Three altitudes of a triangle

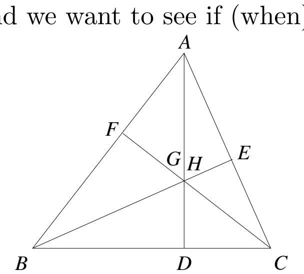
We want to consider here the elementary geometric theorem which says that the three altitudes of a triangle meet in a common point (called the orthocenter) in order to see an example of a primary decomposition of an ideal.

We formulate the problem as follows:

We consider the triangle ABC in an affine plane and we give coordinates to the points: $B = (0, 0)$, $C = (c, 0)$, $A = (a, b)$ (the given coordinates are not restrictive).

Next, we consider the altitudes, so let $D = (a, 0)$, $E = (x_1, x_2)$, $F = (x_3, x_4)$ be three points (D on the line BC, E on the line AC and F on the line AB) such that the lines AD and BC are orthogonal, the lines BE and AC are orthogonal and the lines CF and AB are orthogonal.

We want to see if the three lines AD, BE and CF meet in a common point. In order to see this, we consider the points $G = (a, z_1)$ and $H = (a, z_2)$ which are the intersection points of the line AD with the line BE and CF respectively and we want to see if (when) G and H coincide.



The above conditions give the following equations:

$bc - bx_1 + ax_2 - cx_2 = 0$	points A, C, E collinear
$bx_3 - ax_4 = 0$	points A, B, F collinear
$ac - ax_3 - bx_4 = 0$	lines AB and CF orthogonal
$ax_1 - cx_1 + bx_2 = 0$	lines BE and AC orthogonal
$ax_2 - x_1z_1 = 0$	lines B, E, G collinear
$ax_4 - cx_4 + cz_2 - x_3z_2 = 0$	points A, D, H collinear

Consider now the ideal J generated by the above polynomials. Its primary decomposition (in the polynomial ring $\mathbb{Q}[a, b, c, x_1, \ldots, x_4, z_1, z_2]$) is the following:

$$J = \bigcap_{i=1}^{14} \mathcal{Q}_i$$

where the primary ideals are:

$$
\begin{aligned}
\mathcal{Q}_1 &= \left(x_4, x_3, c, b-z_1, ax_2 - x_1z_1, x_1^2+x_2^2, ax_1+x_2z_1, a^2+z_1^2\right)\\
\mathcal{Q}_2 &= \left(z_1-z_2, c, b-z_2, ax_4-x_3z_2, x_3^2+x_4^2, x_2x_3-x_1x_4, x_1x_3+x_2x_4,\right.\\
&\qquad \left. ax_3+x_4z_2, ax_2-x_1z_2, x_1^2+x_2^2, ax_1+x_2z_2, a^2+z_2^2\right)\\
\mathcal{Q}_3 &= (z_1, x_3, c, b, a)\\
\mathcal{Q}_4 &= (z_2, z_1, c, b, a)\\
\mathcal{Q}_5 &= (x_4, c-x_3, b, a-x_3, x_2x_3-x_1z_1)\\
\mathcal{Q}_6 &= (x_4, x_3, x_2, x_1, c)\\
\mathcal{Q}_7 &= \left(x_2, x_1, c, b-z_2, ax_4-x_3z_2, x_3^2+x_4^2, ax_3+x_4z_2, a^2+z_2^2\right)\\
\mathcal{Q}_8 &= (x_4, x_2, x_1, c-x_3, b)\\
\mathcal{Q}_9 &= (x_2, x_1, b, a, cx_4-cz_2+x_3z_2)\\
\mathcal{Q}_{10} &= (x_3, x_1, c, b, a)\\
\mathcal{Q}_{11} &= (z_2, x_1, c, b, a)\\
\mathcal{Q}_{12} &= (c-x_3, b-x_4, x_4^2, ax_4-x_3x_4, x_2x_3+x_1x_4-x_3x_4-x_1z_1,\\
&\qquad x_1x_3-x_3^2-x_2x_4+x_2z_1+2x_4z_1, ax_3-x_3^2+x_4z_1, ax_2-x_1z_1,\\
&\qquad x_1^2+x_2^2-x_3^2+x_2z_1+2x_4z_1, ax_1-x_3^2+x_2z_1+2x_4z_1, a^2-x_3^2+2x_4z_1)\\
\mathcal{Q}_{13} &= (x_1, b-x_2, ax_4-cx_4+cz_2-x_3z_2, x_3^2+x_4^2-x_4z_2,\\
&\qquad x_2x_3-cx_4+cz_2-x_3z_2, cx_3+x_2x_4-x_4z_2, ax_3+x_2x_4-x_2z_2,\\
&\qquad x_2^2, ax_2, ac-x_2z_2, a^2, cx_2x_4-cx_2z_2+cx_4z_2-cz_2^2+x_3z_2^2,\\
&\qquad c^2x_4-c^2z_2-2x_2x_4z_2+x_4z_2^2)\\
\mathcal{Q}_{14} &= (z_1-z_2, ax_4-cx_4+cz_2-x_3z_2, x_2x_3-cx_4+x_1x_4+cz_2-x_1z_2-x_3z_2,\\
&\qquad x_1x_3-x_3^2-x_2x_4-x_4^2+x_2z_2+x_4z_2, cx_3-x_3^2-x_4^2,\\
&\qquad bx_3-cx_4+cz_2-x_3z_2, ax_3-x_3^2-x_4^2+x_4z_2, ax_2-x_1z_2,\\
&\qquad x_1^2-bx_2+x_2^2-x_3^2-bx_4-x_4^2+x_2z_2+x_4z_2,\\
&\qquad cx_1-bx_2-x_3^2-bx_4-x_4^2+x_2z_2+x_4z_2, ax_1-x_3^2-bx_4-x_4^2+x_2z_2+x_4z_2,\\
&\qquad bc-bx_1-cx_2+x_1z_2, ac-x_3^2-bx_4-x_4^2+x_4z_2,\\
&\qquad a^2-x_3^2-bx_4-x_4^2+bz_2+x_4z_2, cx_2x_4-cx_2z_2+cx_4z_2-2x_1x_4z_2-cz_2^2+x_1z_2^2+x_3z_2^2,\\
&\qquad bx_2x_4-bx_2z_2-bx_4z_2+x_2x_4z_2, c^2x_4-x_3^2x_4-bx_4^2-x_4^3-c^2z_2+x_3^2z_2+2x_4^2z_2)
\end{aligned}
$$

The associated primes are $\mathcal{P}_i$ $(i = 1, \ldots, 14)$ where $P_i = Q_i$ for $i \notin \{12, 13\}$, while Q_{12} and Q_{13} are primary but not prime and

$$
\begin{aligned}
\mathcal{P}_{12} &= (x_4, c-x_3, b, a-x_3, x_2x_3-x_1z_1, x_1x_3-x_3^2+x_2z_1, x_1^2+x_2^2-x_3^2+x_2z_1)\\
\mathcal{P}_{13} &= (x_2, x_1, b, a, cx_4-cz_2+x_3z_2, x_3^2+x_4^2-x_4z_2, cx_3-x_4z_2)
\end{aligned}
$$

The conclusion is that, if we do not consider the degenerate cases, then G and H coincide (i.e. $z_1 = z_2$).

Note that if we perform the computations in $\mathbb{C}$ (or in $\mathbb{Q}\left[\sqrt{-1}\right]$), then the ideals Q_1, Q_2 and Q_7 split in smaller ideals.