1^2+\lambda_2^2) + \frac{a_0}{8}(\lambda_1+\lambda_2) + \frac{a_1}{8} \\ \Phi^*(\mathcal{G}_2) &= \frac{1}{(\lambda_1-\lambda_2)^2}\Big(2\lambda_1\lambda_2 H_0 + (\lambda_1+\lambda_2)H_1 + 2H_2\Big) + \\ &\quad + \frac{\lambda_1\lambda_2}{4(\lambda_1-\lambda_2)^2}\left((\lambda_1^3+\lambda_2^3) - \frac{a_0}{2}(\lambda_1^2+\lambda_2^2) - \frac{a_1}{2}(\lambda_1+\lambda_2)\right) , \end{aligned} \tag{2.119}$$

i.e. they are integrals of motion for the Henon–Heiles system. The action of the map Φ on the Poisson tensor $\tilde{E}$ of the Henon–Heiles system, furnishes a Poisson tensor which is not equal to the Poisson tensor $\tilde{\mathcal{E}}$ of the Garnier system (it is not reported here for the sake of brevity). Moreover the action of Φ on the Poisson tensor $\tilde{P}_0$ is given by

$$\Phi^*\tilde{P}_0\Phi_* = \frac{1}{(\lambda_1-\lambda_2)^2}\begin{bmatrix} 0 & \mathcal{A} & 0 \\ -\mathcal{A}^T & \mathcal{B} & 0 \\ 0 & 0 & 0 \end{bmatrix} , \tag{2.120}$$

where

$$\mathcal{A} = \begin{bmatrix} \frac{(\lambda_1-\lambda_2+\psi_1^2+\psi_2^2)}{\psi_1^2} & -\frac{(\psi_1^2+\psi_2^2)}{(\psi_1\psi_2)} \\ -\frac{(\psi_1^2+\psi_2^2)}{(\psi_1\psi_2)} & \frac{(-\lambda_1+\lambda_2+\psi_1^2+\psi_2^2)}{\psi_2^2} \end{bmatrix} ,$$

$$\mathcal{B} = \begin{bmatrix} 0 & \frac{(\chi_2\psi_1-\chi_1\psi_2)(\psi_1^2+\psi_2^2)}{(\psi_1\psi_2)^2} \\ -\frac{(\chi_2\psi_1-\chi_1\psi_2)(\psi_1^2+\psi_2^2)}{(\psi_1\psi_2)^2} & 0 \end{bmatrix} . \tag{2.121}$$

So the map Φ is not a Poisson morphism. However, according to Eq. (2.118), the orbits of the Henon–Heiles system are mapped into the orbits of the Garnier system.

3. A new integrability structure

3.1. The reduced structures of Henon–Heiles and Garnier systems. In order to have a bi–Hamiltonian hierarchy also in the original (not extended) phase space for the Henon–Heiles system, one can try to perform a geometrical reduction of this structure following the reduction techniques known from the literature [LM, MR]. In particular, two methods can be followed: a *restriction* to the standard phase space or a *projection* onto it.

In the first case, if the restriction submanifold is chosen to be a leaf $S^{(1)}_{a_4}$ of the second natural foliation in M_2, the Hamiltonians $\tilde{H}_j$, the vector fields $\tilde{X}_j$ and the Poisson structure $\tilde{P}_1$ can be trivially restricted respectively to H_j, X_j and E; but it turns out that $\tilde{P}_0$ cannot be restricted. So on $S^{(1)}_{a_4}$ one gets two integrable Hamiltonian vector fields but not a bi–Hamiltonian hierarchy.

In the second case, if $\Pi : M_2 \to S_2, (q_1, q_2, p_1, p_2; a_{2n}) \mapsto (q_1, q_2, p_1, p_2)$ is the projection map, the two Poisson tensors can be projected onto Poisson structures without Casimirs (symplectic structures), but the Hamiltonians $\tilde{H}_j$ and the vector fields $\tilde{X}_j$ cannot be projected onto S_2, because they depend in an essential way on the fiber coordinate. Namely the Poisson tensors $\tilde{P}_0$ and $\tilde{P}_1$ are projected onto:

$$P_H = \Pi_* \tilde{P}_0 \Pi^* = \begin{bmatrix} 0 & A \\ -A^T & B \end{bmatrix} , \qquad E = \Pi_* \tilde{P}_1 \Pi^* , \tag{3.1}$$

with A, B given by (2.78). Because these operators are compatible and invertible they give rise to the following Nijenhuis tensor (hereditary operator) [Ma2, FF]

$$N_H := P_H E^{-1} = \begin{bmatrix} A & 0 \\ B & A^T \end{bmatrix} \tag{3.2}$$

together with the hierarchy of Poisson tensors

$$P_k := N_H^k P_H \qquad k \in \mathbb{Z} \ . \tag{3.3}$$

However these tensors are not invariant along the flow of the Henon–Heiles vector field $X_1 := E\, dH_0$ (with H_0 given by (2.71))

$$X_1 = [p_1, p_2, -3q_1^2 - \frac{1}{2}q_2^2 - a_0 q_1 + \frac{a_1}{4}, -q_1 q_2 + \frac{a}{4q_2^3} - \frac{a_0}{4}q_2]^T . \tag{3.4}$$

In other words X_1 is neither a symmetry of P_0 nor of P_1, so that they cannot generate a bi–Hamiltonian hierarchy starting with X_1.

As in the case of the Henon–Heiles system, one can try to reduce the bi–Hamiltonian structure of the Garnier system with n degrees of freedom and one gets similar results. We report here only the results obtained by the *projection* of the structures on the quotient manifold $\mathcal{S}_n^{(1)}$. If $\Pi : \mathcal{M}_n \to \mathcal{S}_n^{(1)}, (\psi_k, \chi_k; d) \mapsto (\psi_k, \chi_k)$ is the projection map, the Poisson tensor $\tilde{\mathcal{P}}_0$ and $\tilde{\mathcal{P}}_1$ are projected onto:

(3.5)

$$\Pi_* \tilde{\mathcal{P}}_0 \Pi^* = \mathcal{E} \ , \qquad \mathcal{P}_G := \Pi_* \tilde{\mathcal{P}}_1 \Pi^* = \begin{bmatrix} 0 & \Lambda - \psi \otimes \psi \\ -(\Lambda - \psi \otimes \psi)^T & \chi \otimes \psi - \psi \otimes \chi \end{bmatrix} .$$

These are compatible and invertible operators and give rise to the following Nijenhuis tensor

$$\mathcal{N}_G := \mathcal{P}_G \mathcal{E}^{-1} = \begin{bmatrix} \Lambda - \psi \otimes \psi & 0 \\ \chi \otimes \psi - \psi \otimes \psi & \Lambda - \psi \otimes \psi \end{bmatrix} , \tag{3.6}$$

together with the hierarchy of Poisson tensor fields

$$\mathcal{P}_k := \mathcal{N}_G^k \, \mathcal{E} \qquad k \in \mathbb{Z} \ . \tag{3.7}$$

However these tensor fields are not invariant along the flow of the Garnier vector field $\mathcal{Y}_G = \mathcal{E}\, d\mathcal{H}_G$, so they cannot be used to construct a bi–Hamiltonian hierarchy starting with $\mathcal{Y}_G$.

3.2. A new integrability criterion. In Subsect. 2.3 we have constructed a bi–Hamiltonian structure for a Henon–Heiles system with two degrees of freedom in a suitably extended phase space; in the previous subsection we have put into evidence some problems arising when one look for a geometrical reduction of this structure onto the original phase space. Now we introduce a new integrability scheme, weaker than the bi–Hamiltonian one, but living in the standard phase space. We shall define this new structure for a generic Hamiltonian system with n degrees of freedom; for $n = 2$ it coincides with the one introduced in [CRG] for the Henon–Heiles system with the Hamiltonian (2.57) and $a_4 = 0$. In the following two new examples of this integrability structure will be exhibited: the Garnier system with two degrees of freedom and a Henon–Heiles system with four degrees of freedom.

Proposition 3.1. *Let M be a $2n$–dimensional Poisson manifold equipped with a Poisson tensor Q_0, and let Z_0 be Hamiltonian vector field with Hamiltonian h_0: $Z_0 = Q_0\, dh_0$. Let there exist a tensor $\mathcal{N} : TM \to TM$ and a skew–symmetric tensor $Q_1 : T^*M \to TM$ such that*

$$Q_1 = \mathcal{N} Q_0 \ . \tag{3.8}$$

Denote by Z_i the vector fields obtained by the iterated action of the tensor $\mathcal{N}$ on Z_0

$$Z_i := \mathcal{N}^i Z_0 \qquad (i = 1, \ldots, n-1)\ , \tag{3.9}$$

and by α_i the 1–forms obtained by the iterated action of the adjoint $\mathcal{N}^* : T^*M \to T^*M$ on $\alpha_0 := dh_0$

$$\alpha_i := \mathcal{N}^{*^i} \alpha_0 \qquad (i = 1, \ldots, n-1)\ . \tag{3.10}$$

Let there exist $n-1$ independent functions h_i $(i = 1, \ldots, n-1)$ and $(n(n+1)/2-1)$ functions μ_{ij} $(i = 1, \ldots, n-1; 0 \le j \le i)$ with $\mu_{00} = 1$ and $\mu_{ii} \neq 0$ $(i = 1, \ldots, n-1)$, such that the 1-forms α_i can be written as

$$\alpha_i = \sum_{j=0}^{i} \mu_{ij}\, dh_j \qquad (i = 1, \ldots, n-1)\ . \tag{3.11}$$

Under the previous assumptions the following results hold:
i) *the vector fields Z_i satisfy the recursion relations*

$$Z_{i+1} = Q_0 \alpha_{i+1} = Q_1 \alpha_i \qquad (i = 0, \ldots, n-2). \tag{3.12}$$

ii) *the functions h_i are in involution with respect to the Poisson bracket defined by Q_0 and they are constants of motion for the fields Z_k*

$$\{h_i, h_j\}_{Q_0} = 0 \qquad , \qquad \mathcal{L}_{Z_k}(h_i) = 0\ , \tag{3.13}$$

where $\mathcal{L}_{Z_k}$ denotes the Lie derivative with respect to the vector field Z_k.
iii) *the Hamiltonian system corresponding to the vector field Z_0 is Liouville–integrable. In addition if Q_1 is a Poisson tensor field, then also Z_1 is an integrable Hamiltonian*

vector field and the functions h_i are in involution also with respect to the Poisson bracket defined by Q_1.

Proof.
i) From Eq. (3.8) and the skew–symmetry of Q_0 and Q_1 it follows that $Q_0\mathcal{N}^* = \mathcal{N}Q_0$ and $Q_1\mathcal{N}^* = \mathcal{N}Q_1$. Then

$$Z_1 - Q_0\alpha_1 = Z_1 - Q_0\mathcal{N}^*\alpha_0 = Z_1 - \mathcal{N}Q_0\alpha_0 = 0 \tag{3.14}$$

and the first relation (3.12) is proved by induction since it is

$$Z_{i+1} - Q_0\alpha_{i+1} = \mathcal{N}Z_i - Q_0\mathcal{N}^*\alpha_i = \mathcal{N}(Z_i - Q_0\alpha_i) \ . \tag{3.15}$$

The second relation (3.12) follows from

$$Z_{i+1} - Q_1\alpha_i = \mathcal{N}Z_i - Q_1\alpha_i = \mathcal{N}(Z_i - Q_0\alpha_i) \ . \tag{3.16}$$

ii) By (3.11), the gradients dh_k can be expressed for any k in terms of dh_0

$$dh_k = (\sum_{i=0}^{k} \nu_{ki}\mathcal{N}^{*^i})dh_0 \ , \tag{3.17}$$

where ν_{ki} are the elements of the matrix a^{-1}, a being the lower triangular matrix defined by $a_{ij} = \mu_{ij}\,(i \geq j),\ a_{ij} = 0\,(i < j),\ (i, j = 0, \dots , n-1)$. Thus

$$\begin{aligned} \{h_i, h_j\}_{Q_0} :&= \langle dh_i, Q_0 dh_j\rangle \\ &= \sum_{a=0}^{i}\sum_{b=0}^{j} \nu_{ia}\nu_{jb}\langle \mathcal{N}^{*^a} dh_0, Q_0\mathcal{N}^{*^b} dh_0\rangle \\ &= \sum_{a=0}^{i}\sum_{b=0}^{j} \nu_{ia}\nu_{jb}\langle dh_0, \mathcal{N}^{a+b}Q_0 dh_0\rangle \end{aligned} \tag{3.18}$$

and the first relation (4.5) follows from the skew–symmetry of the tensor $\mathcal{N}^m Q_0$ for any m. Furthermore

$$\begin{aligned} \mathcal{L}_{Z_k}(h_i) &= \langle dh_i, Q_0 \alpha_{k-1} \rangle \\ &= \langle dh_i, Q_0 \sum_{j=0}^{k} \mu_{kj} dh_j \rangle \\ &= \sum_{j=0}^{k} \mu_{kj} \{h_i, h_j\}_{Q_0} \\ &= 0 \end{aligned} \tag{3.19}$$

iii) Since Z_0 is a Hamiltonian vector field, it is Liouville–integrable on account of the previous result. Moreover, since it is

$$\begin{aligned} \{h_i, h_j\}_{Q_1} :&= \langle dh_i, Q_1 dh_j \rangle \\ &= \sum_{a=0}^{i} \sum_{b=0}^{j} \nu_{ia} \nu_{jb} \langle \mathcal{N}^{*^a} dh_0, Q_1 \mathcal{N}^{*^b} dh_0 \rangle \\ &= \sum_{a=0}^{i} \sum_{b=0}^{j} \nu_{ia} \nu_{jb} \langle dh_0, \mathcal{N}^{a+b} Q_1 dh_0 \rangle \\ &= 0. \end{aligned} \tag{3.20}$$

it follows that if Q_1 is also a Poisson tensor, $\{,\}_{Q_1}$ is a Poisson bracket, Z_1 is a Hamiltonian vector field and then it is Liouville–integrable. □

Remark 3.1. The recursion scheme and the integrability of the vector field Z_0 do not require that the skew–symmetric tensor Q_1 be a Poisson tensor; so M is a Poisson manifold, not a bi-Hamiltonian one. □

In view of the next applications, it may be worthwhile to remark that the results of Prop. 3.1 hold true if the role of Q_0 and Q_1 are interchanged; to be more precise, one can prove (just as for Prop. 3.1)

Proposition 3.2. *The integrability scheme of Prop. 3.1 is still valid if Q_0 is skew–symmetric, Q_1 is a Poisson tensor and the role of Z_0 is now played by $Z_1 = Q_1 dh_0$. The involution relations* (3.13) *become $\{h_i, h_j\}_{Q_1} = 0$.*

3.3. The integrability structure of Henon–Heiles and Garnier systems. In Subsect. 3.1 we have recovered by projection onto the quotient manifold S_2 the Nijenhuis tensor (3.2) and a hierarchy of compatible Poisson tensors (3.3); however, it is not possible to associate to these tensors and to the Henon–Heiles vector field X_1 (3.4) a bi–Hamiltonian hierarchy of vector fields. Nevertheless it is possible to use these elements to construct an example of the integrability structure introduced in Prop. 3.2. In fact if one takes

i) $Q_1 = E$, the vector field $Z_1 := X_1$ (3.4) with Hamiltonian $h_0 := H_0$ (2.71);
ii) the tensor field $\mathcal{N} := N_H$ (3.2) and $Q_0 := P_{-2} = N_H^{-2} P_H$, with P_H as in (3.1);
iii) the function $h_1 := H_1$ (2.72) and the functions μ_{ij} as $\mu_{10} = 0, \mu_{11} = 1/q_2^2$;

then the conditions of Prop. 3.2 are satisfied. Moreover the vector field $Z_0 := Q_0\, dh_0 = P_{-2}\, dH_0$ is a new integrable vector field:

$$Z_0 = \begin{bmatrix} -2p_1q_1 - p_2q_2 \\ -p_1q_2 \\ -p_2^2 + 6q_1^3 + 2q_1q_2 - \frac{a_4}{4q_2^2} - \frac{a_1}{2}q_1 - 2a_0q_1^2 + \frac{a_0}{4}q_2^2 \\ p_1p_2 + \frac{q_2^3}{2} + 3q_1^2q_2 - \frac{a_1}{4}q_2 + a_0q_1q_2 \end{bmatrix} . \tag{3.21}$$

This integrability structure is related to the one introduced in [CRG], for the Hamiltonian (2.57) with $a_4 = 0$, through the map (2.58).

For the Garnier system with two degrees of freedom one can construct an example of the integrability structures of Prop. 3.1. Indeed if one uses the elements of Subsect. 3.1 and makes the following choices:

i) $\mathcal{Q}_0 := \mathcal{E}$, $h_0 := \tilde{\mathcal{G}}_1$ (2.116), $\mathcal{Z}_0 := \mathcal{Y}_G$ (2.114);
ii) $\mathcal{N} := \mathcal{N}_G^{-1}$, with $\mathcal{N}_G$ given by (3.6), $\mathcal{Q}_1 := \mathcal{P}_{-1} = \mathcal{N}_G^{-1}\mathcal{E}$;
iii) the functions $h_1 := \tilde{\mathcal{G}}_2$ (2.116), $\mu_{10} = 0$, $\mu_{11} = -\frac{\lambda_1\lambda_2}{1-\frac{\psi_1^2}{\lambda_1}-\frac{\psi_2^2}{\lambda_2}}$;

then the conditions of Prop. 3.1 are satisfied. Moreover the vector field $\mathcal{Z}_1 := \mu_{11}\mathcal{Y}_2$ is a new integrable vector field ($\mathcal{Y}_2$ is the restriction to the submanifold of $\mathcal{M}_2$, $d = cost$, of the vector field $\tilde{\mathcal{Y}}_2$ (2.115)).

Now we compute the action, on the recursion operators of the previous integrability structures, of the map between the standard phase spaces of the Henon–Heiles and of the Garnier system, induced by the map (2.117).

Proposition 3.3. *Let us consider the map* $\Phi : (q_1, q_2, p_1, p_2) \mapsto (\psi_1, \psi_2, \chi_1, \chi_2)$*:*

$$\psi_1 = \Big(\frac{\lambda_1^2 + 2\lambda_1q_1 - q_2^2}{(\lambda_1 - \lambda_2)}\Big)^{1/2}, \qquad \psi_2 = \Big(\frac{-\lambda_2^2 - 2\lambda_2q_1 + q_2^2}{(\lambda_1 - \lambda_2)}\Big)^{1/2},$$
$$\chi_1 = \frac{(\lambda_1p_1 - q_2p_2)}{\Big((\lambda_1 - \lambda_2)\,(\lambda_1^2 + 2\lambda_1q_1 - q_2^2)\Big)^{1/2}}, \qquad \chi_2 = \frac{(\lambda_2p_1 - q_2p_2)}{\Big((\lambda_1 - \lambda_2)\,(-\lambda_2^2 - 2\lambda_2q_1 + q_2^2)\Big)^{1/2}} . \tag{3.22}$$

The map Φ *relates the recursion operators of the Henon–Heiles and of the Garnier systems:*

$$\Phi_* N_H = \mathcal{N}_G^{-1}\Phi_* . \tag{3.23}$$

4. A Henon–Heiles system with four degrees of freedom

In order to show the effectiveness of the reduction method presented in Subsect. 2.2, we present a new integrable Hamiltonian system with four degrees of freedom and an indefinite kinetic energy; it can be obtained as a stationary reduction of the ninth-order KdV equation as well as the Henon–Heiles system has been obtained as a reduction of the fifth-order KdV equation. For this reason, the new system will be referred to as a Henon–Heiles system with four degrees of freedom. In the next subsection its bi–Hamiltonian structure in the extended phase space will be constructed and in Subsect. 4.2 its integrability structure will be exhibited as a realization of the criterion introduced in Subsect. 3.1.

Obviously enough, a Henon–Heiles system with n degrees of freedom and an indefinite kinetic energy can be obtained as a stationary reduction of the $(2n+1)$–order KdV equation.

4.1. The bi–Hamiltonian structure. We apply to the case $n = 4$ the general procedure illustrated in Subsect. 2.2. We make the choice

$$a(\lambda) = -\lambda + a_4\lambda^{-4} + a_8\lambda^{-8} \ , \tag{4.1}$$

which, on account of Rem. 2.6, corresponds to the stationary manifold

$$\begin{aligned} M_5 = \Big\{ u \mid \frac{d}{dx}\Big(& u^{(8)} + 18u^{(6)}u + 54u^{(5)}u_x + 114u^{(4)}u_{xx} + 126u^{(4)}u^2 + 69u_{xxx}^2 \\ & + 504u_{xxx}u_xu + 378u_{xx}^2u + 462u_{xx}u_x^2 + 420u_{xx}u^3 + 630u_x^2u^2 + 126u^5\Big) = 0 \Big\} \ . \end{aligned} \tag{4.2}$$

i) The system of ODE's corresponding to Eq. (2.35) becomes

$$p_{0k} - v_{k+1} = 0 \quad (k = 0, 1, 2, 3) \ , \qquad p_{04} = a_4 \ , \tag{4.3}$$

and the one corresponding to Eq. (2.36) differs for the fifth equation, which is

$$p_{44} = a_8 \ . \tag{4.4}$$

ii) The reduced systems corresponding to (2.37) is

$$
(4.5)\qquad
\begin{aligned}
v_{1xx} &= -\frac{3}{2}v_1^2 + v_2\ , \qquad v_{2xx} = +\frac{1}{2}v_1^3 - 2v_1v_2 + \frac{1}{2}v_{1x}^2 + v_3\\
v_{3xx} &= -\frac{1}{2}v_1^4 + \frac{3}{2}v_1^2v_2 - \frac{1}{2}v_{1x}^2v_1 + v_{1x}v_{2x} - \frac{1}{2}v_2^2 - 2v_1v_3 + v_4\\
v_{4xx} &= \frac{1}{2}v_1^5 - 2v_1^3v_2 + \frac{1}{2}v_{1x}^2v_1^2 - \frac{1}{2}v_{1x}^2v_2 + -v_1v_{1x}v_{2x} + \frac{3}{2}v_1v_2^2 +\\
&\quad + \frac{3}{2}v_1^2v_3 - v_1v_{1x}v_{2x} + +v_{2x}^2 + v_{1x}v_{3x} + -v_2v_3 - 2v_1v_4 + \frac{a_4}{2}
\end{aligned}
$$

and the one corresponding to (2.38) differs for the fourth equation which is replaced by

$$
(4.6)\qquad v_{4xx} = \frac{v_{4x}^2}{2v_4} - v_1v_4 + \frac{a_8}{2v_4}\ .
$$

iii) In order to put Eq.s (4.5) in Lagrangian form, let us introduce in (4.5) the coordinates $r_j\,(j = 1,2,3,4)$ given by: $r_1 = v_1/2, r_2 = v_2/2 - v_1^2/8, r_3 = v_3/2 - v_1v_2/4 + v_1^3/16, r_4 = v_4/2 - v_1v_3/4 - v_2^2/8 + 3v_1^2v_2/16 - 3v_1^4/128$. As for the system containing (4.6) we introduce the coordinates $q_j\,(j = 1,2,3,4)$ given by: $q_1 = v_1/2, q_2 = v_2/2 - v_1^2/8, q_3 = v_3/2 - v_1v_2/4 + v_1^3/16, q_4 = \sqrt{-v_4}$. The two previous systems become

$$
(4.7)\qquad
\begin{aligned}
r_{1xx} &= -\frac{5}{2}r_1^2 + r_2\ , \qquad r_{2xx} = \frac{5}{2}r_1^3 - 4r_1r_2 + r_3\\
r_{3xx} &= -\frac{15}{4}r_1^4 + \frac{15}{2}r_1^2r_2 - \frac{3}{2}r_2^2 - 4r_1r_3 + r_4\\
r_{4xx} &= \frac{21}{4}r_1^5 - 15r_1^3r_2 + \frac{15}{2}r_1r_2^2 + \frac{15}{2}r_1^2r_3 - 4r_2r_3 - 5r_1r_4 + \frac{a_4}{4}\ ,
\end{aligned}
$$

$$
(4.8)\qquad
\begin{aligned}
q_{1xx} &= -\frac{5}{2}q_1^2 + q_2\ , \qquad q_{2xx} = \frac{5}{2}q_1^3 - 4q_1q_2 + q_3\\
q_{3xx} &= -\frac{15}{4}q_1^4 + \frac{15}{2}q_1^2q_2 - 2q_2^2 - 5q_1q_3 - \frac{1}{2}q_4^2\ , \quad q_{4xx} = -q_1q_4 + \frac{a_8}{4q_4^3}\ ,
\end{aligned}
$$

with Lagrangian functions

$$
\begin{aligned}
&L_4^{(0)} = r_{1x}r_{4x} + r_{2x}r_{3x} - V_4^{(0)} \ , \\
&V_4^{(0)} = -\frac{7}{8}r_1^6 + \frac{15}{4}r_1^4 r_2 - \frac{15}{4}r_1^2 r_2^2 - \frac{5}{2}r_1^3 r_3 + \frac{1}{2}r_2^3 + 4r_1r_2r_3 + \frac{5}{2}r_1^2 r_4 + \\
&\quad - r_2r_4 - \frac{1}{2}r_3^2 - \frac{a_4}{4}r_1 \ ,
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
&L_4^{(1)} = \frac{1}{2}\left(q_{4x}^2 + 2q_{1x}q_{3x} + q_{2x}^2\right) - V_4^{(1)} \ , \\
&V_4^{(1)} = \frac{3}{4}q_1^5 - \frac{5}{2}q_1^3 q_2 + 2q_1q_2^2 + \frac{5}{2}q_1^2 q_3 + \frac{1}{2}q_1 q_4^2 - q_2q_3 + \frac{a_8}{8q_4^2} \ .
\end{aligned}
\tag{4.10}
$$

The integrals of motion which are obtained by reduction of the GD polynomials are $K_j \equiv -\frac{1}{8}\hat{p}_{jk_{|Y}}$ and $H_j \equiv -\frac{1}{8}\hat{p}_{jk_{|X}}$, $(j = 0, \dots , 4)$:

(4.11)

$$
\begin{aligned}
&K_0 = -\frac{a_4}{8}, \qquad\qquad K_1 = s_1s_4 + s_2s_3 + V_4^{(0)} \ , \\
&K_2 = -s_4^2 r_4 - s_3s_4r_3 - s_2s_4r_2 + s_1s_4r_1 + 2s_2s_3r_1 + s_1s_3 + \frac{1}{2}s_2^2 - \frac{3}{4}r_1^7 + \frac{9}{4}r_1^5 r_2 + \\
&\quad - \frac{5}{4}r_1^4 r_3 - \frac{3}{2}r_1r_2^3 - 2r_1^2 r_2 r_3 + r_1r_3^2 + 3r_1r_2r_4 + + \frac{5}{2}r_2^2 r_3 - \frac{a_4}{8}r_1^2 - r_3r_4 - \frac{a_4}{4}r_2 \ , \\
&K_3 = s_4^2 r_2 r_3 - s_3s_4r_1r_3 - s_2s_4r_1r_2 + s_2s_3r_1^2 - s_2s_4r_3 + \\
&\quad - 2s_3s_4r_4 + s_1s_4r_2 - s_3^2 r_3 + s_2s_3r_2 + s_1s_3r_1 + s_2^2 r_1 + s_1s_2 + \\
&\quad - \frac{3}{2}r_1^6 r_2 + \frac{45}{8}r_1^4 r_2^2 - \frac{1}{4}r_1^5 r_3 - \frac{9}{2}r_1^2 r_2^3 - r_1^3 r_2 r_3 + \frac{5}{4}r_1^4 r_4 + \frac{3}{8}r_2^4 + \frac{1}{2}r_1r_2^2 r_3 + \\
&\quad - \frac{3}{2}r_1^2 r_3^2 - r_1^2 r_2 r_4 - \frac{1}{2}r_4^2 + 2r_2r_3^2 + 3r_1r_3r_4 + \frac{1}{2}r_2^2 r_4 - \frac{a_4}{4}r_1r_2 - \frac{a_4}{4}r_3 \ , \\
&K_4 = \frac{1}{2}s_4^2 r_3^2 + s_3s_4r_2r_3 - s_2s_4r_1r_3 - s_2s_4r_2^2 - s_3^2 r_1 r_3 + s_2s_3r_1r_2 + \frac{1}{2}s_2^2 r_1^2 + \\
&\quad - s_3^2 r_4 - 2s_2s_4r_4 + s_1s_4r_3 + s_1s_3r_2 + s_1s_2r_1 + \frac{1}{2}s_1^2 - \frac{3}{4}r_1^5 r_2^2 - \frac{3}{2}r_1^6 r_3 + \\
&\quad + \frac{5}{2}r_1^3 r_2^3 + 5r_1^4 r_2 r_3 - \frac{3}{2}r_1^5 r_4 + -\frac{3}{2}r_1 r_2^4 - \frac{9}{2}r_1^2 r_2^2 r_3 - 3r_1^3 r_3^2 + 5r_1^3 r_2 r_4 + \\
&\quad + r_2^3 r_3 + 2r_1r_2r_3^2 - 2r_1r_2^2 r_4 - r_1^2 r_3 r_4 + 2r_2r_3r_4 + 2r_1r_4^2 - \frac{a_4}{8}r_2^2 - \frac{a_4}{4}r_1r_3 \ ,
\end{aligned}
$$

and

$$
(4.12)\qquad
\begin{aligned}
H_0 &= \frac{1}{2}\left(p_2^2 + 2p_1p_3 + p_4^2\right) + V_4^{(1)} \ , \\
H_1 &= p_1p_2 + p_2^2q_1 + p_1p_3q_1 + p_4^2q_1 - p_2p_3q_2 - p_3^2q_3 - p_3p_4q_4 + \frac{5}{8}q_1^6 + \\
&\quad + \frac{5}{4}q_1^4q_2 + q_1^2q_2^2 + q_2^3 + 3q_1q_2q_3 - \frac{1}{2}q_3^2 + \frac{a_8q_1}{4q_4^2} \ , \\
H_2 &= \frac{1}{2}p_2^2q_1^2 + \frac{1}{2}p_4^2q_1^2 + \frac{1}{2}p_3^2q_2^2 + p_2p_3q_1q_2 + \frac{1}{2}p_3^2q_4^2 - p_3p_4q_1q_4 + \\
&\quad + 2p_2p_3q_3 + p_4^2q_2 + p_1p_3q_2 + p_1p_2q_1 - p_2p_4q_4 + \frac{1}{2}p_1^2 + \\
&\quad + \frac{5}{4}q_1^5q_2 - 3q_1^3q_2^2 + \frac{1}{2}q_1^3q_4^2 + \frac{5}{4}q_1^4q_3 + q_1q_2^3 - q_1^2q_2q_3 - \frac{1}{2}q_1q_2q_4^2 + \\
&\quad + \frac{1}{2}q_3q_4^2 + +q_2^2q_3 + 2q_1q_3^2 + \frac{a_8}{8q_4^2}(q_1^2 + 2a_8q_2) \ , \\
H_3 &= -p_2p_4q_1q_4 - p_3p_4q_2q_4 + p_2p_3q_4^2 + p_4^2q_1q_2 + p_4^2q_3 - p_1p_4q_4 + \\
&\quad - \frac{5}{8}q_1^4q_4^2 + \frac{3}{2}q_1^2q_2q_4^2 - \frac{1}{2}q_2^2q_4^2 + -q_1q_3q_4^2 - \frac{1}{8}q_4^4 + \frac{a_8}{4q_4^2}(q_1q_2 + q_3) \ , \\
H_4 &= -\frac{a_8}{8}
\end{aligned}
$$

where $s_1 = r_{4x}$, $s_2 = r_{3x}$, $s_3 = r_{2x}$, $s_4 = r_{1x}$ and $p_1 = q_{3x}$, $p_2 = q_{2x}$, $p_3 = q_{1x}$, $p_4 = q_{4x}$ are canonical momenta. The corresponding Hamiltonian vector fields will be respectively denoted by $Y_j := P_0\, dK_j$ and $X_{j+1} := P_1\, dH_j$ ($j = 0, 1, \dots, 4$), where P_0 and P_1 are represented in the corresponding coordinates by the canonical Poisson matrix $E = \left(\begin{smallmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{smallmatrix}\right)$, $\mathbf{0}$ and $\mathbf{1}$ being the 4×4 zero and identity matrices respectively.
iv) From the definition of r_1, r_2, r_3, r_4 and q_1, q_2, q_3, q_4 in terms of v_1 and v_2, given in step **iii)**, and from H_4 (4.12), K_4 (4.11), the map $\Phi : M_4 \to M_4, (r, s; a_4) \mapsto (q, p; a_8)$ in the 9–dimensional extended phase space can be easily obtained:

$$
(4.13)\qquad
\begin{aligned}
&q_1 = r_1, && q_2 = r_2, \quad q_3 = r_3, \quad q_4 = (-r_2^2 - 2r_1r_3 - 2r_4)^{1/2} \ , \\
&p_1 = s_2, && p_2 = s_3, \quad p_3 = s_4, \quad p_4 = -\frac{s_1 + r_1s_2 + r_2s_3 + r_3s_4}{(-r_2^2 - 2r_1r_3 - 2r_4)^{1/2}} \ , \\
&a_8 = -8K_4 \ .
\end{aligned}
$$

In these two charts let us consider the extended Hamiltonians $\tilde{H}_j$, $\tilde{K}_j$, the vector fields $\tilde{X}_j$, $\tilde{Y}_j$ and the Poisson structure $\tilde{E} := \left(\begin{smallmatrix} \mathbf{0} & \mathbf{1} & 0 \\ -\mathbf{1} & \mathbf{0} & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. The following proposition holds

Proposition 4.1. *The action of the map $\Phi : M_4 \to M_4$ defined by (4.13) on the Hamiltonians $\tilde{H}_j$, on the vector fields Y_j and on the Poisson structure $\tilde{P}'_0 := \tilde{E}$ is given by*

$$\Phi^*(\tilde{H}_j) = \tilde{K}_j \ , \qquad \Phi_*(\tilde{Y}_j) = \tilde{X}_j \tag{4.14}$$

$$\tilde{P}_0 := \Phi_* \tilde{P}'_0 \Phi^* = \begin{bmatrix} 0 & A & -8\tilde{X}_4^q \\ -A^T & B & -8\tilde{X}_4^p \\ 8(\tilde{X}_4^q)^T & 8(\tilde{X}_4^p)^T & 0 \end{bmatrix} \ , \tag{4.15}$$

where

$$\begin{aligned} A &= -\begin{bmatrix} 0 & 0 & 0 & \frac{1}{q_4} \\ -1 & 0 & 0 & \frac{q_1}{q_4} \\ 0 & -1 & 0 & \frac{q_2}{q_4} \\ \frac{q_2}{q_4} & \frac{q_1}{q_4} & \frac{1}{q_4} & -\frac{2(q_1q_2+q_3)}{q_4^2} \ , \end{bmatrix} \\ B &= \begin{bmatrix} 0 & 0 & 0 & -\frac{(p_4q_2-p_2q_4)}{q_4^2} \\ 0 & 0 & 0 & -\frac{(p_4q_1-p_3q_4)}{q_4^2} \\ 0 & 0 & 0 & -\frac{p_4}{q_4^2} \\ \frac{(p_4q_2-p_2q_4)}{q_4^2} & \frac{(p_4q_1-p_3q_4)}{q_4^2} & \frac{p_4}{q_4^2} & 0 \end{bmatrix} \end{aligned} \tag{4.16}$$

So the map Φ is not a Poisson morphism.

Thus the Poisson tensor $\tilde{P}'_0$ is mapped into a non canonical Poisson structure $\tilde{P}_0$, which can be verified to be compatible with $\tilde{P}_1 = \tilde{E}$. Both of them give rise to the following bi–Hamiltonian hierarchy

$$\tilde{X}_{j+1} = \tilde{P}_1 \, d\tilde{H}_j = \tilde{P}_0 \, d\tilde{H}_{j+1} = \quad (j = 0, 1, 2, 3) \ , \tag{4.17}$$

the Hamiltonians $\tilde{H}_0$ and $\tilde{H}_4$ being Casimirs of $\tilde{P}_0$ and $\tilde{P}_1$ respectively.

4.2. The integrability structure. In this subsection we construct the integrability structure of the Henon–Heiles system with four degrees of freedom in its eighth–dimensional phase space as an example of the model presented in Prop. 3.2. The first step consists in projecting the two compatible Poisson tensors in the extended phase space via the canonical projection $\Pi : M_4 \to S_4, (q, p; a_8) \mapsto (q, p)$. The projected tensors are given by

$$P_H := \Pi_* \tilde{P}_0 \Pi^* = \begin{bmatrix} 0 & A \\ -A^T & B \end{bmatrix} \ , \qquad E = \Pi_* \tilde{P}_1 \Pi^* \ , \tag{4.18}$$

where A and B are the matrices (4.16). Since these operators are invertible, they give rise to the following Nijenhuis tensor

$$N_H := P_H E^{-1} = \begin{bmatrix} A & 0 \\ B & -A^T \end{bmatrix} , \tag{4.19}$$

together with the hierarchy of Poisson tensors

$$P_k := N_H^k P_H \qquad k \in \mathbb{Z} . \tag{4.20}$$

As a matter of fact, one can use N_H to construct an integrability scheme as in Prop. 3.2 Indeed, if one takes

i) $Q_1 = E$, the vector field $Z_1 := X_1$ with Hamiltonian $h_0 := H_0$(4.12);
ii) the tensor field $\mathcal{N} := N_H$, i.e. the Nijenhuis tensor (4.19), and $Q_0 := P_{-2}$;
iii) the function $h_1 := H_3$, $h_2 := H_2$, $h_3 := H_1$ (4.12);
iv) $\mu_{10} = \mu_{20} = \mu_{30} = 0$ and

$$\begin{aligned} \mu_{11} &= \mu_{22} = \mu_{33} = \frac{1}{q_4^2} \\ \mu_{21} &= \mu_{32} = \frac{2(q_1q_2+q_3)}{q_4^4} \\ \mu_{31} &= \frac{2q_2+q_1^2}{q_4^4} + \frac{4(q_1^2q_2^2+2q_1q_2q_3+q_3^2)}{q_4^6} , \end{aligned} \tag{4.21}$$

then the conditions of Prop. 3.2 are satisfied. Moreover the vector field $Z_0 := Q_0 dh_0 = P_2 dH_0$ is a new integrable vector field:

(4.22)
$$Z_0 = \begin{bmatrix} p_2 - p_3q_1 \\ p_1 - p_3q_2 \\ -p_1q_1 - p_2q_2 - 2p_3q_3 - p_4q_4 \\ -p_3q_4 \\ -p_2^2 - p_1p_3 - p_4^2 + \frac{15}{4}q_1^5 - 10q_1^3q_2 + 6q_1q_2^2 + 10q_1^2q_3 - 3q_2q_3 + \frac{3}{2}q_1q_4^2 - \frac{a_8}{4q_4^2} \\ p_2p_3 - \frac{15}{4}q_1^4 + 10q_1^2q_2 - 3q_2^2 - 5q_1q_3 - \frac{1}{2}q_4^2 \\ p_3^2 + 5q_1^3 - 5q_1q_2 + q_3 \\ p_3p_4 + \frac{5}{2}q_1^2q_4 - q_2q_4 \end{bmatrix} .$$

5. Concluding remarks

In this paper we have derived a bi–Hamiltonian formulation for stationary flows (Prop. 2.2), and for the first restricted flows of the KdV hierarchy (Prop. 2.6). The reduction procedure amounts, respectively, to searching the kernel of the Poisson pencil and of n–Poisson structures extracted from the Poisson pencil of the KdV hierarchy. In this approach the generating function of the GD polynomials plays a relevant role. Moreover it allows us to construct a map between stationary flows and restricted flows; in the case of the fifth-order stationary KdV equation, this map relates solutions of the Henon–Heiles system with solutions of the Garnier system.

However, to obtain these results, one must extend the phase space of the reduced flows by means of some free parameters naturally contained in the corresponding Hamiltonian functions. This difficulty can be bypassed, at least if one analyzes complete integrability of a Hamiltonian system without requiring an explicit knowledge of a bi–Hamiltonian structure. To this purpose, we have introduced a new integrability scheme in the standard phase space, which implies Liouville integrability of the reduced Hamiltonian systems.

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