Advanced Algebra/Istituzioni di Algebra Superiore
academic year: 2019–20
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Basic notions. Rings, ideals, definitions of zero divisors, unitary elements, associate elements in a ring. Prime and maximal ideals. Zorn's lemma. Any commutative, unitary ring has a maximal ideal. Modules over a ring. Direct sum of modules, product of modules, homomorphisms, free modules. Universal property of free modules. Bases for free modules. Every module is a quotient of a free module. Exact sequences of modules. Short exact sequences. Rank of a free module.

Noetherian and artinian modules. Equivalent definitions of noetherian (artinian) modules. Short exact sequences and noetherian (artinian) modules. Direct sum of noetherian (artinian) modules. Definition of noetherian (artinian) rings. Hilbert basis theorem (no proof). Composition series of a module. Length of a composition series. Modules of finite length. A module M has a composition series if and only if it is artinian and noetherian. If in a ring A the zero ideal is a finite product of maximal ideals, then A is an artinian ring if and only if it is noetherian.

PID and UFD. Integral domains. Prime and irreducible elements in a domain. Principal ideal domains. Unique factorization domains. The relation between the notions of prime element and irreducible element. Any PID is a UFD. If A is a UFD, then $A[x]$ is a UFD.

Modules over a PID. Cyclic modules. Annihilator of a module. Torsion modules. Order of an element of a module. If M is a finitely generated torsion module over a PID, M can be decomposed in a unique way as direct sum of cyclic modules. Invariant factor of the module M. Decomposition of a finite abelian group into cyclic groups. The finitely generated torsion module over $K[x]$ associated to an endomorphism of a finite dimensional vector space over a field K. Decomposition of a matrix into the direct sum of companion matrices (over a field K). Cayley-Hamilton theorem. The rational canonical form of a matrix.
Primary modules over a PID. Primary decomposition of a finitely generated, torsion module over a PID. The cyclic primary decomposition theorem. Jordan canonical form of a matrix.
Submodules of free modules of finite rank (over a PID) are free. If a finitely generated module M over a PID is torsionless, then its is free. Any finitely generated module over a PID is the direct sum of a free module and cyclic modules.

Commutative algebra. Local rings. Nilradical of a (commutative, unitary) ring. Jacobson radical of a (commutative, unitary) ring. Operations with ideals: sum, product of ideals; colon of two ideals. Direct product of rings. Radical of an ideal. The Chinese remainder theorem for rings. Extensions and contractions of ideals. Nakayama lemma. Fraction rings: the construction of $S^{-1}A$ (S

a multiplicatively closed subset of A) and its properties. Fraction modules. Localizations. Local properties of modules. Ideals and prime ideals in $S^{-1}A$.

Primary decomposition of ideals. Primary ideals of a (commutative, unitary) ring. Definition of a primary decomposition of an ideal. Minimal primary decomposition of an ideal. First uniqueness theorem of a primary decomposition. Associated primes of a primary decomposition. Minimal (or isolated) primes and embedded primes. Primary ideals of A and of $S^{-1}(A)$ (where S is a multiplicatively closed subset of A). Second uniqueness theorem of a primary decomposition. Irreducible ideals. In a noetherian ring, any ideal is a finite intersection of irreducible ideals. In a noetherian ring, an irreducible ideal is primary. In a noetherian ring, any ideal has a primary decomposition.

Artinian rings. In an artinian ring every prime ideal is maximal. The nilradical of an artinian ring coincides with the Jacobson radical. The Krull dimension of a ring. A ring is artinian if and only if it is noetherian and of dimension zero. Any artinian ring is isomorphic to the direct sum of a finite number of local, artinian rings (in a unique way).

Dedekind domains. Algebraic number fields, rings of algebraic integers. Integrally closed domains. Any UFD is integrally closed. Definition of Dedekind domains (i.e. noetherian, integrally closed domains of Krull dimension 1). A domain A is a Dedekind domain if and only if every ideal of A is a product of prime ideals (in a unique way) (no proof). A PID is a Dedekind domain. The ring of integers O_K over an algebraic number field K is a Dedekind domain (no proof). Example: The ring of algebraic integers of $\mathbb{Q}[\sqrt{-5}]$ is $\mathbb{Z}[\sqrt{-5}]$. The element 6 has two different factorizations: $6 = 2 \cdot 3$ and $6 = (1+\sqrt{-5}) \cdot (1-\sqrt{-5})$ but the ideal (6) has precisely one factorization with prime ideals, which is:

$$(6) = (2, 1+\sqrt{-5})^2 (3, 1+\sqrt{-5})(3, 2+\sqrt{-5}).$$

Basic bibliography

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5. J. J. Rotman, *Advanced Modern Algebra*, Prentice Hall (2003)

6. W. Stein, *Algebraic Number Theory, a computational approach*, preprint (2012).

7. I. Stewart, D. Tall, *Algebraic Number theory and Fermat Last Theorem*, 3rd edition, A K Peters (2002).