The inverse Calderón problem,
ill-posedness and remedies.

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The basic direct problem

Consider the (direct) elliptic Dirichlet problem of finding a weak solution $u \in H^1(\Omega)$ to

$$\begin{cases} \mathrm{div}(\gamma \nabla u) = 0 & \text{in} \quad \Omega \,, \\ u = \varphi & \text{on} \quad \partial\Omega \,, \end{cases}$$

where Ω is a bounded connected open set in $\mathbb{R}^n$, $n \geq 2$, the function $\gamma \in L^\infty$ (conductivity), satisfies

$$0 < \lambda \leq \gamma \leq \lambda^{-1}, \text{ a. e. in } \Omega \,,$$

and $\varphi \in H^{1/2}(\partial\Omega)$ is prescribed.

The Calderón's inverse problem

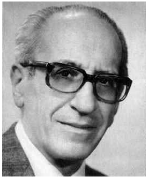


Figure: Alberto P. Calderón, '40-'80.

The *Dirichlet-to-Neumann map* is the operator

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \ni \varphi \to \gamma \nabla u \cdot \nu|_{\partial\Omega} \in H^{-1/2}(\partial\Omega) \ ,$$

where ν is the exterior unit normal to $\partial\Omega$.
The *Calderón's inverse problem* is:

Find γ when Λ_γ is known.

In other words:

Find γ, given all pairs of Cauchy data

Ill-posed?

Figure: Jaques Hadamard, 1902.

A problem is *well-posed* if:

- The solution is unique.
- The solution exists.
- The solution continuously depends upon the data.

Ill-posed?

- Uniqueness:
  Kohn and Vogelius '84,'85, Sylvester and Uhlmann '87, Nachman '96, Astala and Päivärinta 2006 ...
- Existence?
  Reconstruction: Nachman '88, Novikov '88 ...
  Algorithms: ...
- Continuous dependence upon the data.

The Cauchy problem

Hadamard's example, '23

$$\begin{cases} \Delta u = 0, & \text{in } \{(x,y) \in \mathbb{R}^2 \mid y > 0\}, \\ u(x,0) = 0, & \text{for every } x \in \mathbb{R}, \\ u_y(x,0) = A_n \sin nx, & \text{for every } x \in \mathbb{R}. \end{cases}$$

The solution $u = u_n$ is

$$u_n = \frac{A_n}{n} \sin nx \sinh ny.$$

If we choose $A_n = n^{-1}$ then

$$u_{n,y}(x,0) \to 0 \text{ uniformly as } n \to \infty$$

whereas, for any $y > 0$,

$$u_n(x,y) = \frac{A_n}{n} \sin nx \sinh ny \text{ blows up as } n \to \infty.$$

Conditional stability

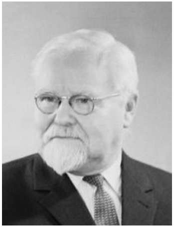


Figure: Andrey N. Tikhonov '43.

Continuous dependence, in ill-posed problems, can be restored in presence of an a-priori bound.

Conditional stability for the Cauchy problem

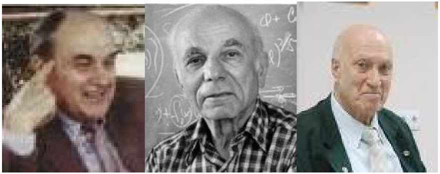


Figure: Carlo Pucci '55, Fritz John '55, Mikhail M. Lavrentev '56.

Optimal stability for the Cauchy problem

Consider a Cauchy problem in a rectangle

$$\begin{cases} \Delta u = 0, & \text{in } (0,\pi) \times (0,1), \\ u(x,0) = 0, & \text{for every } x \in (0,\pi), \\ u_y(x,0) = \psi(x), & \text{for every } x \in (0,\pi), \end{cases}$$

Let us assume the energy bound

$$\iint_{(0,\pi)\times(0,1)} \left(u_x^2 + u_y^2\right) dxdy \leq E^2, \tag{1}$$

for a given $E > 0$.

Let us assume that the following error bound is known

$$\|\psi\|_{H^{-\frac{1}{2}}(0,\pi)} \leq \varepsilon, \tag{2}$$

for some given $\varepsilon > 0$.
Let us choose once more

$$\psi_n(x) = A_n \sin nx,\ n = 1, 2, \ldots$$

and let us select A_n in such a way that equality holds in the energy bound (1). We obtain

$$A_n^2 = \frac{2}{\pi} \frac{2n}{\sinh 2n} E^2.$$

Consequently, in (2) we have equality when $\varepsilon = \varepsilon_n$ is given by

$$\varepsilon_n^2 \sim 4E^2 e^{-2n}, \text{ as } n \to \infty.$$

If we wish to estimate the L^2-norm of u in the rectangle $(0,\pi)\times(0,T)$, for some $T\in(0,1]$, then we see that the solution u_n with the given ψ satisfies

$$\|u_n\|_{L^2((0,\pi)\times(0,T))} \sim \frac{E}{\sqrt{2}}\left(\frac{\varepsilon_n}{2E}\right)^{(1-T)}\left(\log\frac{2E}{\varepsilon_n}\right)^{-1}, \text{ as } n\to\infty.$$

Therefore, if $T<1$, then the stability of the determination of *u up to the level $y=T$* is *at best* of Hölder type.

Whereas, if we want to recover u up to the top side of the rectangle then the best possible rate of stability is logarithmic.

Stability for the Cauchy problem

the state of the art

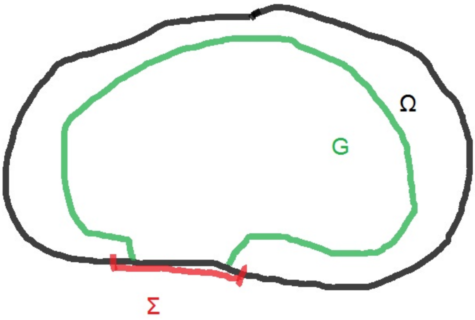
Let $A = \{A_{ij}(x)\}$ be uniformly elliptic and Lipschitz continuous (if $n \geq 3$). Let u solve

$$\begin{cases} div\,(A\nabla u) = 0, & \text{in } \Omega, \\ u = \varphi, & \text{on } \Sigma, \\ A\nabla u \cdot \nu = \psi, & \text{on } \Sigma, \end{cases}$$

with

$$\|\varphi\|_{H^{\frac{1}{2}}(\Sigma)} + \|\psi\|_{H^{-\frac{1}{2}}(\Sigma)} \leq \varepsilon.$$

Stability in the interior



Stability in the interior

Figure: Lawrence E. Payne, '70.

If

$$\|u\|_{L^2(\Omega)} \le E \ ,$$

then

$$\|u\|_{L^2(G)} \le C\varepsilon^{\delta}(E+\varepsilon)^{1-\delta}.$$

Stability up to the boundary

If

$$\|u\|_{H^1(\Omega)} \leq E,$$

then

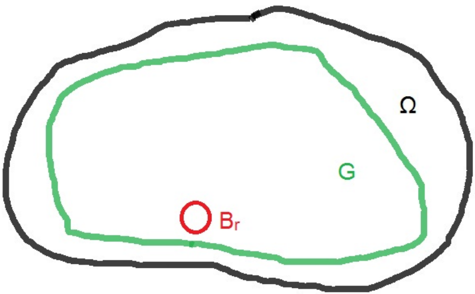
$$\|u\|_{L^2(\Omega)} \leq E\omega\left(\frac{\varepsilon}{E}\right),$$

where

$$\omega(t) \leq C\left(\log\frac{1}{t}\right)^{-\mu}, \quad \text{for } t < 1.$$

Continuation from an open set

Propagation of smallness



Analogous results apply if Cauchy data are replaced with values on a (small) open set.

The three-spheres inequality

Figure: Evgenii M. Landis ’63.

Let u solve

$$div(A\nabla u) = 0, \quad \text{in } B_R \,.$$

Then, for every r_1, r_2, r_3, with $0 < r_1 < r_2 < r_3 \leq R$,

$$\|u\|_{L^2(B_{r_2})} \leq C \left(\|u\|_{L^2(B_{r_1})}\right)^{\alpha} \left(\|u\|_{L^2(B_{r_3})}\right)^{1-\alpha},$$

where $C > 0$ and α, $0 < \alpha < 1$, only depend on ellipticity, Lipschitz regularity of A and $\frac{r_2}{r_1}$ and $\frac{r_3}{r_2}$. Hadamard, 1896!

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The Calderón problem, examples.

Checkerboard

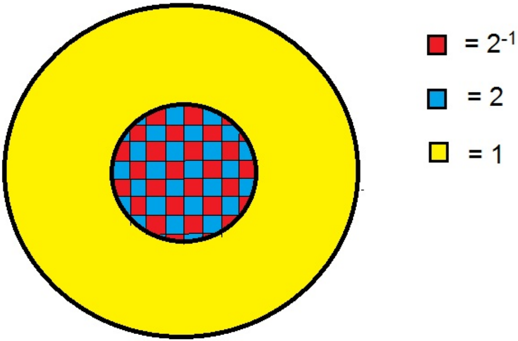
A., Cabib 2008. Let $g = g(x, y)$ be 1–periodic in x and y separately, such that in the unit square $Q = \{(x, y) \in \mathbb{R}^2 | |x| \leq \frac{1}{2} , |y| \leq \frac{1}{2}\}$ is defined as follows

$$g(x, y) = \begin{cases} 2^{-1} & \text{if} \quad xy \geq 0 \quad , \\ 2 & \text{if} \quad xy < 0 \quad . \end{cases}$$

For any $h = 1, 2, \ldots$, we define

$$\gamma_h(x, y) = \begin{cases} 1 & \text{if} \quad (x, y) \in B_1 \setminus B_{\frac{1}{2}} \quad , \\ g(hx, hy) & \text{if} \quad (x, y) \in B_{\frac{1}{2}} \quad , \end{cases}$$

Checkerboard



Checkerboard

We have

$$\lim_{h\to\infty} \Lambda_{\gamma_h} = \Lambda_1 \ ,$$

in the $\mathcal{L}(H^{1/2}(\partial B_1), H^{-1/2}(\partial B_1))$–norm, whereas

$$\lim_{h\to\infty} \int_{B_1} \gamma_h = \frac{17}{16}\pi \neq \pi = \int_{B_1} 1 \ .$$

In fact

$$\gamma_h \overset{*}{\rightharpoonup} 1 + \frac{1}{4}\chi_{B_{\frac{1}{2}}} \ , \text{in } L^\infty$$

whereas

$$\gamma_h \overset{G}{\to} 1 \ .$$

The example by Mandache, 2001

For every $m = 1, 2, \ldots$ there exists $E > 0$ such that for every $\eta > 0$ there exist conductivities γ_1, γ_2 such that

$$\|\gamma_1\|_{C^m}, \|\gamma_2\|_{C^m} \leq E ,$$

$$\|\gamma_1 - \gamma_2\|_\infty \geq \eta ,$$

and

$$\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\| \leq \exp\left(-\eta^{\alpha}\right) .$$

Stability for the Calderón problem

A. '88,'90. Let $n \geq 3$. If

$$\|\gamma_1\|_{C^2}\,, \|\gamma_2\|_{C^2} \leq E\,,$$

and

$$\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\| \leq \varepsilon\,.$$

Then

$$\|\gamma_1 - \gamma_2\|_\infty \leq E\omega\left(\frac{\varepsilon}{E}\right),$$

where

$$\omega(t) = C\left(\log\frac{1}{t}\right)^{-\mu}, \quad \text{for } t < 1.$$

Improvements

- Stability with less and less regularity assumptions. $n = 2$: Liu, '97; J.A. Barceló, T. Barceló, Ruiz, 2001; T. Barceló, Faraco, Ruiz, 2007; Clop, Faraco, Ruiz, 2010. $n \geq 3$: Caro, García, Reyes, 2012.
- Improved stability under stronger regularity assumptions: Novikov 2011. Better estimate of the exponent μ in the modulus of continuity

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Lipschitz stability

A., Vessella 2005
Assume

$$\gamma(x) = \sum_{j=1}^{N} \gamma_j \chi_{D_j}(x),$$

With given D_j internally disjoint, with piecewise smooth boundaries, and $\gamma_1, \ldots, \gamma_N$ unknown constants. Then the map

$$\Lambda_\gamma \mapsto \gamma$$

is Lipschitz continuous.
Beretta, Francini 2011, same result with complex valued γ.
de Hoop, Qiu, Scherzer 2012, when Lipschitz stability is available, then fast convergent recursive algorithm.

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...warning

Consider the map $F : \mathbb{R} \to \mathbb{R}^3$ given by $x = F(t)$, where

$$\begin{aligned} x_1 &= (2+\cos 2\pi\alpha t)\cos 2\pi t\ , \\ x_2 &= (2+\cos 2\pi\alpha t)\sin 2\pi t\ , \\ x_3 &= \sin 2\pi\alpha t\ ,\ \ t \in \mathbb{R}\ , \end{aligned} \tag{3}$$

where α is a parameter. This is a curve winding infinitely many times around the torus

$$T = \left\{ x \in \mathbb{R}^3 \,\middle|\, x_3^2 + \left(\sqrt{x_1^2 + x_2^2} - 2\right)^2 = 1 \right\} .$$

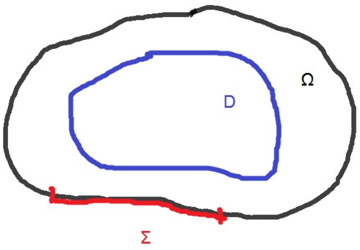
F is smooth, with nonsingular differential and thus it is smoothly locally invertible. If α is *irrational*, F is globally one-to-one, and $F(\mathbb{R})$ is dense into T hence...

...warning

F^{-1} is discontinuous at every point,

If F is restricted to a bounded interval $[-L, L]$, then indeed F^{-1} is globally Lipschitz, but the Lipschitz constant may blow up as α tends to any rational number!

Local measurements



A., K. Kim, 2012. We introduce the local Dirichlet to Neumann map

$$\Lambda_\gamma^\Sigma : \varphi \text{ supported in } \Sigma \mapsto \gamma \nabla u \cdot \nu \text{ restricted to } \Sigma\ .$$

Consider an open subset D strictly contained in Ω.

Local measurements

On the unknown conductivity γ we shall assume that it is precisely known outside D. That is, we assume that we are given a reference conductivity γ_0 and the unknown γ satisfies

$$\gamma = \gamma_0 \text{ in } \Omega \setminus \overline{D}\ .$$

and also the following a-priori regularity bound

$$\|\gamma\|_{C^2} \leq E\ .$$

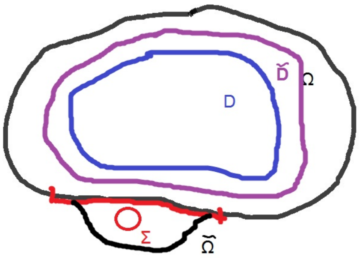
Then the map

$$\Lambda_\gamma^\Sigma \mapsto \gamma$$

is continuous, with a logarithmic modulus of continuity.

Local measurements,

idea of proof



the map

$$\Lambda_\gamma^\Sigma \mapsto \Lambda_\gamma^{\partial \widetilde{D}}$$

is Hölder continuous. Tool: Propagation of smallness.

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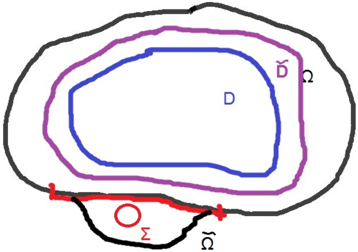

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Partial data,

various formulations

Uniqueness: Bukhgeim and Uhlmann, 2002; Ammari and Uhlmann, 2004; Kenig, Sjöstrand and Uhlmann, 2007; Isakov 2007; . . .
Stability: Heck and Wang, 2006; Fathallah 2007; Caro, Dos Santos Ferreira and Alberto Ruiz 2012.

The end.

¡MUCHAS GRACIAS!

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