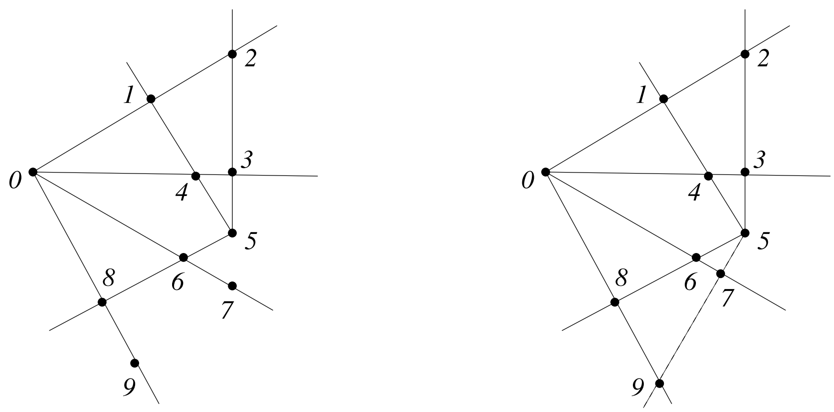

Figure 1: Possible configurations of the 10 points; in the plane (left) and on a cubic curve (right).

Ten points on a cubic curve

Consider the following:
Problem. Let $P_0, \dots P_9$ be 10 distinct points in the projective plane $\mathbb{P}^2_K$ (K a suitable field), which satisfy the following collinearities:

$$L = (0,1,2), (0,3,4), (0,6,7), (0,8,9), (1,4,5), (2,3,5), (5,6,8)$$

((i,j,k) denotes (P_i, P_j, P_k) for short). Is it possible to realize the 10 points on a (smooth) cubic curve?

An example of a possible confiration of the points in the plane is given by Figure 1, left.

Up to a projectivity, we can assume that the points P_0, P_1, P_2, P_3, P_4 have (respectively) the following coordinates:

$$P_0, P_1, P_2, P_3, P_4 = (0,0,1), (1,0,1), (2,0,1), (0,1,1), (0,2,1),$$

the coordinates of the remaining points can be chosen as follows:

$$P_5 = (1, 2t_0, 1+t_0), P_6 = (t_1, t_2, 1), P_7 = (t_1 t_3, t_2 t_3, t_3 + 1)$$

$$P_8 = (t_1 t_4 + 1, t_2 t_4 + 1, t_4 + 3/2), P_9 = (t_1 t_4 t_5 + t_5, t_2 t_4 t_5 + t_5, t_4 t_5 + 3/2 t_5 + 1)$$

where $t_0, \dots, t_5$ are variables. In order to have the above collinearities, the matrices of order 3 whose rows are P_i, P_j, P_k for (i,j,k) in L must have zero determinant. The choice of the coordinates of the points is such that in most of the cases, the determinant is indeed zero, but the variables $t_0, \dots, t_5$ have to be roots of the following polynomials:

$$2t_0 - 1, 2t_0 t_1 + t_0 t_2 - 2t_0 - t_1 - 1/2 t_2 + 1$$

Let $I_0 \subseteq K[t_0, \dots, t_5]$ be the ideal generated by these two polynomials.

Now we want to see if such a configuration can be contained on a smooth cubic curve of the projective plane $\mathbb{P}^2_K$. We impose therefore to the points the condition that they lay on a cubic curve and this is converted into the condition that the variables $t_0, \dots, t_5$ have to satisfy a further condition which is the following:

$$
\begin{aligned}
&4t_1^6t_2^2t_3t_4^4t_5^2 + 2t_1^5t_2^3t_3t_4^4t_5^2 - 12t_1^4t_2^4t_3t_4^4t_5^2 + 2t_1^3t_2^5t_3t_4^4t_5^2 + 4t_1^2t_2^6t_3t_4^4t_5^2 - 4t_1^6t_2^2t_3^2t_4^3t_5\\
&-2t_1^5t_2^3t_3^2t_4^3t_5 + 12t_1^4t_2^4t_3^2t_4^3t_5 - 2t_1^3t_2^5t_3^2t_4^3t_5 - 4t_1^2t_2^6t_3^2t_4^3t_5 - 12t_1^5t_2^2t_3t_4^4t_5^2 + 12t_1^4t_2^3t_3t_4^4t_5^2\\
&+12t_1^3t_2^4t_3t_4^4t_5^2 - 12t_1^2t_2^5t_3t_4^4t_5^2 + 12t_1^5t_2^2t_3^2t_4^3t_5 - 12t_1^4t_2^3t_3^2t_4^3t_5 - 12t_1^3t_2^4t_3^2t_4^3t_5 + 12t_1^2t_2^5t_3^2t_4^3t_5\\
&+4t_1^6t_2t_3t_4^3t_5^2 + 6t_1^5t_2^2t_3t_4^3t_5^2 - 10t_1^4t_2^3t_3t_4^3t_5^2 - 10t_1^3t_2^4t_3t_4^3t_5^2 + 6t_1^2t_2^5t_3t_4^3t_5^2 + 4t_1t_2^6t_3t_4^3t_5^2\\
&+8t_1^4t_2^2t_3t_4^4t_5^2 - 16t_1^3t_2^3t_3t_4^4t_5^2 + 8t_1^2t_2^4t_3t_4^4t_5^2 - 4t_1^6t_2t_3^2t_4^2t_5 - 6t_1^5t_2^2t_3^2t_4^2t_5 + 10t_1^4t_2^3t_3^2t_4^2t_5\\
&+10t_1^3t_2^4t_3^2t_4^2t_5 - 6t_1^2t_2^5t_3^2t_4^2t_5 - 4t_1t_2^6t_3^2t_4^2t_5 - 8t_1^4t_2^2t_3^2t_4^3t_5 + 16t_1^3t_2^3t_3^2t_4^3t_5 - 8t_1^2t_2^4t_3^2t_4^3t_5\\
&-12t_1^5t_2t_3t_4^3t_5^2 + 24t_1^3t_2^3t_3t_4^3t_5^2 - 12t_1t_2^5t_3t_4^3t_5^2 + 12t_1^5t_2t_3^2t_4^2t_5 - 24t_1^3t_2^3t_3^2t_4^2t_5 + 12t_1t_2^5t_3^2t_4^2t_5\\
&+4t_1^5t_2t_3t_4^2t_5^2 + 2t_1^4t_2^2t_3t_4^2t_5^2 - 12t_1^3t_2^3t_3t_4^2t_5^2 + 2t_1^2t_2^4t_3t_4^2t_5^2 + 4t_1t_2^5t_3t_4^2t_5^2 + 8t_1^4t_2t_3t_4^3t_5^2\\
&-8t_1^3t_2^2t_3t_4^3t_5^2 - 8t_1^2t_2^3t_3t_4^3t_5^2 + 8t_1t_2^4t_3t_4^3t_5^2 - 4t_1^5t_2t_3^2t_4t_5 - 2t_1^4t_2^2t_3^2t_4t_5 + 12t_1^3t_2^3t_3^2t_4t_5\\
&-2t_1^2t_2^4t_3^2t_4t_5 - 4t_1t_2^5t_3^2t_4t_5 - 8t_1^4t_2t_3^2t_4^2t_5 + 8t_1^3t_2^2t_3^2t_4^2t_5 + 8t_1^2t_2^3t_3^2t_4^2t_5 - 8t_1t_2^4t_3^2t_4^2t_5\\
&-12t_1^4t_2t_3t_4^2t_5^2 + 12t_1^3t_2^2t_3t_4^2t_5^2 + 12t_1^2t_2^3t_3t_4^2t_5^2 - 12t_1t_2^4t_3t_4^2t_5^2 + 12t_1^4t_2t_3^2t_4t_5 - 12t_1^3t_2^2t_3^2t_4t_5\\
&-12t_1^2t_2^3t_3^2t_4t_5 + 12t_1t_2^4t_3^2t_4t_5 + 8t_1^3t_2t_3t_4^2t_5^2 - 16t_1^2t_2^2t_3t_4^2t_5^2 + 8t_1t_2^3t_3t_4^2t_5^2 - 8t_1^3t_2t_3^2t_4t_5\\
&+16t_1^2t_2^2t_3^2t_4t_5 - 8t_1t_2^3t_3^2t_4t_5
\end{aligned}
$$

Altogether the conditions of ideal I_0 and the above condition, give an ideal I, whose primary decomposition is:

$$I = \bigcap_{i=1}^{11} Q_i$$

where the primary ideals Q_i's are:

$$
\begin{aligned}
&\mathcal{Q}_1 = (2t_0 - 1, t_1t_4 + 1), && \mathcal{Q}_2 = (t_1, 2t_0 - 1),\\
&\mathcal{Q}_3 = (2t_0 - 1, t_1^2 - 2t_1t_2 + t_2^2), && \mathcal{Q}_4 = (t_1 + 2t_2 - 2, 2t_0 - 1),\\
&\mathcal{Q}_5 = (2t_1 + t_2 - 2, 2t_0 - 1), && \mathcal{Q}_6 = (2t_0 - 1, t_4t_5 - t_3),\\
&\mathcal{Q}_7 = (2t_0 - 1, t_2t_4 + 1), && \mathcal{Q}_8 = (t_5, 2t_0 - 1),\\
&\mathcal{Q}_9 = (t_4, 2t_0 - 1), && \mathcal{Q}_{10} = (t_3, 2t_0 - 1),\\
&\mathcal{Q}_{11} = (t_2, 2t_0 - 1).
\end{aligned}
$$

Let $\mathcal{P}_i = \sqrt{\mathcal{Q}_i}$. It turns out that $\mathcal{P}_i = \mathcal{Q}_i$ for all i, except for $i = 3$, in which case $\mathcal{P}_3 = (2t0 - 1, t_1 - t_2)$. Now we analyze the reciprocal position of the points when the variables satisfy the conditions given, respectively, by $\mathcal{P}_1, \dots, \mathcal{P}_{11}$. Many of the conditions given by the ideals $\mathcal{P}_i$ correspond to degenerate cases: for instance, if $i = 1$, we have that the points P_0, P_3, P_4, P_8, P_9 are collinear, hence the cubic curve is reducible; if $i = 8$, $t_5 \in \mathcal{P}_8$ and if this condition is satisfied, P_0 and P_9 coincide, against our assumption.

Only the case $i = 6$ does not give degeneracy conditions but, in this case the points must satisfy a further condition, which is the collinearity of the points P_5, P_7, P_9. In conlcuion we have proved that It turns out that if the points $P_0, \dots, P_9$ are constrained on a cubic curve, then necessarily the points

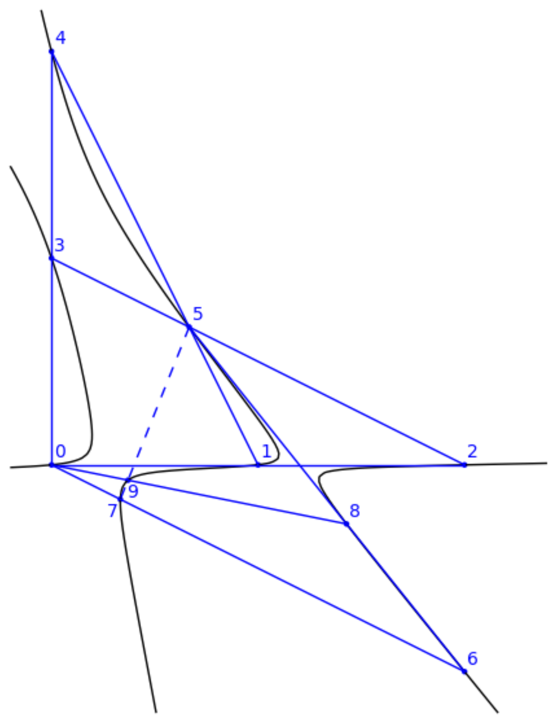

Figure 2: A realization of the configuration of the 10 points and their collinearities (in blue) on a cubic curve (in black).

P_5, P_7, P_9 must be collinear (as in Figure 1, right). See also Figure 2 for a complete example.

The final result is that if we require to 10 points in the plane to satisfy the collinearities given by L and also to lay on a smooth cubic curve, then the new collinearity P_5, P_7, P_9 appears.