**Advanced Algebra/Istituzioni di Algebra Superiore**
academic year: 2023–24
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**Basic notions.** Rings, ideals, definitions of zero divisors, unitary elements, associate elements in a ring. Prime and maximal ideals. Zorn's lemma. Any commutative, unitary ring has a maximal ideal. Modules over a ring. Direct sum of modules, product of modules, homomorphisms, free modules. Universal property of free modules. Bases for free modules. Every module is a quotient of a free module. Exact sequences of modules. Short exact sequences. Rank of a free module.

**Noetherian and artinian modules.** Equivalent definitions of noetherian (artinian) modules. Short exact sequences and noetherian (artinian) modules. Direct sum of noetherian (artinian) modules. Definition of noetherian (artinian) rings. Hilbert basis theorem. Composition series of a module. Length of a composition series. Modules of finite length. A module $M$ has a composition series if and only if it is artinian and noetherian. If in a ring $A$ the zero ideal is a finite product of maximal ideals, then $A$ is an artinian ring if and only if it is noetherian.

**PID and UFD.** Integral domains. Prime and irreducible elements in a domain. Principal ideal domains. Unique factorization domains. The relation between the notions of prime element and irreducible element. Any PID is a UFD. If $A$ is a UFD, then $A[x]$ is a UFD.

**Modules over a PID.** Cyclic modules. Annihilator of a module. Torsion modules. Order of an element of a module. If $M$ is a finitely generated torsion module over a PID, $M$ can be decomposed in a unique way as direct sum of cyclic modules. Invariant factors of the module $M$. Decomposition of a finite abelian group into cyclic groups. The finitely generated torsion module over $K[x]$ associated to an endomorphism of a finite dimensional vector space over a field $K$. Decomposition of a matrix into the direct sum of companion matrices (over a field $K$). Cayley-Hamilton theorem. The rational canonical form of a matrix.
Primary modules over a PID. Primary decomposition of a finitely generated, torsion module over a PID. The cyclic primary decomposition theorem. Jordan canonical form of a matrix.
Submodules of free modules of finite rank (over a PID) are free. If a finitely generated module $M$ over a PID is torsionless, then its is free. Any finitely generated module over a PID is the direct sum of a free module and cyclic modules. Smith normal form of a matrix (with no proofs).

**Commutative algebra.** Local rings. Nilradical of a (commutative, unitary) ring. Jacobson radical of a (commutative, unitary) ring. Operations with ideals: sum, product of ideals; colon of two ideals. Direct product of rings. Radical of an ideal. The Chinese remainder theorem for rings. Extensions and contractions of ideals. Nakayama lemma. Fraction rings: the construction of $S^{-1}A$ ($S$

a multiplicatively closed subset of $A$) and its properties. Fraction modules. Localizations. Local properties of modules. Ideals and prime ideals in $S^{-1}A$.

**Primary decomposition of ideals.** Primary ideals of a (commutative, unitary) ring. Definition of a primary decomposition of an ideal. Minimal primary decomposition of an ideal. First uniqueness theorem of a primary decomposition. Associated primes of a primary decomposition. Minimal (or isolated) primes and embedded primes. Primary ideals of $A$ and of $S^{-1}(A)$ (where $S$ is a multiplicatively closed subset of $A$). Second uniqueness theorem of a primary decomposition. Irreducible ideals. In a noetherian ring, any ideal is a finite intersection of irreducible ideals. In a noetherian ring, an irreducible ideal is primary. In a noetherian ring, any ideal has a primary decomposition.

**Artinian rings.** In an artinian ring every prime ideal is maximal. The nilradical of an artinian ring coincides with the Jacobson radical. The Krull dimension of a ring. A ring is artinian if and only if it is noetherian and of dimension zero. Any artinian ring is isomorphic to the direct sum of a finite number of local, artinian rings (in a unique way).

**Rings of integers and Dedekind domains.** Algebraic extensions. Field of algebraic number. Number fields (i.e. subfields of $\mathbb{C}$ which are finite algebraic extensions of $\mathbb{Q}$). Abel's theorem, i.e. every finite algebraic extension of $\mathbb{Q}$ is a principal extension. Ring of algebraic numbers of a number field. Basis of a number field and integral basis of its ring of algebraic numbers. Discriminant of an integral basis. Existence of integral bases. The ring of integers of a number field is a finitely generated free ablian group. Consequence: it is a noetherian ring. Integral extensions. Integrally closed domains. If $R$ and $S$ are two domains such that $R$ is a subring of $S$ and $S$ is an integral extension of $R$, then $R$ is a field iff $S$ is a field. In particular, the ring of algebraic integers (of a number field) has Krull dimension one. Moreover, the ring of algebraic integers is instegally closed. Definition of a Dedekind domain: a domain of Krull dimension one, integrally closed and noetherian. Some properties of Dedekind domains. In a Dedekind domain every non zero ideal is a product, in a unique way, of prime ideals (no proof). Fracional ideals. The set of fracional ideals in a Dedekind domain if a free abelian group whose basis is given by the prime ideals (no proof).
Example: The ring of algebraic integers of $\mathbb{Q}[\sqrt{-5}]$ is $\mathbb{Z}[\sqrt{-5}]$. The element 6 has two different factorizations in irreducible elements: $6 = 2 \cdot 3$ and $6 = (1+\sqrt{-5}) \cdot (1-\sqrt{-5})$ but the ideal (6) has precisely one factorization with prime ideals, which is:

$$(6) = (2, 1+\sqrt{-5})^2(3, 1+\sqrt{-5})(3, 2+\sqrt{-5}).$$

### Basic bibliography


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6. W. Stein, *Algebraic Number Theory, a computational approach*, preprint (2012).

7. I. Stewart, D. Tall, *Algebraic Number theory and Fermat Last Theorem*, 3rd edition, A K Peters (2002).