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Lexicographic Gröbner bases for transportation problems of format $r \times 3 \times 3$ *

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Abstract

By means of suitable sequences of graphs, we describe the reduced lexicographic Gröbner basis of the toric ideal associated with the 3−dimensional transportation problem of format $r \times 3 \times 3$ (r any integer > 1). In particular, we prove that the bases for $r = 2, 3, 4, 5$ determine all others.

Introduction

In this article we continue the study, begun with [4] (and its larger version [3], available on line), of the reduced lexicographic Gröbner bases related to 3−dimensional transportation problems (for an introduction to these problems, cf. e.g. [10, Chapter 14]).

The new idea introduced in [3] and [4] was the use of sequences of graphs in order to describe the binomials occurring in the mentioned lexicographic Gröbner bases. The same idea is employed here.

The goal of this paper is to give a description of the reduced lexicographic Gröbner basis of the toric ideal associated with the 3−dimensional transportation problem of format $r \times 3 \times 3$ (r any integer > 1). In particular, it turns out that the general case is completely determined by the knowledge of the cases of format $r' \times 3 \times 3$, with r' ranging in $\{2, 3, 4, 5\}$; i.e., there is a stability property of the reduced Gröbner basis, starting from $r = 5$.

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A forthcoming paper will illustrate some geometric applications relative to triangulations of polytopes, computations of Hilbert functions, etc.

As for the Gröbner bases related to transportation problems of format $r \times s \times t$, with fixed s and t but not necessarily 3, we believe that our approach can be profitably used as well, but we are unable to be more specific at this moment.

It is necessary to indicate the relationship between the stability property of our Gröbner bases and the articles [1] and [9] (published after we had completed this paper). Since Markov bases are minimal generating sets of the toric ideal (cf. e.g. the introduction of [9]), Theorem 6.1 below does in fact imply the stability property proved for Markov bases in [1]. Instead one cannot obtain our results from those of [1]. Indeed studying Gröbner bases and studying Markov bases are two different strategies, as explained for example in the introduction of [2]. As for the stability results contained in [9], they apply to Graver bases associated with all formats $r \times s \times t$ for fixed s and t. Hence they do imply, in particular, the stability property of the reduced Gröbner bases with respect to any term order. More specifically, [9] works out the case $r \times 3 \times 3$ as an example and proves that the Graver bases (hence all reduced Gröbner bases) stabilize at $r = 9$.

We sincerely thank the referees for some helpful comments.

1. Recollections

For the convenience of the reader, we recall the approach to 3-dimensional transportation problems introduced in [4]. We restrict to format $r \times 3 \times 3$.

Let $\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}$ indicate the matrix having columns

$$\{\underline{e}_{ij} \oplus \underline{e}_{ik} \oplus \underline{e}'_{jk} \mid i \in \underline{r},\ j \in \underline{3},\ k \in \underline{3}\},$$

where r is an integer ≥ 2,

$$\underline{r} := \{1, 2, \ldots, r\},\ \underline{3} := \{1, 2, 3\},$$

$\{\underline{e}_{ij}\} = \{\underline{e}_{ik}\}$ is the canonical basis of the $\mathbb{Z}$-module of $r \times 3$ integer matrices (denoted by $\mathbb{Z}^{\underline{r}\times\underline{3}}$) and $\{\underline{e}'_{jk}\}$ is the canonical basis of $\mathbb{Z}^{\underline{3}\times\underline{3}}$.

The integer programming problem associated with $\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}$ ("a transportation problem of format $r \times 3 \times 3$") can be solved by studying the toric ideal

$$I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}} := Ker(\Pi_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}),$$

where $\Pi_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ is the following map between polynomial rings:

$$K[x_{ijk}] \to K[u_{ij}, v_{ik}, w_{jk}],$$
$$x_{ijk} \mapsto u_{ij} v_{ik} w_{jk}$$

with $i \in \underline{r}$, $j \in \underline{3}$, $k \in \underline{3}$, and K is any field. We denote the domain of $\Pi_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ by $K[\underline{x}]$.

We think of $\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}$ as of the matrix of the $\mathbb{Z}$-morphism

$$\mathbb{Z}^{\underline{r}\times\underline{3}\times\underline{3}} \longrightarrow \mathbb{Z}^{\underline{r}\times\underline{3}} \oplus \mathbb{Z}^{\underline{r}\times\underline{3}} \oplus \mathbb{Z}^{\underline{3}\times\underline{3}}$$
$$\underline{u} \mapsto \mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}\ \underline{u}$$

where $\mathbb{Z}^{\underline{r}\times\underline{3}\times\underline{3}}$ denotes the $\mathbb{Z}$-module of 3-dimensional integer matrices of format $r \times 3 \times 3$.

Given any integer vector $\underline{u}$, there is a unique way of writing it as the difference of two vectors with non negative entries: $\underline{u} = \underline{u}^+ - \underline{u}^-$. With this notation in mind, let

$$\mathcal{B}_{\underline{r}\times\underline{3}\times\underline{3}} := \{\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-} \mid \underline{u} \in Ker_{\mathbb{Z}}(\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}})\},$$

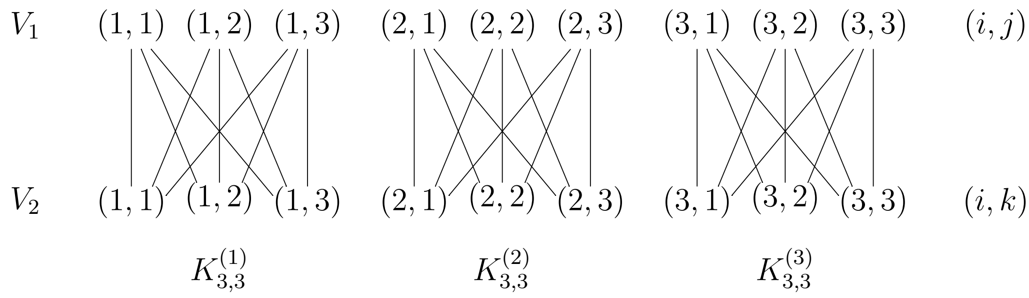
a subset of $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$.

It is a well known fact (cf. e.g. [10]) that if $<$ is any term order on $K[\underline{x}]$, then the reduced Gröbner basis of $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ w.r.t. $<$ consists of a suitable finite subset of $\mathcal{B}_{\underline{r}\times\underline{3}\times\underline{3}}$. As in [4], we are going to use graphs in order to describe $\mathcal{B}_{\underline{r}\times\underline{3}\times\underline{3}}$ and study reduced Gröbner bases.

Let $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ be the bipartite graph having $V_1 := \underline{r} \times \underline{3}$ and $V_2 := \underline{r} \times \underline{3}$ as vertex classes, and $E := \{e_{ijk} \mid i \in \underline{r},\ j \in \underline{3},\ k \in \underline{3}\}$ as set of edges, where

$$e_{ijk} = \{(i, j) \in V_1,\ \ (i, k) \in V_2\}.$$

EXAMPLE 1.1: *$\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$ is the graph:*



It is clear that $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ is the disjoint union of r copies of the complete bipartite graph $K_{3,3}$; we denote them by $K_{3,3}^{(1)}, K_{3,3}^{(2)}, \ldots, K_{3,3}^{(r)}$.

For every choice of i, i' in $\underline{r}$, j in $\underline{3}$ and k in $\underline{3}$, we say that the edges e_{ijk} and $e_{i'jk}$ are parallel (cf. [4, Definition 2.2]).

DEFINITION 1.2: *Let $S := (C_1, C_2, \ldots, C_r)$ be an r-tuple satisfying the following properties:*

(1) *For every $i \in \underline{r}$, either $C_i = \emptyset$ or C_i is a closed path of the subgraph $K_{3,3}^{(i)}$ of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$.*

(2) *For every edge e occurring in S, there are in S as many edges parallel to e that occur in even position as edges parallel to e in odd position.*

(3) *For every $i \in \underline{r}$ such that $C_i \neq \emptyset$, every edge of C_i either is always in odd position ("odd edge"), or is always in even position ("even edge").*

We call S an <u>admissible $r-$tuple of closed paths</u> of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$.

An example of admissible $r-$tuple of closed paths is in [4, Example 3.2].
Another interesting example is in [4, Remark 3.3].

REMARK 1.3: **(1)** *The definition of admissible $r-$tuple of closed paths given above combines together [4, Definition 3.1] and [4, Remark 3.7].*

(2) *Closed paths have to be considered as cyclic structures, with no definite starting point (but still with a division of edges into even and odd).*

The following theorem summarizes the results of [4, Section 3].

THEOREM 1.4: *Let us associate the variable x_{ijk} with the edge e_{ijk} of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$, and viceversa. With every admissible $r-$tuple $S := (C_1, \ldots, C_r)$ of closed paths of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$, we can associate the binomial $\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-}$, where the nonzero entries of $\underline{u}^+$ are given by the multiplicities of all odd edges of S, the nonzero entries of $\underline{u}^-$ by the multiplicities of all even edges, and the multiplicity of an edge e of C_i is the number of times e occurs in C_i. It turns out that $\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-} \in \mathcal{B}_{\underline{r}\times\underline{3}\times\underline{3}}$ and that the application*

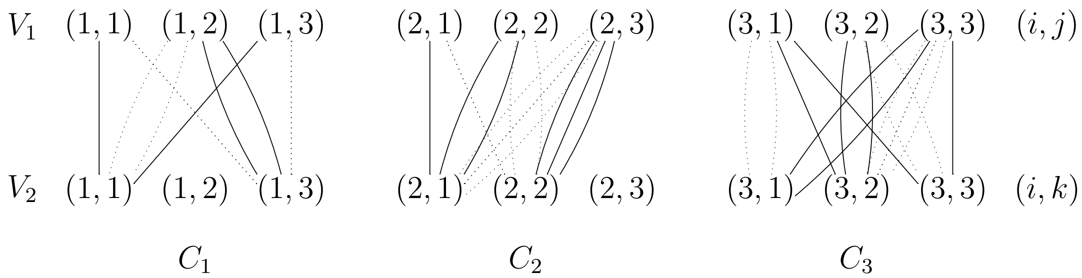
$$\{\textit{admissible } r\textit{-tuples of closed paths of } \mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}\} \to \mathcal{B}_{\underline{r}\times\underline{3}\times\underline{3}}$$

defined in this way is a bijection.

Remark 1.5: **(1)** *In [3] and [4], the word parity is used, instead of multiplicity.*

(2) *If* $\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-} \in \mathcal{B}_{\underline{r}\times\underline{3}\times\underline{3}}$ *, then also* $\underline{x}^{\underline{u}^-} - \underline{x}^{\underline{u}^+} \in \mathcal{B}_{\underline{r}\times\underline{3}\times\underline{3}}$ *, and if* $S := (C_1, \ldots, C_r)$ *is the* $r-$*tuple associated with* $\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-}$*, then* $\underline{x}^{\underline{u}^-} - \underline{x}^{\underline{u}^+}$ *is associated with the* $r-$*tuple obtained from* S *by simply exchanging the roles of even and odd edges.*

Example 1.6: *Let us consider the graph* $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$ *and the following admissible triplet* $S := (C_1, C_2, C_3)$ *of closed paths of* $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$ *(the dotted edges being in even position):*



The binomial associated with S *is*

$$x_{111}x_{123}^2x_{131}x_{211}x_{221}^2x_{232}^3x_{312}x_{313}x_{322}^2x_{331}^2x_{333} - x_{113}x_{121}^2x_{133}x_{212}x_{222}^2x_{231}^3x_{311}^2x_{323}^2x_{332}^3.$$

The corresponding 3-dimensional $\mathbb{Z}$*-matrix* $\underline{u} := \underline{u}^+ - \underline{u}^-$ *is described by the following table:*

u_{111}	*is*	*1*	u_{211}	*is*	*1*	u_{311}	*is*	*-2*
u_{112}		*0*	u_{212}		*-1*	u_{312}		*1*
u_{113}		*-1*	u_{213}		*0*	u_{313}		*1*
u_{121}		*-2*	u_{221}		*2*	u_{321}		*0*
u_{122}		*0*	u_{222}		*-2*	u_{322}		*2*
u_{123}		*2*	u_{223}		*0*	u_{323}		*-2*
u_{131}		*1*	u_{231}		*-3*	u_{331}		*2*
u_{132}		*0*	u_{232}		*3*	u_{332}		*-3*
u_{133}		*-1*	u_{233}		*0*	u_{333}		*1*

One checks that $\underline{u} \in Ker_{\mathbb{Z}}(\mathcal{A}_{\underline{3}\times\underline{3}\times\underline{3}})$.

An example of the inverse bijection can be found in [4, Example 3.8].

2. RG-sequences

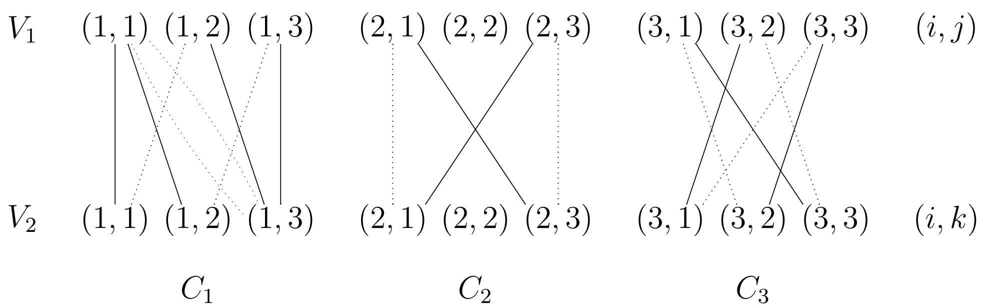
Definition 2.1: *Let* $<$ *be any term order on* $K[\underline{x}]$*. An admissible* $r-$*tuple of closed paths of* $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ *is called an* *RG-sequence of* $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ *w.r.t.* $<$ *if the corresponding element of* $\mathcal{B}_{\underline{r}\times\underline{3}\times\underline{3}}$ *(in the bijection of Theorem 1.4) turns out to belong to the reduced Gröbner basis of* $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ *w.r.t.* $<$.

Let $<_{Lex}$ denote the pure lexicographic term order induced on $K[\underline{x}]$ by

$$x_{ijk} <_{Lex} x_{i'j'k'} \Leftrightarrow (i,j,k) <_{lex} (i',j',k'),$$

where $(i,j,k) <_{lex} (i',j',k')$ if and only if the first nonzero component of the difference vector is negative.

EXAMPLE 2.2: *Let $r = 3$ and choose $<_{Lex}$ on $K[\underline{x}]$. Then the following triplet (C_1, C_2, C_3)*



is an RG−sequence of $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$ w.r.t. $<_{Lex}$.

The edges which are not dotted determine the maximum term (w.r.t. $<_{Lex}$) of the binomial associated with (C_1, C_2, C_3).

REMARK 2.3: *We point out to the reader that, due to an oversight, the triplet described in Example 2.2 (which is (4.5) of [6]) does not appear in the statement of [4, Theorem 5.1] (the only omission in there), but is correctly recorded in [3, Theorem 5.1].*

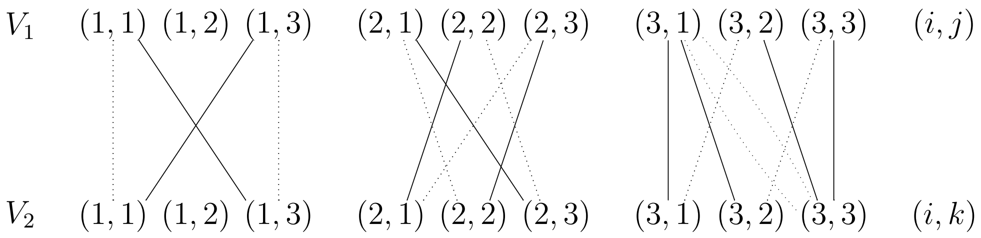
If $<$ is any term order on $K[\underline{x}]$ and $S := (C_1, C_2, \ldots, C_r)$ is an admissible $r-$tuple of (not all empty) closed paths of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$, then we call maximum edge of S (w.r.t. $<$) every edge e_{ijk} of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ such that x_{ijk} occurs in the maximum term (w.r.t. $<$) of the binomial associated with S. (Cf. [4, Definition 4.1].)

Every edge e_{ijk} of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ such that x_{ijk} occurs in the minimum term (w.r.t. $<$) of the above binomial will be called minimum edge of S.

DEFINITION 2.4: *Let $<$ any term order on $K[\underline{x}]$ and $S := (C_1, C_2, \ldots, C_r)$ be an admissible $r-$tuple of closed path of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$. Let $2 \leq r' < r$ and $S' := (D_1, D_2, \ldots, D_{r'})$ an admissible $r'-$tuple of closed path of $\mathcal{G}_{\underline{r'}\times\underline{3}\times\underline{3}}$. We say that the maximum edges of S' (w.r.t. $<$) are among the maximum (resp., minimum) edges of S, if there exist r' indices $1 \leq i_1 < i_2 < \cdots < i_{r'} \leq r$ such that, after embedding every $D_{i'}$ into the corresponding graph $K_{3,3}^{(i_{i'})}$, it turns out that every maximum edge of $D_{i'}$ is a maximum (resp., minimum) edge of $C_{i_{i'}}$, with at least the same multiplicity.*

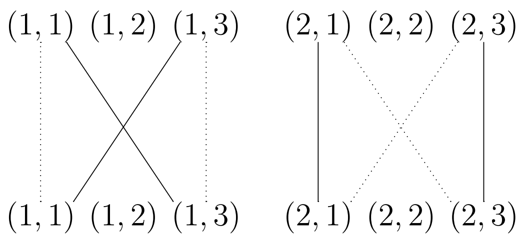
Remark 2.5: *If $g \in I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ and $g' \in I_{\mathcal{A}_{\underline{r'}\times\underline{3}\times\underline{3}}}$ are the binomials associated with S and S', respectively, then - up to an obvious change of the first indices of all the variables occurring in g' - the maximum edges of S' are among the maximum (resp., minimum) edges of S if and only if $in_<(g')$ divides $in_<(g)$ (resp., the minimum monomial of g).*

Example 2.6: *Let $r = 3$ and choose the term order $<_{Lex}$ on $K[\underline{x}]$. Let $S := (C_1, C_2, C_3)$ be the following admissible triplet of closed paths of $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$:*



C_1 C_2 C_3

Let $S' := (D_1, D_2)$ the following admissible pair of closed paths of $\mathcal{G}_{\underline{2}\times\underline{3}\times\underline{3}}$:



D_1 D_2

In both cases, the dotted edges are minimum edges w.r.t. $<_{Lex}$. Hence the maximum edges of S' are among the maximum edges of S (just take $i_1 = 1$ and $i_2 = 3$).

Again let $2 \le r' < r$ and $S' := (D_1, D_2, \ldots, D_{r'})$ an admissible $r'-$tuple of closed paths of $\mathcal{G}_{\underline{r'}\times\underline{3}\times\underline{3}}$. For every choice of r' indices $i_{i'}$ such that $1 \le i_1 < i_2 < \cdots < i_{r'} \le r$, consider the $r-$tuple $S_{i_1,\ldots,\, i_{r'}} := (C_1, \ldots, C_r)$ such that

$$C_h := \begin{cases} D_{i'} & \text{if } h = i_{i'} \\ \emptyset & \text{else.} \end{cases}$$

Clearly, $S_{i_1,\ldots,\, i_{r'}}$ is an admissible $r-$tuple of closed paths of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$.

Let $<$ be any term order on $K[x_{ijk}]$, $i \in \underline{r}$, $j \in \underline{3}$, $k \in \underline{3}$. Let $<$ also denote the obvious term order induced on $K[x_{ijk}]$, $i \in \underline{r'}$, $j \in \underline{3}$, $k \in \underline{3}$.

Proposition 2.7: *In the above conditions, if S' is an $RG-$sequence of $\mathcal{G}_{\underline{r'}\times\underline{3}\times\underline{3}}$ (w.r.t. $<$), then $S_{i_1,\dots,\, i_{r'}}$ is an $RG-$sequence of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ (w.r.t. $<$).*

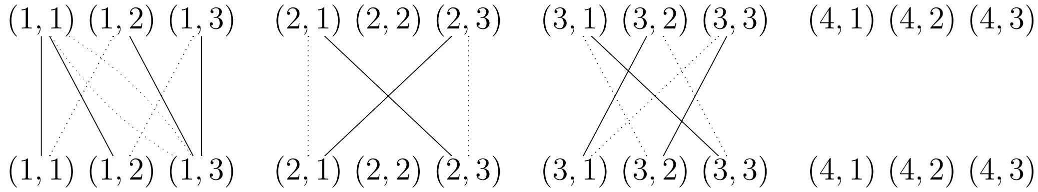
Proof: Let $g' \in I_{\mathcal{A}_{\underline{r'}\times\underline{3}\times\underline{3}}}$ be the binomial associated with S' and $g_{i_1,\dots,\, i_{r'}}$ the binomial associated with $S_{i_1,\dots,\, i_{r'}}$. As observed in Remark 2.5, $g_{i_1,\dots,\, i_{r'}}$ is obtained from g' by suitably changing the first indices of all variables occurring in g'.

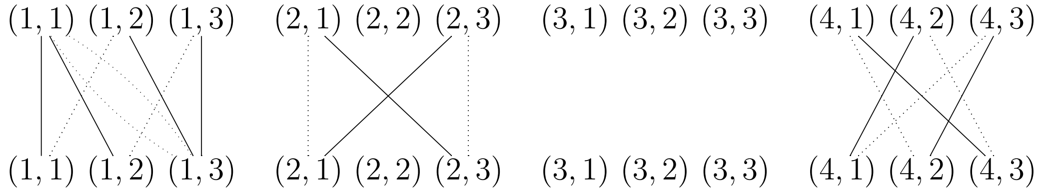
The reduced Gröbner basis of $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$, $\bar{\mathcal{G}}$ say, contains a binomial $\bar{g}$ such that the initial monomial $in_<(\bar{g})$ divides $in_<(g_{i_1,\dots,\, i_{r'}})$. If $\bar{S} := (\bar{C}_1, \bar{C}_2, \dots, \bar{C}_r)$ stands for the $RG-$sequence of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ associated with $\bar{g}$, then the maximum edges of $\bar{S}$ are among the maximum edges of $S_{i_1,\dots,\, i_{r'}}$. It follows that $\bar{C}_h = \emptyset$ whenever $h \notin \{i_1, \dots, i_{r'}\}$. Hence, up to a suitable change of the first indices of all variables involved, we can think of $\bar{g}$ as of an element of $I_{\mathcal{A}_{\underline{r'}\times\underline{3}\times\underline{3}}}$, and $in_<(\bar{g})$ divides $in_<(g')$. But then $in_<(\bar{g}) = in_<(g')$, since g' belongs to the reduced Gröbner basis of $I_{\mathcal{A}_{\underline{r'}\times\underline{3}\times\underline{3}}}$.

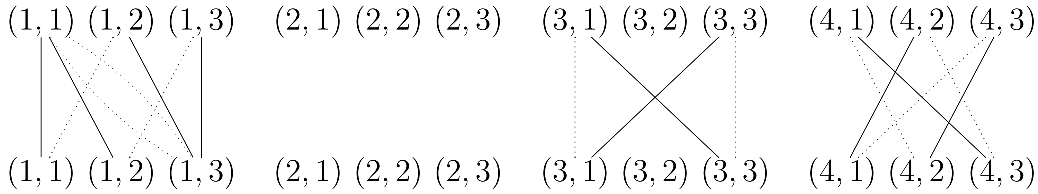
Replacing $\bar{g}$ by $g_{i_1,\dots,\, i_{r'}}$ in $\bar{\mathcal{G}}$, we find another Gröbner basis, $\mathcal{G}$, of $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$. We claim that $\mathcal{G}$ is reduced, so that $\mathcal{G} = \bar{\mathcal{G}}$, and ultimately $\bar{g} = g_{i_1,\dots,\, i_{r'}}$.

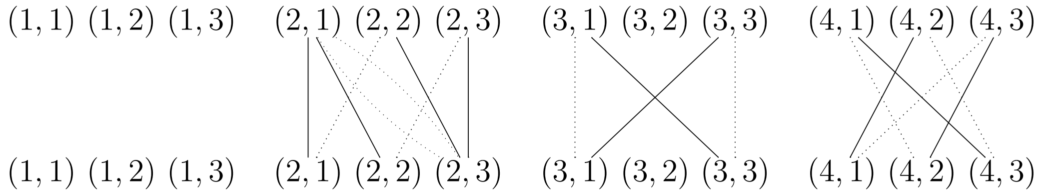
If $\mathcal{G}$ were not reduced, the minimum monomial of $g_{i_1,\dots,\, i_{r'}}$ should be divisible by some $in_<(g)$, with $g \in \bar{\mathcal{G}}$ and $g \neq g_{i_1,\dots,\, i_{r'}}$. As above, we could think of g as of an element of $I_{\mathcal{A}_{\underline{r'}\times\underline{3}\times\underline{3}}}$ and there would be a contradiction with g' being an element of the reduced Gröbner basis of $I_{\mathcal{A}_{\underline{r'}\times\underline{3}\times\underline{3}}}$. $\square$

Example 2.8: *Let $r = 4$ and choose the term order $<_{Lex}$ on $K[\underline{x}]$. Consider the following four admissible $4-$tuples of closed paths of the graph $\mathcal{G}_{\underline{4}\times\underline{3}\times\underline{3}}$ (the dotted edges being the minimum edges):*









Each of them is an $RG-$sequence w.r.t. $<_{Lex}$, because the triplet described in Example 2.2 is an $RG-$sequence of $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$ w.r.t. $<_{Lex}$.

Proposition 2.7 implies that, if we are given an $RG-$sequence $(D_1, D_2, \ldots, D_{r'})$ of $\mathcal{G}_{\underline{r'}\times\underline{3}\times\underline{3}}$ with $D_{i'} \neq \emptyset$ for every $i' \in \underline{r'}$, then for every $r > r'$, we can construct $\binom{r}{r-r'}$ $RG-$sequences of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ by inserting (in all possible ways) $r - r'$ empty paths among the given $D_1, D_2, \ldots, D_{r'}$.

The main result of this paper (to be proven in Section 6, under the assumption that $<_{Lex}$ is the chosen term order on $K[\underline{x}]$) is that whenever $r \geq 6$, every $RG-$sequence of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ is obtained by inserting some empty paths in a suitable $RG-$sequence of $\mathcal{G}_{\underline{r'}\times\underline{3}\times\underline{3}}$, where $r' \in \{2, 3, 4, 5\}$.

Sections 3, 4 and 5 contain all the ingredients needed to prove our claim, always working with $<_{Lex}$.

3. Description of some admissible $r-$tuples of cycles, $r = 4, 5$

From now on, we always assume that our term order on $K[\underline{x}]$ is the term order $<_{Lex}$.

PROPOSITION 3.1: *Let $v_1 \in V_1$ and $v_2 \in V_2$ be two vertices of $K_{3,3}$. Call $K_{v_2}^{v_1}$ the subgraph of $K_{3,3}$ obtained by removing the edge $\{v_1, v_2\}$. Let C_1 and C_2 be the only two length 4 cycles of $K_{v_2}^{v_1}$ having v_1 as first and last vertex. Let C_3 and C_4 be the only two length 4 cycles of $K_{v_2}^{v_1}$ having v_2 as first and last vertex. Then, for every permutation $(C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4})$ of (C_1, C_2, C_3, C_4), there exists exactly one way of turning $(C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4})$ into an admissible $4-$tuple of cycles of $\mathcal{G}_{\underline{4}\times\underline{3}\times\underline{3}}$, such that its odd edges are its maximum edges w.r.t. $<_{Lex}$.*

Proof: For every $j \in \{1, 2, 3, 4\}$, define the map

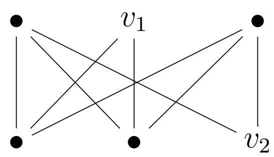
$$p_{i_j} : E(C_{i_j}) \to \{C_1, C_2, C_3, C_4\} \setminus \{C_{i_j}\}$$

which sends each $h \in E(C_{i_j})$ to the unique C_{i_k} ($\neq C_{i_j}$) containing some edge parallel to h. It is surjective, and $p_{i_j}(h) = p_{i_j}(h')$ implies that h and h' are incident.

Let l be the edge of $C_{i_4} \subseteq K_{3,3}^{(4)}$ corresponding to the maximum variable w.r.t. $<_{Lex}$ (the farthest edge to the right) and choose it to be odd. Then the parities of all other edges of C_{i_4} are automatically determined.

But thanks to the surjectivity of p_{i_4}, also the parities of all other edges of $(C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4})$ are forced by the admissibility condition. $\square$

EXAMPLE 3.2: *As usual, we stipulate that the even edges be indicated by dotted lines. Let $K_{v_2}^{v_1}$ be the following:*

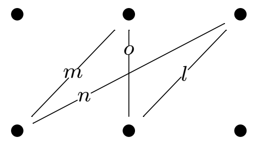


$K_{v_2}^{v_1}$

The corresponding four cycles are:

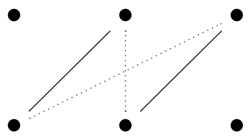
C_1 C_2 C_3 C_4

We consider the $4-$tuple (C_3, C_4, C_1, C_2). In particular:

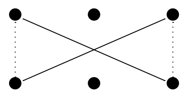
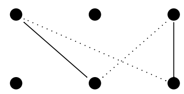
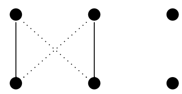


C_2

If l is chosen to be odd, the parities of the edges of C_2 are:



But now $p_2(l) = C_4$, $p_2(n) = C_3$, $p_2(m) = p_2(o) = C_1$. Hence:

C_3 C_4 C_1

PROPOSITION 3.3: *Let C be a length 6 cycle of $K_{3,3}$ and l one of its edges. Call $\bar{C}$ the subgraph of $K_{3,3}$ obtained by drawing the only two chords of C which contain one of the ends of l. Let C_1 and C_2 be the two length 4 cycles of $\bar{C}$ containing l and just one of the two chords. Let C_3 be the only length 4 cycle of $\bar{C}$ containing l and both chords. Then, for every permutation $(C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4})$ of (C_1, C_2, C_3, C), there exists exactly one way of turning $(C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4})$ into an admissible 4−tuple of cycles of $\mathcal{G}_{\underline{4}\times\underline{3}\times\underline{3}}$, such that its odd edges are its maximum edges w.r.t. $<_{Lex}$.*

Proof: For every $j \in \{1, 2, 3, 4\}$, define the map

$$p_{i_j} : E(C_{i_j}) \setminus \{l\} \to \{C_1, C_2, C_3, C\} \setminus \{C_{i_j}\}$$

which sends each $h \in E(C_{i_j}) \setminus \{l\}$ to the unique C_{i_k} ($\neq C_{i_j}$) containing some edge parallel to h.

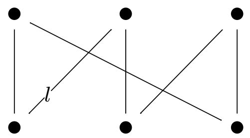
$p_{i_j}(h) = p_{i_j}(h')$ implies that h and h' are incident. Moreover, if $C_{i_j} \neq C$, then $C \in Im(p_{i_j})$; if $C_{i_j} = C$, then p_{i_j} is surjective.

Let m be the edge of C_{i_4} corresponding to the maximum variable w.r.t. $<_{Lex}$ (the farthest edge to the right) and choose it to be odd. Then the parities of all other edges of C_{i_4} are automatically determined.

If $C_{i_4} = C$, then the surjectivity of p_{i_4} and the admissibility condition determine the parities of all other edges of $(C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4})$ as well.

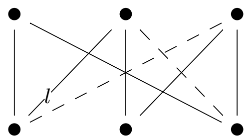
If $C_{i_4} \neq C$, then $C \in Im(p_{i_4})$ says that the parities of the edges of C are determined by those of C_{i_4}. But then, as in the previous case, the parities of all other edges of $(C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4})$ are determined. □

EXAMPLE 3.4: *As usual, we stipulate that the even edges be indicated by dotted lines. Let C and l be as in the following picture:*



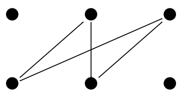
C

Then $\bar{C}$ looks like:

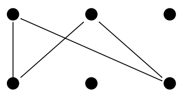


$\bar{C}$

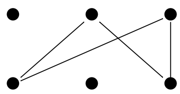
where the two chords are dashed. Hence C_1, C_2 and C_3 are:



C_1

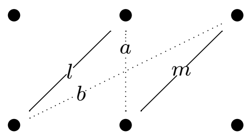


C_2

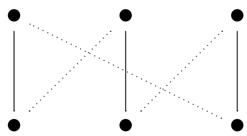


C_3

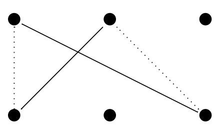
We consider the $4-$tuple (C_2, C, C_3, C_1). If m is chosen to be odd, the parities of the edges of C_1 are:

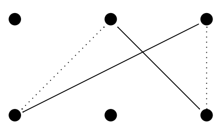


One has $p_1(a) = p_1(m) = C$ and $p_1(b) = C_3$. Then the parities of the edges of C are:



and it follows that the parities of the edges of C_2 and C_3 are:





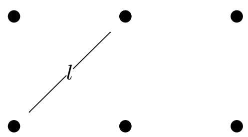
Proposition 3.5: *Let l be an edge of $K_{3,3}$. Let C_1, C_2, C_3 and C_4 be the four length 4 cycles of $K_{3,3}$ containing l. For every C_i, there exists exactly one edge, l_i, different from l and not contained in any other C_j, $j \neq i$. The edges l_1, l_2, l_3 and l_4 form a length 4 cycle, C_5. For every permutation $(C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4}, C_{i_5})$ of $(C_1, C_2, C_3, C_4, C_5)$, there exists exactly one way of turning $(C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4}, C_{i_5})$ into an admissible $5-$tuple of cycles of $\mathcal{G}_{\underline{5}\times\underline{3}\times\underline{3}}$, such that its odd edges are its maximum edges w.r.t. $<_{Lex}$.*

Proof: Similar to the proof of Proposition 3.3. Let m be the edge of C_{i_5} corresponding to the maximum variable w.r.t. $<_{Lex}$, and choose it to be odd. Then the parities of all other edges of C_{i_5} are determined.

If $C_{i_5} = C_5$, then the parity of l_i determines the parities of all edges of C_i ($i = 1, 2, 3, 4$).

If $C_{i_5} \neq C_5$, then the parity of l_{i_5} determines the parities of all edges of C_5, and we are done as in the previous case. $\square$

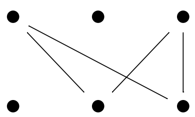
Example 3.6: *As usual, we stipulate that the even edges be indicated by dotted lines. Let l be as follows:*



Then C_1, C_2, C_3 and C_4 are:

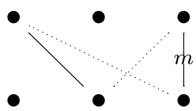
C_1 C_2 C_3 C_4

Hence C_5 looks like:



C_5

We consider the $5-$tuple $(C_1, C_2, C_3, C_4, C_5)$. If m is chosen to be odd, the parities of the edges of C_5 are:



It follows that the parities of the edges of C_1, C_2, C_3 and C_4 are:

4. Reducibility of admissible $r-$tuples of cycles of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$

Let $V_1 := \{a, b, c\}$ and $V_2 := \{d, e, f\}$ be the vertex classes of the bipartite graph $K_{3,3}$. Let σ denote any permutation of a, b, c, d, e, f which belongs to the symmetry group of $K_{3,3}$. σ acts on the edges of $K_{3,3}$ by means of:

$$\sigma(\{v_1, v_2\}) = \{\sigma(v_1), \sigma(v_2)\}.$$

Given a cycle C of $K_{3,3}$, the action of σ on the edges of $K_{3,3}$ turns C into another cycle of $K_{3,3}$, which we denote by $\sigma(C)$.

We stipulate that if $\{v_1, v_2\}$ is an even (=dotted) (resp. odd (= continuous)) edge of C, then $\{\sigma(v_1), \sigma(v_2)\}$ is even (resp. odd).

Proposition 4.1: *If $S := (C_1, \ldots, C_r)$ is an admissible $r-$tuple of cycles of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$, then $\sigma(S) := (\sigma(C_1), \ldots, \sigma(C_r))$ is an admissible $r-$tuple of cycles of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$.*

Proof: It suffices to show that for every edge which occurs in $\sigma(S)$, there are as many edges parallel to it in even position as edges parallel to it in odd position. But this is true because σ acts bijectively on the edges of each $K_{3,3}$, and S is admissible. $\square$

Theorem 4.2: *Let $S := (C_1, \ldots, C_r)$ be an admissible $r-$tuple of cycles of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$. There exists an admissible $r'-$tuple S' of cycles of $\mathcal{G}_{\underline{r'}\times\underline{3}\times\underline{3}}$, $2 \leq r' \leq 5$, such that its maximum edges w.r.t. $<_{Lex}$ are among the maximum edges w.r.t. $<_{Lex}$ of S, in the sense of Definition 2.4.*

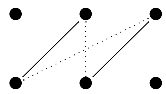
Proof: Consistently with [4, Definition 3.10], we say that two distinct cycles C and $\widehat{C}$ of S are anti-isomorphic if they are equal as subgraphs of $K_{3,3}$, but parallel edges always have opposite parities (i.e. the odd edges of $\widehat{C}$ are precisely the even edges of C).

Without loss of generality, we may assume the following.

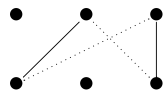
(A) No pair $(\widehat{C}, C)$ of anti-isomorphic cycles occurs in S, for otherwise we may erase all pairs of this kind and work with the remaining shorter admissible sequence.

(B) There exists no (admissible) pair of cycles $S' := (\widehat{C}, C)$ such that the maximum edges of S' w.r.t. $<_{Lex}$ are among the maximum edges of S, and the maximum edges of C w.r.t. $<_{Lex}$ are among the maximum edges of the cycle C_r (the rightmost cycle of S). Otherwise we are done.

The cycle C_r must be one of the following fifteen cycles, whose maximum edges w.r.t. $<_{Lex}$ are the continuous ones, and are assumed to be odd, as usual:



1



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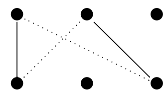

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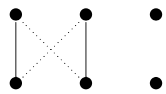

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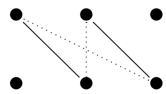

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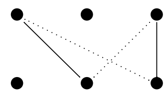

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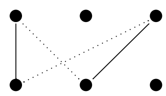

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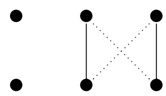

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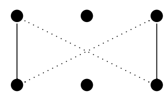

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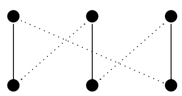

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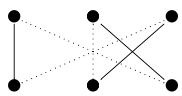

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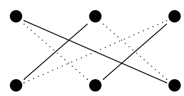

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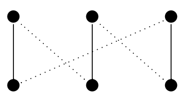
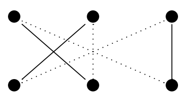


13



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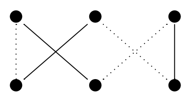


15

Remark that each cycle in the list above has been given a number and will be denoted by that number in the rest of the proof. Also remark that, if $\mathbf{n}$ is one of the cycles $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{15}$, $\widehat{\mathbf{n}}$ denotes the only possible cycle anti-isomorphic to it.

The proof of this theorem is case by case: one case for each one of the possible rightmost cycles $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{15}$ of S. In fact we are going to show that the cases $\mathbf{1}$ and $\mathbf{12}$ determine all others.

Case C_r= cycle $\mathbf{1}$.

We are given any sequence $S := (C_1, \ldots, C_r)$ such that $C_r = \mathbf{1}$ and conditions (A), (B) above are satisfied. We look for a sequence S' as described in the statement. In fact we shall find S' with the property that its rightmost cycle is precisely $\mathbf{1}$.

Since S is admissible, the edge $\{c, e\}$, which occurs in $C_r = \mathbf{1}$ in odd position, must also occur in even position in some other cycle C_i of S, $i \leq r-1$. A priori, there are eight possibilities for C_i : $\widehat{\mathbf{1}}, \mathbf{6}, \widehat{\mathbf{7}}, \mathbf{8}, \mathbf{10}, \widehat{\mathbf{11}}, \widehat{\mathbf{12}}, \mathbf{15}$ (four cases of length 4, four cases of length 6). Since condition (B) holds for S, $\widehat{\mathbf{1}}$ and $\widehat{\mathbf{11}}$

are impossible. Hence we are left with: $\mathbf{6}, \widehat{\mathbf{7}}, \mathbf{8}, \mathbf{10}, \widehat{\mathbf{12}}, \mathbf{15}$. In order to save space, we discuss the subcase $C_i = \mathbf{6}$ completely and leave the reader $\widehat{\mathbf{7}}, \mathbf{8}, \mathbf{10}, \widehat{\mathbf{12}}, \mathbf{15}$ which are very similar, as we have checked working them out.

- Subcase $C_i = \mathbf{6}$.

The edge $\{c, d\}$, which occur in $C_r = \mathbf{1}$ in even position, must occur in odd position in some other cycle C_j of S, $j \leq r-1$. Again thanks to (B), C_j can only be $\widehat{\mathbf{2}}, \widehat{\mathbf{7}}, \widehat{\mathbf{9}}, \widehat{\mathbf{12}}$ and $\widehat{\mathbf{13}}$.

Hence subcase $C_i = \mathbf{6}$ split into the following five sub-subcases: $(\alpha), (\beta), (\gamma), (\delta)$ and (ε).

(α) $C_i = \mathbf{6},\ C_j = \widehat{\mathbf{2}}$. Since the edge $\{b, e\}$, occurring in $C_r = \mathbf{1}$ in even position, must occur in odd position in some further C_k of S, $k \leq r-1$, and C_k can only be $\mathbf{4}, \widehat{\mathbf{5}}, \mathbf{8}, \mathbf{10}$ and $\mathbf{13}$, we are led to the following analysis.

When $C_i = \mathbf{6},\ C_j = \widehat{\mathbf{2}},\ C_k = \widehat{\mathbf{5}}$, we see by Proposition 3.1 that, for a suitable permutation τ on three letters, the 4−tuple $S' = (C_{\tau(i)}, C_{\tau(j)}, C_{\tau(k)}, \mathbf{1})$ does the job.

When $C_i = \mathbf{6},\ C_j = \widehat{\mathbf{2}},\ C_k = \mathbf{8}$, we see by [4, Proposition 4.10] that, for a suitable permutation τ on two letters, the triplet $S' = (C_{\tau(j)}, C_{\tau(k)}, \mathbf{1})$ does the job.

When $C_i = \mathbf{6},\ C_j = \widehat{\mathbf{2}},\ C_k = \mathbf{10}$ or $C_k = \mathbf{13}$, again we are done by the previous sentence, because the odd edges of $\mathbf{8}$ are two of the three odd edges of both $\mathbf{10}$ and $\mathbf{13}$.

We are left with $C_i = \mathbf{6},\ C_j = \widehat{\mathbf{2}},\ C_k = \mathbf{4}$. Since the edge $\{a, f\}$, occurring in $C_i = \mathbf{6}$ in even position, must occur in odd position in some further C_l of S, and, thanks to the previous sentences and both conditions (A) and (B), C_l can only be $\widehat{\mathbf{3}}, \widehat{\mathbf{9}}, \widehat{\mathbf{10}}, \mathbf{12}$, we make the following observations.

When $C_l = \widehat{\mathbf{3}}$, the 5−tuple $S' = (C_{\tau(i)}, C_{\tau(j)}, C_{\tau(k)}, C_{\tau(l)}, \mathbf{1})$ does the job for a suitable permutation τ on four letters (by Proposition 3.5).

When $C_l = \widehat{\mathbf{10}}$ or $C_l = \mathbf{12}$, again we are done by the previous sentence, because the odd edges of $\widehat{\mathbf{3}}$ are two of the three odd edges of both $\widehat{\mathbf{10}}$ and $\mathbf{12}$.

When $C_l = \widehat{\mathbf{9}}$, the 4−tuple $S' = (C_{\tau(i)}, C_{\tau(k)}, C_{\tau(l)}, \mathbf{1})$ does the job for a suitable permutation τ on three letters (by Proposition 3.1).

This completes the sub-subcase (α). We now turn to the other four sub-subcases, adopting lighter notation and skipping some details already illustrated in (α).

(β) $C_i = \mathbf{6},\ C_j = \widehat{\mathbf{7}}$. Due to edge $\{b, e\}$, S must also contain $C_k \in \{\mathbf{4}, \widehat{\mathbf{5}}, \mathbf{8}, \mathbf{10}, \mathbf{13}\}$.

When $C_k = \mathbf{4},\ \mathbf{10},\ \mathbf{13}$, then $S' = (\underline{\mathbf{4}, \widehat{\mathbf{7}}}, \mathbf{1})$ works, where $\underline{\mathbf{4}, \widehat{\mathbf{7}}}$ means $\{\mathbf{4}, \widehat{\mathbf{7}}\}$ up to a permutation.

When $C_k = \widehat{\mathbf{5}},\ S$ must contain $C_l \in \{\widehat{\mathbf{2}}, \mathbf{3}, \mathbf{4}, \mathbf{10}, \widehat{\mathbf{12}}, \widehat{\mathbf{15}}\}$, due to edge $\{b, d\}$.

If $C_l = \widehat{\mathbf{2}}, \widehat{\mathbf{12}}$, then $S' = (\underline{\mathbf{6}, \widehat{\mathbf{5}}, \widehat{\mathbf{2}}}, \mathbf{1})$ works (obvious meaning for $\underline{\mathbf{6}, \widehat{\mathbf{5}}, \widehat{\mathbf{2}}}$).

If $C_l = \mathbf{3}, \widehat{\mathbf{15}}$, then $S' = (\underline{\widehat{\mathbf{7}}, \widehat{\mathbf{5}}, \mathbf{3}}, \mathbf{1})$ works.

If $C_l = \mathbf{4}, \mathbf{10}$, then $S' = (\underline{\widehat{\mathbf{7}}, \mathbf{4}}, \mathbf{1})$ works.

When $C_k = \mathbf{8}$, S must still contain $C_l \in \{\widehat{\mathbf{2}}, \mathbf{3}, \mathbf{4}, \mathbf{10}, \widehat{\mathbf{12}}, \widehat{\mathbf{15}}\}$, due to edge $\{b, d\}$.

If $C_l = \widehat{\mathbf{2}}, \widehat{\mathbf{12}}$, then $S' = (\underline{\mathbf{8}, \widehat{\mathbf{2}}}, \mathbf{1})$ works.

If $C_l = \mathbf{4}, \mathbf{10}$, then $S' = (\underline{\widehat{\mathbf{7}}, \mathbf{4}}, \mathbf{1})$ works.

If $C_l = \widehat{\mathbf{15}}$, then $S' = (\underline{\widehat{\mathbf{7}}, \mathbf{8}, \widehat{\mathbf{15}}}, \mathbf{1})$ works (by Proposition 3.3).

If $C_l = \mathbf{3}$, S must contain a further $C_m \in \{\widehat{\mathbf{2}}, \widehat{\mathbf{9}}, \widehat{\mathbf{10}}, \widehat{\mathbf{13}}, \widehat{\mathbf{15}}\}$, due to edge $\{c, f\}$.

Cycles $\widehat{\mathbf{2}}, \widehat{\mathbf{13}}$ are dealt with by means of $S' = (\underline{\mathbf{8}, \widehat{\mathbf{2}}}, \mathbf{1})$.

Cycle $\widehat{\mathbf{9}}$ by means of $S' = (\underline{\mathbf{8}, \mathbf{3}, \widehat{\mathbf{9}}}, \mathbf{1})$.

Cycle $\widehat{\mathbf{10}}$ by means of $S' = (\underline{\widehat{\mathbf{7}}, \mathbf{8}, \mathbf{3}, \widehat{\mathbf{6}}}, \mathbf{1})$ (obvious meaning for $\underline{\widehat{\mathbf{7}}, \mathbf{8}, \mathbf{3}, \widehat{\mathbf{6}}}$).

Finally, cycle $\widehat{\mathbf{15}}$ by means of $S' = (\underline{\widehat{\mathbf{7}}, \mathbf{8}, \widehat{\mathbf{15}}}, \mathbf{1})$.

This completes the sub-subcase (β).

(γ) $C_i = \mathbf{6}$, $C_j = \widehat{\mathbf{9}}$. Due to edge $\{b, e\}$, S must also contain $C_k \in \{\mathbf{4}, \widehat{\mathbf{5}}, \mathbf{8}, \mathbf{10}, \mathbf{13}\}$.

When $C_k = \mathbf{4}$, $\mathbf{10}$, $\mathbf{13}$, then $S' = (\underline{\mathbf{6}, \widehat{\mathbf{9}}, \mathbf{4}}, \mathbf{1})$ works.

When $C_k = \widehat{\mathbf{5}}$, S must also contain $C_l \in \{\widehat{\mathbf{2}}, \mathbf{3}, \mathbf{4}, \mathbf{10}, \widehat{\mathbf{12}}, \widehat{\mathbf{15}}\}$, due to edge $\{b, d\}$.

If $C_l = \widehat{\mathbf{2}}, \widehat{\mathbf{12}}$, then $S' = (\underline{\mathbf{6}, \widehat{\mathbf{5}}, \widehat{\mathbf{2}}}, \mathbf{1})$ works.

If $C_l = \mathbf{3}, \widehat{\mathbf{15}}$, then $S' = (\underline{\mathbf{6}, \widehat{\mathbf{9}}, \widehat{\mathbf{5}}, \mathbf{3}}, \mathbf{1})$ works.

If $C_l = \mathbf{4}, \mathbf{10}$, then $S' = (\underline{\mathbf{6}, \widehat{\mathbf{9}}, \mathbf{4}}, \mathbf{1})$ works.

When $C_k = \widehat{\mathbf{8}}$, S must still contain $C_l \in \{\widehat{\mathbf{2}}, \mathbf{3}, \mathbf{4}, \mathbf{10}, \widehat{\mathbf{12}}, \widehat{\mathbf{15}}\}$, due to edge $\{b, d\}$.

If $C_l = \widehat{\mathbf{2}}, \widehat{\mathbf{12}}$, then $S' = (\underline{\mathbf{8}, \widehat{\mathbf{2}}}, \mathbf{1})$ works.

If $C_l = \mathbf{3}, \widehat{\mathbf{15}}$, then $S' = (\underline{\widehat{\mathbf{9}}, \mathbf{8}, \mathbf{3}}, \mathbf{1})$ works.

If $C_l = \mathbf{4}, \mathbf{10}$, then $S' = (\underline{\mathbf{6}, \widehat{\mathbf{9}}, \mathbf{4}}, \mathbf{1})$ works.

This completes the sub-subcase (γ).

(δ) $C_i = \mathbf{6}$, $C_j = \widehat{\mathbf{12}}$. Due to edge $\{b, e\}$, S must also contain $C_k \in \{\mathbf{4}, \widehat{\mathbf{5}}, \mathbf{8}, \mathbf{10}, \mathbf{13}\}$.

When $C_k = \mathbf{4}$, $\mathbf{10}$, $\mathbf{13}$, then $S' = (\underline{\mathbf{4}, \widehat{\mathbf{7}}}, \mathbf{1})$ works.

When $C_k = \widehat{\mathbf{5}}$, then $S' = (\underline{\widehat{\mathbf{12}}, \widehat{\mathbf{5}}}, \mathbf{1})$ works (by [4, Proposition 4.13]).

When $C_k = \mathbf{8}$, then $S' = (\underline{\mathbf{8}, \widehat{\mathbf{2}}}, \mathbf{1})$ works.

This completes the sub-subcase (δ).

(ε) $C_i = \mathbf{6},\ C_j = \widehat{\mathbf{13}}$. Due to edge $\{b, e\},\ S$ must still contain $C_k \in \{\mathbf{4}, \widehat{\mathbf{5}}, \mathbf{8}, \mathbf{10}\}$.

The analysis is exactly as the previous one, because the odd edges of $\widehat{\mathbf{13}}$ and the odd edges of $\widehat{\mathbf{12}}$ coincide.

This completes the sub-subcase (ε).

The subcase $C_i = \mathbf{6}$ is now complete.

We remark that many sub-sub-subcases have occurred several times in different sub-subcases. We also remark that $r'-$tuples S' with $r' = 5, 4, 3$ have occurred. ($r' = 2$ is "hidden" under conditions (A) and (B)).

Cases $C_r = \mathbf{2}, \mathbf{3}, \ldots, \mathbf{9}$.

Let $\sigma_{\mathbf{1}}^{\mathbf{n}}$ be any permutation of the vertices $\{a, b, c, d, e, f\}$ of $K_{3,3}$ which turns cycle $\mathbf{1}$ into cycle $\mathbf{n},\ \mathbf{2} \leq \mathbf{n} \leq \mathbf{9}$. For instance, one can choose:

$$\sigma_{\mathbf{1}}^{\mathbf{2}} = (ef); \quad \sigma_{\mathbf{1}}^{\mathbf{3}} = (acb)(ef); \quad \sigma_{\mathbf{1}}^{\mathbf{4}} = (acb); \quad \sigma_{\mathbf{1}}^{\mathbf{5}} = (ac)(df);$$

$$\sigma_{\mathbf{1}}^{\mathbf{6}} = (ab)(def); \quad \sigma_{\mathbf{1}}^{\mathbf{7}} = (ab); \quad \sigma_{\mathbf{1}}^{\mathbf{8}} = (def); \quad \sigma_{\mathbf{1}}^{\mathbf{9}} = (ab)(ef).$$

Thanks to Proposition 4.1, the construction given for cycle $\mathbf{1}$ is transformed by $\sigma_{\mathbf{1}}^{\mathbf{n}}$ into a construction valid for cycle $\mathbf{n},\ \mathbf{2} \leq \mathbf{n} \leq \mathbf{9}$. For $\sigma_{\mathbf{1}}^{\mathbf{n}}$ acts on all edges and cycles having a role in the construction given for cycle $\mathbf{1}$.

Remark

The choice of reducing all cases $\mathbf{2}, \mathbf{3}, \ldots, \mathbf{9}$ to cycle $\mathbf{1}$ is in fact arbitrary. One can start from any length 4 cycle C_r and reduce all other length 4 cases to the chosen one.

We are now going to deal with length 6 cycles $\mathbf{10}, \mathbf{11}, \ldots, \mathbf{15}$. Again we give a construction for one of them, and reduce all other cases to the selected one. However, the selection of the pivotal cycle (cycle $\mathbf{12}$) is not arbitrary, as we shall explain later.

Case C_r= cycle $\mathbf{12}$.

We are given any sequence $S := (C_1, \ldots, C_r)$ such that $C_r = \mathbf{12}$ and conditions (A), (B) are satisfied. We look for a sequence S' as described in the statement. In fact we shall find S' with the property that its rightmost cycle is either $\mathbf{12}$, or $\mathbf{1}$.

The reader will notice an overall resemblance with case $C_r = \mathbf{1}$.

Since S is admissible, the edge $\{c, e\}$, which occurs in $C_r = \mathbf{12}$ in odd position, must also occur in even position in some other cycle C_i of $S,\ i \leq r-1$. Thanks to condition (B), the possibilities for C_i are $\mathbf{6}, \widehat{\mathbf{7}}, \mathbf{8}, \mathbf{10}, \mathbf{15}$. In order to save space, we discuss the subcase $C_i = \mathbf{6}$ completely and leave the reader $\widehat{\mathbf{7}}, \mathbf{8}, \mathbf{10}, \mathbf{15}$ which are very similar, as we have checked working them out. We keep using the lighter notation introduced when dealing with case $C_r = \mathbf{1}$

• Subcase $C_i = \mathbf{6}$.

The edge $\{c, d\}$, which occurs in $C_r = \mathbf{12}$ in even position, must occur in odd position in some other cycle C_j of S, $j \leq r-1$. Again thanks to (B), C_j can only be $\widehat{\mathbf{2}}, \widehat{\mathbf{7}}, \widehat{\mathbf{9}}$ and $\widehat{\mathbf{13}}$.

Subcase $C_i = \mathbf{6}$ splits into the following three sub-subcases: $(\zeta), (\eta)$ and (ϑ).

(ζ) $C_i = \mathbf{6}$, $C_j = \widehat{\mathbf{2}}, \widehat{\mathbf{13}}$. $S' = (\underline{\mathbf{6}, \widehat{\mathbf{2}}}, \mathbf{12})$ works.

(η) $C_i = \mathbf{6}$, $C_j = \widehat{\mathbf{7}}$. Due to edge $\{b, d\}$, S must also contain $C_k \in \{\widehat{\mathbf{2}}, \mathbf{3}, \mathbf{4}, \mathbf{10}, \widehat{\mathbf{15}}\}$.
When $C_k = \widehat{\mathbf{2}}$, then $S' = (\underline{\mathbf{6}, \widehat{\mathbf{2}}}, \mathbf{12})$ works.
When $C_k = \mathbf{3}, \widehat{\mathbf{15}}$, then $S' = (\underline{\widehat{\mathbf{7}}, \mathbf{3}}, \mathbf{12})$ works.
When $C_k = \mathbf{4}, \mathbf{10}$, then $S' = (\underline{\widehat{\mathbf{7}}, \mathbf{4}}, \mathbf{1})$ works.

(ϑ) $C_i = \mathbf{6}$, $C_j = \widehat{\mathbf{9}}$. Due to edge $\{b, d\}$, S must still contain $C_k \in \{\widehat{\mathbf{2}}, \mathbf{3}, \mathbf{4}, \mathbf{10}, \widehat{\mathbf{15}}\}$.
When $C_k = \widehat{\mathbf{2}}$, then $S' = (\underline{\mathbf{6}, \widehat{\mathbf{2}}}, \mathbf{12})$ works.
When $C_k = \mathbf{3}, \widehat{\mathbf{15}}$, then $S' = (\underline{\mathbf{6}, \widehat{\mathbf{9}}, \mathbf{3}}, \mathbf{12})$ works.
When $C_k = \mathbf{4}, \mathbf{10}$, then $S' = (\underline{\mathbf{6}, \widehat{\mathbf{9}}, \mathbf{4}}, \mathbf{1})$ works.

The subcase when $C_i = \mathbf{6}$ is now complete.

Cases $C_r = \mathbf{10}, \mathbf{11}, \mathbf{13}, \mathbf{14}, \mathbf{15}$.

Let $\sigma_{\mathbf{12}}^{\mathbf{n}}$ be any element of the symmetry group of $K_{3,3}$ which turns cycle $\mathbf{12}$ into cycle $\mathbf{n}$, $\mathbf{n} \in \{\mathbf{10}, \mathbf{11}, \mathbf{13}, \mathbf{14}, \mathbf{15}\}$, and such that $\sigma_{\mathbf{12}}^{\mathbf{n}}(\mathbf{1}) \in \{\mathbf{2}, \mathbf{3}, \ldots, \mathbf{9}\}$ (i.e. $\widehat{\sigma_{\mathbf{12}}^{\mathbf{n}}(\mathbf{1})} \notin \{\mathbf{2}, \mathbf{3}, \ldots, \mathbf{9}\}$). For instance, one can choose:

$$\sigma_{\mathbf{12}}^{\mathbf{10}} = (def); \quad \sigma_{\mathbf{12}}^{\mathbf{11}} = (ac)(dfe); \quad \sigma_{\mathbf{12}}^{\mathbf{13}} = (bc)(df);$$

$$\sigma_{\mathbf{12}}^{\mathbf{14}} = (ef); \qquad \sigma_{\mathbf{12}}^{\mathbf{15}} = (ab)(def).$$

We remark that, with this particular choice, we have:

$$\sigma_{\mathbf{12}}^{\mathbf{10}}(\mathbf{1}) = \sigma_{\mathbf{12}}^{\mathbf{13}}(\mathbf{1}) = \mathbf{8}; \quad \sigma_{\mathbf{12}}^{\mathbf{11}}(\mathbf{1}) = \mathbf{3};$$

$$\sigma_{\mathbf{12}}^{\mathbf{14}}(\mathbf{1}) = \mathbf{2}; \qquad \sigma_{\mathbf{12}}^{\mathbf{15}}(\mathbf{1}) = \mathbf{6}.$$

Thanks to Proposition 4.1, the construction given for cycle $\mathbf{12}$ is transformed by $\sigma_{\mathbf{12}}^{\mathbf{n}}$ into a construction valid for cycle $\mathbf{n}$, $\mathbf{n} \in \{\mathbf{10}, \mathbf{11}, \mathbf{13}, \mathbf{14}, \mathbf{15}\}$. For $\sigma_{\mathbf{1}}^{\mathbf{n}}$ acts on all edges and cycles having a role in the construction given for cycle $\mathbf{12}$, and the condition $\sigma_{\mathbf{12}}^{\mathbf{n}}(\mathbf{1}) \in \{\mathbf{2}, \mathbf{3}, \ldots, \mathbf{9}\}$ guarantees that whenever an admissible $r'-$tuple S' such that its maximum edges are among those of S, happens to occur in the construction related to cycle $\mathbf{12}$, then $\sigma_{\mathbf{12}}^{\mathbf{n}}(S')$ has its maximum edges among those of $\sigma_{\mathbf{12}}^{\mathbf{n}}(S)$.

Remark

If one wanted to use another length 6 cycle $\mathbf{n}_0$ as pivotal cycle, instead of $\mathbf{12}$, it would not be possible to guarantee that $\sigma^{\mathbf{n}}_{\mathbf{n}_0}(S')$, $\mathbf{n} \in \{\mathbf{10}, \mathbf{11}, \ldots, \mathbf{15}\} \setminus \{\mathbf{n}_0\}$, has its maximum edges among those of $\sigma^{\mathbf{n}}_{\mathbf{n}_0}(S)$, regardless of all possible requirements put on $\sigma^{\mathbf{n}}_{\mathbf{n}_0}$ with respect to length 4 cycles.

This is why we have selected cycle $\mathbf{12}$ among all length 6 cycles.

This completes the proof of Theorem 4.2. □

5. Conservation of the RG property

Let C be a closed path of $K_{3,3}$ such that every edge either is always in odd position ("odd edge"), or is always in even position ("even edge"). We can think of C as of a coloured multigraph (cf., e.g. [7]) in which every odd (multi-)edge is red, say, and every even (multi-)edge is blue, say.

Lemma 5.1: *C has a decomposition into cycles of length ≥ 4, each one of them having successive edges which alternate in colour.*

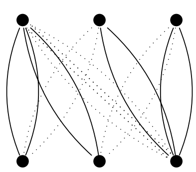
Proof: It suffices to show that C has a decomposition into cycles with alternating colours (it is then obvious that every cycle must have length at least 4). Let $\bar{C}$ be a minimal counterexample. [8, Theorem] says that $\bar{C}$ has at least one cycle D (of length ≥ 4) with alternating colours. But then $\bar{C} \setminus \{D\}$ contradicts the minimality of $\bar{C}$. □

The following proposition is obvious.

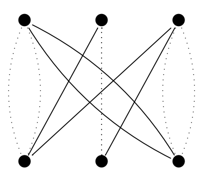
Proposition 5.2: *Let $S := (C_1, \ldots, C_r)$ be an admissible $r-$tuple of closed paths of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$. For every $C_i \neq \emptyset$, let $\mathcal{D}_i$ be a decomposition of C_i into cycles with alternating colours, and call c_i (≥ 1) the cardinality of $\mathcal{D}_i$. Also let $\mathcal{D} := \cup \mathcal{D}_i$. Then every permutation of the elements of $\mathcal{D}$ is an admissible $h-$tuple of cycles of $\mathcal{G}_{\underline{h}\times\underline{3}\times\underline{3}}$, with $h = \sum c_i$.*

Example 5.3: *Let $S := (C_1, C_2, C_3)$ be the following admissible triplet of closed paths of $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$:*

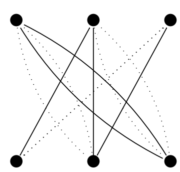
C_1



C_2



C_3



where the continuous (resp., dotted) edges are the red (resp., blue) ones.

$\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are indicated below:

$\mathcal{D}_1$

D_{11} D_{12} D_{13} D_{14}

$\mathcal{D}_2$

D_{21} D_{22}

$\mathcal{D}_3$

D_{31} D_{32}

The $8-$*tuple* $(D_{11}, D_{31}, D_{21}, D_{22}, D_{12}, D_{14}, D_{13}, D_{32})$ *is an admissible sequence of cycles of* $\mathcal{G}_{\underline{8}\times\underline{3}\times\underline{3}}$.

THEOREM 5.4: *Let* $S := (C_1, \ldots, C_r)$ *be an* $RG-$*sequence of* $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ *(w.r.t.* $<_{Lex}$*). Colour the maximum edges red and all the others blue. For every* $C_i \neq \emptyset$*, call* $\mathcal{D}_i$ *a decomposition of* C_i *into cycles with alternating colours and denote the cardinality of* $\mathcal{D}_i$ *by* c_i*. If* $\mathcal{D} := \cup\mathcal{D}_i$*, there exists a permutation of the elements of* $\mathcal{D}$ *with the property that*

(1) *the edge of the rightmost cycle which corresponds to the maximum variable (w.r.t.* $<_{Lex}$*) is red, and*

(2) *the permutation is an* $RG-$*sequence of* $\mathcal{G}_{\underline{h}\times\underline{3}\times\underline{3}}$ *(w.r.t.* $<_{Lex}$*) with* $h = \sum c_i$.

Proof: Let $i_1 < i_2 < \cdots < i_s$ be the indices corresponding to the nonempty paths C_i. For every $j = 1, \ldots, s-1$, let $\mathcal{P}_{i_j}$ denote any permutation of $\mathcal{D}_{i_j}$. Moreover, let $\mathcal{P}_{i_s}$ denote any permutation of $\mathcal{D}_{i_s}$ which satisfies (1). $\mathcal{P}_{i_1}, \mathcal{P}_{i_2}, \ldots, \mathcal{P}_{i_s}$ (in this order) form an admissible sequence, S', which still satisfies (1) and whose maximum edges are all red. We claim that S' satisfies (2) as well.

Assume that S' does not satisfy (2). We are going to show that then S cannot be an RG-sequence: a contradiction.

If S' is not an RG-sequence, then there exists an RG-sequence, S'', of $\mathcal{G}_{\underline{h}\times\underline{3}\times\underline{3}}$ such that its maximum edges are among those of S' (recall Definition 2.4). Since S' is a sequence of cycles, S'' is a sequence of cycles, too. It follows from Theorem

4.2 that S'' consists of at most five cycles. It is not hard to check that in the reduced Gröbner basis of $I_{\mathcal{A}_{\underline{5}\times\underline{3}\times\underline{3}}}$ (w.r.t. $<_{Lex}$) all RG-sequences of cycles involve at most one length 6 cycle, except for the pairs of antisomorphic cycles of length 6. Hence there exist maximum edges of S' which do not occur in S''. For if S' and S'' had the same maximum edges, and S'' were a pair of antisomorphic cycles of length 6, then $S' = S''$ (by admissibility), against the assumption that S' be not an RG-sequence. If S' and S'' had the same maximum edges, and S'' were not a pair of antisomorphic cycles of length 6, then S' should contain all length 4 cycles of S'', hence S' should also contain the only length 6 cycle of S'' (by admissibility), and again $S' = S''$, which is excluded.

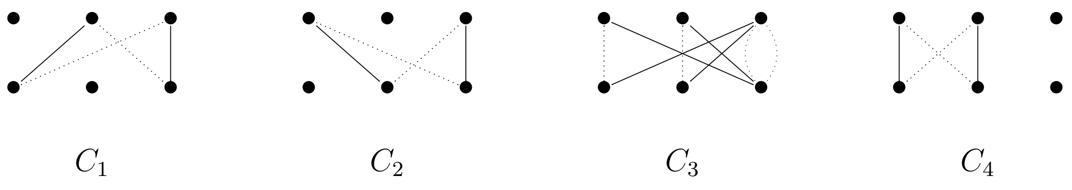
Having ascertained that there exist maximum edges of S' which do not occur in S'', we are able to show that S cannot be an RG-sequence.

Notice that S'' cannot just involve a single path C_i of S, that is, the maximum edges of S'' cannot just be among the maximum edges of the corresponding $\mathcal{P}_i$. If this were the case, then every maximum edge of S'' should occur among the red edges of the mentioned C_i. But if S'' happened to have at least three distinct cycles, then the three (or more) red edges of C_i (coming from three different vertices) would be incident on one and the same vertex, which is impossible. On the other hand, if S'' happened to consist only of two cycles, they would be antisomorphic and C_i should contain a cycle completely coloured red (of length either 4 or 6). C_i would then turn out to be determined (up to repetitions) and its decomposition into cycles with alternating colours would be inconsistent with S'' consisting of two cycles.

Since S'' involves at least two distinct closed paths of S, we can say (possibily patching together the cycles of S'' involving a single C_i, and deleting the edges that in doing so happen to be red and blue at the same time) that there exists an admissible sequence, $\overline{S}$, of closed paths of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ having its maximum edges among the red edges of S. That is, $in_{<_{Lex}}(\overline{g})$ divides $in_{<_{Lex}}(g)$, where $\overline{g}$ (resp., g) stands for the binomial associated with $\overline{S}$ (resp., S).

Recalling the underlined statement above, not all red edges of S occur in $\overline{S}$ with the same multiplicities. Hence $in_{<_{Lex}}(\overline{g})$ properly divides $in_{<_{Lex}}(g)$, and this contradicts the fact that S is an RG-sequence. □

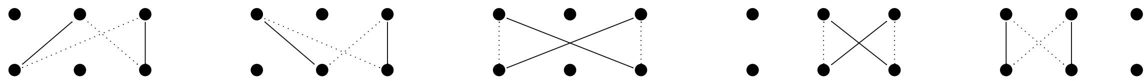
Example 5.5: *Let S be the following $RG-$sequence of $\mathcal{G}_{\underline{4}\times\underline{3}\times\underline{3}}$ (cf. Remark 6.5 below):*



C_1 C_2 C_3 C_4

where the maximum edges (w.r.t. $<_{Lex}$) are the continuous ones, which we assume to be red.

Decomposing C_3 *into cycles with alternating colours, we obtain the following* $5-$*tuple:*

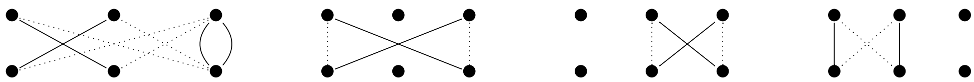


C_1 C_2 C_{31} C_{32} C_4

where the continuous (resp., dotted) edges are coloured red (resp., blue).

The $5-$*tuple* $(C_1,\ C_2,\ C_{31},\ C_{32},\ C_4)$ *is an RG−sequence of* $\mathcal{G}_{\underline{5}\times\underline{3}\times\underline{3}}$. *In fact, every permutation of this* $5-$*tuple gives an RG−sequence, as one sees by means of a simple analysis.*

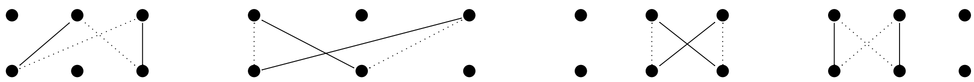
Example 5.6: *Notation as in Example 5.5. Let us take* $(C_1,\ C_2,\ C_{31},\ C_{32},\ C_4)$ *and patch together* C_1 *and* C_2. *We get (no deletion of edges, here):*



$C_1 + C_2$ C_{31} C_{32} C_4

where the continuous (resp., dotted) edges are coloured red (resp., blue). Clearly, every permutation of the $4-$*tuple above is admissible.*

Again, let us take $(C_1,\ C_2,\ C_{31},\ C_{32},\ C_4)$ *and patch together* C_2 *and* C_{31}. *We get (two edges deleted, which happen to be red and blue at the same time):*



C_1 $C_2 + C_{31}$ C_{32} C_4

where the continuous (resp., dotted) edges are coloured red (resp., blue).

Every permutation of the latter $4-$*tuple is admissible.*

6. Main theorem

We prove our main result, announced at the end of Section 2

Theorem 6.1: *Let* $r' \in \{2,3,4,5\}$ *and* $S' := (D_1, \ldots, D_{r'})$ *an RG−sequence of* $\mathcal{G}_{\underline{r'}\times\underline{3}\times\underline{3}}$ *(w.r.t.* $<_{Lex}$*) such that* $D_{i'} \neq \emptyset$ *for every* $i' \in \underline{r'}$. *For every* $r \geq 6$ *and for every choice of indices* $i_1, i_2, \ldots, i_{r'}$ *such that* $1 \leq i_1 < i_2 < \cdots < i_{r'} \leq r$, *consider the* $r-$*tuple* $S_{i_1,\ldots,\,i_{r'}} := (C_1, \ldots, C_r)$ *such that*

$$C_h := \begin{cases} D_{i'} & \textit{if } h = i_{i'} \\ \\ \emptyset & \quad \textit{else,} \end{cases}$$

which is clearly an admissible $r-$tuple of closed paths of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$. Then **all and only** *the $RG-$sequences of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ (w.r.t. $<_{Lex}$) are obtained in this way, when r' ranges over $\{2,3,4,5\}$ and S' ranges over the set of all the $RG-$sequences of $\mathcal{G}_{\underline{r'}\times\underline{3}\times\underline{3}}$ with no empty path.*

Proof: Proposition 2.7 says that $S_{i_1,\dots,\, i_{r'}}$ is an $RG-$sequence of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ (w.r.t. $<_{Lex}$). It remains to show that every $RG-$sequence of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ is of this kind.

Assume for a contradiction that $S := (C_1, \dots, C_r)$ is an $RG-$sequence of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ not of the indicated type. At least six paths of S must be nonempty. If every nonempty C_i is a cycle, Theorem 4.2 says that there exists an admissible $r'-$tuple of cycles of $\mathcal{G}_{\underline{r'}\times\underline{3}\times\underline{3}}$ $2 \leq r' \leq 5$, such that the set of its maximum edges is properly included in the set of the maximum edges of S. This violates the $RG-$property of S (recall Remark 2.5). If some nonempty path of S is not a cycle, Theorem 5.4 produces an $RG-$sequence of $\mathcal{G}_{\underline{h}\times\underline{3}\times\underline{3}}$ $(h > r)$ consisting of all cycles (and at least six of them $\neq \emptyset$). Again a contradiction is obtained by means of Theorem 4.2. □

COROLLARY 6.2: *Let g be an element of the reduced Gröbner basis of $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ w.r.t. $<_{Lex}$ $(r \geq 2)$. Then:*

(1) *The same (distinct) first indices $i_1, \dots, i_{r'}$ occur in the variables of both monomials of g, and $r' \in \{2,3,4,5\}$.*

(2) *Every variable occurring in g can occur with degree at most 2. Furthermore, if $r' = 5$, then both monomials of g must be square-free.*

(3) *The total degree of g is at most 10.*

Proof: **(1)** Let $S := (C_1, \dots, C_r)$ be the $RG-$sequence associated with g. Then the conclusion is obvious (the range of r' coming from Theorem 6.1).

(2) Recall that, in our language, r' is the number of nonempty paths occurring in S and the degree of a variable is the multiplicity of the corresponding edge. If there existed a variable of degree ≥ 3, then some path C_i would contain an edge of multiplicity $m \geq 3$. Hence C_i could be decomposed into at least m cycles (cf. Lemma 5.1) and Theorem 5.4 would imply the existence of an $RG-$sequence of cycles containing at least $2m$ cycles (because of the admissibility property). But $2m \geq 6$ contradicts Theorem 6.1. Furthermore, if $r' = 5$, then a contradiction arises as soon as $m \geq 2$, since one gets an $RG-$sequence of cycles containing at least $m + 4 \geq 6$ cycles.

(3) Notice that Theorem 6.1 implies that, for every $r \geq 6$. the set of the total degrees of all binomials occurring in the reduced Gröbner basis of $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ (w.r.t. $<_{Lex}$) equals the set of the total degrees of all binomials of the reduced Gröbner basis of $I_{\mathcal{A}_{\underline{5}\times\underline{3}\times\underline{3}}}$. Calculation of the latter Gröbner basis shows that no total degree exceeds 10.

□

Remark 6.3: *Notice that Corollary 6.2 - (2) implies that if* $S := (C_1, \dots, C_5)$ *is an RG−sequence of* $\mathcal{G}_{\underline{5}\times\underline{3}\times\underline{3}}$ *(w.r.t.* $<_{Lex}$*) such that all* C_i *are not empty, then* S *must be a* $5-$*tuple of cycles of* $K_{3,3}$.

Remark 6.4: *Theorem 6.1 suggests a purely combinatorial algorithm for the calculation of the reduced Gröbner basis (w.r.t.* $<_{Lex}$*) of* $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$, $r \geq 6$, *once the reduced Gröbner basis of* $I_{\mathcal{A}_{\underline{5}\times\underline{3}\times\underline{3}}}$ *is known.*

Notice that the bound $r = 5$ *is sharp (cf. e.g. Example 5.5).*

Remark 6.5: *The reduced Gröbner basis (w.r.t.* $<_{Lex}$*) of* $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$, $3 \leq r \leq 5$, *has been calculated on a PC with the help of the CoCoA Team of the University of Genoa (cf. [5]), thanks to an improvement of their algorithm TestSet, now implemented in CoCoA* 4.3.

The corresponding outputs (also giving information on the cardinalities of the bases and the running times) are available on the web page of the second author (http://www.dmi.units.it/˜rossif/).

Cases $r = 2$ *and* $r = 3$ *have also been available in [3] for quite a while.*

References

[1] S. Aoki, A. Takemura, (2003). Minimal basis for connected Markov chain over $3 \times 3 \times K$ contingency tables with fixed two-dimensional marginals, *Australian and New Zealand Journal of Statistics*, **45**, 229–249.

[2] S. Aoki, A. Takemura, *Invariant minimal Markov basis for sampling contingency tables with fixed marginals*, Technical Report METR 03-25, Department of Mathematical Engineering and Information Physics, The University of Tokyo, (2003).

[3] G. Boffi, F. Rossi, *Gröbner bases related to 3-dimensional transportation problems*, Quaderni matematici dell'Università di Trieste, **482**, DSM-Trieste, http://www.dmi.units.it/˜rossif/, (2000).

[4] G. Boffi, F. Rossi, (2001). Lexicographic Gröbner bases of 3-dimensional transportation problems, *Contemporary Mathemathics*, **286**, 145–168.

[5] CoCoATeam, CoCoA: a system for doing Computations in Commutative Algebra, Available at http://cocoa.dima.unige.it, (2004).

[6] P. Diaconis, B. Sturmfels, (1998). Algebraic algorithms for sampling from conditional distributions, *The Annals of Statistics*, **26**, 363–397.

[7] R.L. Graham, M. Grötschel, L. Lovász, *Handbook of Combinatorics - Vol.1*, Elsevier, Amsterdam, (1995).

[8] J.W. Grossman, R. Häggkvist, (1983). Alternating Cycles in Edge-Partitioned Graphs, *Journal of Combinatorial Theory, Ser. B*, **34**, 77–81.

[9] F. Santos, B. Sturmfels, (2003). Higher Lawrence configurations, *Journal of Combinatorial Theory, Ser. A*, **103**, 151–164.

[10] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, AMS, Providence RI, (1995).