Gröbner bases related to 3-dimensional transportation problems

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Abstract. This paper illustrates some work in progress on 3-dimensional transportation problems,of format $r\times s\times t$ say. Following Conti and Traverso, a suitable Gröbner basis is sought for, which is hard to be calculated by means of Buchberger algorithm. A different approach involving graph theory makes the calculation tractable when $r = s = t = 3$ (and in fact whenever $3 \in \{r, s, t\}$).

Introduction

The aim of this paper is to illustrate some work in progress on 3-dimensional transportation problems.

A typical problem of this kind goes as follows. Given r production facilities $F_1 \dots F_r$, let a_{ik} $(i := 1 \dots r, k := 1 \dots t)$ denote the number of units of an indivisible good produced by F_i during the k-th month of a fixed period of t months. Assume that there are s outlets $O_1 \dots O_s$ each one demanding a certain number of units per month, say b_{jk} $(j := 1 \dots s, k := 1 \dots t)$. If c_{ijk} stands for the cost associated with transporting one unit from F_i to O_j during the k-th month, one wishes to minimize the total cost of transportation during the whole period of t months.

In mathematical terms, one wishes to solve the integer programming problem associated with the matrix $\mathcal{A}$ whose columns are

$$\{\underline{e}_{ij} \oplus \underline{e}^{'}_{ik} \oplus \underline{e}^{''}_{jk} \mid 1 \le i \le r,\ 1 \le j \le s,\ 1 \le k \le t\},$$

where $\{\underline{e}_{ij}\}$ (resp., $\{\underline{e}^{'}_{ik}\}$; resp., $\{\underline{e}^{''}_{jk}\}$) stands for the canonical basis of the $\mathbb{Z}$-module of $r \times s$ (resp., $r \times t$; resp., $s \times t$) integer matrices.

For 3-dimensional transportation problems see e.g. [**Vl**] and [**St**, Chapter 14].

It is well known that integer programming problems as above can be solved by a method first suggested by Conti and Traverso (cf. [**CT**]), which resorts to the calculation of suitable Gröbner bases (for more information about this method we refer the reader to the survey papers [**HT**] and [**T**]).

More precisely, if $I_{\mathcal{A}}$ denotes the toric ideal canonically associated with the matrix $\mathcal{A}$ described before, then one needs to find the reduced Gröbner basis of $I_{\mathcal{A}}$ relative to some appropriate term order.

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Since it would be hard to use Buchberger algorithm in order to compute such a reduced Gröbner basis, we show in this paper how to perform a calculation along a different route. It is a route involving the use of graph theory.

As for the specific results we obtain, we give a complete description in case $r = s = t = 3$, which in turn outlines a strategy for every case such that $3 \in \{r, s, t\}$. We tend to believe that no essential change takes place when $3 \notin \{r, s, t\}$, but the combinatorics looks much less manageable.

1. Preliminaries

In this section we collect some material on graphs, which is going to be used later. (For basic definitions and notation cf. e.g. [**B**].)

Definition 1.1. Let G be a graph and $C := v_0 v_1 \dots v_n$ be a path of G. If e is an edge of C, we call parity of e relative to C the integer

$$p_C(e) := d - p,$$

where d is the number of times e occurs in C in odd position, and p is the number of times e occurs in C in even position.

Example 1.2. $C := v_0 v_1 v_2 v_3 v_0 v_1$; $e := \{v_0 v_1\}$; $p_C(e) = 2 - 0 = 2$.

We often write $p(e)$ instead of $p_C(e)$, when there is no ambiguity.

Proposition 1.3. *Let G be a bipartite graph and C be a path of G. Then C is closed if, and only if, for every vertex v of C one has*

$$\sum_{e \in C, v \in e} p(e) = 0. \tag{1.1}$$

Proof. Assume that (1.1) holds. If C were not closed, then $v_o \neq v_n$ (notation as in Definition 1.1). It would follow $\sum_{e \in C, v_n \in e} p(e) \neq 0$, for e_{i-1} in odd(even) position in C implies e_i in even(odd) position. Assume now that C is closed; since G is bipartite, C has even length. Let v be a vertex of C, e and edge containing v. If e is in odd position in C, v can be taken as the first vertex of C (i.e., $v = v_0$), and e as the first edge of C. As C has even length, it follows that the "last" edge of C is in even position and contains v. A similar argument works if e is in even position ("last edge of C"). Thus (1.1) holds. □

We denote by A_G the incidence matrix of G. The rows of A_G are indexed by the vertices of G, its columns by the edges, and

$$a_{v,e} := \begin{cases} 1 & \text{if } e \text{ contains } v \\ \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.4. Given a closed path C of G, for every edge e of G we set

$$C_e := \begin{cases} 0 & \text{if } e \notin C \\ \\ p(e) & \text{if } e \in C. \end{cases}$$

We say that $(C_e)_{e \in E}$ is the "sequence associated to C" (Here E stands for the set of edges of G).

Example 1.5. Let $G := K_{3,2}$, whose edges we index by $\{1,2,3\} \times \{1,2\}$. Let C be the closed path pictured bellow (the dotted edges being those in even position):

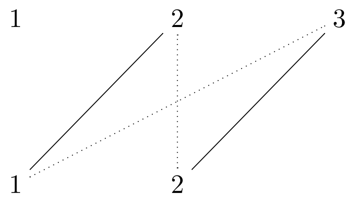


Figure 1

Then: $C_{11} = 0$, $C_{12} = 0$; $C_{21} = 1$, $C_{22} = -1$; $C_{31} = -1$, $C_{32} = 1$. That is, $(0,0,1,-1,-1,1)$ is the sequence associated with C.

Proposition 1.6. *Let G be a bipartite graph.*

(a): *If C is a closed path of G, then the sequence $(C_e)_{e\in E}$ associated with C is an element of $Ker_{\mathbb{Z}}(A_G)$.*

(b): *If $(d_e)_{e\in E}$ is in $Ker_{\mathbb{Z}}(A_G)$, then there exists at least a closed path of G, whose associated sequence is precisely $(d_e)_{e\in E}$.*

Proof.

(a): Let $v \in V := V_1 \cup V_2$. By Proposition 1.3, one gets:

$$\sum_{e\in E} a_{v,e} C_e = \sum_{e\in E, v\in e} C_e = \sum_{e\in C, v\in e} p(e) = 0.$$

(b): If $(d_e)_{e\in E}$ is zero, there is nothing to prove. Assume that there exists $e_1 \in E$ such that $d_{e_1} > 0$; written $e_1 = \{v, v'\}$ ($v \in V_1$, $v' \in V_2$), we set $v_0 := v$, $v_1 := v'$ and think of e_1 as of the first edge of the path. Since $(d_e)_{e\in E} \in Ker_{\mathbb{Z}}(A_G)$, it follows that

$$\sum_{e\in E, v_1\in e} d_e = 0.$$

Recalling $d_{e_1} > 0$, there must be some $e_2 \neq e_1$ such that $v_1 \in e_2$ and $d_{e_2} < 0$. Call $v_2 (\in V_1)$ the further vertex of e_2, and iterate the argument.

We notice that v_4 may coincide with v_0; in such a case, one can choose $e_5 = e_1$ only if $d_{e_1} \geq 2$.

□

Example 1.7. Let $G := K_{3,3}$, with $V_1 = \{1,2,3\}$ and $V_2 = \{4,5,6\}$. $Ker_{\mathbb{Z}}(A_G)$ contains the sequence:

$$d_{14} := -2, d_{15} := 1, d_{16} := 1; d_{24} := 0, d_{25} := 2, d_{26} := -2; d_{34} := 2, d_{35} := -3, d_{36} := 1.$$

As in the proof of Proposition 1.6, one can build the path

$$C = 153416253625341.$$

It is pictured right below (the dotted edges being in even position):

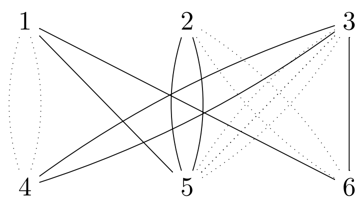


FIGURE 2

2. The graph $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$

In this section we define a special graph of particular importance for what follows. Let $\underline{r} := \{1, 2, \ldots, r\}$, $\underline{s} := \{1, 2, \ldots, s\}$, $\underline{t} := \{1, 2, \ldots, t\}$, where r, s, and t are positive integers. Consider the two disjoint sets

$$V_1 := \underline{r} \times \underline{s} \quad \textit{and} \quad V_2 := \underline{r} \times \underline{t}.$$

We denote by $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$ the graph having $V := V_1 \cup V_2$ as set of vertices, and $E := \{e_{ijk} | i \in \underline{r},\ j \in \underline{s},\ k \in \underline{t}\}$ as set of edges where

$$e_{ijk} = \{(i, j) \in V_1,\ \ (i, k) \in V_2\}.$$

EXAMPLE 2.1. $\underline{r} := \underline{3}$, $\underline{s} := \underline{2}$, $\underline{t} := \underline{3}$. $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$ is the graph:

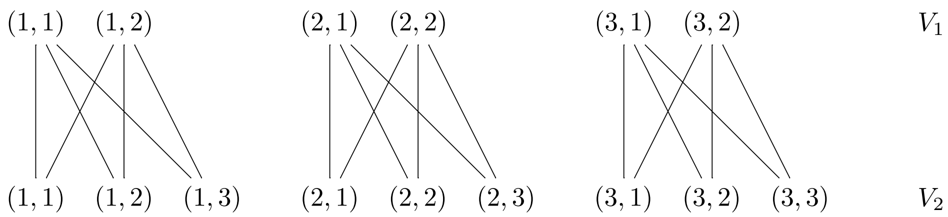


FIGURE 3

PROPOSITION 2.2. *The following hold:*

(a): $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$ *is a bipartite graph, with vertex classes* V_1 *and* V_2*.*

(b): *For every* $i \in \underline{r}$*, the subgraph* $K^{(i)}_{s,t}$ *of* $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$ *induced by the set of vertices*

$$V' := V_{i_1} \cup V_{i_2},$$

$$\textit{with} \quad V_{i_1} := \{i\} \times \underline{s} \quad \textit{and} \quad V_{i_2} := \{i\} \times \underline{t},$$

turns out to be isomorphic to the complete bipartite graph $K_{s,t}$*.*

(c): $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}} = rK_{s,t}$ *(disjoint union of r copies of $K_{s,t}$).*

Proof. Obvious. □

Definition 2.3. For every choice of i, i' in $\underline{r}$, of j in $\underline{s}$, k in $\underline{t}$, we say that the edges e_{ijk} and $e_{i'jk}$ are <u>*parallel*</u>.

We remark that $e_{i'jk} = \varphi^{(i,i')}(e_{ijk})$, where

$$\varphi^{(i,i')} : K_{s,t}^{(i)} \to K_{s,t}^{(i')}$$

is the isomorphism defined by means of

$$\varphi^{(i,i')}(i,j) := (i',j) \qquad \forall j \in \underline{s}$$

and of

$$\varphi^{(i,i')}(i,k) := (i',k) \qquad \forall k \in \underline{t}.$$

3. $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$ and 3-dimensional transportation problems

In this section we give a new approach to 3-dimensional transportation problems by means of the graph $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$. Let $\mathcal{A}$ be the matrix whose columns are

$$\{\underline{e}_{ij} \oplus \underline{e}'_{ik} \oplus \underline{e}''_{jk} \mid i \in \underline{r},\ j \in \underline{s},\ k \in \underline{t}\},$$

where $\{\underline{e}_{ij}\}$ stands for the canonical basis of the $\mathbb{Z}$-module of $r \times s$ integer matrices (denoted by $\mathbb{Z}^{\underline{r}\times\underline{s}}$), $\{\underline{e}'_{ij}\}$ stands for the canonical basis of $\mathbb{Z}^{\underline{r}\times\underline{t}}$ ($r \times t$ integer matrices), and $\{\underline{e}''_{jk}\}$ for the canonical basis of $\mathbb{Z}^{\underline{s}\times\underline{t}}$.

The integer programming problem associated with $\mathcal{A}$ is called "3-dimensional transportation problem" (cf. e.g. [**Vl**], [**St**]). The corresponding toric ideal to be studied is $I_{\mathcal{A}} := Ker(\Pi_{\mathcal{A}})$, where:

$$\Pi_{\mathcal{A}} : K[x_{ijk}] \to K[u_{ij}, v_{ik}, w_{jk}];$$
$$x_{ijk} \mapsto u_{ij}v_{ik}w_{jk}$$

here $i \in \underline{r}$, $j \in \underline{s}$, $k \in \underline{t}$, and K is any field.

Think of $\mathcal{A}$ as of the matrix of the morphism

$$\mathbb{Z}^{\underline{r}\times\underline{s}\times\underline{t}} \to \mathbb{Z}^{\underline{r}\times\underline{s}} \oplus \mathbb{Z}^{\underline{r}\times\underline{t}} \oplus \mathbb{Z}^{\underline{s}\times\underline{t}}$$
$$\underline{u} \mapsto \mathcal{A}\underline{u}$$

where $\mathbb{Z}^{\underline{r}\times\underline{s}\times\underline{t}}$ stands for the $\mathbb{Z}$-module of 3-dimensional matrices of format $r \times s \times t$. It is well known (cf. e.g. [**St**]) that if $<$ is any term order on $K[\underline{x}] := K[x_{ijk}]$, then

$$In_<(I_{\mathcal{A}}) = In_<(\mathcal{B}),$$

where $\mathcal{B} := \{\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-} | \underline{u} \in Ker_{\mathbb{Z}}(\mathcal{A})\} \subseteq I_{\mathcal{A}}$, and $In_<(\mathcal{B})$ stands for the ideal generated by all initial terms $In_<(f)$ with f ranging in the (infinite) set $\mathcal{B}$.

A finite set $Gr \subseteq I_{\mathcal{A}}$ is a Gröbner basis for $I_{\mathcal{A}}$ with respect to $<$ if, and only if, $In_<(f) \in In_<(Gr)$ for every $f \in \mathcal{B}$. Here below, we use $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$ to give a description of $\mathcal{B}$ which will prove helpful for the study of Gröbner bases in the next two sections.

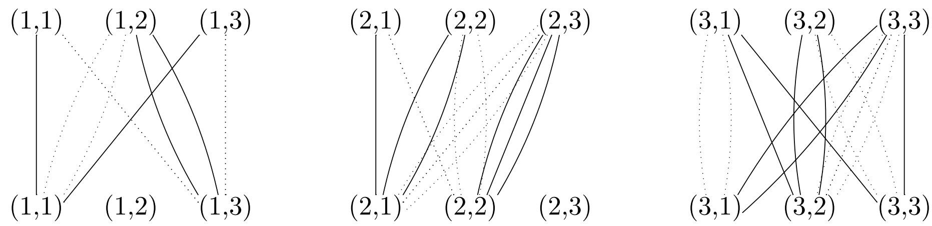
Definition 3.1. Let $S := (C_1, C_2, \dots, C_r)$ be an r-tuple satisfying the following properties:

(1): For every $i = 1, \ldots, r$, either $C_i = \emptyset$ or C_i is a closed path of the subgraph $\mathrm{K}_{s,t}^{(i)}$ of $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$.

(2): Whenever an edge e occurs in C_i in odd (resp., even) position, there exists an edge e' parallel to e occurring in even (resp., odd) position in some $C_{i'}$.

We call S an admissible $r-$tuple of closed paths of $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$.

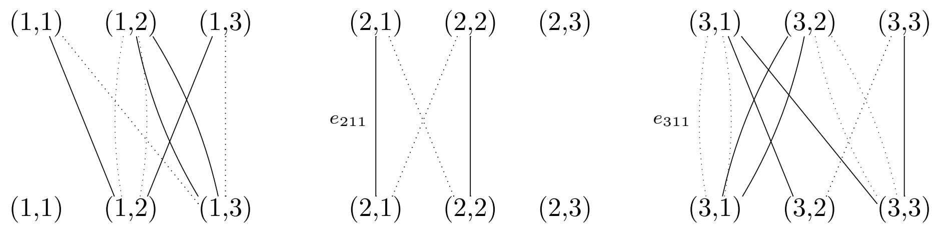
EXAMPLE 3.2. Consider $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$. If C_1, C_2, C_3 are like in the picture below, (C_1, C_2, C_3) is admissible (as usual, the dotted edges are in even position).



C_1 C_2 C_3

FIGURE 4

EXAMPLE 3.3. Again consider $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$. If C_1, C_2, C_3 are like in the picture below, (C_1, C_2, C_3) is NOT admissible. For instance, the edge e_{311} occurs twice in C_3 in even position, but there is just one edge parallel to it (namely, e_{211}) occurring in odd position (in C_2). Other "bad" edges are e_{212}, e_{221} and e_{222}.



C_1 C_2 C_3

FIGURE 5

REMARK 3.4. *Although any closed path can be obtained by "patching cycles together"(and this remark suggests another proof of Proposition 1.6 (a)),it is not true that an admissible r-tuple of closed paths can always be written as a "sum" of admissible r-tuple of cycles. This following picture provides an example.*

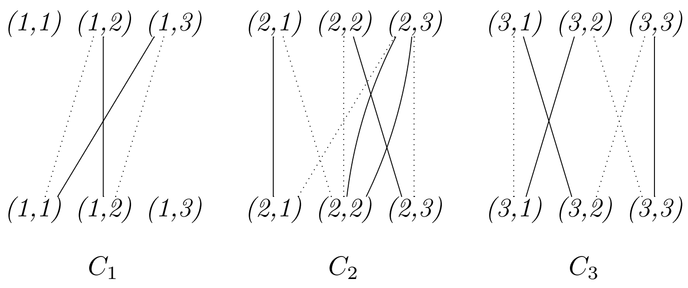


FIGURE 6

DEFINITION 3.5. Let $S := (C_1, C_2, \ldots, C_r)$ be an admissible r-tuple of closed paths of $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$. For every $(i, j, k) \in \underline{r} \times \underline{s} \times \underline{t}$, we set

$$a_{ijk} := \begin{cases} 0 & \text{if } e_{ijk} \text{ does not occur in } C_i \\ p(e_{ijk}) & \text{if } e_{ijk} \text{ occurs in } C_i, \end{cases}$$

where $p(e_{ijk})$ stands for the parity of $p(e_{ijk})$ relative to C_i.

We say that (a_{ijk}) is the sequence associated with S.

REMARK 3.6. *Since e_{ijk} occurs in $K_{s,t}^{(i)}$, e_{ijk} can only occur in C_i.*

The following theorem links $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$ to $Ker_{\mathbb{Z}}(\mathcal{A})$, where $\mathcal{A}$ is the matrix of the 3-dimensional transportation problem introduced at the beginning of this section.

THEOREM 3.7. *The following hold:*

- **(a):** *Let $S := (C_1, C_2, \ldots, C_r)$ be an admissible r-tuple of closed paths of $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$, and let $\underline{a} := (a_{ijk})$ be its associated sequence. Then $\underline{a} \in Ker_{\mathbb{Z}}(\mathcal{A})$.*
- **(b):** *If $\underline{b} := (b_{ijk})$ is in $Ker_{\mathbb{Z}}(\mathcal{A})$, then there exists at least one admissible $r-$tuple of closed paths of $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$, whose associated sequence coincides with $\underline{b}$.*

PROOF.

- **(a):** Let $A_{K_{s,t}^{(i)}}$ be the incidence matrix of the subgraph $K_{s,t}^{(i)}$. We index the columns of $A_{K_{s,t}^{(i)}}$ by all pairs $(j, k) \in \underline{s}\times\underline{t}$, and its rows by all pairs

$(i,j) \in \{i\}\times\underline{s}$ and $(i,k) \in \{i\}\times\underline{t}$. It is clear that $\underline{a}'_{\bar{i}} := (a'_{\bar{i}jk})_{(j,k)\in\underline{s}\times\underline{t}}$ belongs to $Ker_{\mathbb{Z}}(A_{K^{(\bar{i})}_{s,t}})$ if, and only if, one gets:

$$\begin{cases} \sum_{k=1}^{t} a'_{\bar{i}jk} = 0 & \text{for every } j = 1,\dots,s \\ \sum_{j=1}^{s} a'_{\bar{i}jk} = 0 & \text{for every } k = 1,\dots,t. \end{cases} \tag{3.1}$$

For if $\alpha_{(\bar{i},j)(j',k)}$ is the generic entry of row $(\bar{i},j)$ in $A_{K^{(\bar{i})}_{s,t}}$, i.e.

$$\alpha_{(\bar{i},j)(j',k)} := \begin{cases} 0 & \text{if } j' \neq j \\ 1 & \text{if } j' = j\ , \end{cases}$$

then

$$\sum_{(j'k)\in\underline{s}\times\underline{t}} \alpha_{(\bar{i},j)(j',k)} a'_{\bar{i}j'k} = \sum_{k=1}^{t} a'_{\bar{i}jk}.$$

And if $\alpha_{(\bar{i},k)(j,k')}$ is the generic entry of row $(\bar{i},k)$ in $A_{K^{(\bar{i})}_{s,t}}$, then

$$\sum_{(j,k')\in\underline{s}\times\underline{t}} \alpha_{(\bar{i},k)(j,k')} \alpha'_{\bar{i}j'k'} = \sum_{k=1}^{s} a'_{\bar{i}jk}.$$

Let us consider now the sequence $\underline{a}$ associated with S. For every i such that $C_i \neq \emptyset$, the subsequence $\underline{a}_i := (a_{ijk})_{(j,k)\in\underline{s}\times\underline{t}}$ is associated with C_i in the sense of Definition 1.4. Hence Proposition 1.6(a) says that $\underline{a}_i \in Ker_{\mathbb{Z}}(A_{K^{(i)}_{s,t}})$, and (3.1) holds. Since for every i such that $C_i = \emptyset$, 3.1 holds, too, we have shown that the sequence $\underline{a}$ associated with S satisfies the following:

$$\begin{cases} \sum_{k=1}^{t} a_{ijk} = 0 & \text{for every } i = 1,\dots,r \text{ and every } j = 1,\dots,s \\ \sum_{j=1}^{s} a_{ijk} = 0 & \text{for every } i = 1,\dots,r \text{ and every } k = 1,\dots,t. \end{cases} \tag{3.2}$$

In order to complete the proof of (a), it remains to show that:

$$\sum_{i=1}^{r} a_{ijk} = 0 \quad \text{for every } j = 1,\dots,s \text{ and every } k = 1,\dots,t. \tag{3.3}$$

Let $\bar{j}, \bar{k}$ be such that there exist $\bar{i}$ verifying $a_{\overline{ijk}} \neq 0$; then, by definition, $C_{\bar{i}} \neq \emptyset$ and $e_{\overline{ijk}}$ occurs in $C_{\bar{i}}$ with parity $a_{\overline{ijk}}$. Without loss of generality, we may assume that $a_{\overline{ijk}} > 0$, i.e., $e_{\overline{ijk}}$ occurs in $C_{\bar{i}}$ in odd position at least $a_{\overline{ijk}}$ times. Since S is admissible, whenever $e_{\overline{ijk}}$ occurs in $C_{\bar{i}}$ in odd position, there is some $e_{i'\overline{jk}}$ occurring in $C_{i'}$ in even position; hence $\sum_{i=1}^{r} a_{i\overline{jk}} = 0$, and (3.3) follows. This completes the proof of part (a).

(b): Thanks to what we have seen at the beginning of the proof of part (a), if $\underline{b} \in Ker_{\mathbb{Z}}(\mathcal{A})$, then the subsequence $\underline{b}_i := (b_{ijk})_{(j,k)\in\underline{s}\times\underline{t}} \in Ker_{\mathbb{Z}}(A_{K^{(i)}_{s,t}})$

for every i. For $\underline{b} \in Ker_{\mathbb{Z}}(\mathcal{A})$ implies that for every i,

$$\sum_{k=1}^{t} b_{ijk} = 0 \qquad \text{for every} \quad j = 1, \ldots, s$$

$$\sum_{j=1}^{s} b_{ijk} = 0 \qquad \text{for every} \quad k = 1, \ldots, t.$$

Applying Proposition 1.6(b) to $A_{K^{(i)}_{s,t}}$, it follows that for every i, there is a closed path C_i of $K^{(i)}_{s,t}$ such that $\underline{b}_i$ is its associated sequence. If $\underline{b}_i = \underline{0}$, we set $C_i = \emptyset$.

It remains to show that the $r-$tuple $S := (C_1, \ldots, C_r)$ is admissible. Without loss of generality, we may assume that e_{ijk} occurs in C_i in odd position; by the construction of C_i (recall the proof of Proposition 1.6), it follows that $b_{ijk} > 0$, and b_{ijk} is precisely the number of times e_{ijk} occurs in C_i in odd position. Since $\underline{b} \in Ker_{\mathbb{Z}}(\mathcal{A})$, we also have that $\sum_{i=1}^{r} b_{ijk} = 0$, hence, whenever e_{ijk} occurs in C_i in odd position, there is some $e_{i'jk}$ ($i' \neq i$) occurring in $C_{i'}$ in even position (-$b_{i'jk}$ times, since $b_{i'jk}$ is negative).

□

Remark 3.8. *The $r-$tuple $S = (C_1, \ldots, C_r)$ constructed in the proof of Theorem 3.7(b) has the property that every edge of C_i either is always in odd position ("odd edge"), or is always in even position ("even edge"). Hence, from now on, we always assume that our admissible $r-$tuples have such a property.*

Example 3.9.

(1): The sequence associated with (C_1, C_2, C_3) of Example 3.2 looks like this (write ijk instead of a_{ijk}):

111	is	1	211	is	1	311	is	-2
112		0	212		-1	312		1
113		-1	213		0	313		1
121		-2	221		2	321		0
122		0	222		-2	322		2
123		2	223		0	323		-2
131		1	231		-3	331		2
132		0	232		3	332		-3
133		-1	233		0	333		1

It is easy to check that it belongs to $Ker_{\mathbb{Z}}(\mathcal{A})$.

(2): r=s=t=3; take the following element of $Ker_{\mathbb{Z}}(\mathcal{A})$:

111	is	1	211	is	-2	311	is	1
112		0	212		0	312		0
113		-1	213		2	313		-1
121		0	221		1	321		-1
122		-1	222		0	322		1
123		1	223		-1	323		0
131		-1	231		1	331		0
132		1	232		0	332		-1
133		0	233		-1	333		1

The corresponding admissible (C_1, C_2, C_3) constructed in Theorem 3.7(b) is:

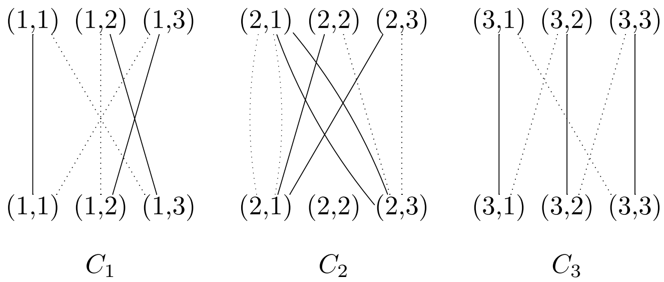


FIGURE 7

Theorem 3.7 and Remark 3.8 should have clarified by now the relationship between $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$ and $\mathcal{B} = \{\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-} | \underline{u} \in Ker_{\mathbb{Z}}(\mathcal{A})\}$, which is summarized in the following (obvious) corollary.

COROLLARY 3.10. *Let us associate the variable x_{ijk} with the edge e_{ijk} of $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$, and viceversa. Then, given $\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-} \in \mathcal{B}$, there exists an admissible $r-$tuple $S := (C_1, \ldots, C_r)$ of closed paths of $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$ such that every edge of C_i is always in odd (even) position, and such that $\underline{u}^+$ is given by the parities of all odd edges of S, $\underline{u}^-$ by the parities of all even edges. Conversely, given an admissible $r-$tuple, a binomial $\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-} \in \mathcal{B}$ is obtained defining $\underline{u}^+$ and $\underline{u}^-$ as above.*

DEFINITION 3.11. Let $S := (C_1, \ldots, C_r)$ be an admissible $r-$tuple of closed paths of $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$. Two paths C_i and $C_{i'}$ $(i \neq i')$ are called anti-isomorphic if they only contain parallel edges with opposite parities.

EXAMPLE 3.12. The following paths of $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$ are anti-isomorphic.

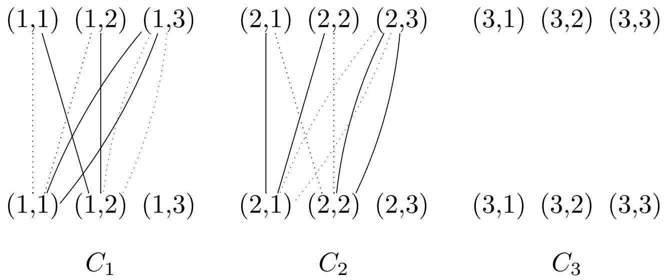


FIGURE 8

PROPOSITION 3.13. *Let $S := (C_1, \dots, C_r)$ be an admissible $r-$tuple of closed paths of $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$.*

(a): *Either $C_i = \emptyset$ for every i, or there are at least two nonempty paths.*
(b): *If there are exactly two nonempty paths, then they are anti-isomorphic.*

PROOF.

(a): If $C_i \neq \emptyset$, then there is an edge e_{ijk} of positive parity. The admissibility of S implies that there is at least an edge $e_{i'jk}$ $(i \neq i')$ of negative parity. Hence $C_{i'} \neq \emptyset$, too.
(b): Obvious.

□

Notice that if $S := (C_1, \dots, C_r)$ is an $r-$tuple of closed paths of $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$, and for every nonempty path C_i there is a path $C_{i'}$ $(i \neq i')$ such that C_i and $C_{i'}$ are anti-isomorphic , then S is admissible.

Let us now introduce in the set $\underline{r} \times \underline{s} \times \underline{t}$ the lexicographic order $<_{lex}$defined by:

$$(i, j, k) <_{lex} (i', j', k')$$

if and only if the first non-zero component of the difference vector is negative.

Then $K[x_{ijk}]$ is endowed with the pure lexicographic term order $<_{Lex}$such that:

$$x_{ijk} <_{Lex} x_{i'j'k'} \Leftrightarrow (i, j, k) <_{lex} (i', j', k').$$

As a first application of the material above, we show how to get in a different way a result by Sturmfels (*cf.* [**St**, Chapter 14]).

PROPOSITION 3.14. *Let $r = 2$, and s, t any integers ≥ 2. The binomials associated with pairs of anti-isomorphic cycles of $\mathcal{G}_{\underline{2}\times\underline{s}\times\underline{t}}$ form a reduced Gröbner basis of $I_{\mathcal{A}}$ relative to $<_{Lex}$.*

PROOF. Let Gr be the set of all binomials associated with pairs of anti-isomorphic cycles. Clearly, $Gr \subseteq \mathcal{B}$ (cf. remark after Proposition 3.13). As mentioned at the beginning of this section, it suffices to show that $In_{<_{Lex}}(\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-}) \in In_{<_{Lex}}(Gr)$ for every $\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-} \in \mathcal{B}$.

It follows from Corollary 3.10 and Proposition 3.13 that we need only consider all pairs of anti-isomorphic closed paths of $\mathcal{G}_{\underline{2}\times\underline{s}\times\underline{t}}$. Let $S := (C_1, C_2)$ be such a pair.

We order the edges of C_2 according to the order of the corresponding variables. Let e be the maximum edge occurring in C_2. Starting with e, we can move along at least one cycle D_2 whose edges are also edges of C_2. Since C_1 and C_2 are anti-isomorphic, C_1 must contain a cycle D_1 anti-isomorphic to D_2. And Corollary 3.10 ends the proof. □

EXAMPLE 3.15. Let S be the following pair of anti-isomorphic closed paths of $\mathcal{G}_{\underline{2}\times\underline{3}\times\underline{3}}$:

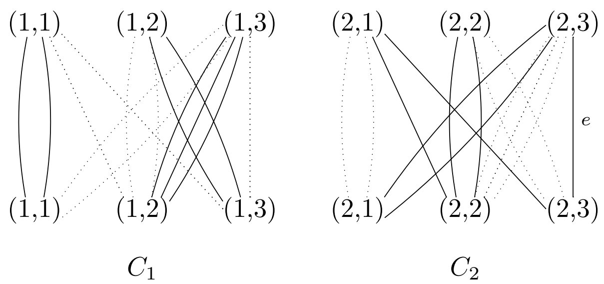


FIGURE 9

where the edges in odd position (the ones which are not dotted) determine $In_{<_{Lex}}$ of the associated binomial $\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-}$.

A corresponding pair (D_1, D_2) is:

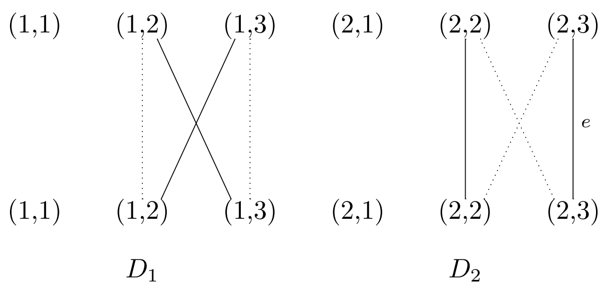


FIGURE 10

Going back to the statement of Proposition 3.14, we point out that, since $r = 2$ implies $\mathcal{A}$ of Lawrence type (cf. [**St**, Proposition 14.11]), it follows from [**St**, Theorem 7.1] that the set of all binomials associated to pairs of anti-isomorphic cycles of $\mathcal{G}_{\underline{2}\times\underline{s}\times\underline{t}}$ also provides the universal Gröbner basis of $I_{\mathcal{A}}$, as well as the Graver basis of $\mathcal{A}$, and a minimal set of generators of $I_{\mathcal{A}}$ (cf. [**St**, Corollary 14.12]).

We end this section by observing that what we have done in it for $rK_{s,t}$ works equally well from $sK_{r,t}$ and $tK_{r,s}$. Hence, for instance, Proposition 3.14 holds for every $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$ such that $2 \in \{r, s, t\}$.

4. r-tuples of cycles

The description of the set $\mathcal{B}$ given in Section 3 by means of the graph $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$ allows us to tackle the study of the reduced Gröbner basis of $I_{\mathcal{A}}$ relative to the term order $<_{Lex}$.

In fact we are going to give a complete description of that basis in the case $r = s = t = 3$, but it will be obvious from the proof that the same strategy applies whenever $3 \in \{r, s, t\}$. We suspect that also when $3 \notin \{r, s, t\}$, no essential change takes place, but the combinatorics involved looks harder.

We are going to proceed in the following way: we describe the basis elements in this section; we show that they are indeed a basis in Section 5. As expected, our basis in the case $3\times3\times3$ turns out to be a subset of the universal basis obtained by Sturmfels by means of "a brute force computation in MACAULAY" ([**St**, Theorem 14.13]).

Definition 4.1. Let $<$ be a term order in $K[\underline{x}]$, and $S := (C_1, \dots, C_r)$ an admissible r-tuple of (not all empty) closed paths of $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$. We call maximum edge of S (relative to $<$) every edge e_{ijk} of $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$ such that x_{ijk} occurs in the maximum term of the binomial associated with S.

If we index $V_1(K_{s,t})$ by elements of $\underline{s}$, and $V_2(K_{s,t})$ by those of $\underline{t}$, then for every $i \in \underline{r}$, an isomorphism

$$\varphi_i : K_{s,t}^{(i)} \to K_{s,t}$$

is defined by means of

$$\varphi_i(i, j) := j \qquad \text{for every } j \in \underline{s}$$

$$\varphi_i(i, k) := k \qquad \text{for every } k \in \underline{t}.$$

Definition 4.2. Let $<$ be a term order in $K[\underline{x}]$, and $S := (C_1, \dots, C_r)$ an admissible r-tuple of (not all empty) closed paths of $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$.

We call associated configuration of S relative to $<$, written $AC_<(S)$, the multiweighted subgraph of $K_{s,t}$ described as follows:

- **(1):** an edge of $K_{s,t}$ belongs to $AC_<(S)$ if it coincides with $\varphi_i(e_{ijk})$ for some i and some maximum edge e_{ijk}.
- **(2):** the multiweight associated with an edge e of $AC_<(S)$ is the set of all indices i such that $\varphi_i(e_{ijk}) = e$ for some maximum edge e_{ijk}.

We say that an edge e of $AC_<(S)$ weighs i in order to indicate that i occurs in the multiweight of e.

Proposition 4.3. *The followings hold:*

- **(a):** *If v is a vertex of $AC_<(S)$, then there are at least two edges incident on v.*
- **(b):** *If e is an edge of $AC_<(S)$, and its multiweight has cardinality h, then $1 \le h \le r - 1$.*

Proof. Let $\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-}$ be the binomial associated with S; without loss of generality we may assume that $In_<(\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-}) = \underline{x}^{\underline{u}^+}$. Then the maximum edges of S are those in odd positions (recall Corollary 3.10).

From now on, we identify with their images all vertices and edges transformed by isomorphisms φ_i.

(a): Let e be an edge incident on v. If e weighs $\bar{\imath}$, e is an edge of $C_{\bar{\imath}}$ in odd position. Since $C_{\bar{\imath}}$ is closed and Remark 3.8 holds, there is an even edge $e' \neq e$ of $C_{\bar{\imath}}$ which is incident on v. Since S is admissible, it follows that exists an edge e'' parallel to e' and occurring in odd position in some $C_{i'}$ with $i' \neq \bar{\imath}$. This means that e'' is an edge of $AC_<(S)$ incident on v and different from e.

(b): h cannot be r since otherwise e would be odd in every C_i.

□

Example 4.4. Recall the admissible (C_1, C_2, C_3) described in Example 3.2. Its associated configuration relative to $<_{Lex}$ is as follows:

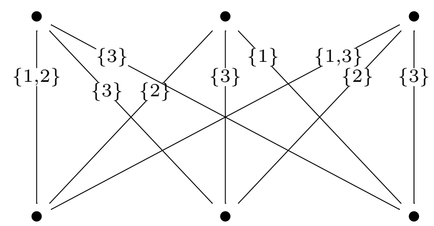


Figure 11

(As in the proof of Proposition 4.3, we assume that the maximum edges are those in odd positions.)

Definition 4.5. An admissible r-tuple of (not all empty) closed paths of $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$, $S := (C_1, \ldots, C_r)$, is called an <u>r-tuple of cycles</u> if every nonempty C_i turns out to be a cycle of $K^{(i)}_{s,t}$.

Proposition 4.6. *Let $<$ be a term order in $K[\underline{x}]$, and S an r-tuple of cycles.*

(a): *If v is a vertex of $AC_<(S)$, and two edges e and e' are incident on v, then the multiweight of e has an empty intersection with the multiweight of e'.*

(b): *If $r = 3$, every edge of $AC_<(S)$ has multiweight of cardinality* 1.

Proof.

(a): If both e and e' weighed i, then C_i would contain two different edges incident of v, either both even or both odd. But this is impossible, for C_i is a cycle of a bipartite graph.

(b): It follows from Proposition 4.3(b) that every edge as multiweight of cardinality at most 2. Suppose that e is an edge with multiweight of cardinality exactly 2; $\{1,2\}$, say. Assuming that the maximum edges are odd, there must exist in C_3 and edge e', parallel to e, which occurs twice in even position; hence C_3 ($\neq \emptyset$) cannot be a cycle: contradiction.

□

In the remaining part of this section, we assume $r = s = t = 3$ and, for the triplets of cycles of $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$, characterize all possible associated configurations relative to $<_{Lex}$ that will be needed in Section 5, when dealing with the reduced Gröbner basis of $I_{\mathcal{A}}$.

Given a triplet of cycles of $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$, say S, the underlying graph of $AC_{<_{Lex}}(S)$ is a subgraph of $K_{3,3}$. We assume $V_1(K_{3,3}) = \{v_1, v_2, v_3\}$ and $V_2(K_{3,3}) = \{v_4, v_5, v_6\}$.

If v is a vertex of $AC_{<_{Lex}}(S)$, we say that its weight is the sum of the weights of all edges incident on v (recall Proposition 4.6(b)).

Corollary 4.7. *Let v be a vertex of $AC_{<_{Lex}}(S)$.*

(a): *There are either two or three edges incident on v.*
(b): *If there are three edges incident on v, the weight of v is 6. If there are two edges incident on v, the weight of v can be 3, 4, 5.*

Proof.

(a): Obvious.
(b): Proposition 4.6 implies that three edges incident on v have weights given by a permutation of 1, 2, 3, while two edges incident on v can have weights 1, 2 or 1, 3 or 2, 3.

□

Proposition 4.8. *Let $S := (C_1, C_2, C_3)$ be a triplet of cycles of $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$ such that $C_i = \emptyset$ for some i.*

(a): *The two cycles other that C_i are nonempty and anti-isomorphic, of length either 4 or 6.*
(b): *$AC_{<_{Lex}}(S)$ has underlying graph a cycle of $K_{3,3}$ of length either 4 or 6, whose vertices all weigh either 3, or 4, or 5.*
(c): *Conversely, if C is a cycle of $K_{3,3}$ of length either 4 or 6, whose vertices all weigh either 3, or 4, or 5, there exists a unique triplet of cycles, $S := (C_1, C_2, C_3)$, such that $C_i = \emptyset$ for some i and $AC_{<_{Lex}}(S) = C$. The latter equality assigns a precise weight to every edge of C.*

Proof.

(a): It follows from Proposition 3.13 that the two cycles other that C_i are nonempty and anti-isomorphic. In particular, they must have equal lengths. But a cycle of $K_{3,3}$ can just have length either 4 or 6.
(b): We only prove the case below (all the other cases being similar):

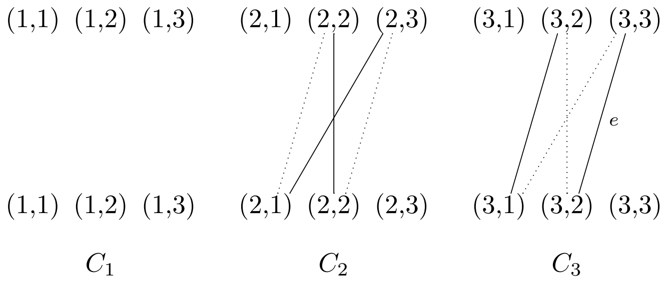


FIGURE 12

According to our conventions, e is a maximum edge, hence the non-dotted edges are those belonging to the maximum monomial. It follows that $AC_{<_{Lex}}(S)$ looks like this:

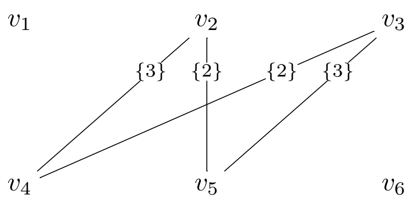


FIGURE 13

The vertices v_2, v_3, v_4 and v_5 all weigh 5.

(c): We only prove the case below (all the other cases being similar):

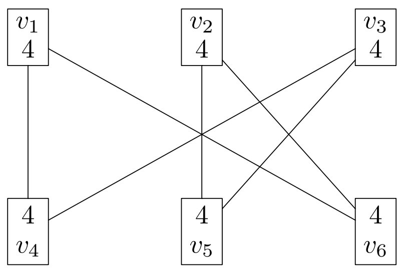


FIGURE 14

In the triplet $S := (C_1, C_2, C_3)$ we wish to construct (if it exist), the edge v_3v_5 must necessarily be a maximum edge relative to $<_{Lex}$, belonging to the nonempty C_i with maximum index (since the two nonempty cycles must be antiisomorphic). It follows that v_3v_5 has weight 3. But then v_3v_4 has weight 1 and the weights of the remaining edges are automatically prescribed:

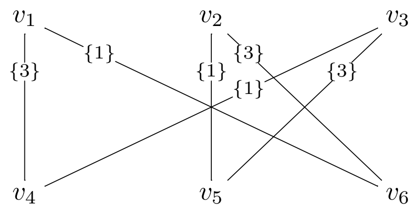


FIGURE 15

S looks like this:

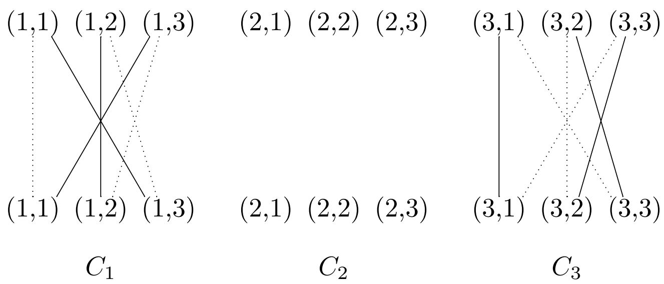


FIGURE 16

□

From now on, given a triplet of cycles, the degree of its associated binomial will also be called the degree of the triplet.

REMARK 4.9. *Proposition 4.8 completely characterizes the degree 4 triplets of cycles, as well as those of degree 6 containing an empty cycle. The complete description of the remaining cases of degree 6 requires the following result.*

PROPOSITION 4.10.

(a): *Let $S := (C_1, C_2, C_3)$ be a degree 6 triplet of cycles, with $C_i \neq \emptyset$ for every i. Then the underlying graph of $AC_{<_{Lex}}(S)$ is the subgraph of $K_{3,3}$ induced by $V_1 \cup V_2 - \{v_a\}$ for some $a \in \{1, \ldots, 6\}$. Moreover, if $v_a \in V_1(resp., V_2)$, the remaining two vertices of $V_1(resp., V_2)$ weigh 6 in $AC_{<_{Lex}}(S)$, while those of $V_2(resp., V_1)$ weigh a permutation of $\{3, 4, 5\}$.*

- **(b):** *Conversely, if* AC *is the subgraph of* $K_{3,3}$ *induced by* $V_1 \cup V_2 - \{v_a\}$, $a \in \{1, \ldots, 6\}$*, further endowed in its vertices with weights observing the same rules given in* (a)*, then there exists a unique degree 6 triplet of cycles, say* $S := (C_1, C_2, C_3)$*, such that* $C_i \neq \emptyset$ *for every* i*, and* $AC_{<_{Lex}}(S) = AC$ *(the latter equality assigning a precise weight to every edge of* AC*).*

The proof of Proposition 4.10 requires the following Lemma, of a more general nature.

Lemma 4.11. *Let* $S := (C_1, \ldots, C_r)$ *be an admissible r-tuple of (not all empty) closed paths of* $\mathcal{G}_{\underline{r}\times\underline{s}\times\underline{t}}$*. Let* $m := \max\{i \in \underline{r} | C_i \neq \emptyset\}$*.*

- **(a):** *If* e *is the maximum edge (relative to* $<_{Lex}$*) occurring in* C_m*,* e *is also a maximum edge of* S *(relative to* $<_{Lex}$*).*
- **(b):** C_m *contains another maximum edge of* S *(relative to*$<_{Lex}$*) which does not intersect* e*.*

Proof.

- **(a):** Clearly, e is the lexicographically maximum edge occurring in all C_i. Without loss of generality, we may assume that e is in odd position in C_m. Then $In_{<_{Lex}}(\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-}) = \underline{x}^{\underline{u}^+}$ and the variable associated with e must occur in $\underline{x}^{\underline{u}^+}$; hence e is a maximum edge of S.
- **(b):** Assume that $e = \{(m, j), (m, k)\}$ is odd (cf. Figure 17 below). An even edge must be incident on (m, j), say $e_{mjk'}$ with $k' < k$ (if $k' > k$, then $e_{mjk'} >_{Lex} e$, contradiction). But then an odd edge, say e',must be incident on (m, k'); e' is a maximum edge of S and, if $e' = \{(m, j'), (m, k')\}$, then $j' < j$ (for otherwise $e' >_{Lex} e$).

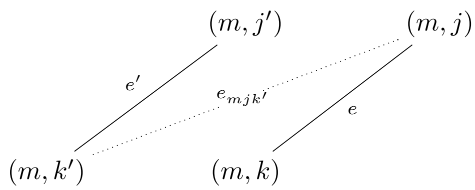


Figure 17

□

Proof of Proposition 4.10.

- **(a):** Since S has degree 6 and $C_i \neq \emptyset$ for every i, every C_i must have length 4. Admissibility then implies that the graph underlying $AC_{<_{Lex}}(S)$ has to be one of the six complete bipartite graphs obtained from $K_{3,3}$ as subgraphs induced by $V_1 \cup V_2 - \{v_a\}$, $a \in \{1, \ldots, 6\}$.

  We continue the proof for one of these cases, pictured below, all the others being similar:

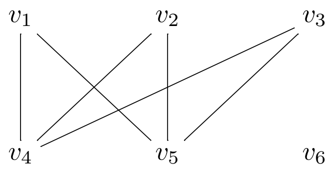


FIGURE 18

Corollary 4.7 (b) says that v_4 and v_5 weigh 6; let us see what are the weights of the remaining vertices, and of all the edges.

Up to a permutation of the cycles, S must look like this:

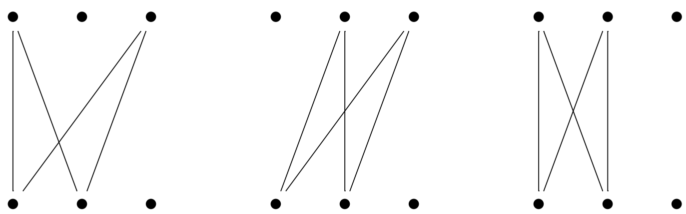


FIGURE 19

The corresponding $AC_{<_{Lex}}(S)$ turns out to be:

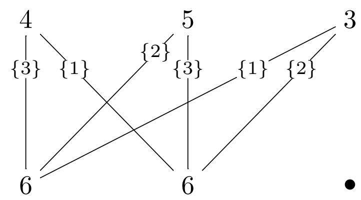


FIGURE 20

which is of the required type.

Also for every permutation of the cycles in Figure 19, the conditions listed in (a) are satisfied.

(b): Without loss of generality, we may assume that AC is:

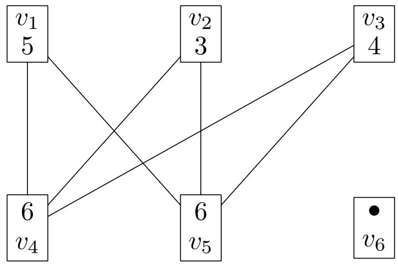


FIGURE 21

Lemma 4.11 says that the maximum edge (relative to $<_{Lex}$) occurring in C_3 can be either v_3v_5 or v_2v_5. But v_2v_5 is ruled out, since v_2 weighs 3; hence it must be v_3v_5.

Lemma 4.11 also says that C_3 contains another maximum edge, which does not intersect v_3v_5. It can be either v_2v_4 or v_1v_4. Since v_2v_4 cannot weigh 3, it turns out to be v_1v_4.

The fact that v_3v_5 and v_1v_4 weigh 3 automatically prescribes the weights of the remaining edges (recall Proposition 4.6):

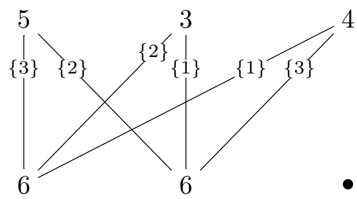


FIGURE 22

In the end, the required S (with the usual conventions) is the following:

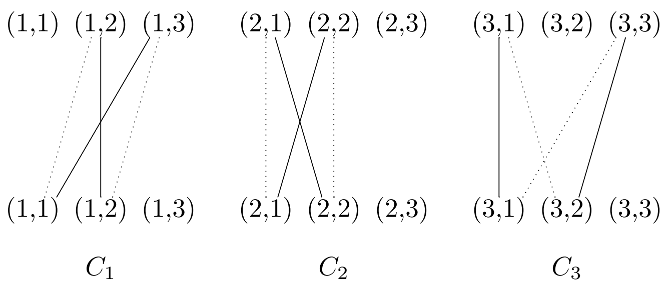


FIGURE 23

□

REMARK 4.12. *Lemma 4.11 can be seen as a Corollary of* [**St**, Theorem 9.1].

We now turn to the degree 7 triplets of cycles.

PROPOSITION 4.13.

(a): *If* $S := (C_1, C_2, C_3)$ *is a degree 7 triplet of cycles, then:*

- **(i):** $C_i \neq \emptyset$ *for every* i*; one of cycles has length 6, the others have length 4;*
- **(ii):** *the underlying graph of* $AC_{<Lex}(S)$ *is a subgraph of* $K_{3,3}$ *consisting of a cycle of length 6 (isomorphic to the unique* C_i *of length 6) plus one of its chords; such a subgraph has two vertices of degree 3 (one in* V_1*, the other in* V_2*), and four vertices of degree 2;*
- **(iii):** *both the vertices of degree 3 weigh 6; the four vertices of degree 2 form two pairs of adjacent vertices; both the vertices belonging to one pair have weight* $a \in \{3, 4, 5\}$*; both the vertices belonging to the other pair have weight* $b \in \{3, 4, 5\}$*, with* $b \neq a$*.*

(b): *If* AC *is a subgraph of* $K_{3,3}$ *consisting of a cycle of length 6 plus one of its chords, and is further endowed in its vertices with weights observing the same rules given in (iii), then there exists a unique degree 7 triplet of cycles, say* $S := (C_1, C_2, C_3)$*, such that* $AC_{<Lex}(S) = AC$ *(the latter equality assigning a precise weight to every edge of* AC*).*

PROOF.

(a): (*i*) and (*ii*) follow immediately from the fact that the only possible cycles have length either 6 or 4; notice that the vertices of degree 3 are those incident on the chord.

(*iii*) Without loss of generality, we may assume that the underlying graph of $AC_{<Lex}(S)$ is:

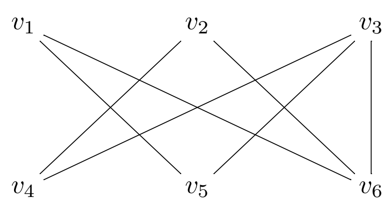


FIGURE 24

Clearly, both v_3 and v_6 weigh 6. It follows that, up to a permutation of the cycles, S must look like this:

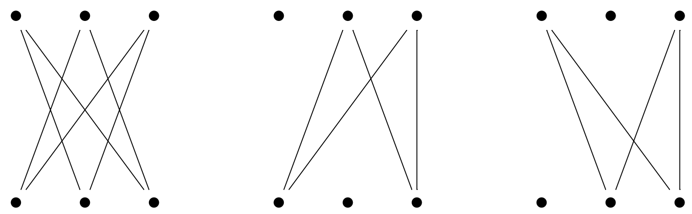


FIGURE 25

The corresponding $AC_{<Lex}(S)$ turns out to be:

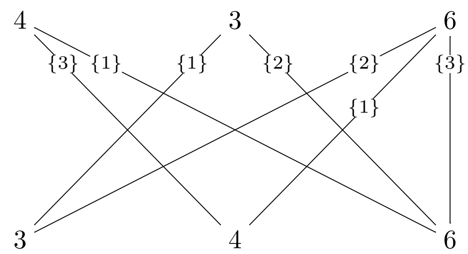


FIGURE 26

which is of the required type.

(b): without loss of generality, we may assume that AC is:

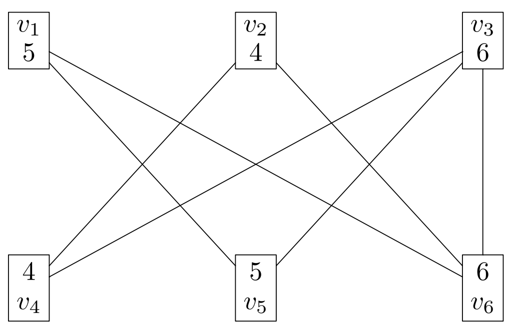


FIGURE 27

The components of S are those in Figure 25. Notice that the cycle of length 6 is obtained by erasing the chord, while the two cycles of length 4 are the only ones containing the chord.

We call A the cycle of length 6, B the cycle of length 4 which is in the middle of Figure 25, and C the other cycle of length 4.

Also thanks to Lemma 4.11, weight 3 can be attributed to either v_3v_6 or v_3v_5.

Assume first that v_3v_6 weighs 3; then both v_1v_5 and v_2v_4 can have weight 3. If v_2v_4 weighs 3, then C_3 must be B, and no further edge can weigh 3; but the choices just made automatically force v_1v_5 to have weight 3: a contradiction. If v_1v_5 weighs 3, another contradiction arises. Hence v_3v_6 cannot weigh 3, and v_3v_5 must have weight 3.

The fact that v_3v_5 weighs 3 implies that $C_3 = A$. Then the weights of all the edges are automatically prescribed:

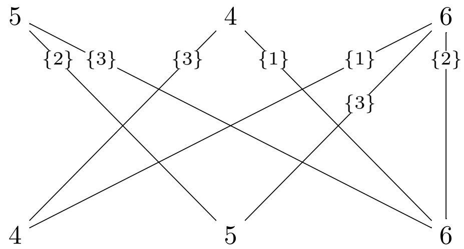


FIGURE 28

It follows that $C_2 = C$ and $C_1 = B$. This completes the proof of (b). □

PROPOSITION 4.14. *Let $AC_{<Lex}(S)$ be the associated configuration of a degree 7 triplet of cycles. If c is the chord of the cycle having length 6, there exists a unique edge of $AC_{<Lex}(S)$, say c', such that $wt(c) = wt(c')$; moreover, c and c' are not adjacent. If C denotes the cycle of length 4 determined by c and c' (a cycle of $AC_{<Lex}(S)$, that is), the non-intersecting edges of C have the same weight, say h; and also the intersecting edges of C have the same weight, say h', with $h' \neq h$.*

DEFINITION 4.15. S is called <u>divisible</u> if $h > h'$.

PROOF OF PROPOSITION 4.14. We know that the chord is the only edge shared by the two length 4 cycles of S; if c weighs $\bar{\imath}$, then $C_{\bar{\imath}}$ is a cycle of length 4, and c' is the other maximum edge of $C_{\bar{\imath}}$. It follows from Proposition 4.6 that c and c' are not adjacent. Proposition 4.13 then implies that the end-vertices of c' have the same weight $a \in \{3, 4, 5\}$. Hence the other two edges of $C_{\bar{\imath}}$ weigh $a - \bar{\imath}$ (which cannot equal a). □

The following Proposition explains Definition 4.15.

Proposition 4.16. *Let $S := (C_1, C_2, C_3)$ be a degree 7 triplet of cycles, and M the maximum term (relative to $<_{Lex}$) of the binomial associated with S.*

M is not divisible by any maximum term of a binomial associated with a degree 6 triplet of cycles.

M is divisible by the maximum term of a binomial associated with a degree 4 triplet of cycles if, and only if, S is divisible.

Proof. Let S' be another triplet of cycles. The maximum term of the binomial associated with S' divides M if, and only if, every edge of $AC_{<_{Lex}}(S')$ is also an edge of $AC_{<_{Lex}}(S)$, with the same weight in both the configurations.

Proposition 4.13 says that $AC_{<_{Lex}}(S)$ has two vertices of weight 6, one in V_1 and the other in V_2; it follows from Proposition 4.10 that M cannot be divisible by the maximum term of the binomial associated with any S' of degree 6 consisting of three nonempty cycles.

If S' is of degree 6 consisting of two nonempty antiisomorphic cycles, again M cannot be divisible by the maximum term of the binomial associated with S', thanks to Propositions 4.8 and 4.14

This completes the proof of the first part of the statement.

As for the part related to the triplets of degree 4, it readily follows from Propositions 4.8, 4.13, and 4.14.

□

Example 4.17. Figure 28 above shows the configuration associated with a non-divisible S (since $3 = h' > h = 2$), while Figure 26 shows the configuration associated with a divisible S (since $1 = h' < h = 3$). In the latter case, the triplet of degree 4 "dividing S" looks like this:

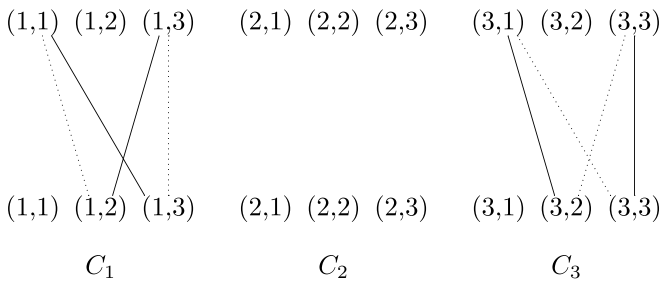


Figure 29

Its associated configuration is:

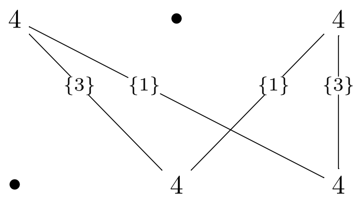


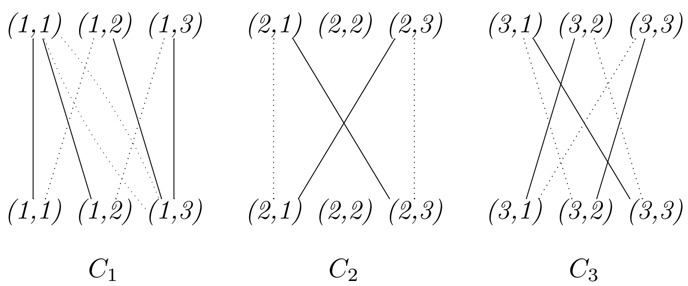
FIGURE 30

We end this section by pointing out that, if $S := (C_1, C_2, C_3)$ is a degree 6 triplet of cycles, then the maximum term of the binomial associated with S cannot be divided by the maximum term of the binomial associated with any degree 4 triplet.

5. A Gröbner basis

In this section we prove the following result.

THEOREM 5.1. *Let $r = s = t = 3$, let $S_0 := (C_1, C_2, C_3)$ be the following admissible triplet of closed paths of $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$:*



and set the following:

$$\begin{aligned}
\mathcal{S}_1 &:= \{\textit{all degree 4 triplets of cycles of } \mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}\},\\
\mathcal{S}_2 &:= \{\textit{all degree 6 triplets of cycles of } \mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}\},\\
\mathcal{S}_3 &:= \{\textit{all non-divisible degree 7 triplets of cycles of } \mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}\},\\
\mathcal{S} &:= \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \{S_0\},\\
Gr &:= \{\textit{all binomial associated with the elements of } \mathcal{S}\}.
\end{aligned}$$

Then Gr is the reduced Gröbner basis of $I_{\mathcal{A}}$ with respect to $<_{Lex}$.

Proof. In view of the previous sections, it suffices to show that, if $S' := (C_1, C_2, C_3)$ is an admissible triplet of (not all empty) closed paths of $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$, and for every i, each edge of C_i either always occurs in even position, or always occurs in odd position, then there exists $S \in \mathcal{S}$ such that

$$AC_{<_{Lex}}(S) \subseteq AC_{<_{Lex}}(S').$$

Here $\subseteq$ means that, if e is an edge of $AC_{<_{Lex}}(S)$ weighing h, then e is also an edge of $AC_{<_{Lex}}(S')$ weighing h.

In case one (and then only one) of the paths C_i is empty, the above statement is an obvious corollary of Propositions 3.13 and 3.14. Hence we assume that $C_i \neq \emptyset$ for every i.

In the remainder of this proof, we resort to a technique of "graph chasing".

Let $K_{3,3}$ be the underlying graph of $AC_{<_{Lex}}(S')$. Assume for instance that v_3v_6 is maximum (w.r.t. $<_{Lex}$) among the edges of $AC_{<_{Lex}}(S')$ weighing 3.

By Lemma 4.11, there exists at least an edge of $AC_{<_{Lex}}(S')$ weighing 3 and not intersecting v_3v_6; we take the minimum such edge (w.r.t. $<_{Lex}$), say v_1v_5.

By assumption, v_3v_5 is an edge of $AC_{<_{Lex}}(S')$; let us suppose that v_3v_5 weighs 3. Then C_3 contains v_1v_5, v_3v_5 and v_3v_6 as maximum edges.

It is not restrictive to assume that the maximum edges of S' are odd; hence the closedness of C_3 forces v_3v_4 and v_2v_5 to be even in C_3; so that v_3v_4 and v_2v_5 cannot weigh 3 in $AC_{<_{Lex}}(S')$.

Hence, with the usual conventions, the following edges are contained in C_3:

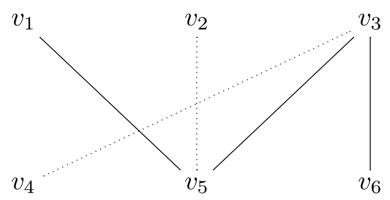


Figure 31

An odd (maximum) edge must be incident on v_4 in C_3: it can only be v_2v_4, given the assumption on v_1v_5. If v_2v_5 and v_3v_4 have a weight l in common (necessarily, l is either 1 or 2), then Proposition 4.8 says that there exists a degree 4 triplet of cycles, S say, such that $AC_{<_{Lex}}(S) \subseteq AC_{<_{Lex}}(S')$. For instance, if $l = 2$, then S is shown in Figure 12 and $AC_{<_{Lex}}(S)$ occurs in Figure 13.

From now on we suppose that v_3v_4 has multiweight $\{1\}$ and v_2v_5 has multiweight $\{2\}$.

Let us assume that v_1v_6 weighs 3. As in the above, it follows that v_2v_6 and v_1v_4 are edges of C_3, but cannot weigh 3. If v_2v_6 weighs 1, since v_3v_4 weighs 1 as well, there exists a degree 4 triplet of cycles, S say, such that $AC_{<_{Lex}}(S) \subseteq AC_{<_{Lex}}(S')$. For $AC_{<_{Lex}}(S')$ turns out to contain the configuration below:

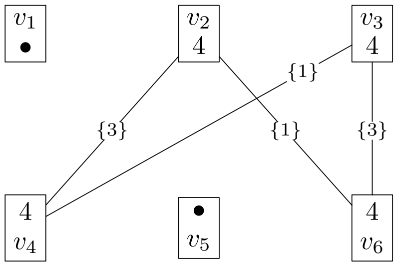


FIGURE 32

And by Proposition 4.8, it is associated with the triplet:

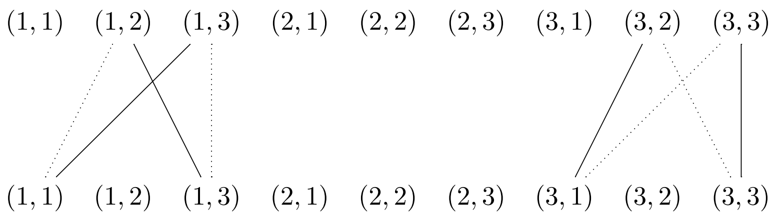


FIGURE 33

If v_2v_6 has multiweight $\{2\}$, the previous argument does not apply. But since both v_2v_5 and v_2v_6 weigh 2, the edge v_2v_4 must be even in C_2. Hence an odd (maximum) edge is incident on v_4 in C_2. It can only be v_1v_4, for v_3v_4 has multiweight $\{1\}$, and can only occur in C_2 in even position. Since v_1v_4 is odd in C_2, v_1v_4 weighs 2 in $AC_{<_{Lex}}(S')$. We then turn to C_1.

We know that v_3v_4 is odd in C_1 (its multiweight is $\{1\}$), so that an even edge is incident on v_4. It can only be v_1v_4, for if it were v_2v_4, then an odd edge should be incident on v_2 in C_1; and this is absurd, since both v_2v_5 and v_2v_6 have multiweight $\{2\}$. Thus an odd edge is incident on v_1 in C_1: either v_1v_5 or v_1v_6.

If it is v_1v_5, then v_1v_5 weighs 1, and $AC_{<_{Lex}}(S')$ turns out to contain:

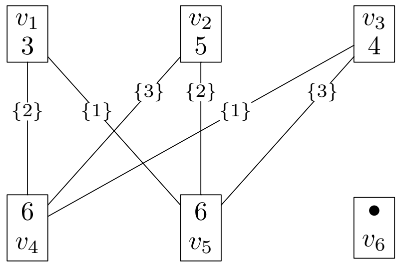


Figure 34

By Proposition 4.10, Figure 34 is the configuration associated with a degree 6 triplet of cycles, and we are done. If the odd edge incident on v_1 in C_1 is v_1v_6, then

$AC_{<_{Lex}}(S')$ contains:

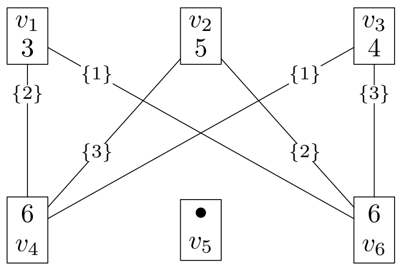


Figure 35

Again, this is the configuration associated with a degree 6 triplet of cycles, and we are through.

The previous analysis proves the statement whenever v_3v_6,v_1v_5,v_3v_5 and v_1v_6 weigh 3 in $AC_{<_{Lex}}(S')$.

Using the notion of admissibility, the same technique also works in all possible cases. In particular, if v_3v_6,v_1v_5 and v_1v_6 weigh 3, but v_3v_5 does not weighs 3, then $AC_{<_{Lex}}(S')$, in some cases, happens to contain the configuration associated with a non-divisible degree 7 triplet of cycles.

Moreover, if v_3v_5 is maximum (w.r.t. $<_{Lex}$) among the edges of $AC_{<_{Lex}}(S')$ weighing 3, and v_2v_4 is minimum (w.r.t. $<_{Lex}$)among the edges weighing 3 and not intersecting v_3v_5, then $AC_{<_{Lex}}(S')$, in some cases, happens to contain the configuration associated with S_0.

Clearly, our graph chasing is simpler when the graph underlying $AC_{<_{Lex}}(S')$ is a proper subgraph of $K_{3,3}$, because there are fewer possibilities to be taken into account.

We end by pointing out that the results obtained in Section 4 immediately imply that Gr is reduced. □

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