# Subharmonic solutions of the prescribed curvature equation[*]

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## Abstract

We study the existence of subharmonic solutions of the prescribed curvature equation

$$-\left(u'/\sqrt{1+u'^2}\right)' = f(t,u).$$

According to the behaviour at zero, or at infinity, of the prescribed curvature $f$, we prove the existence of arbitrarily small classical subharmonic solutions, or bounded variation subharmonic solutions with arbitrarily large oscillations.

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# 1  Introduction

In this paper we are concerned with the existence of periodic, in particular subharmonic, solutions of the quasilinear ordinary differential equation

$$-\left(u'/\sqrt{1+u'^2}\right)' = f(t,u). \tag{1}$$

This equation, together with its $N$-dimensional counterpart

$$-\mathrm{div}\left(\nabla u/\sqrt{1+|\nabla u|^2}\right) = f(x,u),$$

plays a relevant role in various physical and geometrical questions, such as capillarity-type problems, flux limited diffusion phenomena, prescribed mean curvature problems (see, e.g., [16, 22, 19]). The question of the existence of periodic solutions of (1) has received considerable attention in recent years: the existence of classical solutions has been addressed in [2, 5, 3, 6, 7, 4, 23] by using topological methods, whereas the existence of bounded variation solutions has been discussed in [26, 25, 27] by using non-smooth critical point theory. The advisability of considering bounded variation solutions, besides classical solutions, in order to have a complete picture of the solvability patterns of (1), is already evident for the autonomous equation

$$-\left(u'/\sqrt{1+u'^2}\right)' = f(u).$$

Indeed, elementary phase-plane analysis and energy arguments, as in [20, 24, 28, 13], show that any solution $u$, for which $\int_0^{u(t)} f(\xi)d\xi$ exceeds somewhere the threshold 1, exhibits discontinuities and therefore cannot be a solution of (1) in the classical sense. The coexistence of classical and non-classical solutions of (1), according to the terminology introduced in [9, 10, 24, 13, 29], is determined by the specific structure of the curvature operator $\left(u'/\sqrt{1+u'^2}\right)'$, which behaves like the 2-Laplacian $u''$ near zero and like the 1-Laplacian $\left(\mathrm{sgn}(u')\right)'$ at infinity. These considerations lead us to introduce the following concept of periodic solution for equation (1) that will be considered throughout this paper.

**Definition 1.1.** *Let $\tau > 0$ be fixed. We say that a function $u \in BV_{\mathrm{loc}}(\mathbb{R})$ is a $\tau$-periodic solution of (1) if $u$ is $\tau$-periodic, $f(\cdot, u) \in L^1(0,\tau)$ and*

$$\int_0^\tau \frac{(Du)^a\,(D\phi)^a}{\sqrt{1+|(Du)^a|^2}}\,dt + \int_0^\tau \mathrm{sgn}\left(\frac{Du}{|Du|}\right)\frac{D\phi}{|D\phi|}\,|D\phi|^s$$

$$+ \mathrm{sgn}\big(u(0^+) - u(\tau^-)\big)\big(\phi(0^+)\ - \phi(\tau^-)\big) = \int_0^\tau f(t,u)\phi\,dt \tag{2}$$

*holds for every $\phi \in BV_{\mathrm{loc}}(\mathbb{R})$ such that $|D\phi|^s$ is absolutely continuous with respect to $|Du|^s$.*

As usual, for any $v \in BV(a,b)$, $Dv = (Dv)^a dt + (Dv)^s$ is the Lebesgue decomposition of the measure $Dv$ in its absolutely continuous part $(Dv)^a dt$, with density function $(Dv)^a$, and its singular part $(Dv)^s$ with respect to the Lebesgue measure in $\mathbb{R}$, $|Dv|$ denotes the total variation of the measure $Dv$, $|Dv| = |Dv|^a dt + |Dv|^s$ is the Lebesgue decomposition of $|Dv|$, and $\frac{Dv}{|Dv|}$ is the density function of $Dv$ with respect to its total variation $|Dv|$.

It is immediate to verify that if $u$ is a $\tau$-periodic solution of (1) such that $u \in W_{\mathrm{loc}}^{1,1}(\mathbb{R})$, then it is a weak $\tau$-periodic solution of (1), in the sense that

$$\int_0^\tau \frac{u'\phi'}{\sqrt{1+u'^2}}\,dt = \int_0^\tau f(t,u)\phi\,dt$$

for every $\phi \in W^{1,1}(0,\tau)$ with $\phi(0) = \phi(\tau)$. This implies that $u'/\sqrt{1+u'^2} \in W_{\mathrm{loc}}^{1,1}(\mathbb{R})$, $u'/\sqrt{1+u'^2}$ is $\tau$-periodic and $-\left(u'/\sqrt{1+u'^2}\right)' = f(t,u)$ a.e. in $]0,\tau[$. Note that a weak $\tau$-periodic solution $u$ of (1) is continuous, but may present a derivative blow up. However, we have $u' \in C^0([0,\tau],[-\infty,+\infty])$. Hence $u'$ satisfies the periodicity conditions in an extended sense, i.e., with possibly $u'(0) = u'(\tau) = +\infty$ or $u'(0) = u'(\tau) = -\infty$. It is clear that, if $u \in C^1(\mathbb{R})$, then $u$ is a $\tau$-periodic solution of (1) in the Carathéodory sense; if, in addition, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\tau$-periodic in $t$, then $u$ is a classical $\tau$-periodic solution of (1).

There exists a huge literature concerning the existence of periodic, specifically subharmonic, solutions of the semilinear equation

$$-u'' = f(t,u), \tag{3}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is, say, Lipschitz continuous and $T$-periodic in $t$, for some $T > 0$. We refer, e.g., to [11] for a rather exhaustive and updated bibliography on this subject. Although various definitions exist, a subharmonic solution of (3) is usually intended to be a periodic solution of the equation having minimum period $mT$ for some integer $m \geq 2$. If this last information is missing, a subharmonic solution is at least required to be $mT$-periodic, but not $T$-periodic. In case the solution is $mT$-periodic, but not $jT$-periodic, for any $j = 1,2,\ldots,m-1$, then it is referred to as a subharmonic solution of order $m$. As a general rule in this context, one tries to get as much information as possible about the minimality of the period. In particular, in [12] the existence of subharmonic solutions of (3) has been proved assuming that either $f$ is superlinear at 0, i.e.,

$$\lim_{s \to 0^\pm} \frac{f(t,s)}{s} = 0, \tag{4}$$

uniformly in $t$, or sublinear at infinity

$$\lim_{s \to \pm\infty} \frac{f(t,s)}{s} = 0, \tag{5}$$

uniformly in $t$. More precisely, it is shown in [12] that condition (4), even assumed only at $0^+$, or at $0^-$, implies the existence of two sequences of arbitrarily small subharmonic solutions having a prescribed number of zeroes and condition (5), even assumed only at $+\infty$, or at $-\infty$, implies the existence of two sequences of arbitrarily large subharmonic solutions having a prescribed number of zeroes. The proof is performed by a phase-plane analysis and relies on the Poincaré-Birkhoff fixed point theorem; the nodal properties of the solutions are obtained by using the rotation number which counts the number of turns of the solutions around the origin in the phase-plane.

Our aim here is to investigate the existence of subharmonic solutions for (1), taking inspiration from these results, but keeping in mind the behaviour of the curvature operator at 0 and at infinity. The following notions of subharmonic solution of (1) are used in this paper.

**Definition 1.2.** *We say that $u$ is a subharmonic solution of* (1) *if it is a periodic solution of* (1) *having minimum period $\tau = rT$ for some $r \in \mathbb{Q}$ with $r > 1$.*

**Definition 1.3.** *We say that $u$ is a subharmonic solution of order $m$ of* (1) *if it is a $mT$-periodic solution of* (1) *for some $m \in \mathbb{N}$ with $m \geq 2$, but it is not $jT$-periodic, for any $j = 1, 2, \ldots, m - 1$.*

It is easily seen that a subharmonic solution of (1) having minimum period $\tau = \frac{p}{q}T$, for some $p, q \in \mathbb{N}_0$, with $p, q$ coprime and $p > q$, is a subharmonic solution of order $p$ of (1).

Assuming that $f$ is superlinear at 0, we prove in Theorem 3.4 the existence of small classical subharmonic solutions having suitable nodal properties; in this case the proof, which borrows some arguments from [12], is based on the use of the rotation number and on a version of the Poincaré-Birkhoff theorem given in [18, Theorem 8.2] and not requiring uniqueness of solutions for the Cauchy problems associated with (1). In particular, the following result holds.

**Theorem 1.1.** *Assume that*

$(S_0)$ *$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is $T$-periodic in $t$, for some $T > 0$, and continuous,*

$(S_1)$ $\lim\limits_{s \to 0} \dfrac{f(t,s)}{s} = 0$, *uniformly in $t \in [0, T]$,*

$(S_2)$ *there exists $\delta > 0$ such that $f(t, s)s > 0$, for all $t \in [0, T]$ and for all $s \in [-\delta, \delta] \setminus \{0\}$.*

*Then there exists a sequence $(u_k)_k$ of classical subharmonic solutions of* (1) *such that*
$$\lim_{k \to +\infty} \|u_k\|_{C^1} = 0$$
*and whose minimum periods diverge.*

A parallel result concerning the existence of subharmonic solutions of (1) having large oscillations is obtained supposing that the potential $F$ of $f$ is sublinear and coercive at infinity; in this case bounded variation non-classical solutions are expected. The proof makes use of some tools of non-smooth critical point theory, namely a version of the mountain pass lemma in the space of bounded variation functions given in [25, Lemma 2.13], combined with suitable critical value estimates as introduced for the semilinear problem (3) in [17].

**Theorem 1.2.** *Assume that*

$(s_0)$ $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is $T$-periodic in $t$, for some $T > 0$, and satisfies the $L^1$-Carathéodory conditions in $[0, T] \times \mathbb{R}$,*

$(s_1)$ $\lim\limits_{|s| \to +\infty} f(t, s) = 0$ *uniformly a.e. in $t \in [0, T]$,*

$(s_2)$ $\lim\limits_{|s| \to +\infty} \int_0^T F(t, s) \, dt = +\infty$, *where $F(t, s) = \int_0^s f(t, \xi) \, d\xi$,*

$(s_3)$ *there exists $R > 0$ such that $f(t, s)s > 0$ for a.e. $t \in [0, T]$ and every $s$ with $|s| \geq R$.*

*Then there exists a sequence $(u_k)_k$ of subharmonic solutions of (1) such that*

$$\lim_{k \to +\infty} \Big( \operatorname*{ess\,sup}_{\mathbb{R}} u_k - \operatorname*{ess\,inf}_{\mathbb{R}} u_k \Big) = +\infty$$

*and whose minimum periods diverge.*

**Notations**  For any given $a, b \in \mathbb{R}$, with $a < b$, and each $v \in BV(a, b)$ we set, as usual,

$$\int_a^b |Dv| = \sup \Big\{ \int_a^b v w' \, dt : w \in C_0^1(]a, b[), \|w\|_{L^\infty(a,b)} \leq 1 \Big\}$$

and

$$\int_a^b \sqrt{1 + |Dv|^2} = \sup \Big\{ \int_a^b (v w_1' + w_2) \, dt : w_1, w_2 \in C_0^1(]a, b[)$$

$$\text{and } \|w_1^2 + w_2^2\|_{L^\infty(a,b)} \leq 1 \Big\}.$$

Clearly, we have

$$\int_a^b |Dv| \leq \int_a^b \sqrt{1 + |Dv|^2}.$$

The norm in $BV(a, b)$ is defined by

$$\|v\|_{BV(a,b)} = \int_a^b |v| \, dt + \int_a^b |Dv|.$$

We also denote by $v(t_0^+)$ the right trace of $v$ at $t_0 \in [a, b[$ and by $v(t_0^-)$ the left trace of $v$ at $t_0 \in ]a, b]$. Finally, we write $\mathbb{N}_0 = \{n \in \mathbb{N} : n \geq 1\}$ and $\mathbb{R}_0^+ = \{x \in \mathbb{R} : x > 0\}$.

## 2 The autonomous equation

In this section we discuss the existence of periodic solutions of the autonomous equation

$$-\left(u'/\sqrt{1+u'^2}\right)' = f(u), \tag{6}$$

by performing an elementary analysis in the phase-plane. We assume that $f : \mathbb{R} \to \mathbb{R}$ is odd and continuous. We also suppose that $f(s) > 0$ for all $s > 0$ and $\lim\limits_{s \to +\infty} F(s) = +\infty$, with $F(s) = \int_0^s f(\xi)\,d\xi$. Let us define the functions

$$\varphi : \mathbb{R} \to\, ]-1,1[, \quad \varphi(s) = \frac{s}{\sqrt{1+s^2}},$$

$$\psi :\, ]-1,1[ \to \mathbb{R}, \quad \psi(s) = \frac{s}{\sqrt{1-s^2}} \tag{7}$$

and set $\Psi(s) = \int_0^s \psi(\xi)\,d\xi = 1 - \sqrt{1-s^2}$ for every $s \in\, ]-1,1[$. Then equation (6) is equivalent to the planar system

$$\begin{cases} u' = -\psi(v) \\ v' = f(u). \end{cases} \tag{8}$$

The energy function associated with (8) is given by

$$\mathcal{E}(u,v) = \Psi(v) + F(u).$$

Clearly, the solutions $(u,v)$ of (8) parametrize the level curves of $\mathcal{E}$. Let us fix $r > 0$. We know by [30] that there is a unique non-extendible solution $u$ of the Cauchy problem

$$\begin{cases} -\big(\varphi(u')\big)' = f(u) \\ u(0) = r \\ u'(0) = 0, \end{cases} \tag{9}$$

which satisfies

$$\Psi\big(\varphi(u'(t))\big) + F\big(u(t)\big) = F(r)$$

for all $t$ belonging to its domain. Let us define

$$C_r = \{(x,y) \in \mathbb{R}^2 : \Psi\big(\varphi(y)\big) + F(x) = F(r)\}.$$

Note that $\Psi(\varphi(y)) = 1 - \frac{1}{\sqrt{1+y^2}}$. The curve $C_r$ is symmetric with respect to the origin and its topology depends on the value $F(r)$. Indeed, since $C_r$ can be represented in the form

$$C_r = \left\{(x,y) \in \mathbb{R}^2 : y = \pm \frac{\sqrt{F(r) - F(x)}}{\chi(F(r) - F(x))} \right\},$$

with

$$\chi : [0,1] \to \mathbb{R}, \quad \chi(s) = \frac{1-s}{\sqrt{2-s}},$$

we see that $C_r$ is connected if and only if $F(r) < 1$; indeed, under this assumption, $C_r$ is homeomorphic to the circle $S^1$. Otherwise $C_r$ is disconnected and unbounded in the $y$-component; namely, setting $r_\infty = F^{-1}(1) \in ]0,r]$, we have that, if $(x,y) \in C_r$ and $x \to \pm r_\infty^\pm$, then $|y| \to +\infty$.

If $C_r$ is connected and $u$ is a non-extendible solution of (9) such that the trajectory $(u,u')$ parametrizes $C_r$, then $u \in C^2(\mathbb{R})$ and is periodic with minimum period $4T(r)$, where $T(r) \in \mathbb{R}_0^+$ is the first positive zero of $u$, i.e., $u$ is a classical $4T(r)$-periodic solution of (6).

If $C_r$ is disconnected and $u$ is a non-extendible solution of (9) such that the trajectory $(u,u')$ parametrizes $C_r \cap \left(\mathbb{R}_0^+ \times \mathbb{R}\right)$, then there exists $S(r) \in \mathbb{R}_0^+$ such that

$$\lim_{t \to S(r)^-} u(t) = r_\infty \quad \text{and} \quad \lim_{t \to S(r)^-} u'(t) = -\infty$$

and, by symmetry,

$$\lim_{t \to -S(r)^+} u(t) = r_\infty \quad \text{and} \quad \lim_{t \to -S(r)^+} u'(t) = +\infty.$$

Clearly, $u \in W^{1,1}(-S(r), S(r))$. Since, by symmetry, $(-u, -u')$ parametrizes $C_r \cap \left(\mathbb{R}_0^- \times \mathbb{R}\right)$, we can extend $u$ to the interval $] - S(r), 3S(r)[$, by setting $u(t) = -u(t - 2S(r))$ for all $t \in ]S(r), 3S(r)[$, and then by $4S(r)$-periodicity all over $\mathbb{R}$. It is clear that $u \in BV_{\mathrm{loc}}(\mathbb{R})$ and is periodic, with minimum period $4S(r)$. Let us show that $u$ is a $4S(r)$-periodic solution of (6) according to (2). Without restriction we can also replace $u$ with $u(\cdot + S(r))$.

Let $\phi \in BV_{\mathrm{loc}}(\mathbb{R})$ be such that $|D\phi|^s$ is absolutely continuous with respect to $|Du|^s$. Denote by $\phi_1$ and $\phi_2$ the restrictions of $\phi$ to $]0, 2S(r)[$ and to $]2S(r), 4S(r)[$, respectively. By the regularity of $u$ in $]0, 2S(r)[$ and in $]2S(r), 4S(r)[$, we have $|D\phi_1|^s = |D\phi_2|^s = 0$: this implies that $\phi_1 \in W^{1,1}(0, 2S(r))$ and $\phi_2 \in W^{1,1}(2S(r), 4S(r))$. Hence, multiplying (6) by $\phi_j$, $j = 1, 2$, and integrating by parts in $]0, 2S(r)[$ and in $]2S(r), 4S(r)[$, respectively, we obtain

$$-\int_0^{2S(r)} \left(\varphi(u')\right)' \phi_1 \, dt = -\left[\varphi(u')\phi_1\right]_{0^+}^{(2S(r))^-} + \int_0^{2S(r)} \varphi(u')\phi_1' \, dt = \int_0^{2S(r)} f(u)\phi_1 \, dt,$$

$$-\int_{2S(r)}^{4S(r)} \left(\varphi(u')\right)' \phi_2 \, dt = -\left[\varphi(u')\phi_2\right]_{(2S(r))^+}^{(4S(r))^-} + \int_{2S(r)}^{4S(r)} \varphi(u')\phi_2' \, dt = \int_{2S(r)}^{4S(r)} f(u)\phi_2 \, dt.$$

By the properties of $u$, we have

$$\left[\varphi(u')\phi_1\right]_{0^+}^{(2S(r))^-} = -\phi_1\left(0^+\right) - \phi_1\left((2S(r))^-\right),$$
$$\left[\varphi(u')\phi_2\right]_{(2S(r))^+}^{(4S(r))^-} = \phi_2\left((2S(r))^+\right) + \phi_2\left((4S(r))^-\right).$$

Therefore summing up we get

$$\int_0^{2S(r)} \varphi(u')\phi' \, dt + \int_{2S(r)}^{4S(r)} \varphi(u')\phi' \, dt$$

$$+ \left[\phi(0^+) - \phi((4S(r))^-)\right] + \left[\phi((2S(r))^-) - \phi((2S(r))^+)\right] = \int_0^{4S(r)} f(u)\phi \, dt. \tag{10}$$

Notice that

$$\int_0^{2S(r)} \varphi(u')\phi' \, dt + \int_{2S(r)}^{4S(r)} \varphi(u')\phi' \, dt = \int_0^{4S(r)} \varphi((Du)^a)(D\phi)^a \, dt$$

$$= \int_0^{4S(r)} \frac{(Du)^a(D\phi)^a}{\sqrt{1 + |(Du)^a|^2}} \, dt,$$

$$\phi(0^+) - \phi((4S(r))^-) = \operatorname{sgn}\big(u(0^+) - u((4S(r))^-)\big)\big(\phi(0^+) - \phi((4S(r))^-)\big),$$

$$\phi((2S(r))^-) - \phi((2S(r))^+) = -\int_0^{4S(r)} (D\phi)^s = -\int_0^{4S(r)} \frac{(D\phi)^s}{|D\phi|^s} |D\phi|^s$$

$$= -\int_0^{4S(r)} \frac{D\phi}{|D\phi|} |D\phi|^s = \int_0^{4S(r)} \operatorname{sgn}\left(\frac{Du}{|Du|}\right) \frac{D\phi}{|D\phi|} |D\phi|^s.$$

Substituting in (10), we obtain (2) with $\tau = S(r)$, that is, $u$ is a $4S(r)$-periodic solution of (6).

We conclude this section by discussing the existence of periodic solutions of the autonomous equation (6) with reference to two model examples for $f = F'$, at 0 or at $\pm\infty$, respectively. Namely we suppose that

(a) $F(s) = |s|^{p+1}$, for some $p > 1$, in a neighbourhood of 0,

or

(b) $F(s) = |s|^{q+1}$, for some $q \in ]-1, 0[$, in a neighbourhood of $\pm\infty$.

Assume that (a) holds: the expression of the classical time-map $T : ]0, r_\infty[ \to ]0, +\infty[$, with $r_\infty = F^{-1}(1) = 1$, is

$$T(r) = \int_0^r \frac{\chi\big(F(r) - F(s)\big)}{\sqrt{F(r) - F(s)}} \, ds = r^{\frac{1-p}{2}} \int_0^1 \frac{\chi\big(r^{p+1}(1 - s^{p+1})\big)}{\sqrt{1 - s^{p+1}}} \, ds.$$

The concavity of the function $\chi$ implies that

$$\frac{1}{\sqrt{2}}(1 - s) \le \chi(s) \le \frac{1}{\sqrt{2}} \quad \text{in } [0, 1]. \tag{11}$$

Hence we obtain

$$\frac{r^{\frac{1-p}{2}}}{\sqrt{2}} \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}}\, ds \geq T(r) \geq \frac{r^{\frac{1-p}{2}}}{\sqrt{2}} \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}}\, ds - \frac{r^{\frac{3+p}{2}}}{\sqrt{2}} \int_0^1 \sqrt{1-s^{p+1}}\, ds$$

for all $r \in\, ]0,1[$. Then we conclude

$$\lim_{r\to 0^+} T(r) = +\infty.$$

This implies that in case $(a)$ there exists a family of classical periodic solutions of (6), approaching 0 in the $C^1$-norm and having arbitrarily large minimum periods.

Assume that $(b)$ holds: the expression of the non-classical time-map $S : ]r_\infty, +\infty[ \to\, ]0, +\infty[$, with $r_\infty = F^{-1}(1) = 1$, is

$$S(r) = \int_{r_\infty}^r \frac{\chi\big(F(r)-F(s)\big)}{\sqrt{F(r)-F(s)}}\, ds = r^{\frac{1-q}{2}} \int_{r_\infty/r}^1 \frac{\chi\big(r^{q+1}(1-s^{q+1})\big)}{\sqrt{1-s^{q+1}}}\, ds.$$

By using (11), we have

$$\frac{r^{\frac{1-q}{2}}}{\sqrt{2}} \int_{r_\infty/r}^1 \frac{1}{\sqrt{1-s^{q+1}}}\, ds \geq S(r) \geq \frac{r^{\frac{1-q}{2}}}{\sqrt{2}} \int_{r_\infty/r}^1 \frac{1-r^{q+1}(1-s^{q+1})}{\sqrt{1-s^{q+1}}}\, ds$$

$$= \frac{r^{\frac{1-q}{2}}}{(q+1)\sqrt{2}} \int_0^{1/r^{q+1}} \frac{1-r^{q+1}s}{\sqrt{s}}(1-s)^{-\frac{q}{q+1}}\, ds,$$

for all $r > r_\infty$. Taking $r > r_\infty$ sufficiently large, we have $(1-s)^{-\frac{q}{q+1}} \geq \frac{1}{\sqrt{2}}$ for all $s \in [0, \frac{1}{r^{q+1}}]$ and then

$$S(r) \geq \frac{r^{\frac{1-q}{2}}}{2(q+1)} \int_0^{1/r^{q+1}} \frac{1-r^{q+1}s}{\sqrt{s}}\, ds = \frac{2}{3(q+1)r^q}.$$

Hence we conclude that

$$\lim_{r\to+\infty} S(r) = +\infty.$$

This implies that in case $(b)$ there exists a family of periodic solutions of (6) according to (2), having arbitrarily large oscillations and arbitrarily large minimum periods.

These simple observations are the starting point of our study of the general non-autonomous equation (1). In particular, the estimates we have produced on $T(r)$ and on $S(r)$ in the model cases $(a)$ and $(b)$ motivate the introduction of the assumptions of superlinearity of $f$ at 0 and of sublinearity of $F$ at infinity.

## 3 Small classical subharmonic solutions

We start this section with an elementary result concerning a property of the solutions of the first order system in $\mathbb{R}^N$

$$z' = \ell(t, z). \tag{12}$$

We recall that a continuous function $\kappa : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is an Osgood function if

$$\int_0^1 \frac{d\xi}{\kappa(\xi)} = +\infty = \int_1^{+\infty} \frac{d\xi}{\kappa(\xi)}.$$

Let us set, for each $s > 0$,

$$\mathcal{H}(s) = \int_1^s \frac{d\xi}{\kappa(\xi)};$$

it is immediate to see that $\mathcal{H} : \mathbb{R}_0^+ \to \mathbb{R}$ is an increasing diffeomorphism. Then the following conclusion is a consequence of a classical result of I. Bihari [8].

**Lemma 3.1.** *Assume that $\ell : I \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous, with $I \subseteq \mathbb{R}$ an interval, and suppose that there exists an Osgood function $\kappa$ such that*

$$|\ell(t, \zeta) \cdot \zeta| \le \kappa(|\zeta|^2)$$

*for all $t \in I$ and $\zeta \in \mathbb{R}^N$. Then, any non-trivial solution $z$ of (12) is globally defined and satisfies*

$$\mathcal{H}^{-1}(-2|t - t_0| + \mathcal{H}(|z(t_0)|^2)) \le |z(t)|^2 \le \mathcal{H}^{-1}(2|t - t_0| + \mathcal{H}(|z(t_0)|^2)),$$

*for all $t, t_0 \in I$. In particular, $z$ never vanishes.*

Throughout this section we suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies assumptions $(S_0)$, $(S_1)$, $(S_2)$. Let us define $\bar{f} : [0, T] \times \mathbb{R} \to \mathbb{R}$ by setting, for any $t$,

$$\bar{f}(t, s) = \begin{cases} f(t, -\delta) & \text{if } s < -\delta, \\ f(t, s) & \text{if } |s| \le \delta, \\ f(t, \delta) & \text{if } s > \delta. \end{cases}$$

The function $\bar{f}$ satisfies the same assumptions as $f$ does, in particular, $(S_2)$ holds, with $f$ replaced by $\bar{f}$, for all $t \in [0, T]$ and $s \in \mathbb{R} \setminus \{0\}$. Let us also define $\bar{\psi} : \mathbb{R} \to \mathbb{R}$ by

$$\bar{\psi}(s) = \begin{cases} -\psi(\delta) + \psi'(\delta)(s + \delta), & \text{if } s < -\delta, \\ \psi(s) & \text{if } |s| \le \delta, \\ \psi(\delta) + \psi'(\delta)(s - \delta) & \text{if } s > \delta, \end{cases}$$

where $\psi(s) = s/\sqrt{1 - s^2}$ has been defined in (7). It is easily checked that the vector field $\ell(t, \zeta) = (-\bar{\psi}(y), \bar{f}(t, x))$ satisfies the assumptions of Lemma 3.1, where the

Osgood function $\kappa$ can be taken to be a non-zero linear function, say, $\kappa(s) = \kappa s$ and thus $\mathcal{H}(s) = \frac{1}{\kappa}\ln(s)$. Hence Lemma 3.1 guarantees that any Cauchy problem associated with the system

$$\begin{cases} u' = -\bar{\psi}(v) \\ v' = \bar{f}(t, u) \end{cases} \tag{13}$$

has a global solution $z = (u, v) \in C^1(\mathbb{R})$, which, if non-trivial, never vanishes, since it satisfies

$$|z(t_0)|\exp(-\kappa|t - t_0|) \leq |z(t)| \leq |z(t_0)|\exp(\kappa|t - t_0|) \tag{14}$$

for all $t, t_0 \in \mathbb{R}$. This allows, in particular, to represent such a solution in polar coordinates as

$$u(t) = \rho(t)\cos\theta(t), \quad v(t) = \rho(t)\sin\theta(t).$$

Note that the couple $(\rho, \theta)$ satisfies

$$\rho'(t) = \bar{f}\big(t, \rho(t)\cos\theta(t)\big)\sin\theta(t) - \bar{\psi}\big(\rho(t)\sin\theta(t)\big)\cos\theta(t),$$

$$\theta'(t) = \frac{\bar{\psi}\big(\rho(t)\sin\theta(t)\big)\rho(t)\sin\theta(t) + \bar{f}\big(t, \rho(t)\cos\theta(t)\big)\rho(t)\cos\theta(t)}{\rho(t)^2},$$

or equivalently

$$\rho'(t) = \frac{\bar{f}\big(t, u(t)\big)v(t) - \bar{\psi}\big(v(t)\big)u(t)}{\sqrt{u(t)^2 + v(t)^2}},$$

$$\theta'(t) = \frac{\bar{\psi}\big(v(t)\big)v(t) + \bar{f}\big(t, u(t)\big)u(t)}{u(t)^2 + v(t)^2},$$

for all $t \in \mathbb{R}$. For any fixed $t_0, t_1 \in \mathbb{R}$, with $t_0 < t_1$, and assuming that $z(t_0) = z_0$ for some $z_0 \in \mathbb{R}^2 \setminus \{0\}$, we define the rotation number of $z$ in $[t_0, t_1]$ by

$$\text{Rot}(z; [t_0, t_1]) = \frac{\theta(t_1) - \theta(t_0)}{2\pi} = \frac{1}{2\pi}\int_{t_0}^{t_1} \frac{\bar{\psi}\big(v(t)\big)v(t) + \bar{f}\big(t, u(t)\big)u(t)}{u(t)^2 + v(t)^2}\, dt.$$

The rotation number counts the counterclockwise turns of the function $z$ around the origin in the time interval $[t_0, t_1]$.

We notice that the sign conditions satisfied by $\bar{f}$ and $\bar{\psi}$, namely $\bar{f}(t, s)s > 0$ and $\bar{\psi}(s)s > 0$, for all $t \in \mathbb{R}$ and $s \in \mathbb{R} \setminus \{0\}$, imply that the function $\theta$ is strictly increasing, or equivalently that the rotation number is positive for any non-trivial solution $z$ in any compact time interval.

**Lemma 3.2.** *Assume $(S_0)$, $(S_1)$, $(S_2)$. For any $k \in \mathbb{N}_0$, there exist $\tau_k^* > 0$ and $r_k^* \in \,]0, \min\{\frac{1}{k}, \delta\}[$, such that, for any interval $J = [t_0, t_1]$, with $|t_0 - t_1| > \tau_k^*$, and any solution $z$ of (13), with $z(t_0) = z_0$ for some $|z_0| = r_k^*$, we have*

$$\text{Rot}(z; J) > k. \tag{15}$$

*Proof.* By condition $(S_2)$ there exist two continuous functions $g, h : [-\delta, \delta] \to \mathbb{R}$ such that

$$0 < g(s)s \le h(s)s, \quad \text{for all } s \in [-\delta, \delta] \setminus \{0\},$$

and

$$g(s)s \le \bar{f}(t, s)s \le h(s)s, \quad \text{for all } t \in [0, T], s \in [-\delta, \delta]. \tag{16}$$

Indeed, it suffices to define, for all $s \in [-\delta, \delta]$,

$$g(s) = \min_{t \in [0,T]} (\bar{f}(t, s) \operatorname{sgn}(s)), \quad h(s) = \max_{t \in [0,T]} (\bar{f}(t, s) \operatorname{sgn}(s)).$$

Throughout this section we will refer to $g, h$ as some continuous extensions onto $\mathbb{R}$ of the previously defined functions, which satisfy

$$g(s)s > 0, \quad h(s)s > 0, \tag{17}$$

for all $s \in \mathbb{R} \setminus \{0\}$, and

$$\lim_{|s| \to +\infty} G(s) = \lim_{|s| \to +\infty} H(s) = +\infty, \tag{18}$$

where

$$G(s) = \int_0^s g(\xi) \, d\xi, \quad H(s) = \int_0^s h(\xi) \, d\xi.$$

Let us introduce the planar autonomous systems

$$\begin{cases} u' = -\bar{\psi}(v) \\ v' = g(u) \end{cases} \tag{19}$$

and

$$\begin{cases} u' = -\bar{\psi}(v) \\ v' = h(u). \end{cases} \tag{20}$$

The energy functions associated with (19) and (20) are, respectively,

$$\mathcal{E}_G(x, y) = \bar{\Psi}(y) + G(x), \qquad \mathcal{E}_H(x, y) = \bar{\Psi}(y) + H(x),$$

with

$$\bar{\Psi}(s) = \int_0^s \bar{\psi}(\xi) \, d\xi,$$

for any $s \in \mathbb{R}$. By definition of $\bar{\psi}$ and from conditions (17) and (18), it follows that the only equilibrium point of (19) and (20) is $(0, 0)$ and all level curves of $\mathcal{E}_G$ and $\mathcal{E}_H$

are closed curves around $(0,0)$. Hence global existence and uniqueness of solution hold for every Cauchy problem associated with (19) and (20).

Let us introduce two auxiliary functions $M_{\mp} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, defined by

$$M_{-}(x,y) = \begin{cases} \frac{g(x)y - \bar{\psi}(y)x}{g(x)x + \bar{\psi}(y)y}, & \text{if } xy \geq 0, \\ \frac{h(x)y - \psi(y)x}{h(x)x + \psi(y)y}, & \text{if } xy \leq 0 \end{cases}$$

and

$$M_{+}(x,y) = \begin{cases} \frac{g(x)y - \bar{\psi}(y)x}{g(x)x + \psi(y)y}, & \text{if } xy \leq 0, \\ \frac{h(x)y - \psi(y)x}{h(x)x + \psi(y)y}, & \text{if } xy \geq 0 \end{cases}$$

and consider the equations

$$\frac{dr}{d\theta} = r M_{-}(r\cos\theta, r\sin\theta) \tag{21}$$

and

$$\frac{dr}{d\theta} = r M_{+}(r\cos\theta, r\sin\theta). \tag{22}$$

Trajectories associated with non-trivial solutions of (21) and (22) parametrize spirals, surrounding the origin, obtained by alternating the level curves of $\mathcal{E}_G$ and $\mathcal{E}_H$. We see that, for all $(\theta_0, \rho_0)$ with $\theta_0 \in \mathbb{R}$ and $\rho_0 > 0$, uniqueness and global continuability of solutions hold for any Cauchy problem associated with (21) and with (22). Let $r_{-}(\,\cdot\,; \theta_0, \rho_0), r_{+}(\,\cdot\,; \theta_0, \rho_0)$ (sometimes denoted by $r_{\mp}$ for simplicity) be the solutions of (21) and (22), respectively, satisfying $r_{\mp}(\theta_0) = \rho_0$. Note, in particular, that any non-trivial solution $r_{\mp}$ of (21) and (22) satisfies $r_{\mp}(\theta) > 0$ for all $\theta \in \mathbb{R}$.

Let us fix $k \in \mathbb{N}_0$. For any $\rho_0 > 0$, set

$$m_k^*(\rho_0) = \inf_{\substack{\theta_0 \in [0, 2\pi] \\ \theta \in [\theta_0, \theta_0 + 2k\pi]}} r_{-}(\theta; \theta_0, \rho_0) > 0$$

and

$$M_k^*(\rho_0) = \sup_{\substack{\theta_0 \in [0, 2\pi] \\ \theta \in [\theta_0, \theta_0 + 2k\pi]}} r_{+}(\theta; \theta_0, \rho_0).$$

As $\displaystyle\lim_{\rho_0 \to 0^+} M_k^*(\rho_0) = 0$, there exists $r_k^* > 0$ such that

$$0 < m_k^*(r_k^*) \leq r_k^* \leq M_k^*(r_k^*) < ]0, \min\{\tfrac{1}{k}, \delta\}[.$$

Pick $\underline{r_k}, \overline{r_k} > 0$ such that

$$0 < \underline{r_k} < m_k^*(r_k^*) \leq r_k^* \leq M_k^*(r_k^*) < \overline{r_k} < ]0, \min\{\tfrac{1}{k}, \delta\}[.$$

Define a continuous function $\mathcal{G} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ by

$$\mathcal{G}(x, y) = \frac{g(x)x + \bar{\psi}(y)y}{x^2 + y^2}.$$

Let $\mathcal{A} = \{(x, y) \in \mathbb{R}^2 : \underline{r_k} \leq \sqrt{x^2 + y^2} \leq \overline{r_k}\}$ and $\delta_k^* = \min_{\mathcal{A}} \mathcal{G} > 0$. Set $\tau_k^* = \frac{2k\pi}{\delta_k^*}$ and take any interval $J = [t_0, t_1]$, with $\tau = t_1 - t_0 > \tau_k^*$ and a solution $z$ of (13) with $z(t_0) = z_0$, for some $z_0 \in \mathbb{R}^2$ such that $|z_0| = r_k^*$. We want to prove that (15) holds, i.e.,

$$\mathrm{Rot}(z; J) > k.$$

Without loss of generality, we can assume that $\theta(t_0) = \theta_0 \in [0, 2\pi[$. Therefore the thesis amounts to proving that

$$\theta(t_1) - \theta(t_0) > 2k\pi. \tag{23}$$

Set

$$\sigma = \sup\{s \in [t_0, t_1] : \underline{r_k} \leq \rho(t) \leq \overline{r_k} \text{ in } [t_0, s]\}.$$

For all $t \in [t_0, \sigma]$, we have

$$|u(t)| \leq \rho(t) < \delta. \tag{24}$$

Two cases may occur: either $\sigma = t_1$, or $\sigma < t_1$.

If $\sigma = t_1$, by (24) and (16), we have

$$\mathrm{Rot}(z; J) = \frac{1}{2\pi} \int_{t_0}^{\sigma} \frac{\bar{\psi}(v(t))v(t) + \bar{f}(t, u(t))u(t)}{u(t)^2 + v(t)^2} \, dt$$

$$\geq \frac{1}{2\pi} \int_{t_0}^{t_1} \mathcal{G}(u(t), v(t)) \, dt \geq \frac{\tau}{2\pi} \min_{\mathcal{A}} \mathcal{G} > \frac{\delta_k^* \tau_k^*}{2\pi} = k$$

and hence (15) follows.

If $\sigma < t_1$, the maximality of $\sigma$ implies that $\underline{r_k} \leq \rho(t) \leq \overline{r_k}$ for all $t \in [t_0, \sigma]$ and $\rho(\sigma) \in \{\underline{r_k}, \overline{r_k}\}$. Assume that $\rho(\sigma) = \overline{r_k}$, the other case being treated similarly. In order to prove (23), we only need to show that

$$\theta(\sigma) - \theta(t_0) > 2k\pi,$$

since, as already observed, the function $\theta$ is strictly increasing. By contradiction, assume that $\theta(\sigma) - \theta(t_0) \leq 2k\pi$. The monotonicity of $\theta$ also implies

$$\theta(t) \in [\theta_0, \theta_0 + 2k\pi], \tag{25}$$

for all $t \in [t_0, \sigma]$.

On the other hand, $\zeta = (\rho, \theta)$ satisfies

$$\rho'(t) = \bar{f}(t, \rho(t) \cos \theta(t)) \sin \theta(t) - \bar{\psi}(\rho(t) \sin \theta(t)) \cos \theta(t),$$

$$\theta'(t) = \frac{\bar{\psi}(\rho(t) \sin \theta(t)) \sin \theta(t) + \bar{f}(t, \rho(t) \cos \theta(t)) \cos \theta(t)}{\rho(t)}, \tag{26}$$

for all $t \in \mathbb{R}$. In this regard, we introduce some more functions $S, U : J \times \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, defined by

$$S(t, x, y) = \frac{\bar{f}(t, x)y - \bar{\psi}(y)x}{x^2 + y^2},$$

$$U(t, x, y) = \frac{\bar{\psi}(y)y + \bar{f}(t, x)x}{x^2 + y^2},$$

and $R, \Theta : J \times ]0, +\infty[ \times \mathbb{R} \to \mathbb{R}$, defined by

$$R(t, \rho, \theta) = \rho S(t, \rho \cos\theta, \rho \sin\theta),$$
$$\Theta(t, \rho, \theta) = U(t, \rho \cos\theta, \rho \sin\theta).$$

We easily see that

$$\frac{\rho S(t, \rho \cos\theta, \rho \sin\theta)}{U(t, \rho \cos\theta, \rho \sin\theta)} = \frac{R(t, \rho, \theta)}{\Theta(t, \rho, \theta)}$$

holds for all $(\theta, \rho) \in \mathbb{R} \times ]0, +\infty[$, and

$$\frac{S(t, x, y)}{U(t, x, y)} \leq M_+(x, y) \tag{27}$$

holds for all $t \in [t_0, \sigma], (x, y) \in \mathbb{R}^2 \setminus \{0\}$.

Let $\gamma : \mathbb{R} \to ]0, +\infty[$ be the solution of (22) satisfying $\gamma(\theta_0) = r_k^*$. From the definition of $M_k^*$, we have

$$\gamma(\theta) \leq M_k^*(r_k^*) < \overline{r_k},$$

for all $\theta \in [\theta_0, \theta_0 + 2k\pi]$. By continuity of $\gamma$, there exists $\varepsilon > 0$ such that $\gamma(\theta) < \overline{r_k}$ for all $\theta \in ]\theta_0 - \varepsilon, \theta_0 + 2k\pi + \varepsilon[$. From (27) and the positivity of $\gamma$,

$$\gamma'(\theta) = \gamma M_+(\gamma \cos\theta, \gamma \sin\theta) \geq \frac{\gamma S(t, \gamma \cos\theta, \gamma \sin\theta)}{U(t, \gamma \cos\theta, \gamma \sin\theta)} = \frac{R(t, \gamma, \theta)}{\Theta(t, \gamma, \theta)}$$

holds for all $t \in [t_0, \sigma]$ and $\theta \in \mathbb{R}$. Consider again the function $(\rho, \theta)$: when restricted to the interval $[t_0, \sigma]$ and thanks to condition (25), it takes values in $]0, +\infty[ \times ]\theta_0 - \varepsilon, \theta_0 + 2k\pi + \varepsilon[$. Moreover it is a solution of

$$\begin{cases} \rho' = R(t, \rho, \theta) \\ \theta' = \Theta(t, \rho, \theta) \end{cases}$$

with $\rho(t_0) = r_k^* = \gamma(\theta_0) = \gamma(\theta(t_0))$. We know that $\theta'(t) > 0$ for all $t \in [t_0, \sigma]$, so that the function $\theta : [t_0, \sigma] \to [\theta(t_0), \theta(\sigma)]$ is a $C^1$-diffeomorphism, with inverse $s : [\theta(t_0), \theta(\sigma)] \to [t_0, \sigma]$. If we set $\varrho(\theta) = \rho(s(\theta))$, so that $\rho(t) = \varrho(\theta(t))$, we find

$$\frac{d\varrho(\theta)}{d\theta} = \frac{\rho'(t)}{\theta'(t)}\bigg|_{t=s(\theta)} = \frac{R(s(\theta), \rho(s(\theta)), \theta)}{\Theta(s(\theta), \rho(s(\theta)), \theta)}.$$

Hence, $\varrho(\theta)$ satisfies, for $\theta \in [\theta(t_0), \theta(\sigma)]$, the differential inequality

$$\varrho' \leq \varrho M_+(\varrho \cos \theta, \varrho \sin \theta)$$

and, moreover, $\varrho(\theta_0) = \rho(t_0) = \gamma(\theta_0)$. As uniqueness of solutions holds for any Cauchy problem associated with (22), by a classical result on differential inequalities (see, e.g., [21, Section I.6, Theorem 6.1]), we conclude that $\varrho(\theta) \leq \gamma(\theta)$, for all $\theta \in [\theta(t_0), \theta(\sigma)]$, and hence $\rho(t) \leq \gamma(\theta(t))$, for all $t \in [t_0, \sigma]$. In particular, we have $\overline{r_k} = \rho(\sigma) \leq \gamma(\theta(\sigma)) < \overline{r_k}$, which leads to a contradiction. $\qquad \square$

**Lemma 3.3.** *Assume* $(S_0)$, $(S_1)$, $(S_2)$. *Let* $J \subset \mathbb{R}$ *be a compact interval. Then there exists* $r_0 = r_0(J) > 0$ *such that*

$$\mathrm{Rot}(z; J) < 1, \tag{28}$$

*for any solution* $z$ *of* (13), *with* $0 < \min\limits_J |z| \leq r_0$.

*Proof.* Assumptions $(S_1)$ and $(S_2)$ imply in particular that, for any fixed $\varepsilon > 0$, there exists $\delta_\varepsilon \in ]0, \delta[$ such that, for all $t \in \mathbb{R}$ and $s \in [-\delta_\varepsilon, \delta_\varepsilon]$,

$$\bar{f}(t, s)s \leq \varepsilon s^2.$$

Let $J = [t_1, t_2]$ and $z = (u, v)$ be a non-trivial solution of (13) such that, for all $t \in J$,

$$|u(t)| \leq \delta_\varepsilon \tag{29}$$

Denote as usual by $(\rho, \theta)$ the polar coordinates of $z$ and set $\theta_1 = \theta(t_1)$ and $\theta_2 = \theta(t_2)$. We want to prove that

$$\theta_2 - \theta_1 < 2\pi. \tag{30}$$

Assume by contradiction that

$$\theta_2 - \theta_1 \geq 2\pi. \tag{31}$$

As we have, by (29),

$$\bar{f}(t, \rho(t) \cos \theta(t))\rho(t) \cos \theta(t) \leq \varepsilon(\rho(t) \cos \theta(t))^2$$

and

$$\bar{\psi}\big(\rho(t) \sin \theta(t)\big)\rho(t) \sin \theta(t) \leq \psi'(\delta)\big(\rho(t) \sin \theta(t)\big)^2,$$

we obtain, from (26),

$$\frac{\theta'(t)}{\psi'(\delta)\big(\sin \theta(t)\big)^2 + \varepsilon\big(\cos \theta(t)\big)^2} \leq 1,$$

for all $t \in [t_1, t_2]$. Setting $c = \psi'(\delta)$ and integrating over $[t_1, t_2]$ yield

$$
\begin{aligned}
|J| = t_2 - t_1 &\geq \int_{\theta_1}^{\theta_2} \frac{1}{c(\sin s)^2 + \varepsilon(\cos s)^2}\, ds \\
&\geq \int_0^{2\pi} \frac{1}{c(\sin s)^2 + \varepsilon(\cos s)^2}\, ds = 4 \int_0^{\frac{\pi}{2}} \frac{1}{c(\sin s)^2 + \varepsilon(\cos s)^2}\, ds \\
&= \frac{4}{\varepsilon} \int_0^{\frac{\pi}{2}} \frac{1}{(\cos t)^2} \frac{1}{1 + \left(\sqrt{\frac{c}{\varepsilon}}\tan t\right)^2}\, dt = \frac{4}{\sqrt{c\varepsilon}} \int_0^{+\infty} \frac{1}{1 + t^2}\, dt = \frac{2\pi}{\sqrt{c\varepsilon}}.
\end{aligned}
$$

A contradiction is achieved taking $\varepsilon \in \,]0, \frac{4\pi^2}{c|J|^2}[$. Hence (30) follows.

In order to conclude, we use Lemma 3.1: choosing $r_0 > 0$ small enough, any solution $z$, with $0 < \min_J |z| \leq r_0$, by (14) satisfies

$$
\max_J |u| \leq \max_J |z| \leq \delta_\varepsilon.
$$

Hence (30) holds, implying the validity of (28). $\qquad\square$

**Remark 3.1** Lemma 3.3 is still valid replacing assumption $(S_2)$ with

$$
\lim_{s \to 0^+} \frac{f(t,s)}{s} = 0, \quad \text{uniformly in } t \in [0, T]
$$

or

$$
\lim_{s \to 0^-} \frac{f(t,s)}{s} = 0, \quad \text{uniformly in } t \in [0, T].
$$

The proof requires just few minor modifications, as shown in [12, Lemma 3.4].

**Theorem 3.4.** *Assume* $(S_0)$, $(S_1)$, $(S_2)$. *For every* $k \in \mathbb{N}_0$ *there exists* $m_k^* \in \mathbb{N}_0$ *such that for any integer* $m > m_k^*$, *which is coprime with* $k$, *equation* (1) *has a classical subharmonic solution* $u_k$ *of order* $m$ *with precisely* $2k$ *zeroes in* $[0, mT[$.

*Proof.* Fix $k \in \mathbb{N}_0$ and let $m_k^* \in \mathbb{N}_0$, with $m_k^* T > k\, \tau_k^*$, and $r_k^*$ be as in Lemma 3.2. Take $m \in \mathbb{N}_0$ such that $m > m_k^*$ and $r_0 = r_0([0, mT]) < r_k^*$ as in Lemma 3.3. The two results just mentioned guarantee on the one hand that

$$
\text{Rot}(z; [0, mT]) > k
$$

for any solution $z$ with initial value $z_0$ such that $|z_0| = r_k^*$; on the other hand there holds

$$
\text{Rot}(z; [0, mT]) < k
$$

for any solution $z$ with initial value $z_0$ such that $|z_0| = r_0$. Since solutions of the Cauchy problems associated with (13) are globally defined, we can apply the

Poincaré-Birkhoff theorem, in the version of [18, Theorem 8.2]: there exists in particular a point $z_k^* \in \mathbb{R}^2$ such that $r_0 < |z_k^*| < r_k^*$ and a corresponding solution $z_k = (u_k, v_k)$ of (13) which is $mT$−periodic and satisfies

$$\mathrm{Rot}(z_k; [0, mT]) = k.$$

By the previous discussion we know that, denoting by $(\rho_k, \theta_k)$ the polar coordinates of $z_k$, the angular displacement $\theta_k$ is strictly increasing, and thus $u_k$ has exactly $2k$ zeroes in $[0, mT[$. Moreover, since $m, k$ are coprime, $mT$ is the minimum period of $u_k$ among $T, 2T, \ldots, (m-1)T, mT$. Finally, Lemma 3.2 implies that $z_k$ satisfies

$$r_0 < |z_k(t)| < r_k^*$$

for all $t \in \mathbb{R}$. This condition assures that $u_k$ is a classical subharmonic solution of (1) of order $m$, with precisely $2k$ zeroes in $[0, mT[$. $\qquad\square$

**Remark 3.2** Taking $k = 1$ in Theorem we conclude that, for any $m > m_1^*$ there exists at least one subharmonic solution having minimum period $mT$.

We are now in position of proving Theorem 1.1.

**Proof of Theorem 1.1** We keep the same notations as in the proof of Theorem 3.4. Fix any $k \in \mathbb{N}_0$ and take $m_k \in \mathbb{N}_0$, coprime with $k$, such that $m_k > \max\{k^2, m_k^*\}$. From the proof of Theorem 3.4 we know that there exists at least one point $z_k^* \in \mathbb{R}^2$ such that $r_0 < |z_k^*| < r_k^*$ and a corresponding solution $z_k = (u_k, v_k)$ of (13) which is $m_k T$-periodic and satisfies

$$\mathrm{Rot}(z_k; [0, m_k T]) = k.$$

The minimum period $\tau_k$ of $z_k$, and hence of $u_k$, satisfies $\tau_k \geq \frac{m_k}{k} T > kT$. This estimate obviously yields

$$\lim_{k \to +\infty} \tau_k = +\infty.$$

The proof of Theorem 3.4 also guarantees that, for any $k \in \mathbb{N}_0$,

$$\max\{\|u_k\|_\infty, \|(\bar\psi)^{-1}(u_k')\|_\infty\} \leq \|z_k\|_\infty < r_k^*.$$

As we chose $r_k^* < \frac{1}{k}$ in Lemma 3.2, we get

$$\lim_{k \to +\infty} \|u_k\|_{C^1} = 0.$$

This concludes the proof of Theorem 1.1. $\qquad\square$

# 4  Large bounded variation subharmonic solutions

We start with some preparatory results.

**Lemma 4.1.** *Let $\sigma > 0$ and $u \in BV_{\text{loc}}(\mathbb{R})$ be fixed. Then $u$ is a $\sigma$-periodic solution of (1) if and only if $f(\cdot, u) \in L^1(0, \sigma)$ and the following inequality*

$$\mathcal{J}_\sigma(v) - \mathcal{J}_\sigma(u) \geq \int_0^\sigma f(t, u)(v - u)\, dt \tag{32}$$

*holds for all $v \in BV(0, \sigma)$, with*

$$\mathcal{J}_\sigma(v) = \int_0^\sigma \sqrt{1 + |Dv|^2} + |v(0^+) - v(\sigma^-)|.$$

*Proof.* For any $\phi \in BV(0, \sigma)$, let us define $\mathcal{T}_\phi : \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{T}_\phi(s) = \mathcal{J}_\sigma(u + s\phi) - \int_0^\sigma f(t, u)(u + s\phi)\, dt.$$

The function $\mathcal{T}_\phi$ is convex. Let us also set $\mathcal{K}_\sigma : BV(0, \sigma) \to \mathbb{R}$ by

$$\mathcal{K}_\sigma(v) = \int_0^\sigma \sqrt{1 + |Dv|^2}.$$

By [1, Theorem 3.6] the functional $\mathcal{K}_\sigma$ is differentiable in the direction $\phi \in BV(0, \sigma)$ if and only if $|D\phi|^s$ is absolutely continuous with respect to $|Du|^s$ and, under this assumption,

$$\frac{d}{ds}\mathcal{K}_\sigma(u + s\phi)|_{s=0} = \int_0^\sigma \frac{(Du)^a\,(D\phi)^a}{\sqrt{1 + |(Du)^a|^2}}\, dt + \int_0^\sigma \operatorname{sgn}\left(\frac{Du}{|Du|}\right) \frac{D\phi}{|D\phi|}\, |D\phi|^s. \tag{33}$$

Fix now $\phi \in BV(0, \sigma)$ as required in [1, Theorem 3.6]. Then $\mathcal{T}_\phi$ is differentiable at $s = 0$ and the following holds

$$\mathcal{T}_\phi{}'(0) = \frac{d}{ds}\mathcal{J}_\sigma(u + s\phi)|_{s=0} - \frac{d}{ds}\left(\int_0^\sigma f(t, u)(u + s\phi)\, dt\right)\Big|_{s=0}$$

$$= \frac{d}{ds}\mathcal{K}_\sigma(u + s\phi)|_{s=0} + \frac{d}{ds}\left|(u + s\phi)(0^+) - (u + s\phi)(\sigma^-)\right|_{s=0} - \int_0^\sigma f(t, u)\phi\, dt$$

$$= \frac{d}{ds}\mathcal{K}_\sigma(u + s\phi)|_{s=0} + \frac{d}{ds}\left|u(0^+) - u(\sigma^-) + s\big(\phi(0^+) - \phi(\sigma^-)\big)\right|_{s=0} - \int_0^\sigma f(t, u)\phi\, dt$$

$$= \frac{d}{ds}\mathcal{K}_\sigma(u + s\phi)|_{s=0} + \operatorname{sgn}\big(u(0^+) - u(\sigma^-)\big)\big(\phi(0^+) - \phi(\sigma^-)\big) - \int_0^\sigma f(t, u)\phi\, dt.$$

$$\tag{34}$$

From (33) and (34) we obtain

$$\mathcal{T}_\phi{}'(0) = \int_0^\sigma \frac{(Du)^a\,(D\phi)^a}{\sqrt{1+|(Du)^a|^2}}\,dt + \int_0^\sigma \operatorname{sgn}\left(\frac{Du}{|Du|}\right)\frac{D\phi}{|D\phi|}\,|D\phi|^s$$
$$+ \operatorname{sgn}\big(u(0^+) - u(\sigma^-)\big)\big(\phi(0^+) - \phi(\sigma^-)\big) - \int_0^\sigma f(t,u)\phi\,dt. \quad (35)$$

Assume now that $f(\cdot,u) \in L^1(0,\sigma)$ and (32) holds for all $v \in BV(0,\sigma)$. Then, for any given $\phi \in BV(0,\sigma)$, the function $\mathcal{T}_\phi$ has a minimum at $s = 0$. Fix now $\phi \in BV(0,\sigma)$ as required in [1, Theorem 3.6] and such that $\phi(0^+) = \phi(\sigma^-)$ if $u(0^+) = u(\sigma^-)$. Since $s = 0$ is a minimum of $\mathcal{T}_\phi$, we have $\mathcal{T}_\phi'(0) = 0$, or equivalently using (35)

$$\int_0^\sigma \frac{(Du)^a\,(D\phi)^a}{\sqrt{1+|(Du)^a|^2}}\,dt + \int_0^\sigma \operatorname{sgn}\left(\frac{Du}{|Du|}\right)\frac{D\phi}{|D\phi|}\,|D\phi|^s$$
$$+ \operatorname{sgn}\big(u(0^+) - u(\sigma^-)\big)\big(\phi(0^+)\ - \phi(\sigma^-)\big) = \int_0^\sigma f(t,u)\phi\,dt.$$

This means that $u$ is a $\sigma$-periodic solution of (1).

Conversely, let us assume that $u$ is a $\sigma$-periodic solution of (1) and fix $v \in BV(0,\sigma)$. According to [25, Corollary 2.2], there exists a sequence $(v_n)_n$ in $W_\sigma^{1,1}(0,\sigma)$ such that

$$\lim_{n\to+\infty} v_n = v$$

in $L^1(0,\sigma)$ and a.e. in $]0,\sigma[$, and

$$\lim_{n\to+\infty} \mathcal{J}_\sigma(v_n) = \mathcal{J}_\sigma(v).$$

For each $n$, set $\phi_n = v_n - u$. We have that $\phi_n \in BV(0,\sigma)$, with $|D\phi_n|^s = |Du|^s$ and $\phi_n(0^+) - \phi_n(\sigma^-) = u(0^+) - u(\sigma^-)$. As, by assumption, $u$ satisfies the Euler equation (2), we have $\mathcal{T}_{\phi_n}'(0) = 0$; moreover, by convexity of $\mathcal{T}_{\phi_n}$, there holds

$$\mathcal{T}_{\phi_n}(1) \geq \mathcal{T}_{\phi_n}'(0) \cdot 1 + \mathcal{T}_{\phi_n}(0) = \mathcal{T}_{\phi_n}(0),$$

that is,

$$\mathcal{J}_\sigma(u + \phi_n) - \int_0^\sigma f(t,u)(u + \phi_n)\,dt \geq \mathcal{J}_\sigma(u) - \int_0^\sigma f(t,u)u\,dt,$$

or equivalently

$$\mathcal{J}_\sigma(v_n) - \int_0^\sigma f(t,u)v_n\,dt \geq \mathcal{J}_\sigma(u) - \int_0^\sigma f(t,u)u\,dt. \quad (36)$$

As by Lebesgue convergence theorem we have

$$\lim_{n \to +\infty} \int_0^\sigma f(t, u)(v_n) \, dt = \int_0^\sigma f(t, u)v \, dt,$$

we can pass to the limit in (36), as $n$ goes to $+\infty$, obtaining

$$\mathcal{J}_\sigma(v) - \mathcal{J}_\sigma(u) \geq \int_0^\sigma f(t, u)(v - u) \, dt,$$

that is (32). $\qquad\square$

**Lemma 4.2.** *Let $u \in BV_{\mathrm{loc}}(\mathbb{R})$ be a non-constant $\sigma$-periodic function, for some $\sigma > 0$. Then $u$ has a minimum period $\tau > 0$ and $\frac{\sigma}{\tau} \in \mathbb{N}_0$.*

*Proof.* We first prove that $u$ cannot have arbitrarily small periods. Assume by contradiction that there exists a sequence $(\sigma_n)_n$, with $0 < \sigma_n < \frac{\sigma}{n}$, of periods of $u$. As $u$ is $\sigma_n$-periodic, we have by [25, Proposition 2.9]

$$0 < 2(\operatorname*{ess\,sup}_{\mathbb{R}} u - \operatorname*{ess\,inf}_{\mathbb{R}} u) = 2(\operatorname*{ess\,sup}_{[0,\sigma_n]} u - \operatorname*{ess\,inf}_{[0,\sigma_n]} u)$$
$$\leq \int_0^{\sigma_n} |Du| + |u(\sigma_n^-) - u(0^+)| = \frac{1}{n}\left( \int_0^\sigma |Du| + |u(\sigma^-) - u(0^+)| \right),$$

which yields a contradiction by letting $n \to +\infty$.

Let us denote by $\mathcal{T}$ the set of all (positve) periods of $u$ and set $\tau = \inf \mathcal{T}$. We know from the previous step that $\tau > 0$. Let us show that $\tau$ is the minimum period. Let $(\sigma_n)_n$ be a sequence in $\mathcal{T}$ converging to $\tau$, with $\sigma_n > \tau$ for all $n$ . Let $u_r$ denote the right continuous representative of the bounded variation function $u$. As there exists a set $E \subseteq \mathbb{R}$, with zero Lebesgue measure, for which $u_r(t + \sigma_n) = u_r(t)$ for every $n$ and each $t \in \mathbb{R} \setminus E$, we conclude that $u_r(t + \sigma) = u_r(t)$ for each $t \in \mathbb{R} \setminus E$, that is $\tau \in \mathcal{T}$.

It is finally clear that, $\sigma > 0$ being a period of $u$, there holds $\frac{\sigma}{\tau} \in \mathbb{N}_0$. $\qquad\square$

**Lemma 4.3.** *Let $u \in BV_{\mathrm{loc}}(\mathbb{R})$ be a non-constant $\sigma$-periodic solution of* (1) *and let $\tau > 0$ be the minimum period of $u$. Then $u$ is a $\tau$-periodic solution of* (1).

*Proof.* Suppose that $\sigma > \tau$. By definition of $\sigma$-periodic solution of (1), $u$ satisfies

$$\int_0^\sigma \frac{(Du)^a \phi'}{\sqrt{1 + |(Du)^a|^2}} \, dt = \int_0^\sigma f(t, u)\phi \, dt \tag{37}$$

for every $\phi \in C_0^\infty(]0, \sigma[)$.

Let us prove that the function $f(\cdot, u) \in L^1_{\mathrm{loc}}(\mathbb{R})$ is $\tau$-periodic. As $\tau$ is the minimum period of $u$, there exists $N \in \mathbb{N}$, with $N \geq 2$, such that $\sigma = N\tau$. Assume by contradiction that, e.g., $f(t, u(t)) \neq f(t+\tau, u(t+\tau))$ for all $t$ in a subset of $[0, \tau]$ of

positive measure. Take $\phi_1 \in C_0^\infty(]0, \tau[)$ and $\phi_2 \in C_0^\infty(]\tau, 2\tau[)$, with $\phi_1(t) = \phi_2(t+\tau)$ in $[0, \tau]$, such that

$$\int_0^\tau f(t, u)\phi_1 \, dt \neq \int_\tau^{2\tau} f(t, u)\phi_2 \, dt.$$

As $(Du)^a = (Du(\cdot - \tau))^a$ a.e. in $\mathbb{R}$ and $\phi_2' = \phi_1'(\cdot - \tau)$ in $[\tau, 2\tau]$, we get from (37),

$$\int_0^\tau f(t, u)\phi_1 \, dt = \int_0^\tau \frac{(Du)^a \phi_1'}{\sqrt{1 + |(Du)^a|^2}} \, dt$$

$$= \int_\tau^{2\tau} \frac{(Du)^a \phi_2'}{\sqrt{1 + |(Du)^a|^2}} \, dt = \int_\tau^{2\tau} f(t, u)\phi_2 \, dt$$

which is a contradiction.

We next prove that $u$ is a $\tau$-periodic solution of (1). Pick any $w \in BV(0, \tau)$ and let $v_i \in BV((i-1)\tau, i\tau)$ be such that $v_i(t + (i-1)\tau) = w(t)$ for a.e. $t \in [0, \tau]$, for $i = 1, \ldots, N$. Define $v \in BV(0, \sigma)$ by $v(t) = v_i(t)$ for a.e. $t \in [(i-1)\tau, i\tau]$, for $i = 1, \ldots, N$. We have

$$\int_0^\sigma \sqrt{1 + |Dv|^2} + |v(\sigma^-) - v(0^+)| - \int_0^\sigma f(t, u)v \, dt$$

$$= \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \sqrt{1 + |Dv_i|^2}$$

$$+ \sum_{i=1}^{N-1} |v((i\tau)^+) - v((i\tau)^-)| + |v(\sigma^-) - v(0^+)| - \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} f(t, u)v_i \, dt$$

$$= N \int_0^\tau \sqrt{1 + |Dw|^2} + N|w(\tau^-) - w(0^+)| - N \int_0^\tau f(t, u)w \, dt$$

and

$$\int_0^\sigma \sqrt{1 + |Du|^2} + |u(\sigma^-) - u(0^+)| - \int_0^\sigma f(t, u)u \, dt$$

$$= N \int_0^\tau \sqrt{1 + |Du|^2} + N|u(\tau^-) - u(0^+)| - N \int_0^\tau f(t, u)u \, dt.$$

Hence we conclude that

$$\int_0^\tau \sqrt{1 + |Du|^2} + |u(\tau^-) - u(0^+)| - \int_0^\tau f(t, u)u \, dt$$

$$\leq \int_0^\tau \sqrt{1 + |Dw|^2} + |w(\tau^-) - w(0^+)| - \int_0^\tau f(t, u)w \, dt.$$

Therefore $u$ is a $\tau$-periodic solution of (1). $\qquad\square$

The following result guarantees the existence of a sequence of arbitrarily large $kT$-periodic solutions of (1).

**Theorem 4.4.** *Assume $(s_0), (s_2),$*

$(s_1')$ $\displaystyle\lim_{s \to \pm\infty} \frac{F(t,s)}{|s|} = 0$ *uniformly a.e. in $[0,T]$,*

$(s_3')$ *there exists $R > 0$ such that $f(t,s)\, s \geq 0$ for a.e. $t \in [0,T]$ and every $s$ with $|s| \geq R$.*

*Then, there exists a sequence $(u_k)_k$ of $kT$-periodic solutions of (1), satisfying*

$$\lim_{k \to +\infty} \operatorname*{ess\,sup}_{[0,kT]} u_k = +\infty \quad or \quad \lim_{k \to +\infty} \operatorname*{ess\,inf}_{[0,kT]} u_k = -\infty. \tag{38}$$

*Proof.* For any fixed $k \in \mathbb{N}_0$, define a functional $\mathcal{I}_{kT} : BV(0,kT) \to \mathbb{R}$ by setting

$$\mathcal{I}_{kT}(v) = \mathcal{J}_{kT}(v) - \int_0^{kT} F(t,v)\,dt.$$

We also define

$$\mathcal{W}_{kT} = \left\{ w \in BV(0,kT) : \int_0^{kT} w\,dt = 0 \right\}$$

and, for every $v \in BV(0,kT)$, we set

$$r = \frac{1}{kT} \int_0^{kT} v\,dt,$$

so that $w = v - r \in \mathcal{W}_{kT}$.

**Step 1.** $\mathcal{I}_{kT}$ *has a mountain-pass geometry.*

Assumptions $(s_0)$ and $(s_1')$ imply that for every $\varepsilon > 0$ there exists a $T$-periodic function $c_\varepsilon \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that

$$F(t,s) \leq \varepsilon|s| + c_\varepsilon(t), \tag{39}$$

for a.e. $t \in \mathbb{R}$ and every $s \in \mathbb{R}$. For every $w \in \mathcal{W}_{kT}$, we have, using (39) and [25, Corollary 2.7],

$$\mathcal{I}_{kT}(w) = \mathcal{J}_{kT}(w) - \int_0^{kT} F(t,w)\,dt$$

$$\geq \int_0^{kT} |Dw| + |w((kT)^-) - w(0^+)| - \varepsilon \int_0^{kT} |w|\,dt - \int_0^{kT} c_\varepsilon\,dt$$

$$\geq \left(1 - \varepsilon\frac{kT}{4}\right)\left(\int_0^{kT} |Dw| + |w((kT)^-) - w(0^+)|\right) - \int_0^{kT} c_\varepsilon\,dt.$$

This implies that
$$\inf_{\mathcal{W}_{kT}} \mathcal{I}_{kT} > -\infty.$$

On the other hand, by assumption $(s_2)$ there exists $A_k \in \mathbb{R}_0^+$ such that

$$kT - \int_0^{kT} F(t, A_k)\, dt < \inf_{\mathcal{W}_{kT}} \mathcal{I}_{kT}.$$

According to [25, Lemma 2.13], there exist sequences $(v_n)_n$ in $BV(0, kT)$ and $(\varepsilon_n)_n$ in $\mathbb{R}$ such that $\lim\limits_{n \to +\infty} \varepsilon_n = 0$,

$$\mathcal{J}_{kT}(v) - \mathcal{J}_{kT}(v_n) \geq \int_0^{kT} f(t, v_n)(v - v_n)\, dt - \varepsilon_n \|v - v_n\|_{BV(0,kT)}, \qquad (40)$$

for every $v \in BV(0, kT)$, and

$$\lim_{n \to +\infty} \mathcal{I}_{kT}(v_n) = c_k = \inf_{\gamma \in \Gamma_k} \max_{\xi \in [-A_k, A_k]} \mathcal{I}_{kT}\big(\gamma(\xi)\big), \qquad (41)$$

where
$$\Gamma_k = \big\{ \gamma \in C\big([-A_k, A_k], BV(0, kT)\big) : \gamma(\pm A_k) = \pm A_k \big\}. \qquad (42)$$

**Step 2.** *The sequence $(v_n)_n$ is bounded in $BV(0, kT)$.*
Let us write, for each $n \in \mathbb{N}$, $v_n = w_n + r_n$, with $w_n \in \mathcal{W}_{kT}$. Assume by contradiction that, possibly passing to a subsequence that we still denote by $(v_n)_n$,

$$\lim_{n \to +\infty} \|v_n\|_{BV(0,kT)} = +\infty. \qquad (43)$$

Using (39), (41) and [25, Corollary 2.7] again, we get for all large $n$

$$c_k + 1 \geq \mathcal{I}_{kT}(v_n) = \mathcal{J}_{kT}(w_n) - \int_0^{kT} F(t, w_n + r_n)\, dt$$

$$\geq \int_0^{kT} |Dw_n| + |w_n\big((kT)^-\big) - w_n(0^+)| - \varepsilon \int_0^{kT} |w_n|\, dt - \varepsilon kT |r_n| - \int_0^{kT} c_\varepsilon\, dt$$

$$\geq \left(1 - \varepsilon \frac{kT}{4}\right)\left(\int_0^{kT} |Dw_n| + |w_n\big((kT)^-\big) - w_n(0^+)|\right) - \varepsilon kT |r_n| - \int_0^{kT} c_\varepsilon\, dt.$$

Hence we deduce that for every $\eta > 0$, there exists $c_\eta > 0$ such that for all large $n$

$$\frac{1}{2}\left(\int_0^{kT} |Dw_n| + |w_n\big((kT)^-\big) - w_n(0^+)|\right) \leq \eta |r_n| + c_\eta$$

and, by [25, Corollary 2.10],

$$\|w_n\|_{L^\infty(0,kT)} \leq \eta |r_n| + c_\eta. \qquad (44)$$

By (43) and (44), we infer that

$$\lim_{n\to+\infty} |r_n| = +\infty.$$

Possibly passing to a further subsequence that we still denote by $(v_n)_n$, we can suppose that

$$\text{either} \quad \lim_{n\to+\infty} r_n = +\infty \quad \text{or} \quad \lim_{n\to+\infty} r_n = -\infty.$$

Assume that the former case occurs. From (44) we get

$$\lim_{n\to+\infty} \operatorname*{ess\,inf}_{[0,kT]} v_n = +\infty \tag{45}$$

and hence, by $(s_3')$, for all large $n$

$$f\big(t, v_n(t)\big) \geq 0,$$

for a.e. $t \in [0, kT]$. Testing (40) against $v = v_n \pm 1$, we obtain

$$0 \geq \pm \int_0^{kT} f(t, v_n)\, dt - \varepsilon_n kT$$

and then, for all large $n$,

$$\int_0^{kT} |f(t, v_n)|\, dt = \left| \int_0^{kT} f(t, v_n)\, dt \right| \leq \varepsilon_n kT. \tag{46}$$

Now, test (40) against $v = r_n$. We get, using (46) and [25, Corollary 2.7, Corollary 2.10],

$$
\begin{aligned}
\mathcal{J}_{kT}(w_n) &\leq \mathcal{J}_{kT}(r_n) + \int_0^{kT} f(t, v_n) w_n\, dt + \varepsilon_n \|w_n\|_{BV(0,kT)} \\
&\leq kT + \|f(\cdot, v_n)\|_{L^1(0,kT)} \|w_n\|_{L^\infty(0,kT)} + \varepsilon_n \|w_n\|_{BV(0,kT)} \\
&\leq \varepsilon_n \left(1 + \frac{3}{4} kT\right) \left( \int_0^{kT} |Dw_n| + |w_n\big((kT)^-\big) - w_n(0^+)| \right) + kT.
\end{aligned}
\tag{47}
$$

Hence we can easily conclude that there exists a constant $c > 0$ such that for all large $n$

$$\int_0^{kT} |Dw_n| + |w_n\big((kT)^-\big) - w_n(0^+)| \leq c$$

and

$$\mathcal{J}_{kT}(w_n) \leq c.$$

Therefore, using $(s_3')$, we get for all large $n$

$$\mathcal{I}_{kT}(v_n) = \mathcal{J}_{kT}(w_n) - \int_0^{kT} F(t, v_n)\, dt$$

$$= \mathcal{J}_{kT}(w_n) - \int_0^{kT} \left( \int_0^{\operatorname*{ess\,inf}_{[0,kT]} v_n} f(t,s)\, ds \right) dt - \int_0^{kT} \left( \int_{\operatorname*{ess\,inf}_{[0,kT]} v_n}^{v_n(t)} f(t,s)\, ds \right) dt$$

$$\leq c - \int_0^{kT} F\big(t, \operatorname*{ess\,inf}_{[0,kT]} v_n\big)\, dt.$$

Then a contradiction follows, using (45) and $(s_2)$, as we have by (41)

$$\inf_n \mathcal{I}_{kT}(v_n) > -\infty.$$

**Step 3.** *For each $k \in \mathbb{N}_0$, there exists a $kT$-periodic solution $u_k$ of (1), with $\mathcal{I}_{kT}(u_k) = c_k$.*
Fix $k \in \mathbb{N}_0$. Since by Step 2 the sequence $(v_n)_n$ is bounded in $BV(0, kT)$, there exists a subsequence, that we still denote by $(v_n)_n$, and a function $u_k \in BV(0, kT)$, such that

$$\lim_{n \to +\infty} v_n = u_k$$

in $L^1(0, kT)$ and a.e. in $[0, kT]$, and

$$\sup_n \|v_n\|_{L^\infty(0,kT)} < +\infty.$$

Hence, using $(s_0)$, [25, Proposition 2.4] and Lebesgue convergence theorem, we get

$$\lim_{n \to +\infty} f(\cdot, v_n) = f(\cdot, u_k), \quad \text{in } L^1(0, kT),$$

$$\lim_{n \to +\infty} \int_0^{kT} F(t, v_n)\, dt = \int_0^{kT} F(t, u_k)\, dt,$$

$$\liminf_{n \to +\infty} \mathcal{J}_{kT}(v_n) \geq \mathcal{J}_{kT}(u_k),$$

$$\lim_{n \to +\infty} \int_0^{kT} f(t, v_n)(v - v_n)\, dt = \int_0^{kT} f(t, u_k)(v - u_k)\, dt,$$

$$\lim_{n \to +\infty} \varepsilon_n \|v - v_n\|_{BV(0,kT)} = 0,$$

for every $v \in BV(0, kT)$. Accordingly, we obtain from (40)

$$\mathcal{J}_{kT}(v) - \int_0^{kT} f(t, u_k)(v - u_k)\, dt = \mathcal{J}_{kT}(v) - \lim_{n \to +\infty} \int_0^{kT} f(t, v_n)(v - v_n)\, dt$$

$$+ \lim_{n \to +\infty} \varepsilon_n \|v - v_n\|_{BV(0,kT)} \geq \liminf_{n \to +\infty} \mathcal{J}_{kT}(v_n) \geq \mathcal{J}_{kT}(u_k),$$

i.e., $u_k$ is a $kT$-periodic solution of (1). Moreover, testing (40) against $v = u_k$, we get

$$\mathscr{J}_{kT}(u_k) - \int_0^{kT} f(t, u_k)(u_k - v_n) \, dt + \varepsilon_n \|u_k - v_n\|_{BV(0,kT)} \geq \mathscr{J}_{kT}(v_n).$$

Letting $n \to +\infty$, we have

$$\mathscr{J}_{kT}(u_k) \geq \limsup_{n \to +\infty} \mathscr{J}_{kT}(v_n).$$

As

$$\mathscr{J}_{kT}(u_k) \leq \liminf_{n \to +\infty} \mathscr{J}_{kT}(v_n),$$

we conclude that

$$\mathscr{J}_{kT}(u_k) = \lim_{n \to +\infty} \mathscr{J}_{kT}(v_n)$$

and

$$\mathcal{I}_{kT}(u_k) = \lim_{n \to +\infty} \mathcal{I}_{kT}(v_n) = c_k.$$

**Step 4.** *The following limits hold*

$$\lim_{k \to +\infty} \frac{1}{k} \mathcal{I}_{kT}(u_k) = -\infty, \tag{48}$$

$$\lim_{k \to +\infty} \frac{1}{k} \|u_k\|_{L^1(0,kT)} = +\infty,$$

$$\lim_{k \to +\infty} \|u_k\|_{L^\infty(0,kT)} = +\infty. \tag{49}$$

For each $k \in \mathbb{N}_0$, let $\phi_k : [0, kT] \to \mathbb{R}$ be defined by $\phi_k(t) = k\,\mathrm{sgn}\left(t - \dfrac{kT}{2}\right)$. Note that $\phi_k$ is an eigenfunction associated with the second eigenvalue $\frac{kT}{4}$ of the 1-Laplace operator with periodic boundary condition on $[0, kT]$ (cf. [15]). Define a path $\gamma_k : [-A_k, A_k] \to BV(0, kT)$ by

$$\gamma_k(\xi) = \xi + \left(1 - \frac{|\xi|}{A_k}\right)\phi_k.$$

Clearly, we have $\gamma_k \in \Gamma_k$, where $\Gamma_k$ is defined in (42). Let us compute, for each $\xi \in [-A_k, A_k]$,

$$\mathscr{J}_{kT}\big(\gamma_k(\xi)\big) = \mathscr{J}_{kT}\left(\left(1 - \frac{|\xi|}{A_k}\right)\phi_k\right) = 4k\left(1 - \frac{|\xi|}{A_k}\right) + kT \leq (4 + T)k.$$

Hence we obtain, for each $\xi \in [-A_k, A_k]$,

$$\mathcal{I}_{kT}\big(\gamma_k(\xi)\big) \leq (4 + T)k - \int_0^{kT} F\big(t, \gamma_k(\xi)\big) \, dt. \tag{50}$$

Now we want to estimate the last integral for all $k$ large enough. Note that we can assume, without restriction, that $A_k \geq k \geq \max\{4, R\}$. Hence, we see that there exists an interval $[a_k, b_k] \subseteq [0, kT]$ with $a_k = a_k(\xi), b_k = b_k(\xi)$ and $b_k - a_k \geq \lfloor \frac{k}{2} \rfloor T$, such that for each $\xi \in [-A_k, A_k]$

$$\left|\gamma_k(\xi)(t)\right| \geq k,$$

for a.e. $t \in [a_k, b_k]$. Indeed, the following statements hold:

(i) if $\xi = 0$, then

$$\left|\gamma_k(0)(t)\right| = \left|k\,\mathrm{sgn}\left(t - \frac{kT}{2}\right)\right| \geq k,$$

a.e. in $[a_k, b_k] = [0, kT]$;

(ii) if $\xi \in\, ]0, A_k]$, then

$$\left|\gamma_k(\xi)(t)\right| = \left|\xi + k\left(1 - \frac{\xi}{A_k}\right)\mathrm{sgn}\left(t - \frac{kT}{2}\right)\right| = k + \xi\left(1 - \frac{k}{A_k}\right) \geq k,$$

a.e. in $[a_k, b_k] = \left[\lceil \frac{k}{2} \rceil T, kT\right]$.

(iii) if $\xi \in [-A_k, 0[$, then

$$\left|\gamma_k(\xi)(t)\right| = \left|\xi + k\left(1 + \frac{\xi}{A_k}\right)\mathrm{sgn}\left(t - \frac{kT}{2}\right)\right| = -\xi\left(1 - \frac{k}{A_k}\right) + k \geq k,$$

a.e. in $[a_k, b_k] = \left[0, \lfloor \frac{k}{2} \rfloor T\right]$.

As we have assumed $A_k \geq k \geq R$, we have, using $(s_3')$,

$$F(t, s) \geq F(t, k),$$

for a.e. $t \in \mathbb{R}$ and every $s$ with $|s| \geq k$. Moreover, by $(s_0)$, there exists a $T$-periodic function $h \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that

$$F(t, s) \geq -h(t),$$

for a.e. $t \in \mathbb{R}$ and every $s \in \mathbb{R}$. Therefore we obtain, for every $\xi \in [-A_k, A_k]$,

$$
\begin{aligned}
\int_0^{kT} F\left(t, \gamma_k(\xi)\right) dt &= \int_{a_k}^{b_k} F\left(t, \gamma_k(\xi)\right) dt + \int_{[0,kT]\setminus[a_k,b_k]} F\left(t, \gamma_k(\xi)\right) dt \\
&\geq \int_{a_k}^{b_k} F(t, k)\, dt - \int_{[0,kT]\setminus[a_k,b_k]} h\, dt \qquad\qquad (51) \\
&\geq \int_{a_k}^{b_k} F(t, k)\, dt - k\, \|h\|_{L^1(0,T)}\,.
\end{aligned}
$$

Note that, for any $\xi \in [-A_k, A_k]$, we have

$$\int_{a_k}^{b_k} F(t, k) \, dt \geq \left\lfloor \frac{k}{2} \right\rfloor \int_0^T F(t, k) \, dt$$

and then, by $(s_3)$

$$\lim_{k \to +\infty} \frac{1}{k} \int_{a_k}^{b_k} F(t, k) \, dt = +\infty.$$

Therefore we can conclude, from (50) and (51), that

$$\lim_{k \to +\infty} \frac{1}{k} \max_{\xi \in [-A_k, A_k]} \mathcal{I}_{kT}\big(\gamma_k(\xi)\big) \leq \lim_{k \to +\infty} \left( 4 + T + \|h\|_{L^1(0,T)} - \frac{1}{k} \int_{a_k}^{b_k} F(t, k) \, dt \right) = -\infty,$$

which in turn implies that

$$\lim_{k \to +\infty} \frac{1}{k} \mathcal{I}_{kT}(u_k) = -\infty, \tag{52}$$

as

$$\mathcal{I}_{kT}(u_k) = \inf_{\gamma \in \Gamma_k} \max_{\xi \in [-A_k, A_k]} \mathcal{I}_{kT}\big(\gamma(\xi)\big).$$

We finally observe that, as

$$\frac{1}{k} \mathcal{I}_{kT}(u_k) \geq -\frac{1}{k} \int_0^{kT} F(t, u_k) \, dt,$$

then we get, from (52),

$$\lim_{k \to +\infty} \frac{1}{k} \int_0^{kT} F(t, u_k) \, dt = +\infty.$$

Using (39) with $\varepsilon = 1$, we have

$$\frac{1}{k} \int_0^{kT} F(t, u_k) \, dt \leq \frac{1}{k} \int_0^{kT} |u_k| \, dt + \frac{1}{k} \int_0^{kT} |c_1| \, dt$$

$$= \frac{1}{k} \|u_k\|_{L^1(0,kT)} + \|c_1\|_{L^1(0,T)} \leq T \|u_k\|_{L^\infty(0,kT)} + \|c_1\|_{L^1(0,T)},$$

and hence both

$$\lim_{k \to +\infty} \frac{1}{k} \|u_k\|_{L^1(0,kT)} = +\infty$$

and

$$\lim_{k \to +\infty} \|u_k\|_{L^\infty(0,kT)} = +\infty. \qquad \square$$

**Remark 4.1** Under the assumptions of Theorem 4.4 we cannot exclude that all the solutions $u_k$ are constant. This cannot happen if assumption $(s_3')$ is slightly strengthened, replacing it with $(s_3)$, in that case the obtained solutions, if classical, would be subharmonic solutions in the sense, e.g., of [14, p. 426]. This is the content of Theorem 4.5.

**Theorem 4.5.** *Assume $(s_0)$, $(s_1')$, $(s_2)$ and $(s_3)$. Then, there exists a sequence $(u_k)_{k \in \mathbb{N}_0}$ of $kT$-periodic solutions of $(1)$, satisfying*

$$\lim_{k \to +\infty} \left( \operatorname*{ess\,sup}_{[0,\tau_k]} u_k - \operatorname*{ess\,inf}_{[0,\tau_k]} u_k \right) = +\infty. \tag{53}$$

*Moreover, for each $N \in \mathbb{N}_0$, there exists $\bar{k}$ such that, for every $k \geq \bar{k}$, $u_{kN}$ is not a $NT$-periodic solution of $(1)$.*

*Proof.* Theorem 4.4 guarantees the existence of a sequence $(u_k)_k$ of $kT$-periodic solutions of (1) for which (38) holds. Let us prove the validity of (53). Indeed, otherwise from (38) we deduce, possibly passing to a subsequence of $(u_k)_k$, still denoted by $(u_k)_k$, that

$$\lim_{k \to +\infty} \operatorname*{ess\,inf}_{[0,kT]} u_k = +\infty \quad \text{or} \quad \lim_{k \to +\infty} \operatorname*{ess\,sup}_{[0,kT]} u_k = -\infty.$$

Assume that the former case occurs. Hence condition $(s_3)$ implies that, for all large $k$,

$$f(t, u_k(t)) > 0, \tag{54}$$

for a.e. $t \in [0, kT]$. Testing (40) against $v = u_k \pm 1$, we infer

$$\int_0^{kT} f(t, u_k) \, dt = 0.$$

A contradiction then follows from (54).

Next, in order to prove the last conclusion, we suppose by contradiction that, for some $N \in \mathbb{N}_0$, there exists a subsequence $(u_{k_j N})_j$ of $(u_{kN})_k$, such that, for every $j$, $u_{k_j N}$ is $NT$-periodic. Let us denote this subsequence by $(u_{kN})_k$ for simplicity. From condition (48) and the $NT$-periodicity of $u_{kN}$, we have

$$\lim_{k \to +\infty} \frac{1}{N} \mathcal{I}_{NT}(u_{kN}) = \lim_{k \to +\infty} \frac{1}{kN} \mathcal{I}_{kNT}(u_{kN}) = -\infty.$$

This implies that

$$\sup_k \mathcal{I}_{NT}(u_{kN}) = M < +\infty.$$

The same argument employed in Step 2 in the proof of Theorem 4.4 yields the existence of a further subsequence of $(u_{kN})_N$, which we still denote by $(u_{kN})_N$, such that

$$\lim_{k \to +\infty} \operatorname*{ess\,inf}_{[0,NT]} u_{kN} = +\infty \quad \text{or} \quad \lim_{k \to +\infty} \operatorname*{ess\,sup}_{[0,NT]} u_{kN} = +\infty.$$

Now we proceed as above in order to get a contradiction by means of condition $(s_3)$. $\qquad \square$

**Remark 4.2** By a diagonal argument, we see that there exists a sequence $(k_j)_j$ of positive integers, with $k_j \geq j$ for every $j \in \mathbb{N}_0$, such that the corresponding solutions $(u_{k_j})_j$ are $k_j T$-periodic, but not $hT$-periodic for $h = 1, \ldots, j$.

Finally, if both assumptions $(s_1')$ and $(s_3')$ are strengthened into $(s_1)$ and $(s_3)$, respectively, then the obtained solutions exhibit large-amplitude oscillations and have arbitrarily large minimum periods, as stated in Theorem 1.2.

**Proof of Theorem 1.2.** Theorem 4.5 guarantees the existence of a sequence $(u_k)_k$ of $kT$-periodic solutions of (1) for which (53) holds. Since, for all large $k$, $u_k$ is a non-constant $kT$-periodic solution of (1), Lemma 4.2 and Lemma 4.3 imply that $u_k$ has a minimum period $\tau_k > 0$ and it is a $\tau_k$-periodic solution of (1), i.e., for every $v \in BV(0, \tau_k)$,

$$\mathcal{J}_{\tau_k}(v) - \mathcal{J}_{\tau_k}(u_k) \geq \int_0^{\tau_k} f(t, u_k)(v - u_k) \, dt. \tag{55}$$

We want to prove that

$$\lim_{k \to +\infty} \tau_k = +\infty. \tag{56}$$

Assume by contradiction that there exists a subsequence $(\tau_{k_j})_j$ of $(\tau_k)_k$ such that

$$\sup_j \tau_{k_j} = \tau < +\infty.$$

Let us denote $(\tau_{k_j})_j$ simply by $(\tau_k)_k$. Assumptions $(s_0), (s_1), (s_2)$ imply that, for every $\varepsilon > 0$, there exists $c_\varepsilon \in L^1_{\text{loc}}(\mathbb{R})$ such that

$$f(t, s)s \leq \varepsilon|s| + c_\varepsilon(t), \tag{57}$$

for a.e. $t \in \mathbb{R}$ and every $s \in \mathbb{R}$. Testing (55) against $v = 0$ and using (57), we get

$$\mathcal{J}_{\tau_k}(u_k) \leq \int_0^{\tau_k} f(t, u_k)u_k \, dt + \tau_k$$

$$\leq \varepsilon \int_0^{\tau_k} |u_k| \, dt + \int_0^{\tau_k} c_\varepsilon \, dt + \tau_k$$

$$\leq \varepsilon\tau \, \|u_k\|_{L^\infty(0,\tau_k)} + \|c_\varepsilon\|_{L^1(0,\tau)} + \tau.$$

Set $r_k = \frac{1}{\tau_k} \int_0^{\tau_k} u_k \, dt$ and $w_k = u_k - r_k$. By [25, Corollary 2.10], we obtain

$$2 \, \|w_k\|_{L^\infty(0,\tau_k)} \leq \int_0^{\tau_k} |Du_k| + |u_k(\tau_k^-) - u_k(0^+)|$$

$$\leq \varepsilon\tau \, \|w_k\|_{L^\infty(0,\tau_k)} + \varepsilon\tau|r_k| + \|c_\varepsilon\|_{L^1(0,\tau)} + \tau.$$

Hence we conclude that, for every $\eta > 0$, there exists $c_\eta > 0$, which is independent of $k$, such that

$$\|w_k\|_{L^\infty(0,\tau_k)} \leq \eta|r_k| + c_\eta,$$

which is the counterpart of (44) in Step 2 in the proof of Theorem 4.4. We can then proceed as there and obtain, possibly passing to a subsequence of $(u_k)_k$, still denoted by $(u_k)_k$, that

$$\lim_{k \to +\infty} \operatorname*{ess\,inf}_{[0,\tau_k]} u_k = +\infty \quad \text{or} \quad \lim_{k \to +\infty} \operatorname*{ess\,sup}_{[0,\tau_k]} u_k = -\infty.$$

Arguing as in the first part of the proof of Theorem 4.5 we finally get a contradiction by means of condition $(s_3)$. $\qquad \square$

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