The inverse crack problem

open issues and some new result

Giovanni Alessandrini[1]

[1] DMI Università di Trieste

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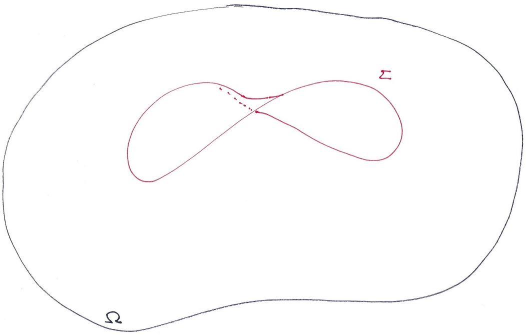
The basic problem



Consider a body $\Omega \subset \mathbb{R}^3$ which might contain an unknown, inaccessible, crack represented by $\Sigma \subset\subset \Omega$, a two-dimensional orientable surface with boundary .

We wish to recover Σ from electrostatic measurements taken on $\partial\Omega$.

The basic problem

Perfectly insulating crack

The direct problem. Given Σ and given ψ on $\partial\Omega$ such that $\int_{\partial\Omega}\psi = 0$, find u such that

$$\begin{cases} \Delta u = 0, & \text{in} & \Omega \setminus \Sigma, \\ \nabla u \cdot \nu^{\pm} = 0, & \text{on either side of} & \Sigma, \\ \nabla u \cdot \nu = \psi, & \text{on} & \partial\Omega. \end{cases}$$

The inverse problem. Find Σ given $u_k|_{\partial\Omega}$, $k = 1, \ldots, K$, with the potentials u_k corresponding to *suitable* finitely many choices of $\psi = \psi_1, \ldots, \psi_K$.

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Variants

Perfectly conducting crack.

$$\begin{cases} \Delta u = 0, & \text{in} \quad \Omega \setminus \Sigma, \\ u = const., & \text{on} \quad \Sigma, \\ \nabla u \cdot \nu = \psi, & \text{on} \quad \partial\Omega. \end{cases}$$

Crack with impedance.

$$\begin{cases} \Delta u = 0, & \text{in} \quad \Omega \setminus \Sigma, \\ -\nabla u \cdot \nu^{\pm} + \gamma^{\pm} u^{\pm} = 0, & \text{on either side of} \quad \Sigma, \\ \nabla u \cdot \nu = \psi, & \text{on} \quad \partial\Omega. \end{cases}$$

here ν^{+}, ν^{-} are the unit outward normal vectors on the two sides of Σ and γ^{+}, γ^{-} are nonnegative functions.

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Further variants

2D model. $\Omega \subset \mathbb{R}^2$, Σ simple arc.

Multiple cracks. Σ disjoint union of finitely many cracks Σ_j .

Full boundary data.

$$N_\Sigma : \nabla u \cdot \nu|_{\partial\Omega} \to u|_{\partial\Omega} \ .$$

or otherwise

$$\Lambda_\Sigma : u|_{\partial\Omega} \to \nabla u \cdot \nu|_{\partial\Omega} \ .$$

Inhomogeneous (known) background. Replace Δ with, for instance, $\mathrm{div}\,(\gamma\nabla\cdot)$, $\gamma = \gamma(x) > 0$ known.

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Results in 2D.

Uniqueness.

Single crack: Friedman-Vogelius '89.

- Two measurements are necessary.
- Two suitable measurements suffice.
- Duality conducting-insulating cracks.

Multiple cracks: Bryan-Vogelius '92. A.-Diaz Valenzuela '96. Kim-Seo '96.

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A. ’93. A.-Rondi ’99, Rondi ’99, Rondi ’05.
Under a-priori regularity (Lipschitz) assumptions on the unknown (multiple) crack the mapping

$$\text{data} \rightarrow \text{crack}$$

is continuous with a logarithmic modulus of continuity.

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Kubo '91. Uniqueness for a planar perfectly insulating crack with 3 suitable measurements.

Eller '96. Uniqueness for a crack with impedance with full boundary data.

A.-DiBenedetto '97.

1. Uniqueness for multiple perfectly conducting cracks with 2 suitable measurements.
2. Stability for a planar perfectly conducting crack.
3. Uniqueness for multiple perfectly insulating planar cracks with 2 suitable measurements.

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Main open problems in 3D.

1. Uniqueness and stability for cracks in known inhomogeneous medium.
2. Uniqueness for curved insulating cracks, with finitely many measurements.
3. Stability for curved cracks, conducting and insulating.

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Conducting crack in 3D.

The result of uniqueness in A.-DiBenedetto is as follows.

Given three distinct points $P, Q_1, Q_2 \in \partial\Omega$, prescribe boundary current densities $\psi_1 = \delta_P - \delta_{Q_1}, \psi_2 = \delta_P - \delta_{Q_2}$. The corresponding boundary potentials $u_1|_{\partial\Omega}, u_2|_{\partial\Omega}$ uniquely determine Σ.

Question. Can this result be extended to inhomogeneous media?

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One crucial step in the proof.

Let S be a regular connected surface in an open set $G \subset \mathbb{R}^3$, let S' be a nonempty open subset of S. Let u and v be two harmonic functions in $G \subset \mathbb{R}^3$. If $u \equiv c = const.$ on S and $v \equiv b = const.$ on S' , then $v \equiv b$ on all of S.

Unique continuation along level surfaces.

Problem (A.-Favaron, '00)

Let S, S' be as above. Let $\mathcal{L}$ be a second order elliptic operator with Lipschitz coefficients in the principal part. Let u, v solve

$$\mathcal{L}u = \mathcal{L}v = 0 \text{ in } G\,.$$

Is it true that, if $u \equiv c = const.$ on S and $v \equiv b = const.$ on S', then $v \equiv b$ on all of S?

Uniqueness in 2D, in a nutshell.

$$\begin{cases} \Delta u = 0, & \text{in} \quad \Omega \setminus \Sigma, \\ \nabla u \cdot \nu^{\pm} = 0, & \text{on either side of} \quad \Sigma, \\ \nabla u \cdot \nu = \psi, & \text{on} \quad \partial\Omega. \end{cases}$$

There exist pairs of boundary current densities ψ_1, ψ_2 such that (no matter which is Σ) the map $U = (u_1, u_2)$ is such that

$$\det DU \neq 0 \text{ everywhere in } \Omega \setminus \Sigma .$$

Question. Can this result be extended to 3D?
Can we find current densities ψ_1, ψ_2, ψ_3 such that (no matter which is Σ) the map $U = (u_1, u_2, u_3)$ is such that

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The ancestor.

Theorem (Radó, Kneser, Choquet)

Let $\Omega \subset \mathbb{R}^2$ be simply connected. Let $D \subset \mathbb{R}^2$ be a convex domain.
Given a homeomorphism $\Phi : \partial\Omega \mapsto \partial D$, consider the solution $U = (u_1, u_2) : \Omega \mapsto \mathbb{R}^2$ to the following Dirichlet problem

$$\begin{cases} \Delta U = 0, & in \quad \Omega, \\ U = \Phi, & on \quad \partial\Omega. \end{cases}$$

then (Radó '26, Kneser '26, Choquet '45)

$$U \text{ is a homeomorphism of } \overline{\Omega} \mapsto \overline{D}$$

and (Lewy '36)

$$\det DU \neq 0 \text{ in } B .$$

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Counterexamples in 3D.

Wood '74. There exists a harmonic homeomorphism $U : \mathbb{R}^3 \mapsto \mathbb{R}^3$ such that $\det DU(0) = 0$.

Melas '93. There exists a harmonic homeomorphism $U : \overline{B} \mapsto \overline{B}$, $B \subset \mathbb{R}^3$ unit ball, such that $\det DU(0) = 0$.

Laugesen '96. $\forall \varepsilon > 0$ $\exists \Phi : \partial B \mapsto \partial B$ homeomorphism, such that $|\Phi(x) - x| < \varepsilon, \forall x \in \partial B$ and the solution U to

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A relaxed question.

In order to prove uniqueness for an insulating crack it would be sufficient to prove that we can find current densities ψ_1, ψ_2, ψ_3 such that, for any Σ and for any regular surface $S \subset \Omega \setminus \Sigma$ there exists one potential u_i, corresponding to the current density ψ_i, such that $\nabla u_i \cdot \nu$ does not identically vanish on S.

A relaxed question.

Problem

Let $U = (u_1, u_2, u_3)$ *be the map whose components solve*

$$\begin{cases} \Delta u_i = 0, & \text{in} & \Omega \setminus \Sigma, \\ \nabla u_i \cdot \nu^{\pm} = 0, & \text{on either side of} & \Sigma, \\ \nabla u_i \cdot \nu = \psi_i, & \text{on} & \partial\Omega. \end{cases}$$

To find ψ_1, ψ_2, ψ_3 *such that, for any* Σ*, the set*

$$\mathcal{S} = \{x \in \Omega \setminus \Sigma | detDU(x) = 0, (DU\nabla detDU)(x) = 0\}$$

has at most dimension 1*.*

Stability with full boundary data.

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$$N_{\Sigma} : \psi \to u|_{\partial\Omega} \ .$$

Theorem (A.-Sincich)

Under a-priori $C^{1,\alpha}$ regularity assumption on Σ, the map

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is continuous with logarithmic modulus of continuity.

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Method of singular solutions.

Uniqueness.

- Isakov '88. Inclusion problem $\mathrm{div}\,((1+\chi_D)\nabla u) = 0$.
- Eller '96.

Stability.

- A. '90, A.-Gaburro '01,'09, Salo '04, A.-Vessella '05, Gaburro-Lionheart '09. Inverse conductivity problem and variants.
- A.-Di Cristo '05. Inclusion problem.
  Di Cristo-Vessella '10. Time dependent inclusion $u_t - \mathrm{div}\,((1+\chi_{D(t)})\nabla u) = 0$.
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- Ikehata, probe method '98 . . . ,
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An identity.

Let Σ_1, Σ_2 be two cracks, u_1, u_2 be corresponding solutions with data ψ_1, ψ_2 and let N_1, N_2 be the associated Neumann-Dirichlet maps.

$$< \psi_1, (N_2 - N_1)\psi_2 >=$$

$$\begin{aligned}
&= \int_{\Sigma_1 \setminus \Sigma_2} \left(u_2[\partial_{\nu_1} u_1]_1 - [u_1]_1 \partial_{\nu_1} u_2\right) d\sigma + \\
&+ \int_{\Sigma_2 \setminus \Sigma_1} \left([u_2]_2 \partial_{\nu_2} u_1 - u_1[\partial_{\nu_2} u_2]_2\right) d\sigma + \\
&+ \int_{\Sigma_1 \cap \Sigma_2} \left([u_2 \partial_{\nu_1} u_1]_1 - [u_1 \partial_{\nu_2} u_2]_2\right) d\sigma \ .
\end{aligned}$$

Here

$$[\ \cdot\ ]_i = \text{ jump across } \Sigma_i \text{ w.r.t. the normal } \nu_i \ .$$

The Robin function.

Fix $\widetilde{\Omega}$ such that $\Omega \subset\subset \widetilde{\Omega}$, for any $y \in \widetilde{\Omega} \setminus \Sigma_i$ consider $R_i(x, y)$ solution to

$$\begin{cases} \Delta R_i(\cdot, y) = -\delta(\cdot - y), \text{ in } \widetilde{\Omega} \setminus \Sigma_i \ , \\ -\nabla R_i(\cdot, y) \cdot \nu^{\pm} + \gamma_i^{\pm} R_i(\cdot, y)^{\pm} = 0, \text{ on either side of } \Sigma \ , \\ \nabla R_i(\cdot, y) \cdot \nu = -\frac{1}{|\partial\widetilde{\Omega}|}, \text{ on } \partial\widetilde{\Omega} \ . \end{cases}$$

For any $y, w \in \widetilde{\Omega} \setminus \overline{\Omega}$ we can choose $u_1 = R_1(\cdot, y), u_2 = R_2(\cdot, w)$ and apply the identity.

$$< \partial_\nu R_1(\cdot, y), (N_2 - N_1)\partial_\nu R_2(\cdot, w) >=$$
$$= \int_{\Sigma_1 \setminus \Sigma_2} (R_2(\cdot, w)[\gamma_1 R_1(\cdot, y)]_1 - [R_1(\cdot, y)]_1 \partial_{\nu_1} R_2(\cdot, w))\, d\sigma +$$
$$+ \int_{\Sigma_2 \setminus \Sigma_1} ([R_2(\cdot, w)]_2 \partial_{\nu_2} R_1(\cdot, y) - R_1(\cdot, y)[\gamma_2 R_2(\cdot, w)]_2)\, d\sigma +$$
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The scheme of the proof.

For any $y, w \in \widetilde{\Omega} \setminus (\Sigma_1 \cup \Sigma_2)$ consider

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\begin{aligned}
&F(y,w) = \\
&= \int_{\Sigma_1 \setminus \Sigma_2} (R_2(\cdot,w)[\gamma_1 R_1(\cdot,y)]_1 - [R_1(\cdot,y)]_1 \partial_{\nu_1} R_2(\cdot,w))\, d\sigma + \\
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- If $y, w \in \widetilde{\Omega} \setminus \overline{\Omega}$ (and away from $\partial\Omega$) $F(y,w)$ is dominated by $\|N_1 - N_2\|$.
- $F(\cdot,w), F(y,\cdot)$ are harmonic in $\widetilde{\Omega} \setminus (\Sigma_1 \cup \Sigma_2)$.
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The end.

from en.wikipedia

THANKS!