Coupled physics inverse problems and Jacobians of σ-harmonic mappings

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Introduction

Since the ’80s, a dominant theme in Inverse Problems has been:
To image the interior of an object from measurements taken in its exterior.
Consider the (direct) elliptic Dirichlet problem of finding a weak solution u to

$$\begin{cases} \operatorname{div}(\sigma \nabla u) = 0 & \text{in} \quad \Omega\,, \\ u = \varphi & \text{on} \quad \partial\Omega\,, \end{cases}$$

where Ω is a bounded connected open set in $\mathbb{R}^n$, $n \geq 2$, and $\sigma = \{\sigma_{ij}(x)\}$ satisfies uniform ellipticity

$$\begin{aligned} \sigma(x)\xi \cdot \xi &\geq K^{-1}|\xi|^2 \quad , \quad \text{for every } x, \xi \in \mathbb{R}^2\,, \\ \sigma^{-1}(x)\xi \cdot \xi &\geq K^{-1}|\xi|^2 \quad , \quad \text{for every } x, \xi \in \mathbb{R}^2\,. \end{aligned}$$

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The *Calderón's inverse problem* (EIT) is:
Find σ, given all pairs of Cauchy data

$$(u|_{\partial\Omega}, \sigma\nabla u \cdot \nu|_{\partial\Omega}) \ .$$

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- If $\sigma = \{\sigma_{ij}(x)\}$, nonuniqueness (Tartar '84).
- If $\sigma = \{\gamma(x)\delta_{ij}\}$, instability (Mandache '01).

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The problem

Monard and Bal '12, '13: reconstruction of σ from $\{H_{ij}\}$, provided $U = (u_1, \ldots, u_n)$ is a σ–harmonic mapping (i.e.: a n–tuple of solutions) such that

$$\det DU > 0, \text{ in } \Omega .$$

Question:
Can we find $\Phi = (\varphi_1, \ldots, \varphi_n)$, independent of σ, such that $\det DU > 0$ everywhere?

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$n = 2$. The Classical Results

Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain and let

$$\Phi = (\varphi_1, \varphi_2) : \partial\Omega \to \partial G,$$

be a homeomorphism. Consider

$$\begin{cases} \Delta U = 0, & \text{in} \quad \Omega, \\ U = \Phi, & \text{on} \quad \partial\Omega. \end{cases}$$

Theorem (H. Kneser '26)

If G is convex, then U is a homeomorphism of $\overline{\Omega}$ onto $\overline{G}$.

Posed as a problem by Radó ('26), rediscovered by Choquet ('45).

Theorem (H. Lewy '36)

If $U : \Omega \to \mathbb{R}^2$ is a harmonic homeomorphism, then it is a diffeomorphism.

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$n = 2$. Variable coefficients.

$$\begin{cases} \operatorname{div}(\sigma \nabla U) = 0, & \text{in} \quad \Omega, \\ U = \Phi, & \text{on} \quad \partial\Omega. \end{cases}$$

Let

$$\Phi : \partial\Omega \to \partial G,$$

be a homeomorphism, and let G be convex.

Theorem (Bauman-Marini-Nesi '01)

Assume Ω, G be $C^{1,\alpha}$–smooth, $\sigma \in C^{\alpha}$ and Φ a $C^{1,\alpha}$ diffeomorphism.

$$\begin{cases} \mathit{div}(\sigma \nabla U) = 0, & \mathit{in} \quad \Omega, \\ U = \Phi, & \mathit{on} \quad \partial\Omega. \end{cases}$$

then $U : \overline{\Omega} \to \overline{G}$ is a diffeomorphism.

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If we only assume $\sigma \in L^\infty$, then U is a homeomorphism of $\overline{\Omega}$ onto $\overline{G}$.

Theorem (A., Nesi '01)

If $U : \Omega \to \mathbb{R}^2$ is a $\sigma-$harmonic homeomorphism, then

$$|\det DU| > 0 \text{ a.e.} \quad .$$

In fact, $|\det DU|$ is a Muckenhoupt weight (A., Nesi '09) .

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An example

Meyers ('63). Fix $\alpha > 0$

$$\sigma(x) = \begin{pmatrix} \frac{\alpha^{-1}x_1^2+\alpha x_2^2}{x_1^2+x_2^2} & \frac{(\alpha^{-1}-\alpha)x_1x_2}{x_1^2+x_2^2} \\ \frac{(\alpha^{-1}-\alpha)x_1x_2}{x_1^2+x_2^2} & \frac{\alpha x_1^2+\alpha^{-1}x_2^2}{x_1^2+x_2^2} \end{pmatrix} .$$

σ has eigenvalues α and α^{-1}. Therefore σ satisfies uniform ellipticity. σ is discontinuous at $(0,0)$ (and only at $(0,0)$) when $\alpha \neq 1$. Denote

$$u_1(x) = |x|^{\alpha-1}x_1 \, , \\ u_2(x) = |x|^{\alpha-1}x_2 \, .$$

A direct calculation shows that $U = (u_1, u_2)$ is σ–harmonic. We compute

$$\det DU = \alpha|x|^{2(\alpha-1)} \, .$$

Therefore det DU vanishes at $(0,0)$ when $\alpha > 1$, when $\alpha \in (0,1)$, it diverges as $z \to 0$.

n=2. Proof sketch

Definition

A function $\varphi \in C(\partial\Omega; \mathbb{R})$ is called unimodal if $\partial\Omega$ can be split into two arcs Γ_1, Γ_2 such that φ is non-decreasing on Γ_1 and non-increasing on Γ_2.

$$\begin{cases} \operatorname{div}(\sigma \nabla u) = 0 & \text{in} \quad \Omega\,, \\ u = \varphi & \text{on} \quad \partial\Omega\,, \end{cases}$$

Lemma

If φ is unimodal, then the level lines of u are formed by simple arcs.

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If

$$\Phi : \partial\Omega \to \partial G\,,\ \ G \text{ convex}\,,$$

is a homeomorphism, then $\varphi = \Phi \cdot \xi$ *is unimodal for all* ξ, $|\xi| = 1$.

Hence

$$DU^T DU\xi \cdot \xi = |DU\xi|^2 = |\nabla(U \cdot \xi)|^2 > 0$$

everywhere and for all ξ, $|\xi| = 1$. Therefore, DU is nonsingular everywhere.

n=2. Quantitative assumptions

Let $\omega : [0,\infty) \to [0,\infty)$ be a continuous strictly increasing function such that $\omega(0) = 0$.

Definition

Given $m, M \in \mathbb{R}$, $m < M$, Given $\varphi \in C^{1,\alpha}(\partial\Omega;\mathbb{R})$ we shall say that it is quantitatively unimodal, if considering the arclength parametrization of $\partial\Omega$, $x = x(s)$, $0 \leq s \leq T = |\partial\Omega|$, the periodic extension of the function $[0,T] \ni s \to \varphi(s) \equiv \varphi(x(s))$ is such that there exists numbers $t_1 \leq t_2 < t_3 \leq t_4 < t_1 + T$ such that

$$\varphi(s) = m \,, s \in [t_1, t_2] \;,\; \varphi(s) = M \,, s \in [t_3, t_4] \,,$$
$$\varphi'(s) \geq \min\{\omega(s - t_2), \omega(t_3 - s)\} \;,\; s \in [t_2, t_3] \,,$$
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We will refer to the quadruple $\{T, m, M, \omega\}$ as to the "character of unimodality" of φ.

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Let

$$\Phi : \partial\Omega \to \mathbb{R}^2$$

be a $C^{1,\alpha}$ one-to-one mapping onto ∂G.

Definition

We say that Φ is quantitatively convex if for every $\xi \in \mathbb{R}^2$, $|\xi| = 1$ the function

$$\varphi = \Phi \cdot \xi$$

is quantitatively unimodal with character of $\{T, m_\xi, M_\xi, \omega\}$ with m_ξ, M_ξ such that $M_\xi - m_\xi \geq D$, for a given $D > 0$. We refer to the triple $\{T, D, \omega\}$ as to the "character of convexity" of Φ.

If ∂G is C^2 with positive curvature, then quantitatively convex mappings $\Phi : \partial\Omega \to \partial G$ are easily constructed.

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n=2. Quantitative bound

Theorem (A., Nesi '15)

Let Ω have $C^{1,\alpha}$ boundary, let σ be uniformly elliptic and C^{α}. Let $\Phi = (\varphi_1, \varphi_2) : \partial\Omega \to \partial G$ be a $C^{1,\alpha}$ quantitatively convex map with character $\{|\partial\Omega|, D, \omega\}$. Let $U = (u_1, u_2)$ solve

$$\begin{cases} div(\sigma \nabla u_i) = 0 & in \quad \Omega\,, \\ u_i = \varphi_i & on \quad \partial\Omega\,. \end{cases}$$

Then there exists $C > 0$, only depending on ellipticity, on the regularity assumptions and on the character of convexity of Φ such that

$$\det DU \geq C > 0 \quad in \quad \overline{\Omega}\,.$$

n=2. Proof sketch

It suffices to obtain a lower bound on $|\nabla u|$ where $u = (U \cdot \xi)$, uniformly w.r.t. ξ, $|\xi| = 1$.

Near the boundary we can use the quantitative unimodality and a Hopf-type lemma (Finn-Gilbarg '57).

In the interior we use the theory of Q.C. mappings. Using complex notation $z = x_1 + ix_2$, $u = \Re e\, f$,

$$f_{\bar{z}} = \mu f_z + \nu \overline{f_z} \quad \text{in } \Omega \ ,$$

where, the so called complex dilatations μ, ν only depend on σ and satisfy the ellipticity condition

$$|\mu| + |\nu| \leq k < 1 \ ,$$

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$$f_{\bar{z}} = \mu f_z + \nu \overline{f_z} \quad \text{in } \Omega \,,$$

Here, being σ Hölder continuous, also μ and ν satisfy a Hölder bound.
Let us denote $g = f^{-1}(w)$, $w \in \mathbb{C}$. A straightforward calculation gives

$$g_{\overline{w}} = -\nu(g) g_w - \mu(g) \overline{g_w} \,.$$

By interior regularity estimates, g_w is locally bounded.

$$\det Df^{-1} = \det Dg = |g_w|^2 - |g_{\overline{w}}|^2 \leq C^2 \,,$$

which can be rewritten as

$$\sigma \nabla u \cdot \nabla u = \det Df \geq C^{-2} \,,$$

at any fixed distance from the boundary.

n>2

Consider $\Omega \subset \mathbb{R}^n$, a bounded domain diffeomorphic to a ball of class $C^{1,\alpha}$. Let σ satisfy uniform ellipticity and Hölder continuity.

Let $G \subset \mathbb{R}^n$ be a convex body with C^2 boundary and having at each point principal curvatures bounded from below by $\kappa > 0$.

Let $\Phi : \partial\Omega \to \partial G$ be an orientation preserving diffeomorphism such that Φ, Φ^{-1} are $C^{1,\alpha}$. Let U be the weak solution to

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Denote $\Omega_\rho = \{x \in \Omega \; dist(x, \partial\Omega) > \rho\}$.

Theorem (A., Nesi '15)

There exists $\rho > 0$ and $Q > 0$ such that U is a diffeomorphism of $\overline{\Omega}\backslash\Omega_\rho$ onto a neighborhood of ∂G, within $\overline{G}$ and we have

$$\det DU \geq Q \quad in \quad \overline{\Omega}\backslash\Omega_\rho\,.$$

Examples

Wood '91:

$$U(x_1, x_2, x_3) = (x_1^3 - 3x_1x_3^2 + x_2x_3, x_2 - 3x_1x_3, x_3)$$

U is a homeomorphism, but $\det DU = 0$ on the plane $\{x_1 = 0\}$.

Laugesen '96:
$\forall \varepsilon > 0 \ \exists \Phi : \partial B \mapsto \partial B$ homeomorphism, such that $|\Phi(x) - x| < \varepsilon, \forall x \in \partial B$ and the solution $U = (u_1, u_2, u_3)$ to

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is not one-to-one.

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Examples

Briane, Milton, Nesi '04:
Set $Q = [0,1]^3 \subset \mathbb{R}^3$, assume σ Q–periodic and consider the *cell* problem

$$\begin{cases} \operatorname{div}(\sigma \nabla U) = 0, & \text{in} \quad \mathbb{R}^3, \\ (U - x)\ Q - \text{periodic} & . \end{cases}$$

There exists an isotropic matrix $\sigma = \gamma I$, with γ taking only two values, with a smooth interface, such that det DU changes its sign in the interior of the cube Q of periodicity.

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Examples

Capdeboscq, March 2015! (elaborating on the Briane-Milton-Nesi example):

Consider

$$\begin{cases} \Delta H = 0, & \text{in} \quad \Omega \subset \mathbb{R}^3, \\ H = \Phi, & \text{on} \quad \partial\Omega. \end{cases}$$

For any Φ such that $\det DH > 0$ everywhere in Ω, there exist $\sigma \in C^\infty$ and isotropic such that, considering

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Capdeboscq, March 2015! (elaborating on the Briane-Milton-Nesi example):

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Open Problem: To control, in terms of the Dirichlet data, the size (or the dimension) of the set of points where the Jacobian may degenerate and possibly evaluate the vanishing rate at such points of degeneration.

Han and Lin, 2000: Let $\sigma \in C^\infty$. If U is nonconstant, then, locally, the set

$$\{rank\ DU = 0\}$$

has finite $n-2$-dimensional Hausdorff measure.
If U is nonconstant, and $U(\Omega)$ is not contained in a straight line, then, locally, the set

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Jin and Kazdan ’91:
$\exists \sigma \in C^\infty$ and a solution $U = (u_1, u_2, u_3)$ to

$$\mathrm{div}\,(\sigma \nabla U) = 0 \ \text{ in } \mathbb{R}^3 \ ,$$

such that

$$\begin{cases} rank\, DU = 2, & \text{for} \quad x_3 \leq 0 \ , \\ det DU > 0, & \text{for} \quad x_3 > 0 \ . \end{cases}$$

Let $a \in C^\infty(\mathbb{R};\mathbb{R})$ and set

$$\sigma(x) = \begin{pmatrix} 1 & a(x_3) & 0 \\ a(x_3) & 1 & 0 \\ 0 & 0 & b(x_3) \end{pmatrix},$$

with

$$\begin{cases} a(x_3) = 0 & \text{for} \quad x_3 \leq 0, \\ a(x_3) \in (0, a_0) & \text{for} \quad x_3 > 0 \quad \text{with} \quad a_0 \in (0,1), \\ b(x_3) = \frac{1}{1-a^2(x_3)} & \text{for} \quad x_3 \in \mathbb{R}. \end{cases}$$

We set

$$U(x) = (x_1, x_2, -x_1 x_2 + v(x_3)) \ ,$$

where v is chosen in such a way that

$$\begin{cases} (bv')' - 2a = 0 \, , & x_3 \in \mathbb{R} \, , \\ v(x_3) = 0 \, , & x_3 < 0 \, . \end{cases}$$

It turns out that $v' > 0$ for $x_3 > 0$ and consequently

$$\det DU = \begin{cases} v' > 0 \, , & \text{for} \quad x_3 > 0 \, , \\ v' = 0 \, , & \text{for} \quad x_3 \leq 0 \, . \end{cases}$$

U maps $\{x_3 \leq 0\}$ into the surface

$$\{x_3 = -x_1 x_2\} \ .$$

The end.

THANKS!