Advanced Algebra/Istituzioni di Algebra Superiore
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Alessandro Logar

Basic notions. Rings, ideals, definitions of zero divisors, unitary elements, associate elements in a ring. Prime and maximal ideals. Zorn's lemma. Any commutative, unitary ring has a maximal ideal. Modules over a ring. Direct sum of modules, product of modules, homomorphisms, free modules. Universal property of free modules. Bases for free modules. Every module is a quotient of a free module. Exact sequences of modules. Short exact sequences. Rank of a free module.

Noetherian and artinian modules. Equivalent definitions of noetherian (artinian) modules. Short exact sequences and noetherian (artinian) modules. Direct sum of noetherian (artinian) modules. Definition of noetherian (artinian) rings. Hilbert basis theorem. Composition series of a module. Length of a composition series. Modules of finite length. A module M has a composition series if and only if it is artinian and noetherian. If in a ring A the zero ideal is a finite product of maximal ideals, then A is an artinian ring if and only if it is noetherian.

PID and UFD. Integral domains. Prime and irreducible elements in a domain. Principal ideal domains. Unique factorization domains. The relation between the notions of prime element and irreducible element. Any PID is a UFD. If A is a UFD, then $A[x]$ is a UFD.

Modules over a PID. Cyclic modules. Annihilator of a module. Torsion modules. Order of an element of a module. If M is a finitely generated torsion module over a PID, M can be decomposed in a unique way as direct sum of cyclic modules. Invariant factors of the module M. Decomposition of a finite abelian group into cyclic groups. The finitely generated torsion module over $K[x]$ associated to an endomorphism of a finite dimensional vector space over a field K. Decomposition of a matrix into the direct sum of companion matrices (over a field K). Cayley-Hamilton theorem. The rational canonical form of a matrix.
Primary modules over a PID. Primary decomposition of a finitely generated, torsion module over a PID. The cyclic primary decomposition theorem. Jordan canonical form of a matrix.
Submodules of free modules of finite rank (over a PID) are free. If a finitely generated module M over a PID is torsionless, then its is free. Any finitely generated module over a PID is the direct sum of a free module and cyclic modules. Smith normal form of a matrix (with no proofs).

Commutative algebra. Local rings. Nilradical of a (commutative, unitary) ring. Jacobson radical of a (commutative, unitary) ring. Operations with ideals: sum, product of ideals; colon of two ideals. Direct product of rings. Radical of an ideal. The Chinese remainder theorem for rings. Extensions and contractions of ideals. Nakayama lemma. Fraction rings: the construction of $S^{-1}A$ (S

a multiplicatively closed subset of A) and its properties. Fraction modules. Localizations. Local properties of modules. Ideals and prime ideals in $S^{-1}A$.

Primary decomposition of ideals. Primary ideals of a (commutative, unitary) ring. Definition of a primary decomposition of an ideal. Minimal primary decomposition of an ideal. First uniqueness theorem of a primary decomposition. Associated primes of a primary decomposition. Minimal (or isolated) primes and embedded primes. Primary ideals of A and of $S^{-1}(A)$ (where S is a multiplicatively closed subset of A). Second uniqueness theorem of a primary decomposition. Irreducible ideals. In a noetherian ring, any ideal is a finite intersection of irreducible ideals. In a noetherian ring, an irreducible ideal is primary. In a noetherian ring, any ideal has a primary decomposition.

Artinian rings. In an artinian ring every prime ideal is maximal. The nilradical of an artinian ring coincides with the Jacobson radical. The Krull dimension of a ring. A ring is artinian if and only if it is noetherian and of dimension zero. Any artinian ring is isomorphic to the direct sum of a finite number of local, artinian rings (in a unique way) (without proof).

Hilbert Function and some applications. Four lectures given by prof. Joan Elias (University of Barcellona). Height of an ideal, Krull dimension,Krull's principal ideal (without proof), some results on the height of ideals. Graded rings and modules, homogeneous multiplicatively closed subsets of a ring. Noetherian graded rings. Associated graded ring A to a ring w.r.t. an ideal of A. The Rees algebra. Primes associated to a quotient M/N of modules. Theorem of Matijtvich-Roberts (without proof). Dimension of a non negative graded ring. Existence of particular chains of graded submodules of a graded module M (if the base ring is noetherian). Definition of the Hilbert function, proof of the Hilbrt theorem. Some applications.

Basic bibliography

1. M. F. Atyah, I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Company (1969).
2. S. Mac Lane, G. Birkhoff, *Algebra*, The Macmillan Company (1965).
3. H. Matsumura, *Commutative Ring Theory*, Cambridge Univ. Press (1990).
4. M. Reid, *Undergraduate Commutative Algebra*, London Math. Soc. % (1995).
5. J. J. Rotman, *Advanced Modern Algebra*, Prentice Hall (2003)
6. W. Stein, *Algebraic Number Theory, a computational approach*, preprint (2012).
7. I. Stewart, D. Tall, *Algebraic Number theory and Fermat Last Theorem*, 3rd edition, A K Peters (2002).