

Advanced Algebra/Istituzioni di Algebra Superiore

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Basic notions. Rings, ideals, definitions of zero divisors, unitary elements, associate elements in a ring. Prime and maximal ideals. Zorn's lemma. Any commutative, unitary ring has a maximal ideal. Modules over a ring. Direct sum of modules, product of modules, homomorphisms, free modules. Universal property of free modules. Bases for free modules. Every module is a quotient of a free module. Exact sequences of modules. Short exact sequences. Rank of a free module.

Noetherian and artinian modules. Equivalent definitions of noetherian (artinian) modules. Short exact sequences and noetherian (artinian) modules. Direct sum of noetherian (artinian) modules. Definition of noetherian (artinian) rings. Hilbert basis theorem. Composition series of a module. Length of a composition series. Modules of finite length. A module M has a composition series if and only if it is artinian and noetherian. If in a ring A the zero ideal is a finite product of maximal ideals, then A is an artinian ring if and only if it is noetherian.

PID and UFD. Integral domains. Prime and irreducible elements in a domain. Principal ideal domains. Unique factorization domains. The relation between the notions of prime element and irreducible element. Any PID is a UFD. If A is a UFD, then $A[x]$ is a UFD.

Modules over a PID. Cyclic modules. Annihilator of a module. Torsion modules. Order of an element of a module. If M is a finitely generated torsion module over a PID, M can be decomposed in a unique way as direct sum of cyclic modules. Invariant factors of the module M . Decomposition of a finite abelian group into cyclic groups. The finitely generated torsion module over $K[x]$ associated to an endomorphism of a finite dimensional vector space over a field K . Decomposition of a matrix into the direct sum of companion matrices (over a field K). Cayley-Hamilton theorem. The rational canonical form of a matrix.

Primary modules over a PID. Primary decomposition of a finitely generated, torsion module over a PID. The cyclic primary decomposition theorem. Jordan canonical form of a matrix.

Submodules of free modules of finite rank (over a PID) are free. If a finitely generated module M over a PID is torsionless, then it is free. Any finitely generated module over a PID is the direct sum of a free module and cyclic modules. Smith normal form of a matrix (with no proofs).

Commutative algebra. Local rings. Nilradical of a (commutative, unitary) ring. Jacobson radical of a (commutative, unitary) ring. Operations with ideals: sum, product of ideals; colon of two ideals. Direct product of rings. Radical of an ideal. The Chinese remainder theorem for rings. Extensions and contractions of ideals. Nakayama lemma. Fraction rings: the construction of $S^{-1}A$ (S

a multiplicatively closed subset of A) and its properties. Fraction modules. Localizations. Local properties of modules. Ideals and prime ideals in $S^{-1}A$.

Primary decomposition of ideals. Primary ideals of a (commutative, unitary) ring. Definition of a primary decomposition of an ideal. Minimal primary decomposition of an ideal. First uniqueness theorem of a primary decomposition. Associated primes of a primary decomposition. Minimal (or isolated) primes and embedded primes. Primary ideals of A and of $S^{-1}(A)$ (where S is a multiplicatively closed subset of A). Second uniqueness theorem of a primary decomposition. Irreducible ideals. In a noetherian ring, any ideal is a finite intersection of irreducible ideals. In a noetherian ring, an irreducible ideal is primary. In a noetherian ring, any ideal has a primary decomposition.

Artinian rings. In an artinian ring every prime ideal is maximal. The nilradical of an artinian ring coincides with the Jacobson radical. The Krull dimension of a ring. A ring is artinian if and only if it is noetherian and of dimension zero. Any artinian ring is isomorphic to the direct sum of a finite number of local, artinian rings (in a unique way).

Rings of integers and Dedekind domains. Algebraic extensions. Field of algebraic number. Number fields (i.e. subfields of \mathbb{C} which are finite algebraic extensions of \mathbb{Q}). Abel's theorem, i.e. every finite algebraic extension of \mathbb{Q} is a principal extension. Ring of algebraic numbers of a number field. Basis of a number field and integral basis of its ring of algebraic numbers. Discriminant of an integral basis. Existence of integral bases. The ring of integers of a number field is a finitely generated free abelian group. Consequence: it is a noetherian ring. Integral extensions. Integrally closed domains. If R and S are two domains such that R is a subring of S and S is an integral extension of R , then R is a field iff S is a field. In particular, the ring of algebraic integers (of a number field) has Krull dimension one. Moreover, the ring of algebraic integers is integrally closed. Definition of a Dedekind domain: a domain of Krull dimension one, integrally closed and noetherian. Some properties of Dedekind domains. In a Dedekind domain every non zero ideal is a product, in a unique way, of prime ideals (no proof). Fractional ideals. The set of fractional ideals in a Dedekind domain is a free abelian group whose basis is given by the prime ideals (no proof).

Example: The ring of algebraic integers of $\mathbb{Q}[\sqrt{-5}]$ is $\mathbb{Z}[\sqrt{-5}]$. The element 6 has two different factorizations in irreducible elements: $6 = 2 \cdot 3$ and $6 = (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5})$ but the ideal (6) has precisely one factorization with prime ideals, which is:

$$(6) = (2, 1 + \sqrt{-5})^2(3, 1 + \sqrt{-5})(3, 2 + \sqrt{-5}).$$

Basic bibliography

1. M. F. Atiyah, I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Company (1969).
2. S. Mac Lane, G. Birkhoff, *Algebra*, The Macmillan Company (1965).

3. H. Matsumura, *Commutative Ring Theory*, Cambridge Univ. Press (1990).
4. M. Reid, *Undergraduate Commutative Algebra*, London Math. Soc. % (1995).
5. J. J. Rotman, *Advanced Modern Algebra*, Prentice Hall (2003)
6. W. Stein, *Algebraic Number Theory, a computational approach*, preprint (2012).
7. I. Stewart, D. Tall, *Algebraic Number theory and Fermat Last Theorem*, 3rd edition, A K Peters (2002).