On the higher dimensional Poincaré – Birkhoff theorem for Hamiltonian flows. 1. The indefinite twist

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Abstract. We propose an extension to higher dimensions of the Poincaré – Birkhoff Theorem which applies to Poincaré time-maps of Hamiltonian systems. Applications to pendulum-type systems and weakly-coupled superlinear systems are given.

1 Introduction

The classical Poincaré – Birkhoff fixed point theorem, also called Poincaré's last geometric theorem, affirms the existence of at least two fixed points for area-preserving homeomorphisms of the planar annulus keeping both boundary circles invariant and twisting them in opposite directions. Going to the universal cover it can be stated as follows (see, e.g. [17]):

Theorem (Poincaré – Birkhoff). *Let $\mathcal{P} : \mathbb{R} \times [a, b] \to \mathbb{R} \times [a, b]$ be an area-preserving homeomorphism of the form*

$$\mathcal{P}(x, y) = (x + \vartheta(x, y), \rho(x, y)) ,$$

where the functions $\vartheta(x, y)$ and $\rho(x, y)$ are 2π-periodic in their first variable x, with $\rho(x, a) = a$ and $\rho(x, b) = b$, for every $x \in \mathbb{R}$. Assume the boundary twist condition:

$$\vartheta(x, a)\, \vartheta(x, b) < 0 \,, \text{ for every } x \in \mathbb{R} \,. \tag{1}$$

Then, $\mathcal{P}$ has at least two fixed points in $[0, 2\pi[\times]a, b[$.

This theorem was conjectured by Poincaré shortly before his death in 1912. The original manuscript [65] contained not only the proof of the theorem in some special cases but also two examples of applications in Dynamics, namely the search of closed geodesic lines on a convex surface, and the study of periodic solutions in the restricted three body problem. The full proof of the theorem is due to Birkhoff [9, 11], who, as Poincaré, had been led to it by its applications to the search of periodic solutions of conservative dynamical systems [10, 13].

The use of this version of the theorem in problems of Dynamics encounters, however, some difficulties due to the requirement that the strip $\mathbb{R}\times[a,b]$ has to be invariant under the Poincaré time map. In other cases it may be interesting to replace the strip $\mathbb{R}\times[a,b]$ by more general regions of the plane, such as the one contained between two ordered curves, or even to generalize the area-preserving condition. Thus, many different generalizations of the Poincaré – Birkhoff theorem have been proposed, see [18, 24, 25, 37, 38, 39, 40, 41, 46, 48, 52, 62, 67, 72]. (See also [34, 51] for recent reviews on the subject.) These extended versions have been used to prove the existence and multiplicity of periodic solutions of non-autonomous planar Hamiltonian systems in a variety of situations.

On the other hand, the possibility of higher dimensional extensions of the Poincaré – Birkhoff theorem was considered an *outstanding question* already by Birkhoff [11, page 299]. Indeed, as Arnold would later write, *attempts to generalize it to higher dimensions are important for the study of periodic solutions of problems with many degrees of freedom* ([3, page 416]). Birkhoff himself was the first author to propose a $2N$-dimensional version of the theorem [12] in which the main assumption was the existence of a manifold, diffeomorphic to the N-torus, where the exact symplectic map preserves the first N coordinates. In this case one can use an idea which goes back to Poincaré [64, Chap. 28] and reduce the problem to that of the critical points of a function on the manifold. Since then, related arguments have been successfully used to prove higher-dimensional versions of the Poincaré – Birkhoff theorem for maps which are close to the identity and also for monotone twist maps [3, 5, 59, 61, 70].

Using a different approach which combined Lyusternik – Schnirelman variational methods with the Conley index theory for flows, Conley and Zehnder [21, Theorem 3] proved, thirty years ago, another version of the Poincaré – Birkhoff Theorem in higher dimensions. Their result concerns the multiplicity of periodic solutions for time-dependent Hamiltonian vector fields provided that the C^2-smooth Hamiltonian function $H = H(t,x,y)$ is periodic in t and the variables x_i, and quadratic on a neighborhood of infinity. Precisely, they assumed

$$|y| \geq R \quad \Rightarrow \quad H(t,x,y) = \tfrac{1}{2}\langle \mathbb{B}y, y\rangle + \langle a, y\rangle\,, \tag{2}$$

for some $R>0$, some vector $a\in\mathbb{R}^N$ and some regular symmetric matrix $\mathbb{B}$. Then, they obtained the existence of at least $N+1$ periodic solutions. Remarkably, their result does not need the Poincaré time-map to be close to the identity, nor to have a monotone twist. The development of infinite-dimensional Lyusternik – Schnirelman methods would allow Szulkin [68, Theorem 4.2] to generalize the Conley and Zehnder theorem by replacing the term $\langle a, y\rangle$ by nonlinearities $G(t,x,y)$ with bounded first-order derivatives. Further results along these lines can be found in [7, 8, 20, 47, 50].

Despite all this ample literature it seems that, for the time being, *there is no genuine generalization of the Poincaré – Birkhoff theorem to higher dimen-*

sions [61, page 140]. The aim of this paper is to take a further step in this direction and propose a new higher-dimensional version of the Poincaré – Birkhoff theorem which will apply to Poincaré time-maps of Hamiltonian systems.

In order to describe our results, let $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$ denote the standard $2N \times 2N$ symplectic matrix, and consider the (time dependent) Hamiltonian system

$$(HS) \qquad\qquad \dot{z} = J\nabla H(t,z)\,,$$

or, what is the same, letting $z = (x,y)$,

$$\begin{cases} \dot{x} = \nabla_y H(t,x,y)\,, \\ \dot{y} = -\nabla_x H(t,x,y)\,. \end{cases}$$

In Theorems 1.1, 1.2, and 1.3 below *we shall assume that the continuous function $H : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $H = H(t,x,y)$ is T-periodic in its first variable t, 2π-periodic in the first N state variables $x_1, \ldots, x_N$, and continuously differentiable with respect to (x,y).* Let the open, bounded and convex subset $D \subseteq \mathbb{R}^N$ be given; if for every initial position $z_0 \in \mathbb{R}^N \times \overline{D}$ there is a unique solution $z(\cdot\,; z_0)$ of (HS) satisfying $z(0; z_0) = z_0$ and, moreover, this solution can be continued to the time interval $[0,T]$, then it makes sense to consider the so-called *Poincaré time map* (on $\mathbb{R}^N \times \overline{D}$); this is the function $\mathcal{P} : \mathbb{R}^N \times \overline{D} \to \mathbb{R}^{2N}$ defined by

$$\mathcal{P}(z_0) = z(T; z_0)\,.$$

It is well known that the function $\mathcal{P}$ preserves the orientation and the Lebesgue measure on $\mathbb{R}^{2N}$, and its fixed points give rise to T-periodic solutions of (HS). Once a T-periodic solution $z(t) = (x(t), y(t))$ has been found, many others appear by just adding an integer multiple of 2π to some of the components $x_i(t)$; for this reason, we will call *geometrically distinct* two periodic solutions of (HS) (or two fixed points of $\mathcal{P}$) which can not be obtained from each other in this way.

In case the set D has a C^1-smooth boundary one may consider the (continuously defined) unit outward normal vector field on ∂D; we shall denote it by $\nu : \partial D \to \mathbb{R}^N$. Let us also recall that a square matrix $\mathbb{B}$ is called *symmetric* if $\mathbb{B}$ equals its transpose $\mathbb{B}^*$, *regular* if $\det \mathbb{B} \neq 0$, and *involutory* if $\mathbb{B}^2 = I_N$. One of the main results of this paper is the following.

Theorem 1.1. *Let $D \subseteq \mathbb{R}^N$ be an open, bounded and convex set. Writing*

$$\mathcal{P}(x,y) = (x + \vartheta(x,y), \rho(x,y))\,, \qquad (x,y) \in \mathbb{R}^N \times \overline{D}\,,$$

let one of the following assumptions hold:

(a) The set D has a C^1-smooth boundary and there is a regular symmetric $N \times N$ matrix $\mathbb{B}$ with

$$\langle \vartheta(x,y), \mathbb{B}\nu(y) \rangle > 0 \,, \quad \text{for every } (x,y) \in \mathbb{R}^N \times \partial D \,.$$

(b) There exists an involutory $N \times N$ matrix $\mathbb{B}$ and some point $d_0 \in D$ with

$$\langle \vartheta(x,y), \mathbb{B}(y - d_0) \rangle > 0 \,, \quad \text{for every } (x,y) \in \mathbb{R}^N \times \partial D \,.$$

Then, $\mathcal{P}$ has at least $N+1$ geometrically distinct fixed points in $\mathbb{R}^N \times D$.

Some comments are in order here:

(i) Condition *(a)* can be thought of as a generalization of the Conley–Zehnder assumption (2). Indeed, their assumption guarantees that the Poincaré time map is well defined, and it can be easily checked that it implies *(a)* for D being a sufficiently large ball centered at the origin.

(ii) In the case $\mathbb{B} = I_N$ condition *(b)* was introduced by Moser and Zehnder in [61, Theorem 2.21]; for this reason, our theorem can also be seen as a generalization in the Hamiltonian case of this result. Notice also that we do not need the monotone twist condition required there.

(iii) One of our assumptions was that the domain D must be convex, and a natural question here is whether this is necessary. We do not know the answer to this question in this generality; however, in the case of $\mathbb{B} = I_N$ this problem will be treated in [36].

(iv) Both conditions *(a)-(b)* require in particular that the matrix $\mathbb{B}$ is regular. However it may fail to be either positive or negative definite; this fact motivates the words 'indefinite twist' in the title of the paper. We do not know whether *(a)-(b)* can be weakened by asking only that $\mathbb{B}$ is regular and dropping the symmetry or involutory assumptions.

As an example, we consider the rectangle

$$\mathcal{R} =]a_1, b_1[\times \cdots \times]a_N, b_N[\,,$$

and we assume that we are dealing with some Hamiltonian system (HS) whose Poincaré time map $\mathcal{P} : \mathbb{R}^N \times \overline{\mathcal{R}} \to \mathbb{R}^N \times \mathbb{R}^N$ is well defined. After smoothing the corners of $\partial\mathcal{R}$, Theorem 1.1 (a) will lead us to the following result:

Theorem 1.2. *Writing*

$$\mathcal{P}(x,y) = (x + \vartheta(x,y), \rho(x,y)) \,,$$

and $\vartheta(x,y) = (\vartheta_1(x,y), \ldots, \vartheta_N(x,y))$, assume that

$$\vartheta_i(x, a_i)\vartheta_i(x, b_i) < 0 \,, \quad \text{for every } x \in \mathbb{R}^N \text{ and } i = 1, \ldots, N.$$

Then, $\mathcal{P}$ has at least $N+1$ geometrically distinct fixed points in $\mathbb{R}^N \times \mathcal{R}$.

We emphasize that the set $\mathbb{R}^N \times \overline{\mathcal{R}}$ is not required to be invariant by $\mathcal{P}$. After a symplectic change of variables we shall show an extension of this result in which a_i and b_i can vary as functions of x_i. This will lead us to Theorem 2.2 and then to Theorem 8.2 below, which is a generalization of some previous versions of the Poincaré – Birkhoff Theorem for planar annuli with star-shaped boundaries [25, 52, 67].

A special situation occurs when the fixed points of $\mathcal{P}$ are *nondegenerate*. We recall that the fixed point $z_0 = \mathcal{P}(z_0)$ is called nondegenerate if 1 is not an eigenvalue of $\mathcal{P}'(z_0)$, or, with other words, if 1 is not a characteristic multiplier of the solution $z(\cdot\,; z_0)$ of (HS). Of course, for this definition to make sense one has to assume that the Hamiltonian function H is twice continuously differentiable with respect to the state variable $z = (x, y)$. Our next result says that, if for some reason the fixed points of $\mathcal{P}$ are known to be nondegenerate, then they appear in a greater number:

Theorem 1.3. *Under the assumptions of Theorems 1.1 or 1.2, if H is twice continuously differentiable with respect to z and all fixed points of $\mathcal{P}$ are nondegenerate, then there are at least 2^N of them.*

Our theorems above apply to maps $\mathcal{P} : \mathbb{R}^N \times \overline{D} \to \mathbb{R}^{2N}$ which are Poincaré time maps of Hamiltonian systems, assumed to be periodic in the x_i variables (or, equivalently, defined on some subset of $T^*(\mathbb{T}^N) \cong \mathbb{T}^N \times \mathbb{R}^N$, the cotangent space of the torus $\mathbb{T}^N = (\mathbb{R}/2\pi\mathbb{Z})^N$). A natural question here concerns to having a way to know when a given map belongs to this class. It is well known that, assuming some smoothness for the Hamiltonian, $\mathcal{P}$ must be a diffeomorphism into its image, differing from the identity on a periodic map in the x_i variables. Besides, it has to be *exact symplectic*, i.e.

$$\mathcal{P}^*\lambda - \lambda = dF\,,$$

for some smooth function $F : \mathbb{T}^N \times \overline{D} \to \mathbb{R}$, where $\lambda = \sum_{i=1}^N y_i dx_i$ is the canonical 1-form. On the other hand, a well known result (cf. [56, Proposition 9.19] or [38, Theorem 58.9]) states that $\mathcal{P}$ is the Poincaré time map of a Hamiltonian system of the type we are dealing with if and only if it can be joined to the identity via a smooth isotopy of exact symplectic maps. However, this criterion could not be easy to check in practical situations. More explicit conditions are available when $\mathcal{P}$ is an exact symplectic *monotone twist* map. Indeed, as Moser has shown [60], in the two dimensional case all such maps are indeed Poincaré time maps of a Hamiltonian system. The higher dimensional case has been treated by Golé [38, Theorem 41.6], assuming that the map is globally defined on $T^*(\mathbb{T}^N)$ and the twist is, in some sense, controlled at infinity.

The paper is organized as follows. In Section 2 we will state our main results in their general form, which apply to Hamiltonian systems which do not necessarily have the property of uniqueness of solutions to initial value problems (so that the Poincaré time map could be multivalued). The results presented in this Introduction will then just be the particularization for systems with uniqueness. Section 3 is devoted to present and prove some facts on the extrinsic geometry of convex bodies which will be needed in the sequel. In Section 4 we present the notion of feasible vector fields, which will provide a unified way to treat both cases *(a)-(b)* appearing in Theorem 1.1. The bulk of the proof is carried out in Sections 5 and 6, using the above-cited result by Szulkin. Then, in Section 7 we shall use an approximation argument to extend Theorem 1.1*(a)* to nonsmooth domains; this will lead us to Theorem 1.2 and some generalizations. Finally, in Section 8 we illustrate some examples of applications, focusing our attention on two types of Hamiltonian systems, *pendulum-like systems* and *weakly-coupled superlinear systems*, for which we extend some classical results.

2 When the Poincaré time map may be multivalued

As described in the Introduction, the Poincaré – Birkhoff theorem has been extensively used to prove existence and multiplicity of periodic solutions for Hamiltonian systems in the plane. Since the theorem applies to (univalued) maps, an important requirement is that there must be uniqueness for initial value problems, something which can be easily ensured by assuming some smoothness (i.e., Lipschitz continuity) on the Hamiltonian vector field. When such a condition fails to hold, in many cases one can still show existence by passing to the limit in a regularization argument; however, multiplicity is usually lost in this procedure because it may not be easy to show that the different solutions do not collapse in the limit. Concerned by this fact, we have developed a generalized version of Theorem 1.1, which does not require uniqueness for initial value problems.

It will be convenient to introduce some terminology. As before we denote by $z = (x, y)$ the vectors in $\mathbb{R}^{2N}$, with $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$. Even though we are interested in time-periodic Hamiltonians and periodic solutions, it will be convenient to have everything defined on a time period interval $[0, T]$. Thus, we shall say that the function $H : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$, $H = H(t, z) = H(t, x, y)$, is an *admissible Hamiltonian* if it is continuous, 2π-periodic in x_i for each $i = 1, \ldots, N$, and it has a continuously defined gradient with respect to z, denoted by ∇H. A solution $z : [0, T] \to \mathbb{R}^{2N}$ of (HS) is said to be T-*periodic* if it satisfies $z(0) = z(T)$. Of course, in case H is the restriction of some function on $\mathbb{R} \times \mathbb{R}^{2N}$ which is $T-$periodic in time, then any T-periodic solution in this sense can be extended to a T-periodic solution defined on $\mathbb{R}$. From now on, unless the contrary is explicitly required, our Hamiltonians will always be admissible.

As mentioned in the Introduction, if $z(t) = (x(t), y(t))$ is a solution of (HS), then, for every choice of integers $m_1, \ldots, m_N$, the function $(x_1(t) + 2\pi m_1, \ldots,$

$x_N(t) + 2\pi m_N \,,\, y(t))$ is still a solution. Thus, two periodic solutions of (HS) are called *geometrically distinct* if they are not related to each other in this way. On the other hand, we notice that initial value problems associated to (HS) may not be uniquely solvable if the Hamiltonian H is merely assumed to be admissible.

Throughout this paper, D will always denote an open, bounded and convex subset of $\mathbb{R}^N$ (such sets will henceforth be called *convex bodies*). Let $\mathcal{F} : \partial D \to \mathbb{R}^N$ be a continuous vector field; we shall say that *the flow of* (HS) *is guided by* $\mathcal{F}$ *on* ∂D if every solution $z(t) = (x(t), y(t))$ of (HS) with $y(0) \in \partial D$ is defined for every $t \in [0, T]$ and satisfies

$$\langle x(T) - x(0), \mathcal{F}(y(0)) \rangle > 0 \,.$$

The main result of this paper is given next. It is a generalized version of Theorems 1.1 and 1.3 which applies to Hamiltonian systems without uniqueness.

Theorem 2.1. *Let the Hamiltonian function* $H : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$ *be admissible, and assume the existence of a convex body* $D \subseteq \mathbb{R}^N$ *such that one of the following conditions hold:*

(a) *The set* D *has a* C^1*-smooth boundary and there exists a symmetric regular* $N \times N$ *matrix* $\mathbb{B}$ *such that the flow of* (HS) *is guided by the vector field* $\mathcal{F}_1(y) = \mathbb{B}\nu(y)$ *on* ∂D.

(b) *There exists an involutory* $N \times N$ *matrix* $\mathbb{B}$ *and a point* $d_0 \in D$ *such that the flow of* (HS) *is guided by the vector field* $\mathcal{F}_2(y) = \mathbb{B}(y - d_0)$ *on* ∂D.

Then, the Hamiltonian system (HS) *has at least* $N{+}1$ *geometrically distinct* T*-periodic solutions* $z^{(0)}, \ldots, z^{(N)}$ *such that, writing* $z^{(k)}(t) = (x^{(k)}(t), y^{(k)}(t))$,

$$y^{(k)}(0) \in D \,, \quad \text{for } k = 0, \ldots, N \,.$$

Moreover, if the Hamiltonian function H *is twice continuously differentiable with respect to* z *and the* T*-periodic solutions with initial condition on* $\mathbb{R}^N \times D$ *are nondegenerate, then there are at least* 2^N *of them.*

As before, in condition *(a)* we are denoting by ν to the (continuously defined) unit outer normal vector field on ∂D. It is clear that, in case there is uniqueness for initial value problems (so that the Poincaré time map $\mathcal{P}$ is well defined on $\mathbb{R}^N \times \overline{D}$) then assumptions *(a)*-*(b)* of Theorem 1.1 become the corresponding assumptions of Theorem 2.1.

We now consider an apparently different situation which, however, will be reduced to the previous one after a suitable change of variables. By a *tube* we mean a set of the form

$$\mathcal{T} = \{(x, y) \in \mathbb{R}^{2N} : a_i(x_i) < y_i < b_i(x_i),\, i = 1, \ldots, N\} \,,$$

where $a_i, b_i : \mathbb{R} \to \mathbb{R}$ are given 2π-periodic continuous functions, for $i = 1, \ldots, N$, with $a_i(s) < b_i(s)$ for every $s \in \mathbb{R}$.

Choose now some $j \in \{1, \ldots, N\}$. The j-th (closed) top face of $\mathcal{T}$ is the set

$$\mathcal{T}_j^+ = \{(x, y) \in \mathbb{R}^{2N} : \, y_j = b_j(x_j) \text{ and } a_i(x_i) \le y_i \le b_i(x_i), \text{ if } i \neq j\} ,$$

while the j-th (closed) bottom face of $\mathcal{T}$ is given by

$$\mathcal{T}_j^- = \{(x, y) \in \mathbb{R}^{2N} : \, y_j = a_j(x_j) \text{ and } a_i(x_i) \le y_i \le b_i(x_i), \text{ if } i \neq j\} .$$

Notice that $\partial\mathcal{T}$ is the union of all the top and bottom faces of $\mathcal{T}$. We will say that the tube $\mathcal{T}$ is *twisted* by the flow of H provided that every solution $z(t) = (x(t), y(t))$ of (HS) with $z(0) \in \partial\mathcal{T}$ is defined of $[0, T]$ and, for any $j = 1, \ldots, N$, either

$$x_j(T) - x_j(0) \begin{cases} > 0 \,, & \text{if } z(0) \in \mathcal{T}_j^+ , \\ < 0 \,, & \text{if } z(0) \in \mathcal{T}_j^- , \end{cases}$$

or

$$x_j(T) - x_j(0) \begin{cases} < 0 \,, & \text{if } z(0) \in \mathcal{T}_j^+ , \\ > 0 \,, & \text{if } z(0) \in \mathcal{T}_j^- . \end{cases}$$

Theorem 2.2. *Let the Hamiltonian function $H : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$ be admissible, and let the tube $\mathcal{T}$ be twisted by the flow of H. Then, the Hamiltonian system (HS) has at least $N + 1$ geometrically distinct T-periodic solutions $z^{(0)}, \ldots, z^{(N)}$, with $z^{(k)}(0) \in \mathcal{T}$, for every $k = 0, \ldots, N$.*

Moreover, if the Hamiltonian function H is twice continuously differentiable with respect to z and the T-periodic solutions with initial condition on $\mathcal{T}$ are nondegenerate, then there are at least 2^N of them.

3 Differential geometry of convex bodies

In this section, we establish some geometrical results which will be needed in the sequel. These results are probably well-known to specialists in the extrinsic geometry of convex bodies; however, we have tried to find a presentation which may be attractive to readers from other areas of mathematics.

Let $D \subseteq \mathbb{R}^N$ be a convex body, i.e., D is open, bounded, and convex. For simplicity we shall say that D is smooth if the boundary ∂D is C^∞-smooth. In this case, the unit normal outer vector field $\nu : \partial D \to \mathbb{R}^N$ is a C^∞-smooth map, and its differential at a given point $p \in \partial D$, denoted by $\nu'(p) : T_p\partial D \to T_p\partial D$, is a selfadjoint endomorphism (see e.g. [27]), which can be identified with the quadratic form

$$\mathit{II}_p[u] := \langle u, \nu'(p)u \rangle, \qquad u \in T_p\partial D \,.$$

Here, $T_p\partial D$ denotes the tangent space to ∂D at the point p.

In order to get some geometrical insight on $I\!I_p$ (usually called the *second fundamental form* of ∂D at p), we consider the height function relative to the tangent hyperplane at this point:

$$h_p(q) := \langle q - p, -\nu(p) \rangle\,, \qquad q \in \partial D\,.$$

Our function attains a critical point at $q = p$, and its Hessian quadratic form there [23, §(16.5.11)] is $\mathrm{Hess}_p h_p = I\!I_p$. This can be easily checked by choosing some vector $u \in T_p \partial D$ and some differentiable curve $\alpha :]-\epsilon, \epsilon[\to \partial D$ with $\alpha(0) = p$ and $\alpha'(0) = u$; then,

$$\mathrm{Hess}_p\, h_p[u, u] = \frac{d^2}{dt^2} h(\alpha(t))\Big|_{t=0} = \langle \alpha''(0), -\nu(p) \rangle = I\!I_p[u]\,, \tag{3}$$

the last equality being a consequence of the fact that $\langle \alpha'(t), \nu(\alpha(t)) \rangle = 0$, for every t. Since the convex set D is supported by its tangent hyperplane at p, we see that h_p attains its minimum at $q = p$, and hence the quadratic form $I\!I_p$ must be positive semidefinite. It is true for every point $p \in \partial D$ and, indeed, this property may be used to characterize those compact hypersurfaces which bound a convex set [28, p. 175]. It motivates the following definition.

The convex body D is said to be *strongly convex* provided that it is smooth and $I\!I_p$ is positive definite for every $p \in \partial D$. With other words, if the eigenvalues of $\nu'(p)$ (usually called the *principal curvatures of* ∂D *at* p) are all of them positive for every $p \in \partial D$. Equivalently, if $\det \nu'(p) \neq 0$ for every $p \in \partial D$.

As an example, assume that the convex body D can be written as a sublevel set $D = \varphi^{-1}(\,]-\infty, c[\,)$, where the C^∞-smooth function $\varphi : \mathbb{R}^N \to \mathbb{R}$ is strongly convex, in the sense that its Hessian matrix $\mathrm{Hess}_p\, \varphi$ is positive definite at every point p. Then, D itself is strongly convex. To see this, we first observe that c must be a regular value of φ, implying that D is smooth; moreover, for every $p \in \partial D$, one has

$$\nu(p) = \frac{\nabla \varphi(p)}{|\nabla \varphi(p)|}\,, \quad \nu'(p) = \left(\frac{1}{|\nabla \varphi(p)|} I_N - \frac{1}{|\nabla \varphi(p)|^3} \nabla \varphi(p) \cdot \nabla \varphi(p)^* \right) \cdot \mathrm{Hess}_p\, \varphi\,.$$

Consequently, the quadratic form $I\!I_p$, being given by

$$I\!I_p[u] = \frac{1}{|\nabla \varphi(p)|} \langle u, (\mathrm{Hess}_p \varphi) u \rangle\,,$$

is positive definite on $T_p \partial D$.

On the other hand, assume instead that there exists a nontrivial segment $[q_1, q_2]$ contained in ∂D. Then D is *not* strongly convex; indeed, $q_2 - q_1 \in \ker \nu'(p)$ for every $p \in [q_1, q_2]$. This is a consequence of (3).

In order to introduce a second notion which will play an important role in this paper, let $D \subseteq \mathbb{R}^N$ be a convex body, not necessarily smooth. The *outer normal cone* at some point $q \in \partial D$ is defined by

$$\mathcal{N}(q) := \Big\{ w \in \mathbb{R}^N : \langle w, q - p \rangle \geq 0 \text{ for every } p \in D \Big\}\,,$$

see e.g. [4]. It can be easily checked that $\mathcal{N}(q)$ is indeed a nontrivial convex cone for every $q \in \partial D$.

For instance, assume that $D = [0,1]^N$ is the unit cube. Then, for every $q = (q_1, q_2, \dots, q_N) \in \partial D$,

$$\mathcal{N}(q) = I(q_1) \times I(q_2) \times \dots I(q_N)\,,$$

where

$$I(q_i) = \begin{cases}]-\infty, 0]\,, & \text{if } q_i = 0\,, \\ [0, +\infty[\,, & \text{if } q_i = 1\,, \\ \{0\}\,, & \text{if } q_i \in]0,1[\,. \end{cases}$$

On the other hand, if the convex body D has a C^1-smooth boundary ∂D and, as usually, we denote by $\nu : \partial D \to \mathbb{R}^N$ the associated unit outward normal vector field, then one easily checks that

$$\mathcal{N}(q) = \{\lambda \nu(q) : \lambda \geq 0\}\,, \qquad q \in \partial D\,.$$

We now show how convex bodies in $\mathbb{R}^N$ can always be approximated from their interior by strongly convex ones (see also [43, 57]).

Lemma 3.1. *Let D be a convex body and let $K \subseteq D$ be a compact set.*

(†) *There exists a strongly convex body D^* such that*

$$K \subseteq D^* \subseteq \overline{D^*} \subseteq D\,.$$

(‡) *Furthermore, given $\varepsilon > 0$ the set D^* can be chosen with the following additional property: for every $p \in \partial D^*$ there exists some $q \in \partial D$ such that*

$$|p - q| < \varepsilon\,, \qquad \mathrm{dist}(\nu^*(p), \mathcal{N}(q)) < \varepsilon\,,$$

where we denote by $\nu^*(p)$ the unit outward normal vector to ∂D^* at $p \in \partial D^*$ and by $\mathcal{N}(q)$ the outer normal cone to ∂D at $q \in \partial D$.

Proof. (†) There is no loss of generality in assuming that $0 \in K$. For any $y \in \mathbb{R}^N \setminus \{0\}$ there exists a unique $\lambda(y) > 0$ such that $\lambda(y)y \in \partial D$. We consider the so-called *gauge* or *Minkowski* function associated to D,

$$f(y) = \begin{cases} 0\,, & \text{if } y = 0\,, \\ \dfrac{1}{\lambda(y)}\,, & \text{if } y \neq 0\,, \end{cases}$$

(see, e.g. [49, Theorem 2, p. 21]). We observe that f is convex, and, moreover

$$f(y) \begin{cases} < 1\,, & \text{if } y \in D\,, \\ = 1\,, & \text{if } y \in \partial D\,, \\ > 1\,, & \text{if } y \in \mathbb{R}^N \setminus \overline{D}\,. \end{cases}$$

We choose some $\delta \in]0, 1 - \max_K f[$. Using a standard convolution argument, we may find a convex C^∞-smooth function $\tilde{f} : \mathbb{R}^N \to \mathbb{R}$ such that

$$|\tilde{f}(y) - f(y)| < \frac{\delta}{2}, \quad \text{for every } y \in \overline{D}.$$

(Indeed, the convolution of a convex function with a nonnegative one is still convex). After replacing $\tilde{f}(y)$ by $\tilde{f}(y) + \rho|y|^2$, for some small positive constant ρ, we may further assume that the Hessian matrix of $\tilde{f}$ is positive definite at every point. Observe now that

$$\tilde{f}(y) < 1 - \frac{\delta}{2}, \text{ for every } y \in K, \tag{4}$$

and

$$\tilde{f}(y) > 1 - \frac{\delta}{2}, \text{ for every } y \in \partial D,$$

so that, since $\tilde{f}$ is convex, we have

$$\tilde{f}(y) > 1 - \frac{\delta}{2}, \text{ for every } y \in \mathbb{R}^N \setminus D. \tag{5}$$

We define $D^* := \tilde{f}^{-1}(]-\infty, 1-\delta/2[)$. Then, D^* is a strongly convex body (see the comments following the definition of strongly convex bodies) and, by (4), it contains K. By (5), the closure of D^* is contained in D, thus concluding the proof.

(‡) In order to show the second part of the lemma we start by letting $K_0 := K$; using (†), we may find some strongly convex body D_1^* with $K_0 \subseteq D_1^* \subseteq \overline{D_1^*} \subseteq D$. We let now $K_1 := \overline{D_1^*} \cup \{q \in D : \operatorname{dist}(q, \partial D) \geq 1\}$; as before we may find a strongly convex body D_2^* with $K_1 \subseteq D_2^* \subseteq \overline{D_2^*} \subseteq D$. We iterate the argument and define, for arbitrary $n \geq 2$, $K_n := \overline{D_n^*} \cup \{q \in D : \operatorname{dist}(q, \partial D) \geq 1/n\}$, after which we choose some strongly convex body D_{n+1}^* with $K_n \subseteq D_{n+1}^* \subseteq \overline{D_{n+1}^*} \subseteq D$. In this way, we have constructed a sequence $\{D_n^*\}_n$ of strongly convex bodies satisfying

$$K \subseteq D_1^* \subseteq D_n^* \subseteq \overline{D_n^*} \subseteq D_{n+1}^*, \qquad \bigcup_{n=1}^{\infty} D_n^* = D.$$

Fix now some $\varepsilon > 0$; we claim that *(ii)* holds by taking $D^* := D_n^*$ for some sufficiently large n. We check this by means of a contradiction argument and assume, on the contrary, the existence of a sequence $(p_n)_n$, with $p_n \in \partial D_n^*$ for every n, and $\operatorname{dist}(\nu_n^*(p_n), \mathcal{N}(q)) \geq \varepsilon$ for every $q \in \partial D$ with $|p_n - q| < \varepsilon$. (Here, $\nu_n^*(p_n)$ denotes the unit outward normal to D_n^* at the point p_n.) After possibly passing to a subsequence, we may assume that $(p_n)_n$ converges to some point $q \in \partial D$ and $\nu_n^*(p_n) \to w$ for some unitary vector $w \in \mathbb{R}^N$. Since $\langle \nu_n^*(p_n), p_n - p \rangle \geq 0$ for every $p \in D_n^*$ and every natural number n, passing to the limit we see that $w \in \mathcal{N}(q)$. Thus, for n large enough, we have $|p_n - q| < \varepsilon$ and $\operatorname{dist}(\nu_n^*(p_n), \mathcal{N}(q)) \leq |\nu_n^*(p_n) - w| < \varepsilon$. This is a contradiction and concludes the proof. $\square$

To conclude this section, let now D be a smooth convex body. The projection map $\pi : \mathbb{R}^N \setminus \overline{D} \to \partial D$, defined by

$$p - \pi p = \operatorname{dist}(p, \partial D)\nu(\pi p)\,,$$

is smooth. Moreover, it is well-known that π is non-expansive, i.e.,

$$\|\pi'(p)\| \leq 1\,, \quad \text{for every } p \in \mathbb{R}^N \setminus \overline{D}\,.$$

This inequality can be improved when D is strongly convex, as follows.

Lemma 3.2. *Let $D \subseteq \mathbb{R}^N$ be a strongly convex body. Then, $\|\pi'(p)\| < 1$ for every $p \in \mathbb{R}^N \setminus \overline{D}$. Moreover, there is a constant $C > 0$ such that*

$$|p|\,\|\pi'(p)\| \leq C\,, \quad \text{for every } p \in \mathbb{R}^N \setminus \overline{D}\,. \tag{6}$$

Proof. We start from the equality

$$\pi(q + t\nu(q)) = q\,, \quad \text{for every } q \in \partial D \text{ and } t \geq 0\,. \tag{7}$$

Differentiating with respect to t, we get

$$\pi'(p)\nu(\pi p) = 0\,, \quad \text{for every } p \in \mathbb{R}^N \setminus \overline{D}\,.$$

Consequently, the norm of the linear map $\pi'(p) : \mathbb{R}^N \to T_{\pi p}\partial D$ coincides with that of its restriction to $T_{\pi p}\partial D$. On the other hand, differentiating with respect to q in (7) gives

$$\pi'(p) \circ [\mathrm{Id}_{T_{\pi p}\partial D} + \operatorname{dist}(p, \partial D)\nu'(\pi p)] = \mathrm{Id}_{T_{\pi p}\partial D}\,, \tag{8}$$

for every $p \in \mathbb{R}^N \setminus \overline{D}$. It means that $\pi'(p)\big|_{T_{\pi p}\partial D} : T_{\pi p}\partial D \to T_{\pi p}\partial D$ is an isomorphism and we have found its inverse:

$$L_p := \left(\pi'(p)\big|_{T_{\pi p}\partial D}\right)^{-1} = \mathrm{Id}_{T_{\pi p}\partial D} + \operatorname{dist}(p, \partial D)\nu'(\pi p)\,.$$

Since $\nu'(\pi p) : T_{\pi p}\partial D \to T_{\pi p}\partial D$ is positive definite for every $p \in \mathbb{R}^N \setminus \overline{D}$, and ∂D is compact, there is a constant $\delta > 0$ (not depending on p) such that

$$\langle L_p u, u\rangle \geq \Big(1 + \delta \operatorname{dist}(p, \partial D)\Big)|u|^2\,, \quad \text{for every } u \in T_{\pi p}\partial D\,,$$

so that, using Schwarz inequality,

$$|L_p u| \geq \Big(1 + \delta \operatorname{dist}(p, \partial D)\Big)|u|\,, \quad \text{for every } u \in T_{\pi p}\partial D\,.$$

Then,

$$\|\pi'(p)\| = \|L_p^{-1}\| \leq \frac{1}{1 + \delta \operatorname{dist}(p, \partial D)}\,,$$

for every $p \in \mathbb{R}^N \setminus \overline{D}$. The result follows. □

4 Feasible vector fields

As usually, D will be a convex body in $\mathbb{R}^N$. In order to treat both cases *(a)*-*(b)* in Theorem 2.1 in a unified way we devote this section to introduce a new class of vector fields on ∂D. Roughly speaking, it will comprise those vector fields $\mathcal{F} : \partial D \to \mathbb{R}^N$ which can be globally extended in such a way that there exists a scalar function, vanishing on D, growing along the integral lines of the extended vector field on $\mathbb{R}^N \setminus D$, and behaving quadratically at infinity.

Precisely, the vector field $\mathcal{F} : \partial D \to \mathbb{R}^N \setminus \{0\}$ is called *feasible* if there exists a continuous extension $\widetilde{\mathcal{F}} : \mathbb{R}^N \to \mathbb{R}^N$ and a C^1-smooth function $h : \mathbb{R}^N \to \mathbb{R}$ satisfying

(i) $h(y) = 0$, for every $y \in D$;

(ii) $\langle \nabla h(y), \widetilde{\mathcal{F}}(y) \rangle > 0$, for every $y \in \mathbb{R}^N \setminus \overline{D}$;

(iii) $\sup_{y \in \mathbb{R}^N} |\nabla h(y) - \mathbb{A} y| < +\infty$, for some regular symmetric matrix $\mathbb{A}$.

We now present some examples of feasible vector fields.

(I) Let $D \subseteq \mathbb{R}^N$ be a strongly convex body, and let $\mathbb{B}$ be a regular symmetric $N \times N$ matrix. Then, the vector field $\mathcal{F}_1 : \partial D \to \mathbb{R}^N$, defined by

$$\mathcal{F}_1(y) = \mathbb{B}\, \nu(y)\,,$$

is feasible.

In order to check this statement, we choose some cutoff C^∞-smooth function $\rho : \mathbb{R} \to \mathbb{R}$, with

$$\rho(s) = \begin{cases} 0\,, & \text{if } s \leq 0\,, \\ 1/2\,, & \text{if } s \geq 1\,, \end{cases} \qquad \rho'(s) > 0\,, \text{ if } s \in]0,1[\,,$$

and define the C^∞-smooth function $\varphi : \mathbb{R}^N \to \mathbb{R}$ by

$$\varphi(y) = \begin{cases} 0\,, & \text{if } y \in \overline{D}\,, \\ \rho(|y - \pi y|)\,, & \text{if } y \in \mathbb{R}^N \setminus \overline{D}\,. \end{cases} \tag{9}$$

By the chain rule,

$$\nabla \varphi(y) = \frac{\rho'(|y - \pi y|)}{|y - \pi y|} (\mathrm{Id} - \pi'(y))^* (y - \pi y)\,,$$

for every $y \in \mathbb{R}^N \setminus \overline{D}$. However, using (8),

$$\ker(\pi'(y)^*) = [\mathrm{Im}(\pi'(y))]^\perp = [T_{\pi y} \partial D]^\perp\,,$$

so that $y - \pi y \in \ker(\pi'(y)^*)$. Hence,

$$\nabla\varphi(y) = \frac{\rho'(|y-\pi y|)}{|y-\pi y|}(y-\pi y)\,, \quad \text{for every } y \in \mathbb{R}^N \setminus \overline{D}\,. \tag{10}$$

We finally define $h : \mathbb{R}^N \to \mathbb{R}$ by

$$h(y) = \begin{cases} 0\,, & \text{if } y \in \overline{D}\,, \\ \varphi(y)\langle y-\pi y, \mathbb{B}(y-\pi y)\rangle\,, & \text{if } y \in \mathbb{R}^N \setminus \overline{D}\,. \end{cases}$$

It is clear that h is a C^1-smooth function satisfying *(i)*. By the chain rule, if $y \in \mathbb{R}^N \setminus \overline{D}$,

$$\nabla h(y) = \langle y-\pi y, \mathbb{B}(y-\pi y)\rangle\nabla\varphi(y) + 2\varphi(y)(\mathrm{Id} - \pi'(y))^*\mathbb{B}(y-\pi y)\,. \tag{11}$$

For $|y|$ large enough, $\varphi(y) = 1/2$ and $\nabla\varphi(y) = 0$, hence

$$|\nabla h(y) - \mathbb{B}y| = |-\mathbb{B}\pi y - \pi'(y)^*\mathbb{B}(y-\pi y)| \le |\mathbb{B}\pi y| + \|\pi'(y)^*\|\,\|\mathbb{B}\|\,|y-\pi y|\,,$$

so that, by (6),

$$\sup_{y\in\mathbb{R}^N} |\nabla h(y) - \mathbb{B}y| < +\infty\,,$$

showing *(iii)* for $\mathbb{A} = \mathbb{B}$. We now continuously extend our vector field $\mathcal{F}_1(y) = \mathbb{B}\nu(y)$ to $\mathbb{R}^N$ by setting

$$\widetilde{\mathcal{F}}_1(y) = \mathbb{B}\nu(\pi y)\,, \ \text{if } y \in \mathbb{R}^N \setminus \overline{D}\,,$$

and with no further requirements (using Tietze Theorem) in D. Recalling (11), we see that, if $y \in \mathbb{R}^N \setminus \overline{D}$,

$$\begin{aligned}\langle\nabla h(y), \widetilde{\mathcal{F}}_1(y)\rangle = {} & \langle y-\pi y, \mathbb{B}(y-\pi y)\rangle\langle\nabla\varphi(y), \mathbb{B}\nu(\pi y)\rangle + \\ & +2\varphi(y)\langle(\mathrm{Id} - \pi'(y))^*\mathbb{B}(y-\pi y), \mathbb{B}\nu(\pi y)\rangle\,.\end{aligned}$$

In view of (10), $\nabla\varphi(y)$ has the same direction as $y - \pi y$. Since $y - \pi y = \mathrm{dist}(y, \partial D)\nu(\pi y)$, the first term in the right hand side of the equality is non-negative. On the other hand, Lemma 3.2 implies that $(\mathrm{Id} - \pi'(y))^*$ is positive definite, for any $y \in \mathbb{R}^N \setminus \overline{D}$, and the second term in the right hand side of the equality is positive. Therefore,

$$\langle\nabla h(y), \widetilde{\mathcal{F}}_1(y)\rangle > 0\,, \ \text{for every } y \in \mathbb{R}^N \setminus \overline{D}\,,$$

showing *(ii)*.

(II) Let $D \subseteq \mathbb{R}^N$ be a strongly convex body, let $d_0 \in D$ be given, and let $\mathbb{B}$ be an involutory $N \times N$ matrix. Then, the vector field $\mathcal{F}_2 : \partial D \to \mathbb{R}^N$, defined by

$$\mathcal{F}_2(y) = \mathbb{B}(y - d_0)\,,$$

is feasible.

Before checking this statement we observe that there is no loss of generality in assuming that $d_0 = 0$. We shall start by proving the result in the special case of $\mathbb{B}$ being orthogonal and symmetric. With this aim, we choose the function $\varphi : \mathbb{R} \to \mathbb{R}$ as in (9), and define $h : \mathbb{R}^N \to \mathbb{R}$ by

$$h(y) = \begin{cases} 0\,, & \text{if } y \in \overline{D}\,, \\ \varphi(y)\langle y - \pi y, \mathbb{B}y\rangle\,, & \text{if } y \in \mathbb{R}^N \setminus \overline{D}\,. \end{cases}$$

It is clear that h is a C^1-smooth function satisfying (i). By the chain rule, if $y \in \mathbb{R}^N \setminus \overline{D}$,

$$\nabla h(y) = \langle y - \pi y, \mathbb{B}y\rangle \nabla\varphi(y) + \varphi(y)\mathbb{B}(y - \pi y) + \varphi(y)(\mathrm{Id} - \pi'(y))^*\mathbb{B}y\,. \tag{12}$$

If $|y|$ is large enough, then $\varphi(y) = \frac{1}{2}$ and $\nabla\varphi(y) = 0$. Therefore,

$$|\nabla h(y) - \mathbb{B}y| = \left|-\tfrac{1}{2}\mathbb{B}\pi y - \tfrac{1}{2}\pi'(y)^*\mathbb{B}y\right| \le \tfrac{1}{2}\|\mathbb{B}\|\,|\pi y| + \tfrac{1}{2}\|\pi'(y)^*\|\,\|\mathbb{B}\|\,|y|\,,$$

which, by (6), implies (iii) for $\mathbb{A} = \mathbb{B}$. It remains to check (ii). With this goal we continuously extend our vector field $\mathcal{F}_2$ by setting $\widetilde{\mathcal{F}}_2(y) = \mathbb{B}y$ for every $y \in \mathbb{R}^N$. Recalling (12) we see that, if $y \in \mathbb{R}^N \setminus \overline{D}$,

$$\begin{aligned}\langle \nabla h(y), \mathbb{B}y\rangle = \langle y - \pi y, \mathbb{B}y\rangle\langle \nabla\varphi(y), \mathbb{B}y\rangle + \varphi(y)\langle \mathbb{B}(y - \pi y), \mathbb{B}y\rangle +\\ +\varphi(y)\langle(\mathrm{Id} - \pi'(y))^*\mathbb{B}y, \mathbb{B}y\rangle\,.\end{aligned}$$

We then see that the first term in the right hand side of the equality is non-negative, the third one is positive, by Lemma 3.2, and, since $\mathbb{B}$ is orthogonal and $0 \in D$,

$$\langle \mathbb{B}(y - \pi y), \mathbb{B}y\rangle = \langle y - \pi y, y\rangle > 0\,, \quad \text{for every } y \in \mathbb{R}^N \setminus \overline{D}\,.$$

So, (ii) is also verified, and the proof is complete in the case of an orthogonal symmetric matrix.

Let us now consider the case of $\mathbb{B}$ being a general involutory matrix. Observe that the only possible (complex) eigenvalues of $\mathbb{B}$ are ± 1. Combining the classical Jordan Decomposition Theorem (see, e.g., [45, Ch. 3]), with the fact that $\mathbb{B}^2 = I$, we see that $\mathbb{B}$ must be diagonalizable. Consequently, $\mathbb{B} = P^{-1}\tilde{\mathbb{B}}P$, for some regular matrix P and some diagonal matrix $\tilde{\mathbb{B}}$ having only ± 1 in the diagonal. We define $\tilde{D} = P(D)$, which is again a strongly convex body containing the origin. Since $\tilde{\mathbb{B}}$ is orthogonal and symmetric we have just proved that there exists a C^1-smooth function $\tilde{h} : \mathbb{R}^N \to \mathbb{R}$ satisfying

$(\widetilde{i})$ $\tilde{h}(y) = 0$, for every $y \in \tilde{D}$;

$(\widetilde{ii})$ $\langle \nabla\tilde{h}(y), \tilde{\mathbb{B}}y\rangle > 0$ for every $y \in \mathbb{R}^N \setminus \overline{\tilde{D}}$;

$(\widetilde{iii})$ $\sup_{y \in \mathbb{R}^N} |\nabla\tilde{h}(y) - \tilde{\mathbb{B}}y| < +\infty$.

We define $h : \mathbb{R}^N \to \mathbb{R}$ by $h(w) = \tilde{h}(Pw)$. It is clear that h is of class C^1, and that $h(w) = 0$ if $w \in D = P^{-1}(\tilde{D})$. Moreover,

$$\nabla h(w) = P^* \nabla \tilde{h}(Pw)\,,$$

for every $w \in \mathbb{R}^N$, and hence

$$\langle \nabla h(w), \mathbb{B}w \rangle = \langle P^* \nabla \tilde{h}(Pw), P^{-1}\tilde{\mathbb{B}}Pw \rangle = \langle \nabla \tilde{h}(Pw), \tilde{\mathbb{B}}Pw \rangle > 0\,,$$

for every $w \in \mathbb{R}^N \setminus \overline{D} = P^{-1}(\mathbb{R}^N \setminus \overline{\tilde{D}})$. Finally, observe that

$$\begin{aligned}\sup_{w\in\mathbb{R}^N} |\nabla h(w) - P^*\tilde{\mathbb{B}}Pw| &= \sup_{w\in\mathbb{R}^N} |P^*\nabla \tilde{h}(Pw) - P^*\tilde{\mathbb{B}}Pw| \\ &\le \|P^*\| \sup_{y\in\mathbb{R}^N} |\nabla \tilde{h}(y) - \tilde{\mathbb{B}}y| < +\infty\,,\end{aligned}$$

showing *(iii)* for $\mathbb{A} = P^*\tilde{\mathbb{B}}P = P^*P\mathbb{B}$. It completes the argument.

We conclude this list by pointing out a third class of feasible vector fields, which will not be further treated in this paper, since the results which could be derived from its use will be contained as particular cases in [36]; nevertheless, the simplicity of the argumentation makes it worth, in our opinion, to devote a few lines to consider it.

(III) *Let $D \subseteq \mathbb{R}^N$ be a smooth convex body and let the continuous vector field $\mathcal{F}_3 : \partial D \to \mathbb{R}^N$ be nowhere tangent, i.e.*

$$\langle \mathcal{F}_3(q), \nu(q) \rangle \neq 0\,, \quad \text{for every } q \in \partial D\,.$$

Then, $\mathcal{F}_3$ is feasible.

We briefly explain how to check this fact now. We observe that there is no loss of generality in assuming that $\langle \mathcal{F}_3(q), \nu(q) \rangle > 0$ for every $q \in \partial D$, and continuously extend $\mathcal{F}_3$ to a vector field $\widetilde{\mathcal{F}}_3 : \mathbb{R}^N \to \mathbb{R}^N$ satisfying $\langle \widetilde{\mathcal{F}}_3(y), \nu(\pi y) \rangle > 0$, for every $y \in \mathbb{R}^N \setminus \overline{D}$. Finally, we define the C^1-smooth function $h : \mathbb{R}^N \to \mathbb{R}$ by $h(y) = \frac{1}{2}\mathrm{dist}(y, D)^2$. The result follows, with $\mathbb{A} = I_N$.

Before closing this section we state a general result concerning Hamiltonian systems whose flow is guided by some feasible vector field on the boundary of a convex body (the notion of a Hamiltonian flow being guided by a vector field was introduced in page 7). Proving this theorem will be the main step towards Theorem 2.1.

Theorem 4.1. *Let the Hamiltonian function $H : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$ be admissible, and assume the existence of a convex body $D \subseteq \mathbb{R}^N$ such that the flow of (HS) is guided by some feasible vector field on ∂D. Then, the same conclusion of Theorem 2.1 holds.*

The proof of this theorem will be organized into two parts. In the next section we introduce an auxiliary class of admissible Hamiltonians, which we shall call *strongly admissible*. Roughly speaking, they are somewhat more regular and can be used to approximate other admissible Hamiltonians, while having the same periodic solutions. In Subsection 6.2 we shall use these facts to show that it suffices to prove Theorem 4.1 for Hamiltonians belonging to this subclass. The proof for this particular case will be carried out in Subsection 6.1.

5 Strongly admissible Hamiltonians

Let $H : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$ be an admissible Hamiltonian, and let $U \subseteq \mathbb{R}^N$ be an open and bounded set (but not necessarily convex). We shall say that H is *strongly admissible with respect to* U provided that:

[1.] There exists a relatively open set $\mathcal{V} \subseteq [0, T] \times \mathbb{R}^N$, containing $\{0\} \times (\mathbb{R}^N \setminus U)$, such that H is C^∞-smooth with respect to the state variables $z = (x, y)$ on the set $\mathcal{V}_\sharp := \{(t, x, y) : (t, y) \in \mathcal{V},\ x \in \mathbb{R}^N\}$;

[2.] There exists some $R > 0$ such that $H(t, x, y) = 0$, if $|y| \geq R$.

Observe that condition **[2.]** implies in particular that ∇H is bounded and the solutions of (HS) cannot explode in finite time. Thus, if H is strongly admissible with respect to some set U, then the solutions of (HS) are defined on the whole time interval $[0, T]$.

The main result of this section is given next. In order to use it in subsequent works (see, e.g., [36]), it is presented in a form which is somewhat more general than needed in this paper. Here, as usually, we denote by (HS) the Hamiltonian system associated to the Hamiltonian H and by $(\widehat{HS})$ the Hamiltonian system associated to $\widehat{H}$.

Proposition 5.1. *Let the admissible Hamiltonian* $H : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$ *and the open and bounded set* $U \subseteq \mathbb{R}^N$ *be given. We assume that all solutions* $z(t) = (x(t), y(t))$ *of* (HS) *starting with* $y(0) \in \partial U$ *are defined on the whole interval* $[0, T]$ *and satisfy* $z(0) \neq z(T)$.

Then, for every $\varepsilon > 0$ *there exists a Hamiltonian* $\widehat{H} : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$ *which is strongly admissible with respect to* U *and satisfies:*

($\diamond$) *The functions* H *and* $\widehat{H}$ *coincide on a relatively open set which contains the graph of any* T*-periodic solution* $\hat{z}(t) = (\hat{x}(t), \hat{y}(t))$ *of* $(\widehat{HS})$ *satisfying* $\hat{y}(0) \in U$.

($\diamond\diamond$) *For any solution* $\hat{z}(t) = (\hat{x}(t), \hat{y}(t))$ *of* $(\widehat{HS})$ *with* $\hat{y}(0) \in \partial U$ *there exists a solution* $z(t) = (x(t), y(t))$ *of* (HS), *with* $y(0) \in \partial U$, *such that*

$$|z(t) - \hat{z}(t)| < \varepsilon\,, \quad \text{for every } t \in [0, T]\,.$$

Before going to the proof of Proposition 5.1 we observe that, under the conditions there, uniqueness for initial value problems associated to (HS) is not guaranteed. Still, the possibly multivalued flow of our Hamiltonian system possesses some properties which evoke continuity. We shall start with the following 'boundedness on compact sets' result.

Lemma 5.2. *Let H and U be under the conditions of Proposition 5.1. Then, there exists some constant $R_1 > 0$ such that any solution $z(t) = (x(t), y(t))$ of (HS) satisfying $y(0) \in \overline{U}$ is defined on $[0, T]$ and satisfies*

$$|y(t)| \leq R_1 \,, \qquad \text{for every } t \in [0, T] \,. \tag{13}$$

Proof. Assume first that $y(0) \in \partial U$. By the periodicity of H, it is sufficient to take $x(0)$ in $[0, 2\pi]^N$. In this case, since the initial values lie in a compact set, the result follows, e.g., from [30, Theorem 5, page 9]. In this way we have shown the existence of some constant $R_1 > 0$ such that (13) holds for every solution $(x(t), y(t))$ of (HS) with $y(0) \in \partial U$.

We now notice that the solutions of (HS) cannot explode in the x component without exploding in the y component, too. This is due to the periodicity of $H(t, x, y)$ in the x_i variables. So, arguing by contradiction, let $z^*(t) = (x^*(t), y^*(t))$ be a solution of (HS), with $y^*(0) \in U$, defined on some interval $[0, T^*] \subseteq [0, T]$, and such that $|y^*(T^*)| > R_1$. We can take T^* so that $|y^*(t)| \leq R_1 + 1$, for every $t \in [0, T^*]$. Along the remaining of the proof, we will only consider times t in $[0, T^*]$.

We denote by $\mathcal{C}^*$ the graph of z^*, i.e.,

$$\mathcal{C}^* = \{(t, z^*(t)) : t \in [0, T^*]\} \,,$$

and define

$$\mathcal{A} = \Big\{(t, z(t)) : t \in [0, T^*],\ z = (x, y) \text{ is a sol. of } (HS) \text{ with } y(0) \in \partial U\Big\} \,.$$

By the choice of R_1, the set $\mathcal{A}$ is contained in $[0, T^*] \times \mathbb{R}^N \times \overline{B}(0, R_1)$. Thus, the sets $\mathcal{C}^*$ and $\mathcal{A}$ do not intersect, as otherwise we could follow first a solution (x, y) departing with $y(0) \in \partial U$ along some interval $[0, t_1]$, and then z^* on $[t_1, T^*]$, so to obtain a solution $z = (x, y)$ of (HS) departing with $y(0) \in \partial U$ and satisfying $y(T^*) > R_1$. Our aim is to deduce from the above that $\mathcal{C}^*$ is disconnected, a contradiction.

We know that $\mathcal{C}^*$ is compact, being the graph of a continuous function, and $\mathcal{A}$ is closed (see, e.g., [30, Theorem 5, page 9]). Therefore, it is possible to find some constant ε, with

$$0 < \varepsilon < |y^*(T^*)| - R_1 \,, \tag{14}$$

such that the 'ε-neighborhood of $\mathcal{A}$', i.e., the set

$$[\mathcal{A}]_\varepsilon = \{(t, w) \in [0, T^*] \times \mathbb{R}^{2N} : \exists w' \in \mathbb{R}^{2N} : (t, w') \in \mathcal{A} \text{ and } |w - w'| < \varepsilon\} \,,$$

is such that $[\mathcal{A}]_\varepsilon \cap \mathcal{C}^* = \emptyset$.

Using a regularization argument, it is possible to construct a sequence $(H_n)_n$ of Hamiltonians $H_n : [0,T^*] \times \mathbb{R}^{2N} \to \mathbb{R}$, $H_n = H_n(t,x,y)$, all of which are 2π-periodic in $x_1, \ldots, x_N$ and C^∞-smooth with respect to $z = (x,y)$, such that ∇H_n converges to ∇H, uniformly on $[0,T^*] \times \mathbb{R}^N \times \overline{B}(0, R_1+1)$, and

$$H_n(t,x,y) = 0\,, \text{ if } |y| \geq R_1 + 2\,.$$

Notice that the associated initial value problems are uniquely solvable, and the corresponding solutions are defined on $[0,T^*]$. Then, we can consider the flow maps $\phi_n : [0,T^*] \times \mathbb{R}^{2N} \to \mathbb{R}^{2N}$, sending each point (t, z_0) to the value at time t of the solution z of

$$J\dot{z} = \nabla H_n(t,z)\,, \tag{15}$$

with $z(0) = z_0$. The maps

$$\Phi_n : [0,T^*] \times \mathbb{R}^{2N} \to [0,T^*] \times \mathbb{R}^{2N}\,, \quad (t,z_0) \mapsto (t, \phi_n(t,z_0))\,,$$

are homeomorphisms. Define the set

$$\mathcal{A}_n = \Phi_n\Big([0,T^*] \times \mathbb{R}^N \times \partial U\Big)\,.$$

The complementary set $([0,T^*] \times \mathbb{R}^{2N}) \setminus \mathcal{A}_n$ is divided into two relatively open sets, precisely

$$Q_n^{in} = \Phi_n\Big([0,T^*] \times \mathbb{R}^N \times U\Big)\,,$$

and

$$Q_n^{out} = \Phi_n\Big([0,T^*] \times R^N \times (\mathbb{R}^N \setminus \overline{U})\Big)\,.$$

By [30, Theorem 1, page 87], for n large enough, $\mathcal{A}_n \subseteq [\mathcal{A}]_\varepsilon$, so that $\mathcal{A}_n \cap \mathcal{C}^* = \emptyset$. Moreover, $\mathcal{A}_n \subseteq [0,T^*] \times \mathbb{R}^N \times B(0, R_1+\varepsilon)$. We claim that, further,

$$Q_n^{in} \subseteq [0,T^*] \times \mathbb{R}^N \times B(0, R_1+\varepsilon)\,. \tag{16}$$

To check this statement we observe that $\mathcal{A}_n$ has empty intersection with the set $\Upsilon := [0,T^*] \times \mathbb{R}^N \times \big(\mathbb{R}^N \setminus B(0,R_1+\varepsilon)\big)$. Thus, the latter set can be written as the disjoint union of the relatively open subsets $\Upsilon \cap Q_n^{in}$ and $\Upsilon \cap Q_n^{out}$. But Υ is connected, and we deduce that either $\Upsilon \subseteq Q_n^{in}$ or $\Upsilon \subseteq Q_n^{out}$. On the other hand, by the periodicity in the x_i variables, we can think of $[0,T^*] \times \mathbb{R}^N \times \overline{U}$ as being compact as a subset of $[0,T^*] \times (\mathbb{R}/2\pi\mathbb{Z})^N \times \mathbb{R}^N$, and so Q_n^{in} is bounded in the y-directions. Since Υ is unbounded in the y-directions, it follows that $\Upsilon \subseteq Q_n^{out}$, which implies the claim.

Now we know that $(0, z^*(0)) \in \mathcal{C}^* \cap Q_n^{in}$ and, by (14) and (16), that $(T^*, z^*(T^*)) \in \mathcal{C}^* \cap Q_n^{out}$. Hence, $\mathcal{C}^*$ is the union of the relatively open nonempty disjoint sets $\mathcal{C}^* \cap Q_n^{out}$ and $\mathcal{C}^* \cap Q_n^{in}$, so that $\mathcal{C}^*$ is disconnected, a contradiction which concludes the proof of this lemma. $\square$

We shall also need a 'continuous dependence' result for our possibly multivalued flow. It is given next:

Lemma 5.3. *Let H and U be under the conditions of Proposition 5.1 and choose some $\varepsilon > 0$. Then, there exists some $\delta > 0$ such that, whenever $\widehat{H} : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$ is an admissible Hamiltonian with*

$$|\nabla \widehat{H}(t,x,y) - \nabla H(t,x,y)| \le \delta\,, \text{ for every } (t,x,y) \in [0,T] \times \mathbb{R}^{2N}\,,$$

then every solution $\hat{z}(t) = (\hat{x}(t), \hat{y}(t))$ of $(\widehat{HS})$ with $\operatorname{dist}(\hat{y}(t_0), \partial U) \le \delta$ for some $t_0 \in [0,\delta]$ can be extended to the whole time interval $[0,T]$ and, moreover,

$$|z(t) - \hat{z}(t)| < \varepsilon\,, \quad \text{for every } t \in [0,T]\,,$$

for some solution $z = (x,y) : [0,T] \to \mathbb{R}^{2N}$ of (HS) satisfying $y(0) \in \partial U$.

Proof. We start by showing the part of the lemma concerning the possibility to extend the solutions to $[0,T]$; moreover, we shall see that, if $\delta > 0$ is small enough, then our solution $\hat{z}(t) = (\hat{x}(t), \hat{y}(t))$ satisfies

$$|\hat{y}(t)| \le R_1 + 1\,, \quad \text{for every } t \in [0,T]\,,$$

where $R_1 > 0$ is the constant given by Lemma 5.2. We check this fact by means of a contradiction argument, and assume instead the existence of a sequence of admissible Hamiltonians $\widehat{H}_n : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$ such that $\nabla \widehat{H}_n(t,x,y) \to \nabla H(t,x,y)$ uniformly with respect to (t,x,y), and a sequence $\hat{z}_n(t) = (\hat{x}_n(t), \hat{y}_n(t))$ of solutions to the respective Hamiltonian systems $(\widehat{HS})_n$ such that $\operatorname{dist}(\hat{y}_n(t_n), \partial U) \to 0$ for some sequence $t_n \to 0$, but $|\hat{y}_n(s_n)| = R_1 + 1$ for some sequence $(s_n)_n \subset [0,T]$. It follows from here that, for n large enough, it has to be $s_n > t_n$, as otherwise our solutions would have to cover a too large distance in a too short time. For the same reason, we see that $(s_n)_n$ is bounded away from 0, at least for large n. Thus, after passing to a subsequence, we may assume that $s_n \to s_* \in]0,T]$ as $n \to \infty$. Moreover, there is no loss of generality in assuming that $|\hat{y}_n(t)| \le R_1 + 1$, for any $t \in [0, s_n]$. Since they are solutions of the corresponding sequence of Hamiltonian systems, we see that $(\hat{y}_n)_n$ is actually bounded in the C^1-sense. Thus, after possibly passing to a subsequence, we may assume that it converges to a solution $z(t) = (x(t), y(t))$ of (HS), satisfying $y(0) \in \partial U$, and $|y(s_*)| = R_1 + 1$. This contradicts the choice of R_1.

To check the second part of the result we use again a contradiction argument and assume, on the contrary, the existence of a sequence of admissible Hamiltonians $\widehat{H}_n : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$ such that $\nabla \widehat{H}_n(t,x,y) \to \nabla H(t,x,y)$ uniformly with respect to (t,x,y), and a sequence $\hat{z}_n = (\hat{x}_n, \hat{y}_n) : [0,T] \to \mathbb{R}^{2N}$ of respective solutions to the corresponding Hamiltonian systems $(\widehat{HS})_n$ such that $\operatorname{dist}(\hat{y}_n(t_n), \partial U) \to 0$ for some sequence $t_n \to 0$, and $\max_{[0,T]} |\hat{z}_n(t) - z(t)| \ge \varepsilon_*$ for any solution $z(t) = (x(t), y(t))$ of (HS), with $y(0) \in \partial U$, and some $\varepsilon_* > 0$. As before, the sequence $\hat{z}_n$ must be bounded in the C^1-topology, and hence the Ascoli–Arzelà Theorem shows that, after possibly passing to a subsequence, one may assume that $\hat{z}_n$ is uniformly converging. Its limit must be a solution $z = (x,y)$ of (HS) with $y(0) \in \partial U$. We thus have a contradiction with the existence of ε_*, which concludes the proof. □

Proof of Proposition 5.1. Let $R_1 > 0$ be as given by Lemma 5.2 above and choose some C^1-smooth cutoff function $a : \mathbb{R} \to \mathbb{R}$, with

$$a(y) = \begin{cases} 1 & \text{if } |y| \le R_1 \,, \\ 0 & \text{if } |y| \ge R := R_1 + 1 \,. \end{cases}$$

After multiplying $H(t,x,y)$ by $a(y)$ we obtain a new Hamiltonian vanishing for large $|y|$, which was condition **[2.]** in the definition of strongly admissible Hamiltonians. Moreover, under this procedure the solutions $z(t) = (x(t), y(t))$ of the Hamiltonian system (HS) which depart from some point $y(0) \in \overline{U}$ are kept the same. Thus, we see that, for the sake of proving Proposition 5.1, there is no loss of generality in assuming that we start from a Hamiltonian H already satisfying **[2.]**. In particular, ∇H is bounded, and consequently all solutions of (HS) are prolongable to $[0,T]$.

Being ∂D compact, a straightforward argument based on Ascoli – Arzelà Theorem implies the existence of some number $\varepsilon_* > 0$ such that $|z(T) - z(0)| \ge \varepsilon_*$ whenever $z(t) = (x(t), y(t))$ is a solution of (HS) departing with $y(0) \in \partial U$. Choose now some $\varepsilon \in]0, \varepsilon_*/2[$ and let $\delta \in]0, T[$ be given by Lemma 5.3. A convolution procedure can be used to build a C^∞-smooth cutoff function $m : \mathbb{R}^N \to [0,1]$ satisfying

$$m(y) = \begin{cases} 1 \,, & \text{if } \operatorname{dist}(y, \mathbb{R}^N \setminus U) \le \delta/2 \,, \\ 0 \,, & \text{if } \operatorname{dist}(y, \mathbb{R}^N \setminus U) \ge \delta \,. \end{cases}$$

Let $M_1 > 0$ be a bound for $|\nabla m|$ on $\mathbb{R}^N$. Using a second convolution argument we can find an admissible Hamiltonian $\bar{H} : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$, which is C^∞-smooth in the (x,y) variables, such that, for every (t,x,y),

$$|\bar{H}(t,x,y) - H(t,x,y)| \le \frac{\delta}{2M_1} \,, \quad |\nabla \bar{H}(t,x,y) - \nabla H(t,x,y)| \le \frac{\delta}{2} \,,$$

and

$$\bar{H}(t,x,y) = 0 \,, \quad \text{if } |y| \ge R + 1 \,.$$

Finally, we choose some continuous cutoff function $n : [0,T] \to [0,1]$, with

$$n(t) = \begin{cases} 1 \,, & \text{if } 0 \le t \le \delta/2 \,, \\ 0 \,, & \text{if } \delta \le t \le T \,, \end{cases}$$

and define $\widehat{H} : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$ by

$$\widehat{H}(t,x,y) := H(t,x,y) + n(t)\, m(y) \Big(\bar{H}(t,x,y) - H(t,x,y) \Big) \,.$$

In this way, $\widehat{H}$ is an admissible Hamiltonian. Furthermore, it is strongly admissible. To check this we firstly observe that $\widehat{H}(t,x,y) = 0$ if $|y| \ge R + 1$, so that **[2.]** holds for $\widehat{H}$. Secondly, we let

$$\mathcal{V} := [0, \delta/2[\times \big\{ y \in \mathbb{R}^N : \operatorname{dist}(y, \mathbb{R}^N \setminus U) < \delta/2 \big\} \,,$$

and we see that $\widehat{H}$ coincides with $\bar{H}$ on $\mathcal{V}_\sharp := \{(t,x,y) : (t,y) \in \mathcal{V},\, x \in \mathbb{R}^N\}$. So, **[1.]** holds for $\widehat{H}$, too, proving that it is strongly admissible.

We further observe that, for any (t,x,y),

$$
\begin{aligned}
&|\nabla\widehat{H}(t,x,y)-\nabla H(t,x,y)|\leq\\
&\leq n(t)\Big(|\nabla m(y)|\,|\bar{H}(t,x,y)-H(t,x,y)|+m(y)|\nabla\bar{H}(t,x,y)-\nabla H(t,x,y)|\Big)\\
&\leq M_1\frac{\delta}{2M_1}+\frac{\delta}{2}=\delta\,,
\end{aligned}
$$

and, moreover, $\widehat{H}$ and H coincide on the relatively open set

$$
\mathcal{O}:=\Big([0,\delta]\times\mathbb{R}^N\times\{y\in\mathbb{R}^N:\operatorname{dist}(y,\mathbb{R}^N\setminus U)>\delta\}\Big)\cup\Big(]\delta,T]\times\mathbb{R}^{2N}\Big)\,.
$$

To check $(\diamond)$, assume that $\hat{z}(t)=(\hat{x}(t),\hat{y}(t))$ is a T-periodic solution of $(\widehat{HS})$ satisfying $\hat{y}(0)\in U$. Let us show that the graph of $\hat{z}$ is contained in $\mathcal{O}$. If not, it has to cross the set

$$
\big([0,T]\times\mathbb{R}^{2N}\big)\setminus\mathcal{O}=[0,\delta]\times\mathbb{R}^N\times\{y\in\mathbb{R}^N:\operatorname{dist}(y,\mathbb{R}^N\setminus U)\leq\delta\}\,.
$$

Then, since $\hat{y}(0)\in U$, the graph of $\hat{z}(t)$ has to enter the set

$$
[0,\delta]\times\mathbb{R}^N\times\{y\in\mathbb{R}^N:\operatorname{dist}(y,\partial U)\leq\delta\}\,.
$$

So, by Lemma 5.3, there is a solution $z(t)=(x(t),y(t))$ of (HS) such that $y(0)\in\partial U$ and $|z(t)-\hat{z}(t)|<\varepsilon$, for every $t\in[0,T]$. Being $\hat{z}(0)=\hat{z}(T)$, we conclude that $|z(T)-z(0)|\leq 2\varepsilon<\varepsilon_*$, a contradiction. It proves $(\diamond)$.

Concerning $(\diamond\diamond)$, it now follows immediately from Lemma 5.3. □

6 Transforming the Hamiltonian into quadratic near infinity

This section is organized in two steps. Firstly, we shall prove Theorem 4.1 (the one concerning feasible vector fields) for Hamiltonians which are strongly admissible with respect to some convex body $D=U$; this will occupy us through most of the section. In the second step we shall use approximation arguments to extend this result and prove, in first place, Theorem 4.1 in its full generality, and finally Theorem 2.1.

6.1 First Step: Theorem 4.1 for strongly admissible Hamiltonians

An important ingredient of our proof is the following theorem due to Szulkin (cf. [68, Theorem 4.2] and [69, Theorem 8.1]), which was proved by variational methods, after a series of previous achievements [1, 2, 20, 50].

Theorem 6.1 (Szulkin). *Let $\mathbb{S}$ be a regular symmetric $N \times N$ matrix, and let $H : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$ be an admissible Hamiltonian, with*

$$H(t,x,y) = \tfrac{1}{2}\langle \mathbb{S}y, y\rangle + G(t,x,y)\,,$$

where the gradient of G with respect to (x,y) is bounded. Then, the Hamiltonian system (HS) has at least $N+1$ geometrically distinct T-periodic solutions.

Moreover, if G is twice continuously differentiable with respect to (x,y) and the T-periodic solutions are known to be nondegenerate, then there are at least 2^N of them.

Since this theorem is not directly applicable to our situation, we will construct a modified Hamiltonian $\widetilde{H} : [0,T]\times\mathbb{R}^{2N} \to \mathbb{R}$, satisfying its assumptions, and being equal to H on a relatively open set $\Omega \subseteq [0,T] \times \mathbb{R}^{2N}$ which contains every T-periodic solution of the modified Hamiltonian system $(\widetilde{HS})$ associated to $\widetilde{H}$.

From now on we operate under the assumptions of Theorem 4.1, i.e., we assume that the flow of (HS) is guided by the feasible vector field $\mathcal{F}$ on the boundary of the convex body D. Furthermore, we assume that the Hamiltonian function $H : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$ is strongly admissible with respect to $U = D$. Then, a compactness argument based on Ascoli–Arzelà Theorem may be used to find some positive constant $\varrho > 0$ such that

$$\langle x(T) - x(0), \widetilde{\mathcal{F}}(y(0))\rangle > 0\,, \tag{17}$$

for every solution $z(t) = (x(t), y(t))$ of (HS) with $\mathrm{dist}(y(0), \partial D) \le \varrho$. As usual, we denote by $\widetilde{\mathcal{F}} : \mathbb{R}^N \to \mathbb{R}^N$ the extension of $\mathcal{F}$ given by the fact that it is feasible.

Let $\mathcal{V}$ be given by **[1.]**; then, taking ϱ small enough, we have that

$$K := \{y \in D : \mathrm{dist}(y, \partial D) \ge \varrho\} \supseteq \{y \in \mathbb{R}^N : (0,y) \notin \mathcal{V}\}\,.$$

The set K is compact and contained in D. Using Lemma 3.1 (†), we may find some smooth convex body $D_* \subseteq \mathbb{R}^N$ with $K \subseteq D_* \subseteq \overline{D}_* \subseteq D$. We choose some constant $c > 0$ and consider the (relatively open) set

$$\Omega := \big(\{0\} \times D_*\big) \cup \big\{(t,y) \in \,]0,T] \times \mathbb{R}^N : \mathrm{dist}(y, D_*) < ct\big\}\,.$$

We denote by $\Omega_\sharp$ the 'augmented set'

$$\Omega_\sharp := \big\{(t,x,y) \in [0,T] \times \mathbb{R}^{2N} : (t,y) \in \Omega\big\}\,.$$

Combining **[1.]** and **[2.]** we see that, if c is large enough, then

$$\big([0,T] \times \mathbb{R}^N\big) \setminus \Omega \subseteq \mathcal{V} \cup \{(t,y) \in [0,T] \times \mathbb{R}^N : |y| > R\}\,,$$

so that

$$H \text{ is } C^\infty\text{-smooth on } \mathcal{G}_\sharp := \big([0,T]\times\mathbb{R}^{2N}\big)\setminus\overline{\Omega}_\sharp \\ = \{(t,x,y)\in[0,T]\times\mathbb{R}^{2N} : \mathrm{dist}(y,D_*) > ct\}\,. \tag{18}$$

Another interesting property of $\Omega_\sharp$, for large $c>0$, is that it is strictly forward-invariant for the flow of (HS), in the sense that

$$(t_0,z(t_0))\in\overline{\Omega}_\sharp \quad\Rightarrow\quad (t,z(t))\in\Omega_\sharp\,,\ \text{for every } t\in\,]t_0,T]\,,$$

where $z=z(t)$ is any solution of (HS). What is more, $\Omega_\sharp$ is strictly forward-invariant for the flow of every Hamiltonian system $(\widetilde{HS})$ which coincides with (HS) on $\Omega_\sharp$. We check these facts below.

Lemma 6.2. *Taking $c>0$ sufficiently large, the set $\Omega_\sharp$ is strictly forward-invariant for the flow of any Hamiltonian system $(\widetilde{HS})$ whose associated (admissible but not necessarily strongly admissible) Hamiltonian $\widetilde{H} : [0,T]\times\mathbb{R}^{2N}\to\mathbb{R}$ coincides with H on $\Omega_\sharp$.*

Proof. Assumption **[2.]** implies in particular that $|\nabla H|$ is bounded, i.e., $|\nabla H(t,x,y)|\le M$ for every $(t,x,y)\in[0,T]\times\mathbb{R}^{2N}$ and some constant $M>0$. Fix now some $c>M$ and let the sets Ω and $\Omega_\sharp$ be defined as above. Let $\widetilde{H}:[0,T]\times\mathbb{R}^{2N}\to\mathbb{R}$ be some admissible Hamiltonian coinciding with H on $\Omega_\sharp$, and let $z(t)=(x(t),y(t))$ be a solution of $(\widetilde{HS})$ with $(t_0,z(t_0))\in\overline{\Omega}_\sharp$ for some $t_0\in[0,T[$. It means that $\mathrm{dist}(y(t_0),D_*)\le ct_0$. Then,

$$|\dot y(t_0)|\le|\dot z(t_0)| = |\nabla\widetilde{H}(t_0,z(t_0))| = |\nabla H(t_0,z(t_0))|\le M<c\,,$$

and, consequently, there exists some $s\in\,]t_0,T]$ such that $(t,z(t))\in\Omega_\sharp$ for every $t\in\,]t_0,s]$. Let s_* be the supremum of the set made by such numbers s; then, $|\dot y(t)|<c$ for every $t\in[t_0,s_*]$ and, consequently,

$$\mathrm{dist}(y(s_*),D_*)\le|y(s_*)-y(t_0)|+\mathrm{dist}(y(t_0),D_*)<c(s_*-t_0)+ct_0=cs_*\,.$$

We conclude that $s_*=T$. The lemma is proved. □

From now on we fix $c>0$ large enough so that (18) and the assertions of Lemma 6.2 hold, and define the sets Ω and $\Omega_\sharp$ accordingly. We consider the set Γ whose elements are those $(t,\zeta)=(t,\xi,\eta)\in[0,T]\times\mathbb{R}^{2N}$ such that the solution $z=z(t)$ of (HS) with initial condition $z(0)=\zeta$ satisfies

$$(s,z(s))\in\mathcal{G}_\sharp\,,\ \text{for every } s\in[0,t]\,.$$

We emphasize the fact that it makes sense to consider the solution of such an initial value problem, since on $\mathcal{G}_\sharp$ uniqueness holds (by (18)). The set Γ is open relative to $[0,T]\times\mathbb{R}^{2N}$, and it contains the closed subsets

$$A_\sharp=\{0\}\times\mathbb{R}^N\times(\mathbb{R}^N\setminus D')\,,\qquad B_\sharp=[0,T]\times\mathbb{R}^N\times(\mathbb{R}^N\setminus B(0,R'))\,,$$

for some $R' \geq R$, and some smooth convex body $D' \subseteq \mathbb{R}^N$ with $\overline{D}_* \subseteq D' \subseteq \overline{D'} \subseteq D$. Moreover, Γ is periodic in the x_i variables, i.e., it could be thought of as a subset of $[0,T] \times (\mathbb{R}/2\pi\mathbb{Z})^N \times \mathbb{R}^N$. Hence, one can find a constant $k > 0$ such that $\overline{\Delta_\sharp} \subseteq \Gamma$, where $\Delta_\sharp := \{(t,\xi,\eta) \in [0,T] \times \mathbb{R}^{2N} : (t,\eta) \in \Delta\}$, and

$$\Delta := \Big\{(t,\eta) \in [0,T] \times \mathbb{R}^N : \mathrm{dist}(\eta, D') > kt\Big\}\,.$$

Let now the C^1-smooth function $h : \mathbb{R}^N \to \mathbb{R}$ be given by the fact that the vector field is admissible. We recall that, by *(i)*, h vanishes on D. This fact will allow us to show the following:

Lemma 6.3. *There is a C^1-smooth function $r : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfying*

($\star$) $r(t,\eta) = 0$, *if* $(t,\eta) \notin \Delta$,

($\star\star$) $\frac{1}{T}\int_0^T r(t,\eta)\,dt = h(\eta)$, *for every* $\eta \in \mathbb{R}^N$,

($\star\star\star$) $r(t,\eta) = h(\eta)$, *if* $|\eta|$ *is sufficiently large.*

Proof. We choose some C^1-smooth function $u : \mathbb{R} \to \mathbb{R}$ with

$$u(s) = 0\,,\ \text{if } s \leq 0\,,\quad u(s) = 1\,,\ \text{if } s \geq 1\,,\quad 0 < u(s) < 1\,,\ \text{if } s \in]0,1[\,,$$

and define $p : [0,T] \times \mathbb{R}^N \backslash \overline{D'} \to \mathbb{R}$ by

$$p(t,\eta) = \frac{u(\mathrm{dist}(\eta,D') - kt)}{\frac{1}{T}\int_0^T u(\mathrm{dist}(\eta,D') - ks)\,ds}\,.$$

The convex body D' being smooth, the function $\eta \mapsto \mathrm{dist}(\eta, D')$ is C^∞-smooth on $\mathbb{R}^N \backslash \overline{D'}$. Thus, we see that p is a C^1-smooth function satisfying

($\cdot$) $p(t,\eta) = 0$, if $(t,\eta) \notin \Delta$,

($\cdot\cdot$) $\frac{1}{T}\int_0^T p(t,\eta)\,dt = 1$, if $\eta \in \mathbb{R}^N \backslash \overline{D'}$,

($\cdot\cdot\cdot$) $p(t,\eta) = 1$, if $|\eta|$ is sufficiently large.

Finally, we define $r : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ by

$$r(t,\eta) = \begin{cases} 0\,, & \text{if } \eta \in \overline{D'}, \\ p(t,\eta)h(\eta)\,, & \text{if } \eta \in \mathbb{R}^N \backslash \overline{D'}\,. \end{cases}$$

It is easily checked that r satisfies all the required properties. It proves the lemma. $\square$

We consider now the flow map $\phi : \Gamma \to \mathbb{R}^{2N}$, $\phi = \phi(t,\zeta)$, giving the position at time t of the solution z of (HS) with $z(0) = \zeta$. Notice that we consider only initial conditions $\zeta = (\xi,\eta)$ with $\eta \in \mathbb{R}^N \backslash \overline{D}_*$, and the solutions satisfy $(s, z(s)) \in \mathcal{G}_\sharp$ for every $s \in [0,t]$, so that uniqueness holds. Hence ϕ is well defined, and the classical theory of ordinary differential equations (see, e.g., [30]) guarantees that the function $\Phi(t,\zeta) = (t, \phi(t,\zeta))$ is a C^∞-smooth diffeomorphism between Γ and its image $\Phi(\Gamma)$.

Lemma 6.4. $\Phi(\Gamma) = \mathcal{G}_\sharp$.

Proof. We know that $\Phi(\Gamma) \subseteq \mathcal{G}_\sharp$, by the very definition of the set Γ. To prove the opposite inclusion, let (t_0, z_0) be an element of $\mathcal{G}_\sharp$. If $t_0 = 0$, then surely $(0, z_0) \in \Gamma$, and $\Phi(0, z_0) = (0, z_0)$, so that $(0, z_0) \in \Phi(\Gamma)$. Assume now $t_0 \in]0, T]$. If z is the solution of (HS) with $z(t_0) = z_0$, we can go back in time until we find $z(0) = \zeta_0$, for some $\zeta_0 \in \mathbb{R}^{2N}$. Notice that, by Lemma 6.2, one has that $(s, z(s)) \in \mathcal{G}_\sharp$, for every $s \in [0, t_0]$. So, $(t_0, \zeta_0) \in \Gamma$, and $\Phi(t_0, \zeta_0) = (t_0, z_0)$. Hence, $(t_0, z_0) \in \Phi(\Gamma)$, and the equality is proved. $\square$

We now consider the function $r_\sharp : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$ defined by

$$r_\sharp(t, \xi, \eta) = r(t, \eta)\,,$$

and we define $\mathcal{R} : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$ as

$$\mathcal{R}(t, z) = \begin{cases} r_\sharp(\Phi^{-1}(t, z))\,, & \text{if } (t, z) \in \mathcal{G}_\sharp\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

It can be checked that $\mathcal{R}$ is C^1-smooth. Let

$$\widetilde{H}(t, z) = H(t, z) + \lambda \mathcal{R}(t, z)\,,$$

where $\lambda > 0$ is a constant, to be determined later. We observe that, for $|y|$ large enough,

$$\widetilde{H}(t, x, y) = \lambda \mathcal{R}(t, x, y) = \lambda r(t, y) = \lambda h(y)\,,$$

and in view of assumption *(iii)* in page 13, $\nabla \widetilde{H}(t, x, y) - \lambda \mathbb{A} y$ is bounded on $[0, T] \times \mathbb{R}^{2N}$. Thus, we may apply Szulkin's Theorem 6.1 with $\mathbb{S} = \lambda \mathbb{A}$ to the perturbed Hamiltonian system $(\widetilde{HS})$ associated to $\widetilde{H}$, to get at least $N + 1$ geometrically distinct T-periodic solutions of $(\widetilde{HS})$, or 2^N of them if the Hamiltonian is twice continuously differentiable with respect to (x, y) and the periodic solutions are nondegenerate. In what follows, we will prove that, if λ is chosen large enough, these are actually T-periodic solutions of (HS); indeed, they do not cross $\mathcal{G}_\sharp$. In view of Lemma 6.2 we only have to show that they must depart from some initial condition $\tilde{z}(0) = (\tilde{x}(0), \tilde{y}(0)) \in \mathbb{R}^N \times \overline{D}_*$.

Thus, from now on we fix some solution $\tilde{z} = (\tilde{x}, \tilde{y}) : [0, T] \to \mathbb{R}^{2N}$ of $(\widetilde{HS})$, with $\tilde{y}(0) \notin \overline{D}_*$. We will show that such a solution cannot be T-periodic. We observe that $(0, \tilde{z}(0)) \in \mathcal{G}_\sharp$ and define

$$\omega = \sup \Big\{ t \in [0, T] : (t, \tilde{z}(t)) \in \mathcal{G}_\sharp \Big\}\,.$$

Since $\mathcal{G}_\sharp$ was open relative to $[0, T]$ we see that $\omega \in]0, T]$. Moreover, in view of Lemma 6.2,

$$(t, \tilde{z}(t)) \in \mathcal{G}_\sharp\,, \quad \text{for any } t \in [0, \omega[\,, \qquad (t, \tilde{z}(t)) \in \Omega_\sharp\,, \quad \text{for any } t \in]\omega, T]\,.$$

Remembering Lemma 6.4, we can now define the curve $\zeta : [0,\omega[\, \to \mathbb{R}^{2N}$ by

$$(t,\zeta(t)) = \Phi^{-1}(t,\tilde{z}(t))\,.$$

In this way,

$$(t,\zeta(t)) \in \Gamma\,, \text{ and } \tilde{z}(t) = \phi(t,\zeta(t))\,, \text{ for every } t \in [0,\omega[\,. \tag{19}$$

Observe also that $\zeta(0) = \tilde{z}(0)$. Hence, writing $\zeta = (\xi,\eta)$, we have that $\eta(0) \in \mathbb{R}^N \setminus \overline{D}_*$.

Lemma 6.5. *The function $\zeta : [0,\omega[\, \to \mathbb{R}^{2N}$ is a solution of the Hamiltonian system*

$$\dot{\zeta} = \lambda J \nabla r_\sharp(t,\zeta)\,. \tag{20}$$

Proof. The map $\Phi : \Gamma \to \mathcal{G}_\sharp$ being a diffeomorphism, ζ is continuously differentiable. Differentiating in the equality $\tilde{z}(t) = \phi(t,\zeta(t))$ we get

$$\dot{\tilde{z}} = \frac{\partial\phi}{\partial t}(t,\zeta) + \frac{\partial\phi}{\partial\zeta}(t,\zeta)\dot{\zeta}\,,$$

so that

$$\frac{\partial\phi}{\partial\zeta}(t,\zeta)\dot{\zeta} = J\nabla\widetilde{H}(t,\tilde{z}) - J\nabla H(t,\tilde{z}) = \lambda J\nabla\mathcal{R}(t,\tilde{z})\,. \tag{21}$$

Being ϕ the flow map associated with a Hamiltonian system, $\phi(t,\cdot)$ is canonical, i.e.,

$$\Big(\frac{\partial\phi}{\partial\zeta}(t,\zeta(t))\Big)^* J\,\frac{\partial\phi}{\partial\zeta}(t,\zeta(t)) = J\,,$$

for every $t \in [0,\tilde{\omega}[\,$. Recalling that $\mathcal{R}(t,\phi(t,\zeta)) = r_\sharp(t,\zeta)$, multiplying both sides of (21) by $-J\Big(\frac{\partial\phi}{\partial\zeta}(t,\zeta(t))\Big)^* J$ we get

$$\dot{\zeta} = \lambda J\Big(\frac{\partial\phi}{\partial\zeta}(t,\zeta)\Big)^* \nabla\mathcal{R}(t,\phi(t,\zeta)) = \lambda J\nabla r_\sharp(t,\zeta)\,.$$

The lemma is thus proved. □

Since $\sup_{[0,T]\times\mathbb{R}^N} |\nabla r(t,\eta) - \mathbb{A}\eta| < \infty$, we see that our solution $\zeta : [0,\omega[\, \to \mathbb{R}^{2N}$ of (20) can be (uniquely) extended to a solution $\zeta : [0,T] \to \mathbb{R}^{2N}$ of the same system. We now explore the behavior of this solution at times $t \geq \omega$.

Lemma 6.6. *Either $\omega = T$, or $(\omega,\zeta(\omega)) \notin \Gamma$. In this last case,*

$$\zeta(t) = \zeta(\omega)\,, \text{ for every } t \in [\omega,T]\,.$$

Proof. Let us assume instead that $(\omega,\zeta(\omega)) \in \Gamma$. Then, by continuity arguments,

$$(\omega,\tilde{z}(\omega)) = \lim_{t\to\omega^-}(t,\tilde{z}(t)) = \lim_{t\to\omega^-}\Phi(t,\zeta(t)) = \Phi(\omega,\zeta(\omega))\,,$$

and Lemma 6.4 states that $(\omega, \tilde{z}(\omega)) \in \mathcal{G}_\sharp$. Since $\mathcal{G}_\sharp$ is open, the definition of ω then implies that $\omega = T$, and the proof of the first part is completed. Concerning the second part, we observe that, as a consequence of its definition, the set Γ has the following property:

$$(t_0, \zeta_0) \notin \Gamma \Rightarrow (t, \zeta_0) \notin \Gamma \text{ for every } t \geq t_0 \,.$$

The result now follows from the fact that r vanishes outside Γ. □

In particular, $\zeta(T) = \zeta(\omega)$. In the following lemma we compare our solution $\tilde{z} = (\tilde{x}, \tilde{y})$ with a solution $z = (x, y)$ of (HS) which arrives at the same position at time T.

Lemma 6.7. *There exists a solution $z = (x, y) : [0, T] \to \mathbb{R}^{2N}$ of (HS) for which*

$$(x(0), y(0)) = (\tilde{x}(0) + \lambda T \nabla h(\tilde{y}(0))\,, \tilde{y}(0))\,, \quad (x(T), y(T)) = (\tilde{x}(T), \tilde{y}(T))\,.$$

Proof. Since $r(t, \zeta)$ does not depend on ξ, we see that, for the solution $\zeta = (\xi, \eta)$ of (20) defined above,

$$\eta(\cdot) \textit{ is constant},$$

and we denote by η_0 its constant value. So, $\eta_0 = \tilde{y}(0) \in \mathbb{R}^N \backslash \overline{D}$. On the other hand, by Lemma 6.3,

$$\begin{aligned} \xi(\omega) - \xi(0) = \xi(T) - \xi(0) &= \lambda \int_0^T \frac{\partial r}{\partial \eta}(t, \xi(t), \eta(t))\, dt \\ &= \lambda \int_0^T \frac{\partial r}{\partial \eta}(t, \eta(t))\, dt = \lambda \frac{\partial}{\partial \eta} \int_0^T r(t, \eta)\, dt \bigg|_{\eta = \eta_0} = \lambda T \nabla h(\eta_0)\,. \end{aligned}$$

Choose an increasing sequence $(\omega_n)_n$ in $]0, \omega[$ such that $\omega_n \to \omega$. In view of (19), we have that $(\omega_n, \zeta(\omega_n)) \in \Gamma$, for every n, and it makes sense to consider the functions $z_n = \phi(\cdot, \zeta(\omega_n))$, defined on $[0, \omega_n]$, which are solutions of (HS). We observe that $z_n(0) = \zeta(\omega_n) \to \zeta(\omega)$, and $z_n(\omega_n) = \phi(\omega_n, \zeta(\omega_n)) = \tilde{z}(\omega_n) \to \tilde{z}(\omega)$. Since ∇H is bounded, after passing to a subsequence we have that $(z_n)_n$ converges, uniformly on compact subsets of $[0, \omega[$, to some solution $z : [0, \omega] \to \mathbb{R}^{2N}$ of (HS). Moreover, $z(0) = \zeta(\omega)$ and $z(\omega) = \tilde{z}(\omega)$. If $\omega < T$, we extend z to $[0, T]$ by setting

$$z(t) = \tilde{z}(t)\,, \text{ if } \omega < t \leq T\,.$$

In this way, z is a solution of (HS) because $\tilde{z}$ is a solution of (HS) on $[\omega, T]$. So, in any case, $z(T) = \tilde{z}(T)$, and

$$z(0) - \tilde{z}(0) = \zeta(\tilde{\omega}) - \zeta(0) = (\lambda T \nabla h(\eta_0), 0)\,,$$

thus concluding the proof. □

The next lemma leads to the conclusion of the proof.

Lemma 6.8. *If $\lambda > 0$ is large enough (uniformly with respect to $\tilde{z}$), we have*

$$\langle \tilde{x}(T) - \tilde{x}(0), \widetilde{\mathcal{F}}(\tilde{y}(0)) \rangle > 0 \,.$$

Proof. According to Lemma 6.7, we can write

$$\tilde{x}(T) - \tilde{x}(0) = x(T) - x(0) + \lambda T \nabla h(\tilde{y}(0)) \,.$$

We will now make use of item (ii) in the definition of feasible vector fields. We distinguish three cases.

Case 1 : $\mathrm{dist}(\tilde{y}(0), \partial D) < \varrho$.

In this case, by (17),

$$\langle \tilde{x}(T) - \tilde{x}(0), \widetilde{\mathcal{F}}(\tilde{y}(0)) \rangle = \langle x(T) - x(0), \widetilde{\mathcal{F}}(\tilde{y}(0)) \rangle + \lambda T \langle \nabla h(\tilde{y}(0)), \widetilde{\mathcal{F}}(\tilde{y}(0)) \rangle > 0 \,.$$

Case 2 : $|\tilde{y}(0)| > R$.

In this case, since $H(t, x, y) = 0$ for $|y| \geq R$, we have that $x(T) = x(0)$, so

$$\langle \tilde{x}(T) - \tilde{x}(0), \widetilde{\mathcal{F}}(\tilde{y}(0)) \rangle = \lambda T \langle \nabla h(\tilde{y}(0)), \widetilde{\mathcal{F}}(\tilde{y}(0)) \rangle > 0 \,.$$

Case 3 : $\mathrm{dist}(\tilde{y}(0), \partial D) \geq \varrho$ and $|\tilde{y}(0)| \leq R$.

By compactness, there are two real constants c, c' such that, if $y \in \mathbb{R}^N \setminus \overline{D}$ satisfies $\mathrm{dist}(y, \partial D) \geq \varrho$ and $|y| \leq R$, then

$$|\widetilde{\mathcal{F}}(y)| \leq c \,, \quad \left| \frac{\partial H}{\partial y}(t, x, y) \right| \leq c \,, \quad \text{for every } (t, x) \in [0, T] \times \mathbb{R}^N \,,$$

and

$$\langle \nabla h(y), \widetilde{\mathcal{F}}(y) \rangle \geq c' > 0 \,.$$

Then, $|x(T) - x(0)| \leq Tc$, so that, taking $\lambda > c^2/c'$,

$$\langle \tilde{x}(T) - \tilde{x}(0), \widetilde{\mathcal{F}}(\tilde{y}(0)) \rangle \geq \lambda T \langle \nabla h(\tilde{y}(0)), \widetilde{\mathcal{F}}(\tilde{y}(0)) \rangle - |x(T) - x(0)| \, |\widetilde{\mathcal{F}}(\tilde{y}(0))| > 0 \,.$$

The lemma is therefore proved. $\square$

A particular consequence of Lemma 6.8 is that $\tilde{z}$ cannot be T-periodic. The proof of Theorem 4.1 for strongly admissible Hamiltonians is thus concluded.

6.2 From the strongly admissible case to the general result: approximation arguments

We are finally ready to complete the proofs of Theorems 4.1 and 2.1.

Proof of Theorem 4.1. Let the admissible Hamiltonian $H : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$ and the convex body $D \subseteq \mathbb{R}^N$ be such that the flow of (HS) is guided by some feasible vector field $\mathcal{F}$ on ∂D. Then we use Proposition 5.1, with $U = D$ and some small $\varepsilon > 0$, and find a Hamiltonian $\widehat{H}$, which is strongly admissible with respect to D, in such a way that all T-periodic solutions $\hat{z}(t) = (\hat{x}(t), \hat{y}(t))$ of the corresponding Hamiltonian system $(\widehat{HS})$ departing with $\hat{y}(0) \in D$ are also solutions of (HS). Moreover, if ε is small enough, the flow of $(\widehat{HS})$ will still be guided by the feasible vector field $\mathcal{F}$ on ∂D. For this new Hamiltonian system, we have the desired conclusion, as shown in the previous subsection. Since the T-periodic solutions found are also solutions of (HS), we have thus concluded the proof. □

Proof of Theorem 2.1. Let the admissible Hamiltonian $H : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$ and the convex body $D \subseteq \mathbb{R}^N$ satisfy either condition *(a)* or condition *(b)* of Theorem 2.1. Observe that we cannot directly apply Theorem 4.1, because D may not be strongly convex, and thus we cannot ensure that the vector field which guides our flow is feasible. So, we need to find a strongly convex set D_*, contained in D, for which the same assumptions hold.

With this purpose we first observe that in case *(a)* we may find some $\rho_* > 0$ such that

$$\langle x(T) - x(0), \mathbb{B}\nu(y(0)) \rangle \geq \rho_* \,,$$

for every solution $(x(t), y(t))$ of (HS) departing with $y(0) \in \partial D$. Similarly, in case *(b)* we may find some $\rho_* > 0$ such that

$$\langle x(T) - x(0), \mathbb{B}(y(0) - d_0) \rangle \geq \rho_* \,,$$

for every solution $(x(t), y(t))$ of (HS) departing with $y(0) \in \partial D$. Both things follow from the combination of Lemma 5.2 and the Ascoli – Arzelà Theorem.

Then, we may apply Lemma 5.3 with $U = D$, $\widehat{H} = H$ and some small $\epsilon > 0$, and we find that there exists some $\delta > 0$ with the property that, whenever a smooth convex body D_* is such that

- $\{y \in D : \mathrm{dist}(y, \partial D) \geq \delta\} \subseteq D_* \subseteq D$,

and, in case of assumption *(a)*,

- for every $q \in \partial D_*$ there exists some $p \in \partial D$ with $|p - q| < \delta$ and $|\nu(p) - \nu_*(q)| < \delta$,

then our assumption *(a)* or *(b)* keeps its validity for D_* instead of D. Remembering Lemma 3.1, we can actually find a strongly convex set D_* satisfying the conditions above. But then, as shown in *(I)* and *(II)* of Section 4, our Hamiltonian flow is guided by a feasible vector field on ∂D_*. The result follows now from Theorem 4.1. □

7 Nonsmooth sets and the theorem for tubes

In this section we are going to prove Theorem 2.2 and the associated Theorem 1.2. With this goal we shall first extend Theorem 2.1*(a)* in two directions. Firstly, we would like to have a version of this result for general convex bodies, not necessarily C^1-smooth ones. With this aim, we will have to replace the outer normal vector field $\nu = \nu(y)$ by the outer normal cone $\mathcal{N}(y)$ at each point $y \in \partial D$. Secondly, we would like to generalize the notion of admissible Hamiltonians to allow them to be periodic in the x variable with respect to some basis of $\mathbb{R}^N$ which is not necessarily the usual one. More specifically, we shall say that the Hamiltonian function $H : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$, $H = H(t,x,y)$ is *admissible with respect to the basis* $\mathcal{B} = \{b_1, \ldots, b_N\}$ of $\mathbb{R}^N$ provided that, besides the usual regularity assumption, it satisfies

$$H(t, x + b_i, y) = H(t,x,y)\,, \quad \text{for every } (t,x,y) \in [0,T] \times \mathbb{R}^{2N}\,.$$

We are now prepared to show the following

Theorem 7.1. *Let the Hamiltonian $H = H(t,x,y)$ be admissible with respect to some basis of $\mathbb{R}^N$. Let the $N \times N$ matrix $\mathbb{B}$ be regular and symmetric, let $D \subset \mathbb{R}^N$ be a convex body, and assume that every solution $z(t) = (x(t), y(t))$ of (HS) departing with $y(0) \in \partial D$ is defined for every $t \in [0,T]$ and satisfies*

$$\langle x(T) - x(0), \mathbb{B}w \rangle > 0\,, \quad \text{for every } w \in \mathcal{N}(y(0)) \setminus \{0\}\,. \tag{22}$$

Then, the same conclusion of Theorem 2.1 holds.

Proof. We first prove the result in case D is smooth but the Hamiltonian H is admissible with respect to some basis $\mathcal{B}$ of $\mathbb{R}^N$. In this case, (22) becomes the usual condition $\langle x(T) - x(0), \mathbb{B}\nu(y(0)) \rangle > 0$. Let P be the (nonsingular) matrix whose columns are the elements of $\mathcal{B}$ and consider the canonical change of variables

$$x_1 = P^{-1}x, \qquad y_1 = P^* y\,. \tag{23}$$

It transforms (HS) into another Hamiltonian system (HS_1), whose associated Hamiltonian

$$H_1(t, x_1, y_1) := H(t, Px_1, (P^*)^{-1} y_1)\,, \tag{24}$$

is now admissible with respect to the usual basis. Moreover, the set $D_1 := P^*(D)$ is again a smooth convex body, with unit normal vector

$$\nu_1(y_1) = \frac{P^{-1}\nu((P^*)^{-1}y_1)}{|P^{-1}\nu((P^*)^{-1}y_1)|}\,,$$

and one easily checks that every solution $(x_1(t), y_1(t))$ of (HS_1) departing with $y_1(0) \in \partial D_1$ is defined for every $t \in [0,T]$ and satisfies

$$\langle x_1(T) - x_1(0), \mathbb{B}_1 \nu_1(y_1(0)) \rangle > 0\,,$$

where ν_1 denotes the unit outward normal field on ∂D_1, and $\mathbb{B}_1 = P^* \mathbb{B} P$. Thus, since $\mathbb{B}_1$ is symmetric, the result follows from Theorem 2.1.

In the general case, the convex body could be nonsmooth; however, we observe first that there must exist some positive constant $\rho_* > 0$ such that $\langle x(T) - x(0), \mathbb{B}w \rangle \geq \rho_*$ for every solution $(x(t), y(t))$ of (HS) departing with $y(0) \in \partial D$ and every $w \in \mathcal{N}(y(0))$ with $|w| = 1$. This is a consequence of Lemma 5.2 and the Ascoli–Arzelà Theorem. The result follows from an approximation argument similar to the one used in the final part of the proof of Theorem 2.1 (page 30). We omit the details, for briefness. □

Proof of Theorem 2.2. We first assume that $\mathcal{T} = \mathbb{R}^N \times]-1, 1[^N$ is twisted by the flow of H, i.e., that $a_1 \equiv -1$ and $b_i \equiv 1$, for every $i = 1, \ldots, N$. Then, if $(x(t), y(t))$ is a solution of (HS) starting with $y(0)$ on the boundary of $]-1, 1[^N$, we see that

$$\langle x(T) - x(0), \mathbb{B}v \rangle > 0\,, \quad \text{for every } v \in \mathcal{N}(y(0)) \setminus \{0\}\,,$$

where $\mathbb{B}$ is a diagonal matrix, whose elements on the diagonal are $+1$ or -1. The result then follows from Theorem 7.1. (Indeed, we have thus proved a slightly more general result than the one claimed, namely, that it holds for Hamiltonians H which are admissible with respect to some basis $\mathcal{B}$.)

We now treat the general case. After an approximation argument based on the Féjer Theorem and Lemma 5.3, there is no loss of generality in assuming that the functions a_i, b_i are C^∞-smooth. We define the functions $c_i, l_i : \mathbb{R} \to \mathbb{R}$ by

$$c_i(s) = \frac{a_i(s) + b_i(s)}{2}\,, \qquad l_i(s) = \frac{b_i(s) - a_i(s)}{2}\,,$$

for $i = 1, \ldots, N$, and let $\Psi : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ be defined by

$$\Psi(x, y) = \left(\int_0^{x_1} l_1(s)\, ds\,, \ldots, \int_0^{x_N} l_N(s)\, ds\,, \frac{y_1 - c_1(x_1)}{l_1(x_1)}\,, \ldots, \frac{y_N - c_N(x_N)}{l_N(x_N)} \right).$$

It can be verified that Ψ is a symplectic diffeomorphism. Hence, the change of variables $(\hat{x}, \hat{y}) = \Psi(x, y)$ (which, for $N = 1$ was proposed in [61, Exercise 1, p. 132]), transforms our Hamiltonian system (HS) into a new one, with Hamiltonian function

$$\widehat{H}(t, \hat{z}) = H(t, \Psi^{-1}(\hat{z}))\,.$$

The new Hamiltonian is still periodic in the variables $x_1, \ldots, x_N$, but the corresponding periods have changed and are now $T_1 = \int_0^{p_1} l_1(s)\, ds$, $\ldots$, $T_N = \int_0^{p_N} l_N(s)\, ds$, respectively. With other words, $\widehat{H}$ is now admissible with respect to the basis $\mathcal{B} = \{T_1 b_1, \ldots, T_N b_n\}$, where $\{b_1, \ldots, b_N\}$ is the canonical basis of $\mathbb{R}^N$. Moreover, the change of variables transforms the tube $\mathcal{T}$ into $\mathbb{R}^N \times]-1, 1[^N$, which is now twisted by the flow of $\widehat{H}$. We are thus reduced to the first step, and the result follows. □

8 Applications

In this section, we illustrate how our theorems may be applied to two types of situations, which we call *pendulum-like systems*, and *weakly-coupled superlinear systems*. For briefness, we only concentrate in the search of T-periodic solutions, but the experienced reader will recognize the possibility of proving the existence of periodic solutions of the second kind, for the pendulum-like systems, and of subharmonic solutions, for the superlinear systems. The stated results provide existence of $N+1$ solutions. Needless to say, the number of solutions we find will be 2^N in the nondegenerate situation.

8.1 Pendulum-like systems

One year after the publication of the 1983 paper by Conley and Zehnder [21], Mawhin and Willem [55] studied some pendulum-like scalar second order differential equations, by the use of a variational method. They proved that, if the T-periodic forcing term has zero mean value, then there are at least two T-periodic solutions. They thus improved previous results by Hamel [42], Dancer [22] and Willem [71], where the existence of one periodic solution had been proved, in the same setting. The papers by Conley – Zehnder and Mawhin – Willem attracted a lot of attention. They were further extended in [6, 16, 29, 32, 33, 44, 50, 53, 54, 66, 68], always using variational methods. As observed by Rabinowitz [66] in 1988, the Mawhin – Willem result could have been obtained (in the smooth case) from the Conley – Zehnder theorem, after a suitable modification of the nonlinearity. Alternatively, as noticed in [35, 37], it could also have been obtained directly from some generalized version of the Poincaré – Birkhoff theorem.

In this subsection, we exploit this idea and study Hamiltonian systems whose behavior reminds that of pendulum-like equations. Our main result will be the following

Theorem 8.1. *Let the Hamiltonian $H : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$ be admissible, and assume that*

$$\lim_{|y|\to\infty} \frac{\nabla_x H(t,x,y)}{|y|} = 0\,, \quad \text{uniformly in } (t,x) \in [0,T] \times \mathbb{R}^N\,. \tag{25}$$

If, moreover, there are two positive constants r, ρ and a regular $N \times N$ matrix $\mathbb{A}$, having only real eigenvalues, such that

$$|y| \geq r \quad \Rightarrow \quad \langle \nabla_y H(t,x,y), \mathbb{A}y \rangle > \rho\, |\nabla_y H(t,x,y)|\, |y|\,, \tag{26}$$

then the Hamiltonian system (HS) has at least $N+1$ distinct T-periodic solutions.

Proof. Let $z(t) = (x(t), y(t))$ be a solution of (HS), with $z(0) = z_0 = (x_0, y_0)$. Even if no uniqueness is assumed, we will denote such a solution by $z(t; z_0)$, and similarly for its components. By (25) and the periodicity of H in the x_i variables, such a solution has to be globally defined on $[0,T]$. Let us first prove that

$$\lim_{|y_0|\to\infty} \frac{y(t;x_0,y_0)-y_0}{|y_0|} = 0\,, \qquad \text{uniformly in } (t,x_0)\in[0,T]\times\mathbb{R}^N\,. \tag{27}$$

Fix $\varepsilon>0$. By (25), there is a $\bar\alpha>0$ such that

$$|y|\geq\bar\alpha \quad\Rightarrow\quad |\nabla_x H(t,x,y)|\leq\varepsilon|y|\,, \qquad \text{for every } (t,x)\in[0,T]\times\mathbb{R}^N.$$

Since the solutions of the initial value problems are globally defined, we can find a $\bar\beta>\bar\alpha$ such that, if $|y_0|\geq\bar\beta$, then $|y(t;x_0,y_0)|\geq\bar\alpha$, for every $t\in[0,T]$ and $x_0\in\mathbb{R}^N$. Hence,

$$|y_0|\geq\bar\beta \quad\Rightarrow\quad |\dot y(t;x_0,y_0)|\leq\varepsilon|y(t;x_0,y_0)|\,,$$

for every $(t,x_0)\in[0,T]\times\mathbb{R}^N$.Then,

$$|y(t;x_0,y_0)|\leq|y_0|+\int_0^t|\dot y(s;x_0,y_0)|\,ds\leq|y_0|+\varepsilon\int_0^t|y(s;x_0,y_0)|\,ds\,,$$

so that, by the Gronwall Lemma,

$$|y(t;x_0,y_0)|\leq|y_0|e^{\varepsilon t}\leq|y_0|e^{\varepsilon T}\,, \qquad \text{for every } (t,x_0)\in[0,T]\times\mathbb{R}^N\,.$$

Then,

$$|y(t;x_0,y_0)-y_0|\leq\int_0^t|\dot y(s)|\,ds\leq\varepsilon\int_0^t|y(s)|\,ds\leq\varepsilon Te^{\varepsilon T}|y_0|\,,$$

for every $(t,x_0)\in[0,T]\times\mathbb{R}^N$, thus proving (27). We can then easily see that

$$\lim_{|y_0|\to\infty}\left(\frac{y(t;x_0,y_0)}{|y(t;x_0,y_0)|}-\frac{y_0}{|y_0|}\right)=0\,, \qquad \text{uniformly in } (t,x_0)\in[0,T]\times\mathbb{R}^N\,. \tag{28}$$

We now first assume the matrix $\mathbb{A}$ to be symmetric. Since the solutions of the initial value problems are globally defined, we can find an $R>r$ such that, if $(x(t),y(t))$ is a solution of (HS) with $(x(0),y(0))=(x_0,y_0)$ and $|y_0|\geq R$, then $|y(t)|\geq r$, for every $t\in[0,T]$. Then, by (26) and (28), if R is sufficiently large,

$$\begin{aligned}\left\langle\frac{\nabla_yH(t,x(t),y(t))}{|\nabla_yH(t,x(t),y(t))|}\,,\mathbb{A}\frac{y_0}{|y_0|}\right\rangle&=\left\langle\frac{\nabla_yH(t,x(t),y(t))}{|\nabla_yH(t,x(t),y(t))|}\,,\mathbb{A}\frac{y(t)}{|y(t)|}\right\rangle+\\&\quad+\left\langle\frac{\nabla_yH(t,x(t),y(t))}{|\nabla_yH(t,x(t),y(t))|}\,,\mathbb{A}\left(\frac{y(t)}{|y(t)|}-\frac{y_0}{|y_0|}\right)\right\rangle>0\,.\end{aligned}$$

Consequently,

$$\begin{aligned}\langle x(T)-x(0),\mathbb{A}y(0)\rangle&=\int_0^T\langle\nabla_yH(t,x(t),y(t)),\mathbb{A}y(0)\rangle\,dt\\&=\int_0^T\left\langle\frac{\nabla_yH(t,x(t),y(t))}{|\nabla_yH(t,x(t),y(t))|}\,,\mathbb{A}\frac{y_0}{|y_0|}\right\rangle|y_0|\,|\nabla_yH(t,x(t),y(t))|\,dt>0\,.\end{aligned}$$

The conclusion follows from Theorem 2.1*(a)*, taking $D=B(0,R)$.

Let us now treat the case when $\mathbb{A}$ is diagonalizable. Let Q be a regular matrix for which $Q^{-1}\mathbb{A}Q$ is diagonal, and set $P = (Q^*)^{-1}$. With the canonical change of variables (23) we get a new Hamiltonian system, with Hamiltonian function H_1 defined as in (24). It is then easily seen that (25) implies

$$\lim_{|y_1|\to\infty} \frac{\nabla_{x_1} H_1(t, x_1, y_1)}{|y_1|} = 0\,, \quad \text{uniformly in } (t, x_1) \in [0, T] \times \mathbb{R}^N\,. \tag{29}$$

On the other hand, (26) implies the existence of some positive constants r_1, ρ_1 such that

$$|y_1| \geq r_1 \quad \Rightarrow \quad \langle \nabla_{y_1} H_1(t, x_1, y_1), Q^{-1}\mathbb{A}Q y_1 \rangle > \rho_1\, |y_1|\, |\nabla_{y_1} H_1(t, x_1, y_1)|\ ,$$

so that we are reduced to the case of a diagonal (hence symmetric) matrix.

Finally, let $\mathbb{A}$ be any regular matrix having only real eigenvalues. Then, $\mathbb{A}$ can be approximated by diagonalizable matrices: there is a sequence $(\mathbb{A}_n)_n$ of matrices, all of which are regular and have distinct real eigenvalues, which converges to $\mathbb{A}$ in the usual operator norm topology. This can be easily seen using the Jordan canonical form (see, e.g., [45, Ch. 3]). Then, if $|y| \geq r$, taking n large enough, we have

$$\begin{aligned}
&\left\langle \nabla_y H(t, x, y), \mathbb{A}_n \frac{y}{|y|} \right\rangle = \\
&\qquad = \left\langle \nabla_y H(t, x, y), \mathbb{A} \frac{y}{|y|} \right\rangle + \left\langle \nabla_y H(t, x, y), (\mathbb{A}_n - \mathbb{A}) \frac{y}{|y|} \right\rangle \\
&\qquad > \frac{\rho}{2}\, |\nabla_y H(t, x, y)|\,,
\end{aligned}$$

so that we are back to the previous case. The proof is thus completed. □

As a possible example of application, we can deal with second order systems of the type

$$\ddot{x} + \nabla F(t, x) = e(t)\,,$$

where $F(t, x_1, \ldots, x_N)$ is 2π-periodic with respect to each variable $x_1, \ldots, x_N$ (so ∇F is bounded), and $e : \mathbb{R} \to \mathbb{R}^N$ is a T-periodic forcing with zero mean value, i.e.,

$$\int_0^T e(t)\, dt = 0\,. \tag{30}$$

(As an example, if $N = 1$, we have in mind the pendulum equation.) Writing the equivalent Hamiltonian system

$$\dot{x} = y + E(t)\,, \quad \dot{y} = -\nabla F(t, x)\,,$$

with $E(t) = \int_0^t e(s)\, ds$, we see that Theorem 8.1 directly applies, taking as $\mathbb{A}$ the identity matrix. Similar results have been obtained in [53, 66].

Another example is given by equations of the type

$$\frac{d}{dt}\left(\nabla\Phi \circ \dot{x}\right) + \nabla F(t,x) = e(t)\,, \tag{31}$$

where Φ is a real valued, strictly convex C^1-smooth function defined on a ball $B(0,a) \subseteq \mathbb{R}^N$, with $\nabla\Phi : B(0,a) \to \mathbb{R}^N$ being a homeomorphism, and $\nabla\Phi(0) = 0$. Denoting by Φ^* the Legendre−Fenchel transform of Φ, we can write the equivalent Hamiltonian system

$$\dot{x} = \nabla\Phi^*(y + E(t))\,, \quad \dot{y} = -\nabla F(t,x)\,. \tag{32}$$

Recall that $\nabla\Phi^* = (\nabla\Phi)^{-1} : \mathbb{R}^N \to B(0,a)$ and, since Φ^* is strictly convex and coercive, it satisfies

$$\liminf_{|y|\to\infty} \frac{\langle \nabla\Phi^*(y), y\rangle}{|y|} > 0\,.$$

So, assuming (30), Theorem 8.1 easily applies, again with $\mathbb{A} = I_N$. We thus obtain as a corollary a recent result by Mawhin [54], generalizing previous existence results in [6, 16]. As a particular case, one can take $\Phi(y) = 1 - \sqrt{1-|y|^2}$ (leading to the so-called 'relativistic operator').

A rather similar situation is encountered in a result by Golé [38, Theorem 42.2], where the Hamiltonian function is assumed to be *uniformly optical.* Under his assumptions, the gradient of the Hamiltonian with respect to the first state variable turns out to be bounded, while the gradient with respect to the second one satisfies our condition (26), for some positive definite matrix $\mathbb{A}$. Hence, his result can also be obtained from our theorem.

A variant of the above concerns the case when Φ is a strictly convex C^1-smooth function defined on the whole $\mathbb{R}^N$, with $\nabla\Phi : \mathbb{R}^N \to B(0,a)$ being a homeomorphism, and $\nabla\Phi(0) = 0$. As a particular case, one can take $\Phi(y) = 1 - \sqrt{1+|y|^2}$ (leading to the so-called 'mean curvature operator'). Let $h : [0,T] \to \mathbb{R}$ be such that

$$|\nabla F(t,x)| \le h(t)\,, \text{ for every } t \in [0,T] \text{ and } x \in \mathbb{R}^N.$$

Writing equation (31) as the equivalent system (32), if we want to apply Theorem 2.1, with $D = B(0, \frac{1}{2}a)$, we must be careful that the solutions $z(t) = (x(t), y(t))$ starting with $y(0) \in \overline{D}$ remain with $y(t)$ in a compact set contained in $B(0,a)$, for $t \in [0,T]$, so that the Hamiltonian function can be modified outside this set and extended to the whole space. This will be guaranteed if

$$2(\|h\|_1 + \|E\|_\infty) < a\,.$$

We thus generalize a result obtained in [35] for the scalar equation (see also [63], where bounded variation solutions are obtained).

To conclude this subsection, we observe that Theorem 2.1 can also be applied to recover a result by Josellis [47] where, besides (25), it was assumed that

$$\lim_{|y|\to\infty} \frac{\nabla_y H(t,x,y) - A(t)y}{|y|} = 0\,, \quad \text{uniformly in } (t,x) \in [0,T] \times \mathbb{R}^N\,,$$

and that the matrix $\mathbb{A} := \int_0^T A(t)\,dt$ be regular and symmetric. We omit the details, for briefness.

8.2 Generalized annuli and weakly-coupled superlinear systems

In this last part of the paper we present a theorem involving a generalized annulus, i.e., a product of planar annuli. We start recalling that, if $z : [t_1, t_2] \to \mathbb{R}^2$ is a continuous path such that $z(t) \neq (0,0)$ for every $t \in [t_1, t_2]$, its (counter-clockwise) *rotation number* around the origin is defined as

$$\mathrm{Rot}\,(z(t); [t_1, t_2]) = \frac{\theta(t_1) - \theta(t_2)}{2\pi}\,,$$

where $-\theta : [t_1, t_2] \to \mathbb{R}$ is a continuous determination of the argument along z, i.e. $z(t) = |z(t)|(\cos\theta(t), -\sin\theta(t))$. Since we are dealing with curves which are not closed in general, the rotation number may not be integer and can instead take arbitrary real values.

We shall work with Hamiltonian systems (HS) where the (continuous) Hamiltonian function $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$, $H = H(t,x,y)$ is T-periodic in time and continuously differentiable with respect to the state variables (x,y); however, in contrast with what was our framework until now, *we shall not assume any periodicity in the state variables* x_i. Instead, we assume that for each $i = 1, \dots, N$ we have selected two strictly star-shaped Jordan curves around the origin $\Gamma_1^i, \Gamma_2^i \subseteq \mathbb{R}^2$, such that

$$0 \in \mathcal{D}(\Gamma_1^i) \subseteq \overline{\mathcal{D}(\Gamma_1^i)} \subseteq \mathcal{D}(\Gamma_2^i)\,.$$

Here we denote by $\mathcal{D}(\Gamma)$ the open bounded region delimited by the Jordan curve Γ. We consider the generalized annular region

$$\mathcal{A} = \left[\overline{\mathcal{D}(\Gamma_2^1)} \setminus \mathcal{D}(\Gamma_1^1)\right] \times \cdots \times \left[\overline{\mathcal{D}(\Gamma_2^N)} \setminus \mathcal{D}(\Gamma_1^N)\right] \subset \mathbb{R}^{2N}\,.$$

Theorem 8.2. *Under the framework above, denoting $z_i(t) = (x_i(t), y_i(t))$, assume that every solution $z(t) = (z_1(t), \dots, \dots z_N(t))$ of (HS) departing from $z(0) \in \partial\mathcal{A}$, is defined on $[0,T]$ and satisfies*

$$z_i(t) \neq (0,0) \text{ for every } t \in [0,T] \text{ and } i = 1, \dots, N\,.$$

Assume finally that there are integer numbers $\nu_1, \ldots, \nu_N \in \mathbb{Z}$ such that, for each $i = 1, \ldots, N$ either

$$\mathrm{Rot}\,(z_i(t);[0,T]) \begin{cases} < \nu_i\,, & \text{if } z_i(0) \in \Gamma_1^i\,, \\ > \nu_i\,, & \text{if } z_i(0) \in \Gamma_2^i\,, \end{cases} \tag{33}$$

or

$$\mathrm{Rot}\,(z_i(t);[0,T]) \begin{cases} > \nu_i\,, & \text{if } z_i(0) \in \Gamma_1^i\,, \\ < \nu_i\,, & \text{if } z_i(0) \in \Gamma_2^i\,. \end{cases} \tag{34}$$

Then, the Hamiltonian system (HS) has at least $N+1$ distinct T-periodic solutions $z^0(t), \ldots, z^N(t)$, with $z^0(0), \ldots, z^N(0) \in \mathcal{A}$, such that

$$\mathrm{Rot}\,(z_i^k(t);[0,T]) = \nu_i\,, \text{ for every } k = 0, \ldots, N \text{ and } i = 1, \ldots, N\,.$$

Proof. Since the solutions $z(t)$ departing from $z(0) \in \mathcal{A}$ are defined on $[0,T]$ and none of their components attain the origin, we can find a constant $\delta_0 > 0$ such that $|z_i(t)| > 2\delta_0$, for every $t \in [0,T]$ and $i = 1, \ldots, N$, for each of those solutions. We now modify the Hamiltonian function near the origin, as follows. Let $\omega : \mathbb{R} \to \mathbb{R}$ be a C^∞-smooth function such that

$$\omega(r) = \begin{cases} 0\,, & \text{if } r \le \delta_0\,, \\ 1\,, & \text{if } r \ge 2\delta_0\,. \end{cases}$$

Then, we consider the new Hamiltonian system

$$\dot z = J\nabla \bar H(t,z)\,,$$

with

$$\bar H(t,z) = \omega\big(\min\{|z_i| : i = 1, \ldots, N\}\big) H(t,z)\,,$$

so that $\bar H(t,z) = 0$ when one of the components of z is too near the origin. This will not affect the solutions starting from $\mathcal{A}$. We now consider the (time-dependent) change of variables

$$x_i = \sqrt{2\rho_i}\,\cos\big(\theta_i - (2\pi/T)\nu_i t\big)\,, \quad y_i = -\sqrt{2\rho_i}\,\sin\big(\theta_i - (2\pi/T)\nu_i t\big)\,, \tag{35}$$

so to get the Hamiltonian system

$$\dot\theta_i = \frac{\partial \mathcal{H}}{\partial \rho_i}(t,\rho,\theta)\,, \quad \dot\rho_i = -\frac{\partial \mathcal{H}}{\partial \theta_i}(t,\rho,\theta)\,,$$

defined for $\theta = (\theta_1, \ldots, \theta_N) \in (\mathbb{R}/2\pi\mathbb{Z})^N$ and $\rho = (\rho_1, \ldots, \rho_N) \in \mathbb{R}^N$ with $\rho_i > 0$ for every i. Here,

$$\mathcal{H}(t,\theta,\rho) := \bar H(t,x,y) + \sum_{i=1}^N (2\pi/T)\nu_i \rho_i\,,$$

the variables x, y in the argument of $\bar{H}$ being related to θ, ρ by (35). Notice that the change of variables is justified if $z(0) \in \mathcal{A}$, since then $z_i(t) \neq (0,0)$ for every $t \in [0,T]$ and $i = 1, \dots, N$. This system can now be extended also when $\rho_i \leq 0$ for some i, by simply setting $\mathcal{H}(t,\theta,\rho) := \sum_{i=1}^{N} (2\pi/T)\nu_i \rho_i$ there. Now, Theorem 2.2 applies. Indeed, the star-shaped curves Γ_1^i, Γ_2^i are transformed into the continuous and 2π-periodic functions a_i, b_i, and the twist condition follows from (33), (34). Going back to the original variables, the proof is easily concluded. □

As an application, we consider the system

$$\begin{cases} \ddot{x}_1 + g_1(x_1) = \dfrac{\partial \mathcal{U}}{\partial x_1}(t, x_1, \dots, x_N)\,, \\ \dots \\ \ddot{x}_N + g_N(x_N) = \dfrac{\partial \mathcal{U}}{\partial x_N}(t, x_1, \dots, x_N)\,, \end{cases} \tag{36}$$

where the continuous function $\mathcal{U} : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is continuously differentiable in $x_1, \dots, x_N$. The result presented below generalizes the first part of [15, Theorem 3.1]. It can also be seen as a version for systems of the main theorem of [26]. Possibly, it can be adapted to situations where the retractive forces $g_1, \dots, g_N$ can have one or two singularities (cf. [31]).

Theorem 8.3. *Assume that*

$$\lim_{|s|\to\infty} \frac{g_i(s)}{s} = +\infty\,,$$

and that there is a constant $K > 0$ such that

$$\left| \frac{\partial \mathcal{U}}{\partial x_i}(t, x_1, \dots, x_N) \right| \leq K\,,$$

for every $i = 1, \dots, N$ and $(t, x_1, \dots, x_N) \in [0,T] \times \mathbb{R}^N$. Then, there is a positive integer $\bar{\nu}$ with the following property: for any fixed integers $\nu_1, \dots, \nu_N \geq \bar{\nu}$, system (36) *has at least $N+1$ distinct T-periodic solutions*

$$x^{(k)}(t) = (x_1^{(k)}(t), \dots, x_N^{(k)}(t))\,, \qquad k = 0, \dots, N\,,$$

such that each $x_i^{(k)}(t)$ has exactly $2\nu_i$ simple zeros in $[0, T[$.

Proof. We consider the equivalent Hamiltonian system

$$\dot{x}_i = y_i\,, \quad \dot{y}_i = -\frac{\partial \mathcal{U}}{\partial x_i}(t, x_1, \dots, x_N)\,, \qquad i = 1, \dots, N\,, \tag{37}$$

corresponding to the Hamiltonian function $H(t,x,y) := \frac{1}{2}|y|^2 + \mathcal{U}(t,x,y)$. Using the arguments from [19, Lemma 1], one checks that the solutions of our system are globally defined. Moreover, following the lines in [19, Lemma 2] one checks that, for every $r > 0$ there is some $R(r) > r$ such that, if a solution $z(t) = (x(t), y(t))$ of (37) satisfies $x_i(0)^2 + y_i(0)^2 \geq R(r)$ for some $i = 1, \dots, N$, then $x_i(t)^2 + y_i(t)^2 \geq r$ for every $t \in [0, T]$. In particular, $z_i(t) \neq (0, 0)$, for every $t \in [0, T]$ (i.e., the zeroes of x_i are simple), and we can therefore compute the rotation number $\mathrm{Rot}\,(z_i(t); [0, T])$. It is standard to show (see [19, Lemma 3]) that the superlinear growth of g_i implies that the negative angular speed of $z_i(t)$ grows to infinity as the amplitude $|z_i(t)|$ increases. Thus, arguing as in [19, Lemma 4], taking e.g. $\bar{R} = R(1)$, there is an integer $\bar{\nu} \geq 1$ such that

$$(i) \text{ if } |z_i(0)| = \bar{R}, \text{ then } -\mathrm{Rot}\,(z_i(t); [0, T]) < \bar{\nu}, \qquad i = 1, \dots, N\,.$$

Choose numbers $\nu_1, \dots, \nu_N \geq \bar{\nu}$; there is a constant $\widehat{R} > \bar{R}$ such that

$$(ii) \text{ if } |z_i(0)| = \widehat{R}, \text{ then } -\mathrm{Rot}\,(z_i(t); [0, T]) > \nu_i, \qquad i = 1, \dots, N\,.$$

Applying Theorem 8.2, we find $N + 1$ distinct T-periodic solutions whose i-th component satisfies $\mathrm{Rot}\,(z_i(t); [0, T]) = -\nu_i$, for every $i = 1, \dots, N$. It implies that x_i has $2\nu_i$ simple zeroes on $[0, T[$, thus concluding the proof. □

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