An example of a decomposition
of a f.g. torsion module over a PID

Let $A = K[x]$ be the polynomial ring over a field K. Let U be the submodule of $A \oplus A$ generated by $(x^2, 0)$ and $(0, x^3)$. Let $M = (A \oplus A)/U$.

- M is an A finitely generated A-module (for instance by $m_1 = [(1, 0)]$ and $m_2 = [(0, 1)]$);
- The order of m_1 is x^2 and the order of m_2 is x^3.
- The minimal annihilator of M is $\nu = x^3$. In particular M is a torsion module.
- We can represent the elements of M in a canonical way as follows: if $[(a, b)] \in M$, we can divide a by x^2 (the first component of the first generator of U) and we get $a = px^2 + r$, where r is a polynomial in x, $\deg(r) \leq 1$. Analogously, we divide b by x^3 and we get $b = qx^3 + s$, where $\deg(s) \leq 2$. Hence $[(a, b)] = [(r, s)]$. Therefore every element of M is represented by an element of the form $[(a_0 + a_1 x, b_0 + b_1 x + b_2 x^2)]$ $(a_0, a_1, b_0, b_1, b_2 \in K)$ in a unique way.
- It is useful to see the A-module M as a K-vector space. From the previous point it follows that a K-basis of M is:

$$[(1, 0)],\ [(x, 0)],\ [(0, 1)],\ [(0, x)],\ [(0, x^2)]$$

- In order to decompose M as a direct sum of cyclic modules we need an element of order ν. We can choose for instance $c_1 = [(0, 1)]$, so we define $C_1 = \langle c_1 \rangle$. Therefore $M = C_1 \oplus L$ for a suitable L that has to be determined.
- The module C_1 is a K-vector space. Its basis is given by

$$[(0, 1)],\ [(0, x)],\ [(0, x^2)]$$

- It follows that L is the submodule of M that, as a K vector space, has a basis given by $[(1, 0)], [(x, 0)]$. As an A-module, L is generated by $c_2 = [(1, 0)]$, therefore is cyclic.
- As a consequence of the previous points, we see that, if we set $C_2 = \langle c_2 \rangle$, we decompose M as follows:

$$M = C_1 \oplus C_2$$

where the order μ_1 of C_1 is x^3, the order μ_2 of C_2 is x^2. Moreover, μ_1 is a multiple of μ_2 so μ_1 and μ_2 are the invariant factors of M.

- The above decomposition of M is dependent of the choice of the element c_1 of order ν. We can find another decomposition of M starting from another element of order ν. For instance take $c_1' = [(1,1)]$ (which, indeed, is of order ν).

- Let $C_1' = \langle c_1' \rangle$ be the cyclic module generated by c_1'. Its K-basis is $[(1,1)], [(x,x)], [(0,x^2)]$. Indeed: every element of C_1' is of the form $[(f,f)]$, $f \in A$, let $f = qx^3 + s$, then

$$[(f,f)] = [(qx^3 + s, s)] = [(s,s)] = [(b_0 + b_1 x, b_0 + b_1 x + b_2 x^2)]$$

- Again, we have: $M = C_1' \oplus L$, for a suitable A-module L. Since the element $c_2' = [(1,0)]$ is not a K-linear combination of the K-basis of C_1', we can assume that $c_2' \in L$. Let $C_2' = \langle c_2' \rangle$ be the cyclic module generated by c_2' (of order x^2), we have that $L = C_2'$ and we get the decomposition $M = C_1' \oplus C_2'$. The invariant factors are again x^3, x^2 and $C_1 \simeq C_1'$, $C_2 \simeq C_2'$.