Let R be a commutative, unitary ring. Then, from Zorn's lemma, it follows that R has a maximal ideal $\mathcal{M}$, therefore $K = R/\mathcal{M}$ is a field.

Recall that a free module F over R can be defined as a module which admits a basis, i.e. in F there exists a subset $B = (b_i)_{i\in I}$ such that the elements of B are a system of generators of F and are linearly independent over R.

Theorem 1 *Let F be a free module over a ring R. Then all the bases of F have the same cardinality.*

Proof. Let $\mathcal{M}$ be a maximal ideal of the ring R. By $\mathcal{M}F$ we denote the submodule of F generated by the elements of the form ma where $m \in \mathcal{M}$ and $a \in F$. It is easy to see that, if $B = (b_i)_{i\in I}$ is a basis of F over R, then $\mathcal{M}F$ is generated by the elements mb where $m \in \mathcal{M}$ and $b \in B$. The quotient module $F/\mathcal{M}F$ (which is an R-module) can also be considered as a $K = R/\mathcal{M}$-vector space, as soon as we define the product $K \times F/\mathcal{M}F \longrightarrow F/\mathcal{M}F$ by: $([\lambda], [a]) \mapsto [\lambda a]$. The set $\bar{B} = ([b_i])_{i\in I}$ is a basis of $F/\mathcal{M}F$ as a K-vector space. Indeed

- $\bar{B}$ is a system of generators:
  if $[a] \in F/\mathcal{M}F$, then $a = \sum_{i\in I} \lambda_i b_i$ so $[a] = \sum_{i\in I}[\lambda_i][b_i]$;
- the elements of $\bar{B}$ are linearly independent over K:
  if $\sum_{i\in I}[\lambda_i][b_i] = 0$, then $\sum_{i\in I} \lambda_i b_i \in \mathcal{M}F$ so $\sum_{i\in I} \lambda_i b_i = \sum_{i\in I} \mu_i m_i b_i$, where $\mu_i \in R$ and $m_i \in \mathcal{M}$, therefore $\sum_{i\in I}(\lambda_i - \mu_i m_i)b_i = 0$. Since the elements b_i are linearly independent, it follows that $\lambda_i - \mu_i m_i = 0$, therefore $\lambda_i \in \mathcal{M}$, so $[\lambda_i] = 0$ for all i.

Since in a vector space all the bases have the same cardinality, the result follows.