On the higher dimensional Poincaré – Birkhoff theorem for Hamiltonian flows. 2. The avoiding rays condition

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Abstract. We propose a higher dimensional generalization of the Poincaré – Birkhoff Theorem which applies to Poincaré time maps of Hamiltonian systems. The maps under consideration are neither required to be close to the identity nor to have a monotone twist. The annulus is replaced by the product of an N-dimensional torus and the interior of an embedded sphere in $\mathbb{R}^N$; on the other hand, the classical boundary twist condition is replaced by an *avoiding rays condition.*

1 Introduction

The so-called Poincaré – Birkhoff theorem is a classical topological result, first inferred by Poincaré shortly before his death [32], and proved in its full generality by Birkhoff some years later [5, 6]. In broad terms, it states the existence of at least two fixed points of an area-preserving homeomorphism of the (closed) planar annulus, provided that it keeps both boundary circles invariant, while rotating them in opposite senses. It has been extended in different directions, and subsequently widely applied to the study of the dynamics of (planar) Hamiltonian systems. See, e.g., [10] or [31, Ch. 2] for a more precise description of this result, and [17, 23] for two recent review papers, including corresponding lists of references.

The efforts to generalize this theorem to higher dimensions go back to Birkhoff himself [7], and have been later continued in many works, including, for instance [2, 8, 14, 18, 21, 29, 31, 36, 38]. Out of these extensions, we shall be particularly interested in the version due to Moser and Zehnder [31, Theorem 2.21, p. 135], which is depicted next. Let $\mathcal{S} \subseteq \mathbb{R}^N$ be a smooth, compact and convex hypersurface bounding some open region $\operatorname{int}\mathcal{S}$, and let the smooth map $\mathcal{P} : \mathbb{R}^N \times \overline{\operatorname{int}\mathcal{S}} \to \mathbb{R}^N \times \mathbb{R}^N$ be given. It is assumed to be an exact symplectic diffeomorphism into its image and to have the form

$$\mathcal{P}(x,y) = (x + \vartheta(x,y), \rho(x,y))\,, \qquad (x,y) \in \mathbb{R}^N \times \overline{\operatorname{int}\mathcal{S}}\,, \tag{1}$$

where both maps ϑ, ρ are 2π-periodic in each of the first N variables $x_1, \dots, x_N$. Suppose further that there exists some $c \in \operatorname{int}\mathcal{S}$ such that

$$\langle \vartheta(x,y), y - c\rangle > 0\,, \quad \text{for every } (x,y) \in \mathbb{R}^N \times \mathcal{S}\,. \tag{2}$$

Under the additional condition that $\mathcal{P}$ is either close to the identity or satisfies a monotone twist condition, the Moser – Zehnder theorem ensures the existence of at least $N+1$ fixed points of $\mathcal{P}$ in $\mathbb{R}^N \times \operatorname{int}\mathcal{S}$. Incidentally, we observe that this result (and the proof given by Moser and Zehnder) keeps its validity if the sign of the inequality (2) is reversed.

A natural way to build exact symplectic diffeomorphisms such as those considered above is by using time maps of Hamiltonian systems. In this Hamiltonian setting, the closeness to the identity and the monotone twist condition have been recently shown to be unessential [18, Theorem 1.1*(b)*]. We now plan to further extend this result in two directions. Firstly, we shall consider compact surfaces $\mathcal{S}$ which are not necessarily convex, but are only required to be diffeomorphic to the sphere. And secondly, we are going to replace the twist condition (2) by a more general one, which, loosely speaking, requires that $\vartheta(x,y)$ misses either the inner or the outer normal ray coming from $\mathcal{S}$ whenever y belongs to this set. We have called it the *avoiding inner/outer rays* condition.

To be more precise, we consider the Hamiltonian system

$$(HS) \qquad \dot{z} = J\nabla H(t,z)\,.$$

Here, $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$ denotes the standard $2N \times 2N$ symplectic matrix, and the continuous function $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$, $H = H(t,z) = H(t,x,y)$ is T-periodic in its first variable t, 2π-periodic in the first N state variables $x_1, \dots, x_N$, and continuously differentiable with respect to $z = (x,y)$.

Let $\mathcal{S} \subseteq \mathbb{R}^N$ be a (C^1-smooth) embedded sphere; i.e., a C^1-submanifold which is C^1-diffeomorphic to the standard sphere $\mathbb{S}^{N-1}$. It is well-known that $\mathcal{S}$ separates $\mathbb{R}^N$ into two open connected components; a bounded one, usually called the *interior of* $\mathcal{S}$ and denoted by $\operatorname{int}\mathcal{S}$, and an unbounded one, the *exterior of* $\mathcal{S}$, denoted by $\operatorname{ext}\mathcal{S}$. Correspondingly, there is a well-defined unit outward normal vectorfield, which will be written as $\nu : \mathcal{S} \to \mathbb{R}^N$.

Let us assume, for the moment being, that for every initial position $z_0 \in \mathbb{R}^N \times \overline{\operatorname{int}\mathcal{S}}$ there is a unique solution $z(\cdot\,; z_0)$ of (HS) satisfying $z(0; z_0) = z_0$ and, moreover, this solution can be continued to the time interval $[0,T]$. Then it makes sense to consider the so-called *Poincaré time map* $\mathcal{P} : \mathbb{R}^N \times \overline{\operatorname{int}\mathcal{S}} \to \mathbb{R}^{2N}$, defined by

$$\mathcal{P}(z_0) = z(T; z_0)\,.$$

It is clear that $\mathcal{P}$ has the form (1), the functions θ, ρ being 2π-periodic in each variable x_i. Observe also that the fixed points of $\mathcal{P}$ correspond to T-periodic solutions of (HS). Once such a T-periodic solution $z(t) = (x(t), y(t))$ has been found, many others appear by just adding an integer multiple of 2π to some of the components $x_i(t)$; for this reason, we will call *geometrically distinct* two periodic solutions of (HS) (or two fixed points of $\mathcal{P}$) which can not be

obtained from each other in this way. If the Hamiltonian $H = H(t, z)$ is C^2-smooth with respect to z, a fixed point z_0 of $\mathcal{P}$ is called *nondegenerate* if 1 is not an eigenvalue of $\mathcal{P}'(z_0)$.

At a point y of the hypersurface $\mathcal{S}$, the inward and outward rays are defined by

$$\mathscr{R}_-(y) := \big\{ -\lambda\nu(y) : \lambda \geq 0 \big\}, \qquad \mathscr{R}_+(y) := \big\{ \lambda\nu(y) : \lambda \geq 0 \big\},$$

respectively. We shall say that $\mathcal{P}$ satisfies the *avoiding inward rays condition* provided that $\vartheta(x, y) \notin \mathscr{R}_-(y)$ for every $(x, y) \in \mathbb{R}^N \times \mathcal{S}$, and we shall say that $\mathcal{P}$ satisfies the *avoiding outward rays condition* provided that $\vartheta(x, y) \notin \mathscr{R}_+(y)$ for every $(x, y) \in \mathbb{R}^N \times \mathcal{S}$. We shall show the following

Theorem 1.1. ***(a).*** *Assume either the avoiding inward rays condition or the avoiding outward rays condition; then, $\mathcal{P}$ has at least $N + 1$ geometrically distinct fixed points in $\mathbb{R}^N \times \operatorname{int} \mathcal{S}$.* ***(b).*** *If, furthermore, $H = H(t, z)$ is twice continuously differentiable with respect to z and all fixed points of $\mathcal{P}$ on $\mathbb{R}^N \times \operatorname{int} \mathcal{S}$ are nondegenerate, then there are at least 2^N of them.*

We emphasize that in this theorem we do not assume the invariance of the domain $\mathbb{R}^N \times \overline{\operatorname{int} \mathcal{S}}$, nor that our map should be close to the identity, nor any monotone twist condition on $\mathbb{R}^N \times \operatorname{int} \mathcal{S}$. We did presuppose uniqueness for initial value problems, at least for solutions departing from $\mathbb{R}^N \times \overline{\operatorname{int} \mathcal{S}}$, so that the Poincaré time map is well defined. This assumption may fail if the Hamiltonian function is not smooth enough (say, for Hamiltonians of class C^1), reducing the applications of our theorem; for this reason, it will also be dropped in Theorem 2.1, which is the main result of this paper.

Throughout this paper, $N \geq 1$ is a fixed natural number. The case $N = 1$ is special, because then the embedded sphere $\mathcal{S}$ becomes a two-point set $\{a, b\}$, with $a < b$, its interior is the open interval $]a, b[$, and the unit outward normal vector field is the function $\nu : \{a, b\} \to \mathbb{R}$ defined by $\nu(a) = -1,\ \nu(b) = 1$. Thus, for $N = 1$, Theorem 1.1 becomes the particularization for Hamiltonian systems of the standard planar Poincaré – Birkhoff theorem (for non necessarily invariant annuli). Also the more general Theorem 2.1 is known in this case (see [18, Theorem 2.1]). On the other hand, some of the arguments of this paper (in particular those in Section 3) become much simpler if $N = 1$. For these reasons we shall be concerned mainly with dimensions $N \geq 2$.

Theorem 1.1 applies to Poincaré time maps of Hamiltonian systems which are periodic in the x_i variables. It leads to the question of finding an alternative way to characterize such maps. Clearly, the transformations $\mathcal{P} : \mathbb{R}^N \times \overline{\operatorname{int} \mathcal{S}} \to \mathbb{R}^{2N}$ which we are considering differ from the identity on some map which is periodic in the x_i variables. In addition, assuming the Hamiltonian to be twice continuously differentiable, $\mathcal{P}$ must be an exact symplectic diffeomorphism into its image. Indeed, it can be seen (cf. [21, Theorem 58.9] or [27, Proposition 9.19]) that $\mathcal{P}$ is the Poincaré time map of a Hamiltonian system of the type

we are dealing with if and only if it can be joined to the identity via a smooth homotopy of exact symplectic diffeomorphisms. However, this criterion could not be easy to check in practical situations. More explicit conditions are available when $\mathcal{P}$ is an exact symplectic *monotone twist* map. Indeed, Moser [30, Theorem 1] has shown that, when $N = 1$, all such maps are indeed Poincaré time maps of a Hamiltonian system. A higher dimensional version of this result has been obtained by Golé [21, Theorem 41.6], assuming that the map $\mathcal{P}$ is globally defined on $\mathbb{R}^N \times \mathbb{R}^N$ and the twist is, in some sense, controlled at infinity.

Notice that Theorem 1.1 states the existence of fixed points for certain maps defined on $\mathbb{R}^N \times \overline{\operatorname{int}\mathcal{S}}$, the embedded sphere $\mathcal{S} \subseteq \mathbb{R}^N$ being C^1-smooth. The generalized Schoenflies theorem [9] implies that $\overline{\operatorname{int}\mathcal{S}}$ is homeomorphic to the unit ball; however, there are indications that in some cases it may not be diffeomorphic [28, p. 1069].

Another result presented in this paper is Theorem 2.2. Here, the avoiding rays condition is replaced by some assumptions near infinity, and we shall refer to it as *our basic Hamiltonian Theorem*. It will be used in Section 4 to prove Theorem 2.1 by suitably modifying the Hamiltonian; in order to do so we shall need a couple of technical properties of embedded spheres which are developed in Section 3.

Having completed the passage to Theorem 2.1 from our basic Hamiltonian Theorem 2.2, we devote the three last sections of the paper to obtain this last result. In Section 5 we set a variational framework for our problem. The periodic solutions of our Hamiltonian system then become the critical points of a (strongly indefinite) functional, and the corresponding abstract theorem is proved in Section 6 (for the general, possibly degenerate case) and in Section 7 (in the nondegenerate case).

The avoiding rays conditions have implications on the Brouwer degree of the maps $\vartheta(x,\cdot) : \overline{\operatorname{int}\mathcal{S}} \to \mathbb{R}^N$. Indeed, under the avoiding *inward* rays condition, $\deg\big(\vartheta(x,\cdot), \operatorname{int}\mathcal{S}, 0\big) = 1$ for every $x \in \mathbb{R}^N$, while if the avoiding *outward* rays condition holds, $\deg\big(\vartheta(x,\cdot), \operatorname{int}\mathcal{S}, 0\big) = (-1)^N$ for every $x \in \mathbb{R}^N$. This is easy to check, since the maps $\vartheta(x,\cdot) : \overline{\operatorname{int}\mathcal{S}} \to \mathbb{R}^N$ can be connected by homotopies to other maps which coincide on $\mathcal{S}$ with the outer normal vectorfield ν (in the case of the avoiding inward rays condition) or the inner normal vectorfield $-\nu$ (in the case of the avoiding outward rays condition); subsequently, the degrees can be computed, e.g., by using the main theorem of [1]. We do not know whether the avoiding rays conditions in Theorem 1.1 can be replaced by these more general assumptions on the topological degrees.

2 Hamiltonian systems without uniqueness

The main result of this paper is a version of Theorem 1.1 which does not require uniqueness for initial value problems. In order to describe it precisely,

it will be convenient to introduce some terminology. We shall say that the function $H : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$, $H = H(t,z) = H(t,x,y)$, is an *admissible Hamiltonian* if it is continuous, 2π-periodic in x_i for each $i = 1, \dots, N$, and it has a continuously defined gradient with respect to z, denoted by ∇H. Observe that under these conditions the Poincaré time map could be multivalued (and thus not defined in the usual sense). All Hamiltonians appearing in this paper will be admissible.

A solution $z : [0,T] \to \mathbb{R}^{2N}$ of (HS) is said to be *T-periodic* if it satisfies $z(0) = z(T)$. Of course, in case H is the restriction of some function on $\mathbb{R} \times \mathbb{R}^{2N}$ which is T-periodic in time, then any T-periodic solution in this sense can be extended to a T-periodic solution defined on $\mathbb{R}$. When $H = H(t,z)$ is twice differentiable with respect to z, a T-periodic solution of (HS) is said to be *nondegenerate* if the linearized system does not have nontrivial T-periodic solutions. Equivalently, if the corresponding fixed point of the Poincaré map is nondegenerate.

The avoiding inward/outward rays conditions considered in the Introduction can be easily adapted to our situation. We shall say that the flow of the Hamiltonian system (HS) satisfies the *avoiding inward [resp. outward] rays condition* relatively to $\mathcal{S}$ if every solution $z(t) = (x(t), y(t))$ of (HS) with $y(0) \in \mathcal{S}$ is defined for every $t \in [0,T]$ and satisfies

$$x(T) - x(0) \notin \mathscr{R}_-(y(0))\,, \quad [\text{resp. } x(T) - x(0) \notin \mathscr{R}_+(y(0))]\,.$$

We are now ready to state our main result. Once more, we emphasize that we do not assume any invariance of the domain, nor any closeness to the identity, nor monotone twist condition.

Theorem 2.1. ***(a).*** *Let the Hamiltonian function $H : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$ be admissible, and assume the existence of an embedded sphere $\mathcal{S} \subseteq \mathbb{R}^N$ such that the flow of (HS) satisfies the avoiding inward [resp. outward] rays condition relatively to $\mathcal{S}$. Then, the Hamiltonian system (HS) has at least $N+1$ geometrically distinct T-periodic solutions $z^{(0)}, \dots, z^{(N)}$ such that, writing $z^{(k)}(t) = (x^{(k)}(t), y^{(k)}(t))$,*

$$y^{(k)}(0) \in \operatorname{int} \mathcal{S}\,, \text{ for } k = 0, \dots, N\,.$$

(b). *Moreover, if the Hamiltonian function $H = H(t,z)$ is twice continuously differentiable with respect to z and the T-periodic solutions with initial condition on $\mathbb{R}^N \times \operatorname{int} \mathcal{S}$ are nondegenerate, then there are at least 2^N of them.*

An important ingredient to prove Theorem 2.1 is given below. It is reminiscent of a theorem due to Szulkin (cf. [36, Theorem 4.2] and [37, Theorem 8.1]), but there are some differences. In contrast to Szulkin's results, we deal with Hamiltonians which are not quadratic or coercive in the y directions, but have a finite limit as $|y| \to \infty$. On the other hand, we shall assume that $H(t,x,y)$ does not depend on t, x for $|y|$ big enough.

Theorem 2.2. *Let the Hamiltonian function $H : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$ be admissible. Assume that*

[$\boldsymbol{H}_1$] *there exists some $R_0 > 0$ such that $H(t,x,y) \equiv h(y)$ does not depend on t, x whenever $|y| \geq R_0$,*

[$\boldsymbol{H}_2$] *the function h has a finite limit ℓ as $|y| \to \infty$; furthermore, $h(y) \neq \ell$ for $|y|$ sufficiently large,*

[$\boldsymbol{H}_3$] $\lim_{|y|\to\infty} \nabla h(y) = 0$.

Then, system (HS) has at least $N+1$ geometrically distinct T-periodic solutions. If, in addition:

[$\boldsymbol{H}_4$] *H is C^2-smooth with respect to (x,y),*

[$\boldsymbol{H}_5$] $\lim_{|y|\to\infty} \operatorname{Hess} h(y) = 0$,

[$\boldsymbol{H}_6$] *the T-periodic solutions of (HS) are nondegenerate,*

then (HS) has at least 2^N geometrically different T-periodic solutions.

In this result, assumption [$\boldsymbol{H}_1$] looks quite strong, and it seems plausible that it could be avoided if one replaces the limits in [$\boldsymbol{H}_{2,3,5}$] by their analogues referred to $H(t,x,y)$, assumed uniform with respect to (t,x). However, for the purposes of this paper we shall only need the result in the form above; indeed, in Section 4 we are going to use Theorem 2.2 as the main tool to prove Theorem 2.1. We shall first need a couple of facts about embedded spheres in $\mathbb{R}^N$; they will be the objective of the next section.

3 Embedded spheres in $\mathbb{R}^N$

We devote this section to establish two geometrical facts on embedded spheres in $\mathbb{R}^N$. The first one is an approximation result:

Lemma 3.1. *Let $\mathcal{S} \subseteq \mathbb{R}^N$ be a C^1-smooth embedded sphere and let $K \subseteq \operatorname{int} \mathcal{S}$ be a compact set.*

(†) *There exists a C^∞-smooth embedded sphere $\mathcal{S}_* \subseteq \mathbb{R}^N$ with*

$$K \subseteq \operatorname{int} \mathcal{S}_* \subseteq \overline{\operatorname{int} \mathcal{S}_*} \subseteq \operatorname{int} \mathcal{S}\,.$$

(‡) *Furthermore, given $\varepsilon > 0$, the embedded sphere $\mathcal{S}_*$ can be chosen with the following additional property: for any $q \in \mathcal{S}_*$ there is some $p \in \mathcal{S}$ with*

$$|q-p| < \varepsilon\,, \qquad |\nu_*(q) - \nu(p)| < \varepsilon\,.$$

Here, ν and ν_* denote, respectively, the unit normal outward vectorfields on $\mathcal{S}$ and $\mathcal{S}_*$.

Proof. The outer unit normal vector field $\nu : \mathcal{S} \to \mathbb{R}^N$ is continuously defined. Using a regularization argument we can find a C^1-smooth vector field $X : \mathcal{S} \to \mathbb{R}^N$ with $|X(p)| = 1$ for every $p \in \mathcal{S}$ and

$$\langle X(p), \nu(p) \rangle > 0\,, \quad \text{for all } p \in \mathcal{S}\,.$$

In order to prove (‡), choose some C^1-smooth diffeomorphism $\sigma : \mathbb{S}^{N-1} \to \mathcal{S}$ and some number $\delta \in]0, \varepsilon[$, and consider the map $\varphi : \mathbb{S}^{N-1} \times]-\delta, \delta[\to \mathbb{R}^N$ defined by

$$\varphi(\theta, t) := \sigma(\theta) + tX(\sigma(\theta))\,.$$

If $\delta \in]0, \operatorname{dist}(\mathcal{S}, K)[$ is small enough, this is a C^1-smooth diffeomorphism into its (open) image. Moreover,

$$\varphi(\mathbb{S}^{N-1} \times \{0\}) = \mathcal{S}\,, \quad \varphi(\mathbb{S}^{N-1} \times]0, \delta[) \subseteq \operatorname{ext} \mathcal{S}\,, \quad \varphi(\mathbb{S}^{N-1} \times]-\delta, 0[) \subseteq (\operatorname{int} \mathcal{S}) \setminus K\,.$$

Observe that

$$(i_\triangle) \qquad |\varphi(\theta, -\delta/2) - \varphi(\theta, 0)| < \varepsilon/2\,, \text{ for any } \theta \in \mathbb{S}^{N-1}.$$

We denote $\mathcal{S}_\triangle := \varphi(\mathbb{S}^{N-1} \times \{-\delta/2\})$, which is a C^1-smooth embedded sphere contained into $\operatorname{int} \mathcal{S}$ and containing K into its interior. After possibly replacing δ by an smaller number there is no loss of generality in further assuming that

$$(ii_\triangle) \qquad |\nu_\triangle(\varphi(\theta, -\delta/2)) - \nu(\varphi(\theta, 0))| < \varepsilon/2\,, \text{ for any } \theta \in \mathbb{S}^{N-1}.$$

(We call $\nu_\triangle : \mathcal{S}_\triangle \to \mathbb{R}^N$ the unit outer normal vector field on $\mathcal{S}_\triangle$.) Using again a regularization argument, one finds a C^∞-smooth map $\phi_* : \mathbb{S}^{N-1} \to \mathbb{R}^N$ which is close, in the C^1-sense, to $\varphi(\cdot\,, -\delta/2)$. Then, the C^∞-smooth embedded sphere $\mathcal{S}_* := \phi_*(\mathbb{S}^{N-1})$ will still be contained in $\operatorname{int} \mathcal{S}$ and will still contain K in its interior. Moreover,

$$(i_*) \qquad |\phi_*(\theta) - \varphi(\theta, -\delta/2)| < \varepsilon/2\,, \text{ for any } \theta \in \mathbb{S}^{N-1},$$

$$(ii_*) \qquad |\nu_*(\phi_*(\theta)) - \nu_\triangle(\varphi(\theta, -\delta/2))| < \varepsilon/2\,, \text{ for any } \theta \in \mathbb{S}^{N-1}.$$

Combining $(i_\triangle)$ with (i_*), and $(ii_\triangle)$ with (ii_*), we deduce that $|\phi_*(\theta) - \varphi(\theta, 0)| < \varepsilon$ and $|\nu_*(\phi_*(\theta)) - \nu(\varphi(\theta, 0))| < \varepsilon$, for any $\theta \in \mathbb{S}^{N-1}$. The result follows. □

We shall be particularly interested in the following consequence of Lemma 3.1(‡). Let the (admissible) Hamiltonian function H be given and assume that the avoiding inward/outward rays condition (with respect to the embedded sphere $\mathcal{S}$) holds. Then the modified embedded sphere $\mathcal{S}_*$ may be taken so that the avoiding rays condition perdures. This is the content of the following

Lemma 3.2. *Let $\mathcal{S} \subseteq \mathbb{R}^N$ be a C^1-smooth embedded sphere and let $K \subseteq \operatorname{int} \mathcal{S}$ be a compact set. Let H be an admissible Hamiltonian and assume that either the avoiding inward rays condition or the avoiding outward rays condition (relatively to $\mathcal{S}$) holds. Then, the embedded sphere $\mathcal{S}_*$ given by Lemma 3.1(†) can be chosen so that the relative inward/outward avoiding rays condition holds.*

Proof. Using [18, Lemma 5.2], we see that all solutions $(x(t), y(t))$ of (HS) starting with $y(0) \in \overline{\operatorname{int} \mathcal{S}}$ are defined on $[0, T]$, and indeed, the set of such solutions is uniformly bounded in the y component. Consequently, there are also bounds in the variation of the x component, i.e.,

$$|x(T) - x(0)| \leq M \,, \tag{3}$$

for every solution $(x(t), y(t))$ of this type, the constant $M > 0$ not depending on the particular solution. In addition, assuming for instance the avoiding *inward* rays condition (relative to $\mathcal{S}$), a compactness argument shows the existence of some $\epsilon_0 > 0$ such that

$$\operatorname{dist}(x(T) - x(0), \mathscr{R}_-(p)) \geq \epsilon_0 \,, \tag{4}$$

whenever $p \in \mathcal{S}$, $(x(t), y(t))$ is a solution starting with $y(0) \in \overline{\operatorname{int} \mathcal{S}}$, and $|y(0) - p| < \epsilon_0$.

Recalling Lemma 3.1(‡) we may find an embedded sphere $\mathcal{S}_* \subseteq \operatorname{int} \mathcal{S}$ with $K \subseteq \operatorname{int} \mathcal{S}_*$ and with the following property: for every $q \in \mathcal{S}_*$ there is some $p \in \mathcal{S}$ with

$$|q - p| < \epsilon_0 \,, \qquad |\nu_*(q) - \nu(p)| < \epsilon_0 / M \,. \tag{5}$$

(we denote by ν_* the unit outer normal vectorfield associated with $\mathcal{S}_*$). To conclude the proof it suffices to check the avoiding (inward) rays condition, relative to this embedded sphere $\mathcal{S}_*$. We see this by a contradiction argument and assume instead that there is some solution $(x(t), y(t))$ of (HS) starting with $y(0) = q \in \mathcal{S}_*$ and such that $x(T) - x(0) = -\lambda \nu_*(q)$ for some $\lambda \geq 0$. By (3), we see that $\lambda \in [0, M]$. We choose some point $p \in \mathcal{S}$ satisfying (5) and observe that

$$\begin{aligned} \operatorname{dist}(x(T) - x(0), \mathscr{R}_-(p)) &= \operatorname{dist}(-\lambda \nu_*(q), \mathscr{R}_-(p)) \\ &\leq \lambda |-\nu_*(q) + \nu(p)| < M(\epsilon_0 / M) = \epsilon_0 \,, \end{aligned}$$

contradicting (4). It concludes the proof. □

In the second part of this section we continue our study of embedded spheres in $\mathbb{R}^N$, but with a different goal, which this time will be Lemma 3.5. We shall start with an apparently unrelated result stating that any two points in an open domain $B \subseteq \mathbb{R}^N$ can be sent into each other by a diffeomorphism which leaves the exterior of the domain pointwise fixed.

Lemma 3.3. *Let $B \subseteq \mathbb{R}^N$ be open and connected. Then, for every $p, q \in B$ there exists a C^∞-smooth diffeomorphism $\mathscr{G}_{pq} : \mathbb{R}^N \to \mathbb{R}^N$ such that $\mathscr{G}_{pq}(p) = q$, and $\mathscr{G}_{pq}(y) = y$ for every $y \in \mathbb{R}^N \setminus B$.*

Proof. It is divided into three steps. Firstly, we check the statement in the special case when B is a convex subset of $\mathbb{R}^N$ and the points p, q belong to the first coordinate axis $\mathbb{R} \times \{0\} \subseteq \mathbb{R} \times \mathbb{R}^{N-1} \equiv \mathbb{R}^N$; in this situation the result is due to Huebsch and Morse [22]. These additional assumptions are subsequently removed in the second and third steps.

First Step: B is convex and $p = (a, 0), q = (b, 0) \in \mathbb{R} \times \{0\}$. Since one can take $\mathscr{G}_{qp} := \mathscr{G}_{pq}^{-1}$ there is no loss of generality in assuming that $a < b$. The open set B contains the segment $[p, q] = \{(t, 0) : a \leq t \leq b\}$, and hence, there are numbers $a' < a < b < b'$ and $e > 0$ such that the cylindrical region $\{(t, u) \in \mathbb{R} \times \mathbb{R}^{N-1} : a' < t < b', |u| < e\}$ is contained in B. The result now follows from [22, Lemma 2.1].

Second Step: B is convex and $p, q \in B$ are arbitrary. Choose some affine isomorphism $\mathscr{J} : \mathbb{R}^N \to \mathbb{R}^N$ sending both points p, q into the first coordinate axis $\mathbb{R} \times \{0\}$. The open set $\mathscr{J}(B)$ is convex, so that, by the first step there is a C^∞-smooth diffeomorphism $\mathscr{G}_{p'q'} : \mathbb{R}^N \to \mathbb{R}^N$ which sends $p' = \mathscr{J}(p)$ into $q' = \mathscr{J}(q)$ while leaving the points of $\mathbb{R}^N \setminus \mathscr{J}(B)$ pointwise fixed. It then suffices to set $\mathscr{G}_{pq} := \mathscr{J}^{-1} \circ \mathscr{G}_{p'q'} \circ \mathscr{J}$.

Third Step: the general case. We define an equivalence relation R on B by the rule

$$p \, R \, q \;\Leftrightarrow\; \text{there exists } \mathscr{G}_{pq} : \mathbb{R}^N \to \mathbb{R}^N \text{ under the conditions of the lemma.}$$

It follows from the second step (applied to balls) that every equivalence class is open. Moreover, any two different equivalence classes are disjoint. Thus, the connected set B is the disjoint union of the (open) equivalence classes, and we deduce that there is just one class. It completes the proof. □

In [28], Morse proved a differentiable version of the Schoenflies Theorem which states that the interior of an embedded sphere in $\mathbb{R}^N$ is, excepting for a possible singular point, diffeomorphic to the pointed ball. The fact that, in view of Lemma 3.3, the singular points can be prescribed, leads to a variant of this result in which the exterior of the standard sphere is mapped into the exterior of an embedded sphere. We denote by $\mathbb{B}^N$ the open unit ball in $\mathbb{R}^N$.

Lemma 3.4. *Given a C^2-smooth embedded sphere $\mathcal{S} \subseteq \mathbb{R}^N$, there are open sets $\mathcal{U} \supset \mathbb{R}^N \setminus \mathbb{B}^N$ and $\mathcal{V} \supset \overline{\operatorname{ext} \mathcal{S}}$, and a C^2-smooth diffeomorphism $\mathcal{F} : \mathcal{U} \to \mathcal{V}$ such that $\mathcal{F}(\mathbb{S}^{N-1}) = \mathcal{S}$ and $\mathcal{F}(\mathbb{R}^N \setminus \overline{\mathbb{B}^N}) = \operatorname{ext} \mathcal{S}$.*

Proof. There is no loss of generality in assuming that $0 \in \operatorname{int} \mathcal{S}$. We define

$$\mathscr{S} := \left\{ \frac{y}{|y|^2} : y \in \mathcal{S} \right\} .$$

This is again a C^2-smooth embedded sphere in $\mathbb{R}^N$, and [28, Theorem 1.1] states the existence of open sets $\mathscr{U} \supset \overline{\mathbb{B}^N}$ and $\mathscr{V} \supset \overline{\operatorname{int} \mathscr{S}}$, together with a homeomorphism $\mathscr{F} : \mathscr{U} \to \mathscr{V}$ such that $\mathscr{F}(\mathbb{S}^{N-1}) = \mathscr{S}$ and, for some point $p \in \mathbb{B}^N$, the restriction $\mathscr{F} : \mathscr{U} \setminus \{p\} \to \mathscr{V} \setminus \{\mathscr{F}(p)\}$ is a C^2-smooth diffeomorphism. But, in view of Lemma 3.3 (applied to the open sets $B_1 = \mathbb{B}^N$ and $B_2 = \operatorname{int} \mathcal{S}$), it is not restrictive to assume that $p = 0 = \mathscr{F}(p)$. It suffices now to consider the sets

$$\mathcal{U} := \left\{ \frac{y}{|y|^2} : y \in \mathscr{U} \setminus \{0\} \right\} , \qquad \mathcal{V} := \left\{ \frac{y}{|y|^2} : y \in \mathscr{V} \setminus \{0\} \right\} ,$$

and define the C^2-smooth diffeomorphism $\mathcal{F} : \mathcal{U} \to \mathcal{V}$ by the rule

$$\mathcal{F}(u) := \frac{1}{\left|\mathscr{F}\left(u/|u|^2\right)\right|^2} \mathscr{F}\left(u/|u|^2\right) .$$

The lemma follows. □

At this moment we are ready to show the second main result of this section:

Lemma 3.5. *Let $\mathcal{S} \subseteq \mathbb{R}^N$ be an embedded sphere of class C^2. Then, there exists a C^2-smooth function $h : \mathbb{R}^N \to [0, 1[$ satisfying:*

(i) $h(y) = 0$, for every $y \in \operatorname{int} \mathcal{S}$;

(ii) $\nabla h(y) \neq 0$, for every $y \in \operatorname{ext} \mathcal{S}$;

(iii) $\displaystyle\lim_{\substack{y \to y_0 \\ y \in \operatorname{ext} \mathcal{S}}} \frac{\nabla h(y)}{|\nabla h(y)|} = \nu(y_0)$, for every $y_0 \in \mathcal{S}$;

(iv) $\displaystyle\lim_{|y| \to \infty} h(y) = 1 ; \qquad \lim_{|y| \to \infty} \nabla h(y) = 0 ; \qquad \lim_{|y| \to \infty} \operatorname{Hess} h(y) = 0 .$

Proof. Choose the open neighborhoods of infinity $\mathcal{U}, \mathcal{V} \subseteq \mathbb{R}^N$ and the C^2-smooth diffeomorphism $\mathcal{F} : \mathcal{U} \to \mathcal{V}$ as given by Lemma 3.4. It can be assumed that $0 \notin \mathcal{U}$. We consider the function $f : \mathcal{V} \to \mathbb{R}$ defined by

$$f(y) = |\mathcal{F}^{-1}(y)| - 1 .$$

Observe that $f \in C^2(\mathcal{V})$. Moreover,

$$f(y) \begin{cases} > 0 , & \text{if } y \in \operatorname{ext} \mathcal{S} , \\ = 0 , & \text{if } y \in \mathcal{S} , \\ < 0 , & \text{if } y \in (\operatorname{int} \mathcal{S}) \cap \mathcal{V} . \end{cases}$$

In addition, f is coercive, i.e. $\lim_{|y| \to \infty} f(y) = +\infty$. This means that its level sets are compact, allowing us to define the function $m : [0, +\infty[\to \mathbb{R}$ by

$$m(r) := r + \max\{|\nabla f(y)| + \|\operatorname{Hess} f(y)\| : y \in f^{-1}(r)\} .$$

Observe that m is continuous and positive. It is therefore possible to find a C^2-smooth function $g : \mathbb{R} \to \mathbb{R}$, with

$$\begin{array}{lcl} (\star) & g(r) = 0 , & \text{if } r \leq 0 , \\ (\star\star) & 0 < g'(r) , & \text{if } r > 0 , \\ (\star\star\star) & g'(r) < \dfrac{1}{rm(r)} \text{ and } -\dfrac{1}{rm(r)^2} < g''(r) < 0 , & \text{if } r > 1 . \end{array}$$

Combining the first part of ($\star\star\star$) with the fact that $m(r) > r$ for any $r > 0$, we see that the limit $\ell = \lim_{r\to+\infty} g(r)$ is finite. We finally define $h : \mathbb{R}^N \to \mathbb{R}$ by

$$h(y) = \begin{cases} 0\,, & \text{if } y \in \operatorname{int}\mathcal{S}\,, \\ \dfrac{1}{\ell}\, g(f(y))\,, & \text{if } y \in \mathcal{V}\,. \end{cases}$$

Observe that, since on $(\operatorname{int}\mathcal{S}) \cap \mathcal{V}$ both definitions coincide, the function h is well defined and it is C^2-smooth. Properties (i)–(iv) are now easily checked. $\square$

4 Modifying the Hamiltonian: the proof of the main theorem

In this section we carry out the proof of Theorem 2.1, assuming for this purpose the validity of our basic Hamiltonian Theorem 2.2. Some of the arguments have already been employed in [18]; thus, we will be brief while going through them, and will concentrate instead on the differences.

The concept of *strongly admissible Hamiltonian*, considered in [18, Section 5], will play an important role here. We recall the definition from there; the Hamiltonian function $H : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$ is said to be strongly admissible (with respect to the set $U = \operatorname{int}\mathcal{S}$) provided that it is admissible and the following two additional conditions hold:

[1.] there exists a relatively open set $\mathcal{W} \subseteq [0,T] \times \mathbb{R}^N$, containing $\{0\} \times \overline{\operatorname{ext}\mathcal{S}}$, such that H is C^∞-smooth with respect to the state variables $z = (x,y)$ on the 'augmented set' $\mathcal{W}_\sharp := \{(t,x,y) : (t,y) \in \mathcal{W},\ x \in \mathbb{R}^N\}$;

[2.] there exists some $R_1 > \max_{y\in\mathcal{S}} |y|$ such that $H(t,x,y) = 0$, if $|y| \geq R_1$.

We shall start by observing that it suffices to prove Theorem 2.1 under the additional assumptions that the Hamiltonian H is strongly admissible (with respect to $\operatorname{int}\mathcal{S}$) and the embedded sphere $\mathcal{S}$ is C^2-smooth. With this aim, let the admissible Hamiltonian H and the C^1-smooth embedded sphere $\mathcal{S} \subseteq \mathbb{R}^N$ lie under the framework of Theorem 2.1, i.e., either the avoiding inward rays condition or the avoiding outward rays condition relative to $\mathcal{S}$ holds. As usual, we denote by $(\widehat{HS})$ the Hamiltonian system associated with the modified Hamiltonian $\widehat{H}$.

Lemma 4.1. *Let H and $\mathcal{S}$ satisfy the assumptions of Theorem 2.1. Then, it is possible to find a strongly admissible Hamiltonian $\widehat{H}$ and a C^2-smooth embedded sphere $\widehat{\mathcal{S}} \subseteq \operatorname{int}\mathcal{S}$, such that:*

($\ast$) *H and $\widehat{H}$ coincide on some relatively open set containing the graph of every T-periodic solution $(\widehat{x},\widehat{y})$ of $(\widehat{HS})$ starting with $\widehat{y}(0) \in \operatorname{int}\widehat{\mathcal{S}}$;*

($\ast\ast$) *the flow of* $\widehat{H}$ *satisfies the avoiding inward rays condition (resp., the avoiding outward rays condition) relative to* $\widehat{\mathcal{S}}$.

Proof. In order to fix the ideas we assume, for instance, that the Hamiltonian H satisfies the avoiding *inward* rays condition (with respect to $\mathcal{S}$). Using [18, Lemma 5.2] we see that there is a uniform bound for the solutions (x, y) of (HS) starting with $y(0) \in \mathcal{S}$. Then, a compactness argument shows the existence of some $\varepsilon > 0$ such that every solution $(x(t), y(t))$ of (HS) starting with $y(0) \in \mathcal{S}$ satisfies $\operatorname{dist}(x(T) - x(0), \mathscr{R}_-(y(0))) \geq \varepsilon$. Applying [18, Proposition 5.1] with $U = \operatorname{int}\mathcal{S}$ we see that there is a strongly admissible (with respect to $\operatorname{int}\mathcal{S}$) Hamiltonian $\widehat{H}$ which coincides with H on a relatively open set containing the graph of any T-periodic solution $\widehat{z}(t) = (\widehat{x}(t), \widehat{y}(t))$ of $(\widehat{HS})$ satisfying $\widehat{y}(0) \in \operatorname{int}\mathcal{S}$, and such that the avoiding inner rays condition (relative to $\mathcal{S}$) still holds. Recalling now Lemma 3.2 we see that we can replace $\mathcal{S}$ by a C^2-smooth embedded sphere $\widehat{\mathcal{S}} \subseteq \mathbb{R}^N$ with

$$\{v \in \mathbb{R}^N : (0, v) \notin \mathcal{W}\} \subseteq \operatorname{int}\widehat{\mathcal{S}} \subseteq \overline{\operatorname{int}\widehat{\mathcal{S}}} \subseteq \operatorname{int}\mathcal{S}\,,$$

such that the avoiding inward rays condition still holds. The result follows. □

Thus, from now on we assume, without loss of generality, that H *is strongly admissible (with respect to* $\operatorname{int}\mathcal{S}$*), and* $\mathcal{S}$ *is* C^2*-smooth.* As before, in order to fix ideas we assume that the avoiding *inward* rays condition with respect to $\mathcal{S}$ holds. In view of **[2.]** all solutions of (HS) are defined on $[0, T]$ and, moreover, the set of solutions starting from any given compact set is compact. Let the function $h : \mathbb{R}^N \to \mathbb{R}$ be given by Lemma 3.5 for the embedded sphere $\mathcal{S}$; there must be some $\varrho > 0$ such that, whenever $(x(t), y(t))$ is a solution of (HS),

$$\operatorname{dist}(y(0), \partial\mathcal{S}) < \varrho \quad \Rightarrow \quad x(T) - x(0) \notin \{-r\nabla h(y(0)) : r \geq 0\}\,. \tag{6}$$

Let $\mathcal{W}$ be given by **[1.]**; after possibly replacing ϱ by an smaller number we have that

$$\{y \in \mathbb{R}^N : (0, y) \notin \mathcal{W}\} \subseteq K := \{y \in \operatorname{int}\mathcal{S} : \operatorname{dist}(y, \mathcal{S}) \geq \varrho\}\,.$$

The set K is compact and contained inside $\operatorname{int}\mathcal{S}$, and using Lemma 3.1(†), we may find C^∞-smooth embedded spheres $\mathcal{S}', \mathcal{S}_* \subseteq \mathbb{R}^N$ with

$$K \subseteq \operatorname{int}\mathcal{S}' \subseteq \overline{\operatorname{int}\mathcal{S}'} \subseteq \operatorname{int}\mathcal{S}_* \subseteq \overline{\operatorname{int}\mathcal{S}_*} \subseteq \operatorname{int}\mathcal{S}\,.$$

We choose some constant $c > 0$ and consider the (relatively open) set

$$\Omega := \big(\{0\} \times \operatorname{int}\mathcal{S}'\big) \cup \big\{(t, y) \in]0, T] \times \mathbb{R}^N : \operatorname{dist}(y, \operatorname{int}\mathcal{S}') < ct\big\}\,.$$

Denote $\Omega_\sharp := \big\{(t, x, y) \in [0, T] \times \mathbb{R}^{2N} : (t, y) \in \Omega\big\}$. Combining **[1.]** and **[2.]** we see that, if c is large enough, then H is C^∞-smooth on the set

$$\mathcal{G}_\sharp := \big([0, T] \times \mathbb{R}^{2N}\big) \setminus \overline{\Omega}_\sharp = \big\{(t, x, y) \in [0, T] \times \mathbb{R}^{2N} : \operatorname{dist}(y, \operatorname{int}\mathcal{S}') > ct\big\}\,.$$

Moreover, using similar arguments to those carried out in the proof of [18, Lemma 6.2], we see that, for large $c > 0$, the set $\Omega_\sharp$ is strictly forward-invariant for the flow of (HS), in the sense that

$$(t_0, z(t_0)) \in \overline{\Omega}_\sharp \quad \Rightarrow \quad (t, z(t)) \in \Omega_\sharp\,, \text{ for every } t \in]t_0, T]\,, \tag{7}$$

where $z = z(t)$ is any solution of (HS). What is more, $\Omega_\sharp$ is strictly forward-invariant for the flow of every Hamiltonian system $(\widetilde{HS})$ whose (admissible but not necessarily strongly admissible) Hamiltonian $\widetilde{H}$ coincides with H on $\Omega_\sharp$.

Consider the set Γ, whose elements are those $(t, \zeta) \in [0, T] \times \mathbb{R}^{2N}$ such that the solution z of (HS) with $z(0) = \zeta$ satisfies $(s, z(s)) \in \mathcal{G}_\sharp$, for every $s \in [0, t]$. The set Γ is open relatively to $[0, T] \times \mathbb{R}^{2N}$, and it contains the sets

$$A_\sharp = \{0\} \times \mathbb{R}^N \times (\operatorname{ext} \mathcal{S}')\,, \qquad B_\sharp = [0, T] \times \mathbb{R}^N \times (\mathbb{R}^N \setminus \mathbb{B}^N_{R_1})\,,$$

where $\mathbb{B}^N_{R_1}$ denotes the ball in $\mathbb{R}^N$ centered at the origin and having radius R_1, the constant given by **[2.]**. Choose now some C^2-smooth function $h_{\text{♪}} : \mathbb{R}^N \to \mathbb{R}$, with

$$h_{\text{♪}}(\eta) = 0\,, \text{ if } \eta \in \operatorname{int} \mathcal{S}_*\,; \quad h_{\text{♪}}(\eta) > 0\,, \text{ if } \eta \in \operatorname{ext} \mathcal{S}_*\,; \quad \lim_{|\eta| \to \infty} h_{\text{♪}}(\eta) = +\infty\,.$$

(For instance, one can take $h_{\text{♪}}(\eta) := h_*(\eta) + \max\{(|\eta| - R_1)^3, 0\}$, the function h_* being given by Lemma 3.5 for the embedded sphere $\mathcal{S}_*$). Pick next some constant $k > 0$ large enough so that $\overline{\Delta}_\sharp \subseteq \Gamma$, where $\Delta_\sharp := \{(t, \xi, \eta) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N : (t, \eta) \in \Delta\}$, and

$$\Delta := \Big\{(t, \eta) \in [0, T] \times \mathbb{R}^N : h_{\text{♪}}(\eta) > kt\Big\}\,.$$

We now recall that the function $h : \mathbb{R}^N \to \mathbb{R}$ was given by Lemma 3.5 for the embedded sphere $\mathcal{S}$. The result below states the existence of a function $r = r(t, \eta)$ whose time average is $h = h(\eta)$ and having support contained in $\overline{\Delta}$.

Lemma 4.2. *There is a C^2-smooth function $r : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfying*

($\star$) $r(t, \eta) = 0$, *if* $(t, \eta) \notin \Delta$,

($\star\star$) $\frac{1}{T} \int_0^T r(t, \eta)\, dt = h(\eta)$, *for every* $\eta \in \mathbb{R}^N$,

($\star\star\star$) $r(t, \eta) = h(\eta)$, *if* $|\eta|$ *is sufficiently large.*

Proof. We choose a C^∞-smooth function $u : \mathbb{R} \to \mathbb{R}$ satisfying

$$u(s) = 0\,, \text{ if } s \le 0\,; \qquad u(s) = 1\,, \text{ if } s \ge 1\,; \qquad 0 < u(s) < 1\,, \text{ if } s \in]0, 1[\,.$$

Then, we define $p : [0, T] \times (\operatorname{ext} \mathcal{S}_*) \to \mathbb{R}$ by

$$p(t, \eta) = \frac{u(h_{\text{♪}}(\eta) - kt)}{\frac{1}{T} \int_0^T u(h_{\text{♪}}(\eta) - ks)\, ds}\,.$$

We observe that p is a C^2-smooth function satisfying

$(\circledast)$ $p(t,\eta)=0$, if $(t,\eta)\notin\Delta$,

$(\circledast\circledast)$ $\frac{1}{T}\int_0^T p(t,\eta)\,dt=1$, if $\eta\in\operatorname{ext}\mathcal{S}_*$,

$(\circledast\circledast\circledast)$ $p(t,\eta)=1$, if $|\eta|$ is sufficiently large.

Finally, we define $r:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ by

$$r(t,\eta)=\begin{cases}0\,, & \text{if } \eta\in\overline{\operatorname{int}\mathcal{S}_*},\\ p(t,\eta)h(\eta)\,, & \text{if } \eta\in\operatorname{ext}\mathcal{S}_*\,.\end{cases}$$

It is easily checked that r satisfies all the required properties. It proves the lemma. □

Proof of Theorem 2.1. Since $H(t,x,y)$ is periodic in x and is equal to zero for $|y|$ large, there is some constant $c>0$ such that

$$\left|\frac{\partial H}{\partial y}(t,x,y)\right|\le c\,,\quad \text{for every } (t,x,y)\in[0,T]\times\mathbb{R}^{2N}\,. \tag{8}$$

On the other hand, recalling *(ii)* in Lemma 3.5, we may find another constant $c'>0$ such that

$$|\nabla h(y_0)|\ge c'\,,\quad \text{if } y_0\in\operatorname{ext}\mathcal{S} \text{ satisfies } \operatorname{dist}(y_0,\mathcal{S})\ge\varrho \text{ and } |y_0|\le R_1\,, \tag{9}$$

the constants ϱ and R_1 having been introduced in (6) and assumption **[2.]**, respectively. Consider the flow map $\phi:\Gamma\to\mathbb{R}^{2N}$, $\phi=\phi(t,\zeta)$, giving the position at time t of the solution z of (HS) with $z(0)=\zeta$. Using arguments similar to those in [18, Lemma 6.4], it can be seen that the function $\Phi:\Gamma\to\mathcal{G}_\sharp$, defined as $\Phi(t,\zeta)=(t,\phi(t,\zeta))$, is a C^∞-smooth diffeomorphism. Let $r:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ be given by Lemma 4.2 above, let $r_\sharp:[0,T]\times\mathbb{R}^{2N}\to\mathbb{R}$ be defined by

$$r_\sharp(t,\xi,\eta):=r(t,\eta)\,,$$

and let the Hamiltonian $\mathcal{R}:[0,T]\times\mathbb{R}^{2N}\to\mathbb{R}$ be given as

$$\mathcal{R}(t,z)=\begin{cases}r_\sharp(\Phi^{-1}(t,z))\,, & \text{if } (t,z)\in\mathcal{G}_\sharp\,,\\ 0\,, & \text{otherwise}\,.\end{cases}$$

It can be checked that $\mathcal{R}$ is C^2-smooth. Fix some constant

$$\lambda>c/c', \tag{10}$$

and define

$$\widetilde{H}(t,z)=H(t,z)+\lambda\mathcal{R}(t,z)\,.$$

Observe that $\widetilde{H}$ coincides with H on the set $\overline{\Omega}_\sharp$. Letting $z=(x,y)$ and combining **[2.]** with $(\star\star\star)$ in Lemma 4.2, for $|y|$ big enough one has

$$\Phi^{-1}(t,z)=(t,z)\,,\qquad \mathcal{R}(t,z)=h(y)\,.$$

Thus, we immediately obtain condition $[\boldsymbol{H}_1]$, and $[\boldsymbol{H}_{2,3,5}]$ follow easily from item *(iv)* of Lemma 3.5, and the fact that $h(y) < 1$, for every $y \in \mathbb{R}^N$. Applying Theorem 2.2, we see that the modified system $(\widetilde{HS})$ has at least $N+1$ geometrically distinct T-periodic solutions, while, in the nondegenerate case, their number is at least 2^N.

Let us show now that these are indeed T-periodic solutions of (HS). With this aim, fix some solution $\tilde{z} = (\tilde{x}, \tilde{y}) : [0,T] \to \mathbb{R}^{2N}$ of $(\widetilde{HS})$, with $\tilde{y}(0) \in \operatorname{ext} \mathcal{S}'$. Arguing as in [18, Lemmas 6.5, 6.6, 6.7] we see that there exists a solution $z = (x, y) : [0,T] \to \mathbb{R}^{2N}$ of (HS) such that $y(t)$ and $\tilde{y}(t)$ coincide at $t = 0, T$; $x(t)$ and $\tilde{x}(t)$ coincide at $t = T$, and

$$x(0) = \tilde{x}(0) + \lambda T \nabla h(y(0)) \,.$$

In particular, we have

$$\tilde{x}(T) - \tilde{x}(0) = x(T) - x(0) + \lambda T \nabla h(y(0)) \,.$$

To conclude the argument we shall see that $\tilde{z}$ is not T-periodic because $\tilde{x}(T) \neq \tilde{x}(0)$. Equivalently,

$$x(T) - x(0) \neq -\lambda T \nabla h(y(0)) \,. \tag{11}$$

This fact will end the proof of Theorem 2.1, since it implies, by (7), that all T-periodic solutions of $(\widetilde{HS})$ lie in $\overline{\Omega}_\sharp$, hence they are T-periodic solutions of (HS).

We distinguish three cases.

Case 1 : $\operatorname{dist}(y(0), \mathcal{S}) < \varrho$. In this situation, (11) follows directly from the choice of ϱ in (6).

Case 2 : $|y(0)| > R_1$. Since $H(t,x,y) = 0$ for $|y| \geq R_1$, we now have that $x(T) = x(0)$, and the result follows.

Case 3 : $\operatorname{dist}(y(0), \mathcal{S}) \geq \varrho$ and $|y(0)| \leq R_1$. Recalling (8), we have $|x(T) - x(0)| \leq Tc$, so that, by (9) and (10),

$$|x(T) - x(0)| \leq Tc < T\lambda c' \leq |-\lambda T \nabla h(y(0))| \,,$$

and hence (11) holds. The proof of Theorem 2.1 is complete. □

5 Critical point theory

5.1 Counting the critical points of a function on a finite-dimensional manifold

In this subsection we recall some classical facts relating the topology of a compact manifold $\mathcal{V}$ and the number of critical points of a function defined on it. Let $\mathcal{V}$ be a finite-dimensional, compact, connected C^2-smooth manifold

without boundary, and let $f : \mathcal{V} \to \mathbb{R}$ be a function of class C^1. The maximum and the minimum of f on $\mathcal{V}$ are critical values of f; in particular, f has at least two different critical points. However, if one assumes some additional 'complexity' in the topology of $\mathcal{V}$, in some cases it is possible to combine algebraic topology with critical point theory methods to predict the existence of more critical points. For instance, it is well known that any real-valued function defined on the 2-torus $\mathbb{T}^2$ must have at least three different critical points, and four if they are nondegenerate. Results of the former type motivated Ljusternik and Schnirelmann [25] to develop the concept of *category*; on the other hand, the latter statement is a well-known example of the consequences of *Morse theory* (see, e.g., [3]).

We recall that $\mathrm{cat}(\mathcal{V})$, the *category* of $\mathcal{V}$, is the minimum number of closed, contractible subsets whose union is $\mathcal{V}$. If no such a finite covering of $\mathcal{V}$ by closed contractible subsets exists, then $\mathrm{cat}(\mathcal{V}) := +\infty$. The importance of this topological invariant is ensured by the Ljusternik – Schnirelmann theorem: any C^1-smooth function on $\mathcal{V}$ has at least $\mathrm{cat}(\mathcal{V})$ critical points. See e.g. [13, Section 5.2.2] or [35, Chapter V] for more details.

In general, the category of a given manifold may not be easy to compute directly. For this reason, it is usual to consider also other topological invariants, such as the so-called cuplength of $\mathcal{V}$, denoted $\mathrm{cl}(\mathcal{V})$. It is the largest integer k for which there are elements $\alpha_j \in H^{q_j}(\mathcal{V}),\ j = 1, \ldots, k$ (the singular cohomology vector spaces with real coefficients), such that $q_j \geq 1$, and the cup product $\alpha_1 \cup \ldots \cup \alpha_k$ does not vanish (see, e.g., [35, p. 161] or [36, p. 732]). It can be used to estimate $\mathrm{cat}(\mathcal{V})$; indeed,

$$\mathrm{cat}(\mathcal{V}) \geq \mathrm{cl}(\mathcal{V}) + 1\,.$$

Finally, a third relevant number associated with our manifold $\mathcal{V}$ is $\mathrm{sb}(\mathcal{V})$, the sum of its Betti numbers. With other words,

$$\mathrm{sb}(\mathcal{V}) = \sum_{n=0}^{+\infty} \dim[H_n(\mathcal{V})]\,,$$

where $H_n(\mathcal{V})$ denotes the usual n-th homology vector space, whose elements are equivalence classes of n-dimensional chains with zero boundary and real coefficients. Observe that all homology vector spaces $H_n(\mathcal{V})$ are finitely generated, and they vanish for $n > \dim \mathcal{V}$, so that the sum is finite. The importance of this number arises from its connection with the critical point theory of Morse functions, i.e., C^2-functions with only nondegenerate critical points. Indeed, a well known result in this context is the so-called Morse inequality, which implies that the number of critical points of any Morse function on $\mathcal{V}$ is at least $\mathrm{sb}(\mathcal{V})$ (see, e.g. [3, Section 3.4]).

In this paper, we shall be particularly interested in the case of $\mathcal{V}$ being the N-torus $\mathbb{T}^N = (\mathbb{R}/2\pi\mathbb{Z})^N$, for which one has

$$\mathrm{cl}(\mathbb{T}^N) = N\,, \qquad \mathrm{cat}(\mathbb{T}^N) = N + 1\,, \qquad \mathrm{sb}(\mathbb{T}^{\mathbb{N}}) = 2^N\,.$$

(The first two equalities are proposed as an exercise in [35, p. 161]; the last one is a well-known consequence of the so-called Künneth formula for the homology with coefficients on a field, see, e.g., [12, p. 5], or [34, p. 235]). Thus, any C^1-smooth function on the torus $\mathbb{T}^N$ has at least $N+1$ critical points, and 2^N if the function is C^2-smooth and the critical points are nondegenerate.

5.2 Bounded perturbations of strongly indefinite, quadratic functionals with nontrivial kernel

We shall develop results of the kind described above for a certain class of functionals defined on the product $\mathcal{M} = E \times \mathcal{V}$, the Hilbert space E being possibly infinite-dimensional. Our functionals will display a 'saddle-like' geometry in the first variable, in line with other results which were amply studied in the literature some 25 years ago, see e.g. [11, 20, 24, 36, 37]. However, these works treat cases in which the (global) Palais – Smale condition holds, and we are here interested in allowing the existence of degenerate directions of the quadratic part, along which compactness may fail. Our results will nevertheless ensure the existence of at least $\mathrm{cl}(\mathcal{V}) + 1$ different critical points, and $\mathrm{sb}(\mathcal{V})$ in the nondegenerate case.

Precisely, let E be a separable real Hilbert space, endowed with the scalar product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. Let $L : E \to E$ be a bounded selfadjoint linear operator, and assume that E splits as the orthogonal direct sum

$$E = E_- \oplus E_0 \oplus E_+ \,, \tag{12}$$

where $E_0 = \ker L \neq \{0\}$ is finite-dimensional and $E_\pm$ are closed subspaces which are invariant for L. We further assume that L is positive definite on E_+ and negative definite on E_-, i.e., there is some constant $\varepsilon_0 > 0$ such that

$$\begin{cases} \langle Le_-, e_- \rangle \leq -\varepsilon_0 \|e_-\|^2, & \text{for every } e_- \in E_- \,, \\ \langle Le_+, e_+ \rangle \geq \varepsilon_0 \|e_+\|^2, & \text{for every } e_+ \in E_+ \,. \end{cases} \tag{13}$$

Let $\mathcal{V}$ be a finite-dimensional compact C^2-smooth manifold without boundary, let $\mathcal{M} := E \times \mathcal{V}$, and let $\psi : \mathcal{M} \to \mathbb{R}$ be given. We shall be interested in the associated *complemented functional* $\varphi : \mathcal{M} \to \mathbb{R}$, defined by

$$\varphi(e, v) = \tfrac{1}{2} \langle Le, e \rangle + \psi(e, v) \,. \tag{14}$$

We observe that the finite-dimensional compact manifold $\mathcal{V}$ admits many Riemannian structures, which are however equivalent. From now on we see $\mathcal{V}$ as endowed with such a Riemannian structure; it allows us to work with the gradient maps

$$\nabla_{\mathcal{M}} \varphi \equiv (\nabla_E \varphi, \nabla_{\mathcal{V}} \varphi) \,, \quad \nabla_{\mathcal{M}} \psi \equiv (\nabla_E \psi, \nabla_{\mathcal{V}} \psi) \,,$$

which are defined on $\mathcal{M}$ and take values in $T\mathcal{M} \equiv E \times T\mathcal{V}$. In view of (14),

$$\nabla_{\mathcal{M}}\varphi(e,v) = \big(Le + \nabla_E\psi(e,v), \nabla_{\mathcal{V}}\psi(e,v)\big)\,, \qquad (e,v) \in E \times \mathcal{V}\,.$$

In addition, if this functional is assumed to be C^2-smooth, it is possible to identify its second differential at a critical point $(e,v) \in \mathcal{M}$ with the associated Hessian linear map $\mathrm{Hess}_{\mathcal{M}}\varphi(e,v) \in \mathscr{L}(E \times T_v\mathcal{V})$, see, e.g., [16, §(16.5.11)]. As usual, this critical point is called nondegenerate provided that $\mathrm{Hess}_{\mathcal{M}}\varphi(e,v)$ is a topological automorphism.

In order to introduce the assumptions on ψ, we shall say that it belongs to the class $\mathscr{A}$ provided that:

$[\boldsymbol{\psi}_1]$ ψ is bounded, there exists some $\ell \in \mathbb{R}$ such that

$$\psi(e,v) \neq \ell\,, \quad \text{for every } (e,v) \in \mathcal{M}\,,$$

and

$$\lim_{\substack{\|e_0\|\to\infty \\ e_0\in E_0}} \psi(e_0+b,v) = \ell\,, \tag{15}$$

uniformly with respect to b belonging to bounded subsets of E and $v \in \mathcal{V}$;

$[\boldsymbol{\psi}_2]$ ψ is C^1-smooth, its partial gradient map $\nabla_E\psi$ is bounded and completely continuous, and

$$\lim_{\substack{\|e_0\|\to\infty \\ e_0\in E_0}} \nabla_E\psi(e_0+b,v) = 0\,, \tag{16}$$

uniformly with respect to b belonging to bounded subsets of E and $v \in \mathcal{V}$.

We shall say that ψ belongs to the class $\mathscr{A}^+$ provided that it not only belongs to the class $\mathscr{A}$, but further satisfies:

$[\boldsymbol{\psi}_3]$ there exists some $R > 0$ such that $\nabla_{\mathcal{M}}\psi(e_0,v) \in E_0$ whenever $(e_0,v) \in E_0 \times \mathcal{V}$, $\|e_0\| \geq R$;

$[\boldsymbol{\psi}_4]$ ψ is C^2-smooth, its partial Hessian map $\mathrm{Hess}_E\psi : \mathcal{M} \to \mathscr{L}(E)$ is globally compact (i.e., its image is relatively compact in $\mathscr{L}(E)$), and

$$\lim_{\substack{\|e_0\|\to\infty \\ e_0\in E_0}} \mathrm{Hess}_E\psi(e_0+b,v) = 0\,, \tag{17}$$

uniformly with respect to b belonging to bounded subsets of E and $v \in \mathcal{V}$;

$[\boldsymbol{\psi}_5]$ all critical points of φ are nondegenerate.

Since the subspaces $E_\pm$ are allowed to be infinite-dimensional, (13) and assumption $[\boldsymbol{\psi}_1]$ tell us that, in some sense, the functional φ has a (strongly indefinite) global saddle geometry. This fact will allow us to estimate the number of its critical points from certain topological invariants of the compact manifold $\mathcal{V}$. Indeed, we shall show the following:

Theorem 5.1. ***(a).*** *Assume that ψ belongs to the class $\mathscr{A}$; then, φ has at least* $\mathrm{cl}(\mathcal{V})+1$ *critical points.* ***(b).*** *If ψ belongs to the class $\mathscr{A}^+$, then φ has at least* $\mathrm{sb}(\mathcal{V})$ *critical points.*

The two parts of this theorem will be proved, respectively, in Sections 6 and 7. We just remark now that, using an observation made in [4, Remark 1.10] (see also [36, p. 732]), there will be no loss of generality in assuming that[1]

$$L(e) = e_+ - e_- \,, \quad \text{for every } e \in E \,. \tag{18}$$

Indeed, otherwise we may define the linear map $\mathcal{L} : E \to E$ by $\mathcal{L}(e) = e_+ - e_-$, and introduce an equivalent scalar product on E by setting $\prec e|\mathfrak{e} \succ := \langle Le_+ - Le_- + e_0, \mathfrak{e}\rangle$, so that $\langle Le, e\rangle = \prec \mathcal{L}e \mid e \succ$.

5.3 An abstract framework for periodic solutions of Hamiltonian systems

In this subsection we shall see how to obtain our basic Hamiltonian Theorem 2.2 from the abstract Theorem 5.1. To do so we need to write the T-periodic solutions of our spatially-periodic Hamiltonian system (HS) as the critical points of a suitable functional defined on the cartesian product of a Hilbert space E and the N-torus $(\mathbb{R}/2\pi\mathbb{Z})^N = \mathcal{V}$. The arguments are mostly well-known (see, e.g. in [33, Chapter 6]); for this reason we recall them only briefly.

There is no loss of generality in assuming that $T = 1$. We shall borrow the notation from complex Fourier analysis and rewrite 1-periodic functions $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^N$ as

$$f(t) = \sum_{m=-\infty}^{+\infty} f_m \exp(2\pi m t\, i)\,,$$

where $f_m = \overline{f_{-m}} \in \mathbb{C}^N$, for every $m \in \mathbb{Z}$. If $\sum_{m=1}^{+\infty} |m|\,|f_m|^2 < +\infty$, such a function is said to belong to $H^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$, and this space is endowed with a Hilbert structure by setting

$$\left\langle \sum_{m=-\infty}^{+\infty} f_m \exp(2\pi m t\, i), \sum_{m=-\infty}^{+\infty} g_m \exp(2\pi m t\, i) \right\rangle := (f_0|g_0) + \sum_{m=-\infty}^{+\infty} |m|(f_m|g_m)\,.$$

Henceforth we denote by $(\cdot|\cdot)$ the usual scalar product in $\mathbb{C}^N$, i.e. $(z|w) = \sum_{i=1}^N z_i\, \overline{w_i}$.

Consider the bounded linear operator $\mathscr{T} : H^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N) \to H^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$ defined by

$$\mathscr{T}\left(\sum_{m=-\infty}^{+\infty} f_m \exp(2\pi m t\, i) \right) := \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} i\,\mathrm{sign}(m)\, f_m \exp(2\pi m t\, i)\,.$$

[1] When the symbols e_+, e_-, e_0 appear in combination with e (assumed to be some element of E), they denote the corresponding orthogonal projection on E_+, E_- or E_0, respectively.

It is easy to check that, whenever $f, g \in H^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$,

$$\langle \mathscr{T} f, g \rangle = \frac{1}{2\pi} \int_0^1 (f'(t)|g(t))\, dt\,. \tag{19}$$

Observe also that $\ker \mathscr{T}$ is the space of constant functions (which can be identified to $\mathbb{R}^N$), while its image is the subspace $\widetilde{H}^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$ made of functions with zero mean.

The solutions of our Hamiltonian system (HS) will be sought in the set of couples $z = (x, y) \in H^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N) \times H^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$. Two such couples $(x_1, y_1), (x_2, y_2)$ will be identified provided that $y_1 = y_2$ and $x_2 - x_1$ differ on an element of $(2\pi\mathbb{Z})^N$. We shall formalize this procedure by rewriting the couple (x, y) in the form $((\tilde{x}, y), \bar{x})$, where $\bar{x} = \int_0^1 x(t)\, dt$ is seen as an element of $\mathcal{V} = (\mathbb{R}/2\pi\mathbb{Z})^N$, and $(\tilde{x}, y)$ belongs to

$$E = \widetilde{H}^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N) \times H^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)\,.$$

The set E becomes a separable Hilbert space after being endowed with the scalar product (which we shall continue to denote by $\langle \cdot, \cdot \rangle$) inherited from $H^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N) \times H^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$. We consider the continuous linear operator $L : E \to E$ given by

$$L(\tilde{x}, y) := 2\pi\, (-\mathscr{T} y,\ \mathscr{T} \tilde{x})\,, \qquad (\tilde{x}, y) \in E\,.$$

In view of (19), for any $(\tilde{x}, y), (\tilde{u}, v) \in E$ one has

$$\left\langle L \begin{pmatrix} \tilde{x} \\ y \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ v \end{pmatrix} \right\rangle = \int_0^1 \Big((\dot{\tilde{x}}(t)|v(t)) - (\dot{y}(t)|\tilde{u}(t)) \Big)\, dt\,,$$

implying that L is selfadjoint. Observe also that its kernel is the finite-dimensional subspace $E_0 := \{0\} \times \mathbb{R}^N \subseteq E$. Moreover, setting

$$E_+ := \left\{ \begin{pmatrix} \tilde{x} \\ y \end{pmatrix} \in E : y = \mathscr{T} \tilde{x} \right\}, \quad E_- := \left\{ \begin{pmatrix} \tilde{x} \\ y \end{pmatrix} \in E : y = -\mathscr{T} \tilde{x} \right\},$$

we see that $\langle L e_+, e_+ \rangle \geq 2\pi \|e_+\|^2$ for any $e_+ \in E_+$, while $\langle L e_-, e_- \rangle \leq -2\pi \|e_-\|^2$ for any e_- in E_-. Finally, observing that $\mathscr{T}^2 = -\mathrm{Id}$ on $\widetilde{H}^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$, one checks that, for any $(\tilde{x}, \tilde{y}) \in \widetilde{H}^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N) \times \widetilde{H}^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$,

$$(\tilde{x}, \tilde{y}) = \frac{1}{2}\Big(\tilde{x} + \mathscr{T} \tilde{y}\,, -\mathscr{T} \tilde{x} + \tilde{y} \Big) + \frac{1}{2}\Big(\tilde{x} - \mathscr{T} \tilde{y}\,, \mathscr{T} \tilde{x} + \tilde{y} \Big) \in E_- + E_+\,,$$

and it easily follows that E splits as the orthogonal direct sum $E = E_- \oplus E_0 \oplus E_+$. Thus, the Hilbert space E and the operator L satisfy all the requirements of Subsection 5.2.

We consider the Nemytskiĭ functional $\psi : \mathcal{M} := E \times \mathcal{V} \to \mathbb{R}$ defined by

$$\psi((\tilde{x}, y), \bar{x}) = \int_0^1 H\Big(t, \bar{x} + \tilde{x}(t), y(t)\Big) dt\,. \tag{20}$$

Well-known arguments show that ψ is C^1-smooth; moreover, a point $((\tilde{x}, y), \bar{x})$ in $\mathcal{M}$ is critical for the complemented functional φ (defined as in (14)) if and only if $z(t) = (\bar{x}+\tilde{x}(t), y(t))$ is a 1-periodic solution of (HS). Thus, we are now (almost) ready to deduce Theorem 2.2 from Theorem 5.1. We shall prepare the proof with a couple of results, the first of which comes from linear functional analysis:

Lemma 5.2. *Let X, Y, Z be real Banach spaces and $K : X \to Y$ a completely continuous linear map. Then, the composition operator $\mathscr{L}(Y, Z) \to \mathscr{L}(X, Z)$, $M \mapsto M \circ K$, is completely continuous.*

Proof. Let $\{M_n\}_n \subseteq \mathscr{L}(Y, Z)$ be bounded. After restricting them to the compact set $C := \overline{K(\mathbb{B}^X)}$, (the closure of the image by K of the unit ball in X), we get a sequence of continuous maps $M_n : C \to Z$. This sequence is uniformly bounded and equicontinuous, and hence the Ascoli-Arzelà theorem guarantees the existence of a subsequence which converges uniformly on C. Then, $\{M_n \circ K\}_n$ converges, along a subsequence, to some continuous linear operator from X to Z. The proof is complete. □

Assume now either $[\boldsymbol{H}_{1-3}]$ or $[\boldsymbol{H}_{1-6}]$, and let the function $h : \mathbb{R}^N \setminus \mathbb{B}^N_{R_0} \to \mathbb{R}$ and the constant ℓ be given by these assumptions. After replacing R_0 by a bigger constant and changing, if necessary, the sign of the Hamiltonian, one may assume that

$$h(y) = H(t, x, y) < \ell\,, \text{ whenever } |y| \geq R_0\,. \tag{21}$$

The aim of the lemma below consists in showing that it suffices to prove Theorem 2.2 assuming that this inequality holds on the whole extended phase space $[0, 1] \times \mathbb{R}^N \times \mathbb{R}^N$, i.e.

$$H(t, x, y) < \ell\,, \quad \text{for any } (t, x, y) \in [0, 1] \times \mathbb{R}^N \times \mathbb{R}^N\,. \tag{22}$$

Lemma 5.3. *Let the (admissible) Hamiltonian H satisfy either assumptions $[\boldsymbol{H}_{1-3}]$ or $[\boldsymbol{H}_{1-6}]$ in Theorem 2.2, and (21). Then, there exists a modified (admissible) Hamiltonian $\breve{H}$ which satisfies these same assumptions and also (22), and such that the associated Hamiltonian systems have the same T-periodic solutions.*

Proof. Choose some constant $\breve{\ell} > \sup_{[0,1]\times\mathbb{R}^N\times\mathbb{R}^N} H$, together with some C^2-smooth function $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying

$$\alpha(s) = s\,, \quad \text{if } s \leq \max_{|y|=R_0} h(y)\,; \quad \alpha'(s) > 0\,, \quad \text{for all } s \in \mathbb{R}\,; \quad \alpha(\ell) = \breve{\ell}\,.$$

Then, define

$$\breve{H}(t,x,y) := \begin{cases} H(t,x,y)\,, & \text{if } |y| \le R_0\,, \\ \breve{h}(y) := \alpha(h(y))\,, & \text{if } |y| \ge R_0\,. \end{cases}$$

It is clear that (22) now holds for $\breve{H}$ and $\breve{\ell}$. Observe that the periodic solutions $z(t) = (x(t), y(t))$ of the Hamiltonian systems associated with either $\breve{H}$ or H are of two classes; those which stay in the region $\{|y| \le R_0\}$, and those for which $y(t)$ has a constant value $y_0 \in \mathbb{R}^N \setminus \overline{\mathbb{B}^N_{R_0}}$, which must be a critical point of h. Consequently, both Hamiltonian systems have the same 1-periodic solutions, and assumptions $[\boldsymbol{H}_{1-3}]$ (or $[\boldsymbol{H}_{1-6}]$) are inherited by $\breve{H}$. This concludes the proof. □

Proof of Theorem 2.2. As shown by Lemma 5.3, there is no loss of generality in assuming (22), which, in view of the definition of ψ in (20), implies the first part of $[\boldsymbol{\psi}_1]$. In order to check (15), choose sequences $\{e_0^n = \bar{y}^n\}_n \subseteq E_0 = \mathbb{R}^N$, $\{b^n = (\tilde{x}^n, y^n)\}_n \subseteq E = \widetilde{H}^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N) \times H^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$ and $\{v^n = \bar{x}^n\}_n \subseteq \mathcal{V} = \mathbb{T}^N$, satisfying $\|\bar{y}^n\| \to \infty$ and $\sup_n(\|\tilde{x}^n\| + \|y^n\|) < \infty$; since the inclusion $H^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N) \subseteq L^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$ is compact, after passing to a subsequence there is no loss of generality in assuming that both sequences $\{\tilde{x}^n\}$ and $\{y^n\}$ converge in $L^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$ and indeed pointwise for almost every $t \in \mathbb{R}/\mathbb{Z}$. Equality (15) now follows from $[\boldsymbol{H}_{1-2}]$ and Lebesgue's dominated convergence theorem.

A similar reasoning can be used to deduce (16) from $[\boldsymbol{H}_3]$. Concerning the boundedness and the complete continuity of $\nabla_E \psi$, it follows from well-known arguments, based on the boundedness of ∇H and the fact that the inclusion $E \subseteq L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2N})$ is compact.

Assume finally $[\boldsymbol{H}_{1-6}]$. Then $[\boldsymbol{\psi}_3]$ is an immediate consequence of $[\boldsymbol{H}_1]$. On the other hand, in view of $[\boldsymbol{H}_{4-5}]$ we see that H is C^2-smooth with respect to $z = (x, y)$ and its associated Hessian map $\operatorname{Hess} H : [0,T] \times \mathbb{R}^{2N} \to \mathcal{M}_{2N}(\mathbb{R})$ is bounded. We deduce that ψ is twice continuously differentiable in its first variable and, for any $(\tilde{x}, y), e, \mathfrak{e} \in E$ and $\bar{x} \in \mathcal{V}$,

$$\big\langle \operatorname{Hess}_E \psi((\tilde{x}, y), \bar{x}) e, \mathfrak{e} \big\rangle = \int_0^1 \Big(\operatorname{Hess} H\big(t, \bar{x} + \tilde{x}(t), y(t)\big) e(t) \Big| \mathfrak{e}(t) \Big) dt\,. \tag{23}$$

Thus, $\operatorname{Hess}_E \psi : \mathcal{M} \to \mathscr{L}(E)$ can be written as the composition of the map $\mathfrak{h} : \mathcal{M} \to \mathscr{L}(L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2N}), E)$ defined by

$$\big\langle \mathfrak{h}((\tilde{x}, y), \bar{x}) e, \mathfrak{e} \big\rangle = \int_0^1 \Big(\operatorname{Hess} H\big(t, \bar{x} + \tilde{x}(t), y(t)\big) e(t) \Big| \mathfrak{e}(t) \Big) dt\,,$$

and the 'restriction operator' $\mathscr{L}(L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2N}), E) \to \mathscr{L}(E)$ sending M into $M\big|_E$. Since $\mathfrak{h}$ is continuous and globally bounded, and the inclusion $E \subseteq L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2N})$ is compact, it follows from Lemma 5.2 that $\operatorname{Hess}_E \psi$ is globally compact.

It remains to check (17). Equivalently, (letting $b = (\tilde{x}, y) \in E$, $v = \bar{x}$ and $e_0 = \bar{y}$),

$$\lim_{|\bar{y}|\to\infty} \mathrm{Hess}_E \psi((\tilde{x}, \bar{y} + y), \bar{x}) = 0\,,$$

uniformly with respect to $(\tilde{x}, y)$ belonging to bounded subsets of E and $\bar{x} \in \mathbb{T}^N$. However, in view of (23), for any $(\tilde{x}, y), e, \mathfrak{e} \in E$ and $\bar{x} \in \mathcal{V}$, one has

$$\Big|\big\langle \mathrm{Hess}_E \psi((\tilde{x}, y), \bar{x}) e, \mathfrak{e}\big\rangle\Big| \le \big\|\mathrm{Hess}\, H\big(\cdot, \bar{x} + \tilde{x}, y\big) e\big\|_{L^2} \|\mathfrak{e}\|_{L^2}\,;$$

moreover, since E is continuously embedded in $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2N})$, there is some constant $K > 0$ such that $\|\mathfrak{e}\|_{L^2} \le K\|\mathfrak{e}\|_E$, and hence,

$$\big\|\mathrm{Hess}_E \psi((\tilde{x}, y), \bar{x}) e\big\|_E \le K \big\|\mathrm{Hess}\, H\big(\cdot, \bar{x} + \tilde{x}, y\big) e\big\|_{L^2}\,,$$

for any $(\tilde{x}, y), e \in E$ and $\bar{x} \in \mathcal{V}$. Thus, it suffices to check that $\|\mathrm{Hess}\, H\big(\cdot, \bar{x} + \tilde{x}, \bar{y} + y\big) e\big\|_{L^2} \to 0$ as $|\bar{y}| \to \infty$, uniformly with respect to $(\tilde{x}, y), e$ belonging to bounded subsets of E and $\bar{x} \in \mathcal{V}$.

With this aim, choose bounded sequences $\{(\tilde{x}^n, y^n)\}_n$, $\{e^n\}_n \subseteq E$, and $\{\bar{x}^n\}_n, \{\bar{y}^n\}_n \subseteq \mathbb{R}^N$ with $|\bar{y}^n| \to \infty$. The inclusion of E into $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2N})$ being compact, after possibly passing to a subsequence we may assume that, for some $(\tilde{x}, y)$, $e \in E$,

$$(\tilde{x}^n, y^n) \to (\tilde{x}, y) \;\text{ and }\; e^n \to e\,, \;\text{ in } L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2N})\,.$$

Moreover, there is no loss of generality in assuming that this convergence holds pointwise at almost every point. By the triangle inequality,

$$\begin{aligned}
&\|\mathrm{Hess}\, H\big(\cdot, \bar{x}^n + \tilde{x}^n, \bar{y}^n + y^n\big) e^n\|_{L^2} \le \\
&\le \|\mathrm{Hess}\, H(\cdot, \bar{x}^n + \tilde{x}^n, \bar{y}^n + y^n)(e^n - e)\|_{L^2} + \|\mathrm{Hess}\, H(\cdot, \bar{x}^n + \tilde{x}^n, \bar{y}^n + y^n) e\|_{L^2}\,.
\end{aligned}$$

The first term in the right side converges to zero because $e^n \to e$ and $\mathrm{Hess}\, H$ is uniformly bounded, while the second term also converges to zero, by $[\boldsymbol{H}_5]$ and Lebesgue's theorem. The proof is complete. □

6 Relative category and multiplicity of critical points

The goal of this section is to prove the first part of Theorem 5.1: if ψ belongs to the class $\mathscr{A}$ then the complemented functional φ has at least $\mathrm{cl}(\mathcal{V}) + 1$ critical points. Accordingly, we go back to the framework and assumptions of this result, and consider a Hilbert space E splitting as in (12), a selfadjoint operator $L : E \to E$ having the form (18) and a compact connected C^2-smooth manifold $\mathcal{V}$. Finally, we choose a functional $\psi : \mathcal{M} = E \times \mathcal{V} \to \mathbb{R}$ in the class $\mathscr{A}$, and construct its complemented functional φ by (14). Our tools will be mainly topological, and more precisely, will come from algebraic topology. Many of these arguments were developed by Szulkin [36], and it is from this paper that we borrow the notation and, to some extent, the line of argument.

The set of critical points of φ is $\Xi = \{z \in \mathcal{M} : \nabla_{\mathcal{M}}\varphi(z) = 0\}$. We recall that a function $V : \mathcal{M} \setminus \Xi \to E \times T\mathcal{V}$ is said to be a *pseudogradient* vector field for φ if it is locally Lipschitz continuous and satisfies

$$\|V(z)\| \leq 2\|\nabla_{\mathcal{M}}\varphi(z)\|\,, \qquad \langle \nabla_{\mathcal{M}}\varphi(z), V(z)\rangle \geq \tfrac{1}{2}\|\nabla_{\mathcal{M}}\varphi(z)\|^2\,,$$

for every $z \in \mathcal{M} \setminus \Xi$. The result below has been extracted from [36, pp. 730–731]:

Lemma 6.1. *There exists a pseudogradient vector field of the form*

$$V(e,v) = \Big(V_1(e,v) = Le + W(e,v)\,, V_2(e,v)\Big)\,,$$

where $W : \mathcal{M} \setminus \Xi \to E$ is bounded and completely continuous (i.e., it maps bounded subsets of $\mathcal{M} \setminus \Xi$ into relatively compact subsets of E).

From now on, let V be fixed under the conditions of the above lemma. We consider the initial value problem

$$\frac{d\gamma}{d\tau} = -V(\gamma)\,, \quad \gamma(0,z) = z\,. \tag{24}$$

For each initial position $z \in \mathcal{M} \setminus \Xi$, the unique solution $\gamma(\cdot\,,z)$ is defined on some maximal interval $]\tau_-(z), \tau_+(z)[$.

6.1 An admissible class of deformations of $\mathcal{M}$

By a *deformation* of $\mathcal{M}$ we mean a continuous map $\eta : [0,1] \times \mathcal{M} \to \mathcal{M}$ such that $\eta(0,z) = z$ for every $z \in \mathcal{M}$. Following [36, Definition 2.3], a class $\mathcal{D}$ of deformations of $\mathcal{M}$ will be called *admissible* if it contains the trivial deformation $\eta(t,z) = z$ and, whenever $\eta, \tilde{\eta}$ belong to $\mathcal{D}$, their superposition

$$\eta \star \tilde{\eta}(t,z) = \begin{cases} \eta(2t,z)\,, & \text{for } 0 \leq t \leq 1/2\,, \\ \tilde{\eta}(2t-1, \eta(1,z))\,, & \text{for } 1/2 \leq t \leq 1 \end{cases}$$

also belongs to $\mathcal{D}$.

Inspired by Szulkin [36, Definition 3.5], we consider the class $\mathcal{D}$ of deformations of $\mathcal{M}$ defined as follows: the deformation η belongs to $\mathcal{D}$ provided that:

($\odot$) for every $z \in \mathcal{M} \setminus \Xi$, the flow $\eta(\cdot\,,z)$ moves forward along the integral curve $\gamma(\cdot\,,z)$ of (24), in the sense that, for any $t_1 < t_2$ in $[0,1]$, there are $\tau_1 \leq \tau_2$ in $[0, \tau_+(z)[$ such that $\eta(t_1,z) = \gamma(\tau_1,z)$ and $\eta(t_2,z) = \gamma(\tau_2,z)$;

($\odot\odot$) there are continuous functions $q_\pm : [0,1] \times \mathcal{M} \to \mathbb{R}$ which are bounded on bounded sets and satisfy $q_\pm(0,e,v) = 1$, and a completely continuous map $K : [0,1] \times \mathcal{M} \to E$ with $K(0,e,v) = e_0$, such that $\eta(t,e,v) = \big(\eta_1(t,e,v)\,, \eta_2(t,e,v)\big)$, with

$$\eta_1(t,e,v) = q_-(t,e,v)e_- + q_+(t,e,v)e_+ + K(t,e,v)\,, \tag{25}$$

for every $(t,e,v) \in [0,1] \times E \times \mathcal{V}$.

It is easy to check that this class of deformations is admissible in the sense given above. We remark that it is not exactly the class of deformations considered in [36], since it is further required there (for $Le = e_+ - e_-$), that both functions $q_\pm$ are bounded away from zero, and satisfy $q_-(t,e,v)q_+(t,e,v) \equiv 1$. So, our class $\mathcal{D}$ actually contains more deformations than that the one considered by Szulkin; however, we will keep the essential properties of Szulkin's relative category.

A closed set $\mathcal{N} \subseteq \mathcal{M}$ is said to be invariant for the class $\mathcal{D}$ if $\eta([0,1] \times \mathcal{N}) \subseteq \mathcal{N}$ for every $\eta \in \mathcal{D}$. In view of property $(\odot)$ with which our class of deformations $\mathcal{D}$ has been defined, a sufficient condition for the set $\mathcal{N}$ to be invariant for the class $\mathcal{D}$ is that it be forward-invariant for the flow of (24). The result below, which follows along the lines of [36, Theorem 3.8], establishes the existence of a family of sets which satisfy these conditions.

Lemma 6.2. *If $R > 0$ is big enough, the set*

$$\mathcal{N}(R) = \big((E_- \setminus \mathbb{B}_R^{E_-}) \oplus E_0 \oplus E_+\big) \times \mathcal{V} \tag{26}$$

is invariant for the class $\mathcal{D}$.

Proof. In view assumption $[\boldsymbol{\psi}_2]$, our functional ψ has a bounded gradient. This implies the existence of some $R > 0$ such that $\Xi \cap \mathcal{N}(R) = \emptyset$. Furthermore, since also W is bounded, after possibly replacing R by a bigger number we may assume that $\langle Le + W(e,v), e_- \rangle < 0$ for every $(e,v) \in \mathcal{N}(R)$. Then $\mathcal{N}(R)$ is forward-invariant for the flow of (24), and the result follows. □

There is a well-known deformation lemma which provides the existence of many nontrivial deformations in the class $\mathcal{D}$. Given some level $c \in \mathbb{R}$ we write

$$\varphi_c = \{z \in \mathcal{M} : \varphi(z) \le c\}\,, \qquad \Xi_c = \{z \in \Xi : \varphi(z) = c\}\,.$$

Lemma 6.3. *Let φ satisfy the Palais–Smale condition at a given level $c \in \mathbb{R}$. If U is an open neighborhood of Ξ_c, then there exists a deformation $\eta \in \mathcal{D}$ such that $\eta(1, \varphi_{c+\varepsilon} \setminus U) \subseteq \varphi_{c-\varepsilon}$, for some $\varepsilon > 0$.*

In view of (18), this result follows from straightforward adaptations in the arguments of [33, Theorem A.4−Proposition A.18].

6.2 Relative category

The notion of (absolute) category was already considered in Subsection 5.1. The generalized concept of *relative category* was introduced by Fournier and Willem [19], and adapted by Szulkin [36, Definition 2.4], so to treat infinite-dimensional variational problems. We shall follow almost exactly Szulkin's approach, but using the modified class of deformations $\mathcal{D}$ introduced in the previous subsection.

It will be convenient to begin by clarifying the terminology and setting the notation. Let X be a metric space and $A \subseteq X$ a closed set; by a *deformation* of A in X we mean a continuous map $\eta : [0,1] \times A \to X$ such that $\eta(0,a) = a$ for every $a \in A$. We shall say that A is *contractible in* X if there exists a deformation η of A in X such that $\eta(1,A) = \{p\}$ is a singleton. If Y is another closed subset of X, we shall say that A is of category $k \geq 0$ relative to Y, denoted $\mathrm{cat}_{X,Y}(A) = k$, provided that k is the smallest integer such that

$$A = A_0 \cup A_1 \cup \ldots \cup A_k \,, \tag{27}$$

where all A_j are closed in X, all A_j with $j \geq 1$ are contractible in X, and there exists a deformation η_0 of $A_0 \cup Y$ in X satisfying

$$\eta_0(1, A_0) \subseteq Y \,, \quad \text{and} \quad \eta_0(t, Y) \subseteq Y \,, \text{ for every } t \in [0,1] \,. \tag{28}$$

(If no such a k exists, $\mathrm{cat}_{X,Y}(A) = +\infty$.)

We recall that the closed subset $Y \subseteq X$ is called a *retract* of X if there exists a continuous map $r : X \to Y$ such that $r(y) = y$ for every $y \in Y$. We shall be interested in the following properties of relative category.

Lemma 6.4. *The following hold:*

(♮) *Let $Z \subseteq Y \subseteq X$ be closed, and assume that Y is a retract of X. Then,* $\mathrm{cat}_{Y,Z}(Y) \leq \mathrm{cat}_{X,Z}(Y)$.

(♮♮) *Let $\mathcal{V}$ be a finite-dimensional, compact, connected C^2-smooth manifold without boundary, and let $m \geq 1$ be an integer. Then,*

$$\mathrm{cat}_{\overline{\mathbb{B}^m} \times \mathcal{V}, \mathbb{S}^{m-1} \times \mathcal{V}}(\overline{\mathbb{B}^m} \times \mathcal{V}) \geq \mathrm{cl}(\mathcal{V}) + 1 \,.$$

Proof. Item (♮) follows easily from the definitions. On the other hand, (♮♮) is a consequence of [36, Proposition 2.6 and Lemma 3.7]. □

In order to deal with infinite-dimensional problems it is convenient to restrict the family of deformations η_0 allowed in (28). Thus, assume now that $X = \mathcal{M} = E \times \mathcal{V}$ is the ambient space considered in Subsection 5.3, and let $\mathcal{D}$ be the class of deformations defined in the previous subsection. Given two closed sets $A, \mathcal{N} \subseteq \mathcal{M}$ we shall say that A is of category $k \geq 0$ relative to $\mathcal{N}$ and $\mathcal{D}$, written $\mathrm{cat}^{\mathcal{D}}_{\mathcal{M},\mathcal{N}}(A) = k$, provided that k is the smallest integer such that (27) holds: here, all sets A_j are required to be closed in $\mathcal{M}$, all A_j with $j \geq 1$ should be contractible in $\mathcal{M}$, and (28) must hold for some deformation $\eta_0 \in \mathcal{D}$. Again, $\mathrm{cat}^{\mathcal{D}}_{\mathcal{M},\mathcal{N}}(A)$ is defined as $+\infty$ if no such a k exists.

This second concept of relative category satisfies again many properties for which we refer to [36, Propositions 2.8 and 2.9]. We have extracted two of them, which will be needed in the sequel; the proofs are immediate from the definitions, and are hence omitted.

Lemma 6.5. *The following hold:*

(ɦ) $\mathrm{cat}^{\mathcal{D}}_{\mathcal{M},\mathcal{N}}(A_2) \leq \mathrm{cat}^{\mathcal{D}}_{\mathcal{M},\mathcal{N}}(A_1)$, *for any closed sets* $A_2 \subseteq A_1 \subseteq \mathcal{M}$.

(ɦɦ) *Let* $\mathcal{N}_2 \subseteq \mathcal{N}_1$ *be closed and assume that* $\mathcal{N}_1$ *is invariant for the class* $\mathcal{D}$. *Then,* $\mathrm{cat}^{\mathcal{D}}_{\mathcal{M},\mathcal{N}_1}(A) \leq \mathrm{cat}^{\mathcal{D}}_{\mathcal{M},\mathcal{N}_2}(A)$, *for any closed set* $A \subseteq \mathcal{M}$.

A well-known result in the framework of the Ljusternik – Schnirelmann theory is that, under suitable compactness conditions, the relative category of two level sets can be used to estimate the number of critical points which lie in between. This fact continues to hold for the relative category with respect to our class $\mathcal{D}$ of deformations.

Lemma 6.6. *Let the levels* $a < b$ *be such that the Palais – Smale condition (PS)*$_c$ *holds for every* $c \in [a, b]$. *Then,* φ *has at least* $\mathrm{cat}^{\mathcal{D}}_{\mathcal{M},\varphi_a}(\varphi_b)$ *critical points in* $\varphi^{-1}([a, b])$.

Proof. It suffices to transcribe the proof of [36, Proposition 3.2], in which the Palais – Smale condition is assumed for all levels and not only for the candidate critical ones. □

A key step in our argument towards the proof of Theorem 5.1***(a)*** will be a refinement of [36, Proposition 3.6], which we examine next. For any positive number $R > 0$ we consider the sets $\mathcal{N}(R)$, defined as in (26), and

$$A(R) = \overline{\mathbb{B}^{E_-}_R} \times \mathcal{V}\,.$$

Lemma 6.7. *The following inequality holds:*

$$\mathrm{cat}^{\mathcal{D}}_{\mathcal{M},\mathcal{N}(R)}\big(A(R)\big) \geq \mathrm{cl}(\mathcal{V}) + 1\,.$$

Proof. There is no loss of generality in assuming $R = 1$, and, in order to simplify the notation, we shall just write $\mathcal{N}, A$ in the place of $\mathcal{N}(1), A(1)$. Using a contradiction argument, assume that

$$\mathrm{cat}^{\mathcal{D}}_{\mathcal{M},\mathcal{N}}(A) \leq \kappa = \mathrm{cl}(\mathcal{V})\,.$$

Then, the set A can be decomposed in the form (27), where all A_j are closed subsets of $\mathcal{M}$, all A_j with $j \geq 1$ are contractible in $\mathcal{M}$, and there exists a deformation $\eta_0 \in \mathcal{D}$ with (28), for $Y = \mathcal{N}$. We write $\eta_0 = (\eta_1, \eta_2)$, with η_1 having the form (25) for some continuous functions $q_\pm : [0, 1] \times \mathcal{M} \to \mathbb{R}$ with $q_\pm(0, e, v) = 1$, and a completely continuous map $K : [0, 1] \times \mathcal{M} \to E$ with $K(0, e, v) = e_0$, for every $(e, v) \in \mathcal{M}$.

The map K being completely continuous, there are finite-dimensional subspaces $F_\pm \subseteq E_\pm$, and a continuous map $C : [0, 1] \times \mathcal{M} \to F := F_- \oplus E_0 \oplus F_+$, with

$$\|K(t, e, v) - C(t, e, v)\| < \tfrac{1}{2}\,, \quad \text{for every } (t, e, v) \in [0, 1] \times A\,. \tag{29}$$

Furthermore, since $K(0, e, v) = e_0$, after possibly replacing the function C by $C(t, e, v) - C(0, e, v) + e_0$, we see that C can be taken satisfying $C(0, e, v) = e_0$, for every $(e, v) \in \mathcal{M}$.

We consider the sets

$$\mathcal{M}^* := F_- \times \mathcal{V}\,, \qquad \mathcal{N}^* := \mathbb{S}^{F_-} \times \mathcal{V}\,, \qquad A^* := A \cap \mathcal{M}^* = \overline{\mathbb{B}^{F_-}} \times \mathcal{V}\,,$$

and $A_j^* = A_j \cap \mathcal{M}^*$, for $j = 0, 1, \ldots, \kappa$. Observe that $A^* = A_0^* \cup A_1^* \cup \cdots \cup A_\kappa^*$, and A_j^* is either empty or contractible in $\mathcal{M}^*$, for every $j = 1, \ldots, \kappa$. We define $\eta^* : [0,1] \times \mathcal{M}^* \to \mathcal{M}^*$ by the rule

$$\eta^*(t, f_-, v) = \Big(\eta_1^*(t, f_-, v) = q_-(t, f_-, v) f_- + \pi C(t, f_-, v)\,,\, \eta_2(t, f_-, v)\Big)\,,$$

where $\pi : F \to F_-$ is the orthogonal projection $\pi(f_- + e_0 + f_+) = f_-$. Observe that

$$\eta^*(0, f_-, v) = (f_-, v)\,, \quad \text{for every } (f_-, v) \in \mathcal{M}^*,$$

and, letting $\mathscr{O} := F_- \setminus \mathbb{B}_{1/2}^{F_-}$, the combination of (28) for $Y = \mathcal{N}$ and (29) gives

$$\eta_1^*(1, A_0^*) \subseteq \mathscr{O}\,, \quad \text{and} \quad \eta_1^*(t, \mathcal{N}^*) \subseteq \mathscr{O}\,, \quad \text{for every } t \in [0,1]\,.$$

We consider the continuous function

$$d : \mathscr{O} \to \mathbb{R}\,, \qquad d(f_-) := \min\Big\{1, \big|\|f_-\| - 1\big|\Big\}\,,$$

which measures the distance from f_- to the unit sphere, with a maximum of one. The Tietze Extension Theorem implies the existence of continuous maps $\alpha : [0,1] \times F_- \to F_-$ and $\delta : [0,1] \times \mathcal{M}^* \to [0,1]$, with

$$\alpha(t, f_-) = \begin{cases} f_-\,, & \text{if } t = 0\,, \\ f_-/\|f_-\|\,, & \text{if } f_- \in \mathscr{O} \text{ and } d(f_-) \le t \le 1\,, \end{cases}$$

$$\delta(t, f_-, v) = \begin{cases} 0\,, & \text{if } t = 0\,, \\ d(\eta_1^*(t, f_-, v))\,, & \text{if } (t, f_-, v) \in \big(\{1\} \times A_0^*\big) \cup \big([0,1] \times \mathcal{N}^*\big)\,. \end{cases}$$

We define $h_1 : [0,1] \times \mathcal{M}^* \to F_-$ by

$$h_1(t, f_-, v) = \alpha(\delta(t, f_-, v), \eta_1^*(t, f_-, v))\,,$$

and observe that

$$h_1(0, f_-, v) = f_-\,, \qquad h_1(1, A_0^*) \subseteq \mathbb{S}^{F_-}, \qquad h_1(t, \mathcal{N}^*) \subseteq \mathbb{S}^{F_-}\,,$$

for every $(f_-, v) \in \mathcal{M}^*$ and every $t \in [0,1]$. Consider now the continuous map $h : [0,1] \times \mathcal{M}^* \to \mathcal{M}^*$, defined by

$$h(t, f_-, v) := (h_1(t, f_-, v), \eta_2(t, f_-, v))\,.$$

Then,

$$h(0, f_-, v) = (f_-, v)\,, \qquad h(1, A_0^*) \subseteq \mathcal{N}^*, \qquad h(t, \mathcal{N}^*) \subseteq \mathcal{N}^*\,,$$

for every $(f_-, v) \in \mathcal{M}^*$ and every $t \in [0,1]$. Since, as observed above, A_j^* is either empty or contractible in $\mathcal{M}^*$, for every $j = 1, \dots, \kappa$, we conclude that

$$\mathrm{cat}_{\mathcal{M}^*, \mathcal{N}^*}(A^*) \leq \kappa \,.$$

The finite-dimensional space F_- can now be identified with $\mathbb{R}^m$, the integer m being its dimension. We see that

$$\mathrm{cat}_{\mathbb{R}^m \times \mathcal{V}, \mathbb{S}^{m-1} \times \mathcal{V}}(\overline{\mathbb{B}^m} \times \mathcal{V}) \leq \kappa \,.$$

Consider now the usual retraction $r : \mathbb{R}^m \to \overline{\mathbb{B}^m}$, sending $\mathbb{R}^m \setminus \mathbb{B}^m$ into the sphere $\mathbb{S}^{m-1}$. By Lemma 6.4 (♃),

$$\mathrm{cat}_{\overline{\mathbb{B}^m} \times \mathcal{V}, \mathbb{S}^{m-1} \times \mathcal{V}}(\overline{\mathbb{B}^m} \times \mathcal{V}) \leq \mathrm{cat}_{\mathbb{R}^m \times \mathcal{V}, \mathbb{S}^{m-1} \times \mathcal{V}}(\overline{\mathbb{B}^m} \times \mathcal{V}) \leq \kappa = \mathrm{cl}(\mathcal{V}) \,.$$

However, this contradicts Lemma 6.4 (♃♃). The proof of the lemma is thus concluded. □

Proof of Theorem 5.1 ***(a).*** After possibly replacing φ by $-\varphi$, we may assume that $\psi(e, v) < \ell$, for every $(e, v) \in \mathcal{M}$.

Step 1. The Palais – Smale condition $(\mathrm{PS})_c$ holds for every level $c \neq \ell$.

Indeed, choose such a number c and assume the existence of a sequence $\{(e^n, v^n)\}_n$ in $\mathcal{M}$ with $\varphi(e^n, v^n) \to c$ and $\nabla_{\mathcal{M}} \varphi(e^n, v^n) \to 0$. In particular,

$$\nabla_E \varphi(e^n, v^n) = Le^n + \nabla_E \psi(e^n, v^n) \to 0 \,, \tag{30}$$

and since $\nabla_E \psi$ is bounded, we see that $\{e_-^n\}_n$ and $\{e_+^n\}_n$ are bounded. Assume for a moment that $\{e_0^n\}_n$ were also bounded. Then, the compactness of $\nabla_E \psi$ implies, when combined with (30), that $\{e_\pm^n\}_n$ have converging subsequences. Also the bounded, finite-dimensional sequence $\{e_0^n\}_n$ must have a converging subsequence, and we see that c must be a critical value of φ.

Thus, we assume on the contrary that, for a subsequence, $\|e_0^n\| \to +\infty$. Then, by (16), we have that $\nabla_E \psi(e^n, v^n) \to 0$, and (30) implies that $e_\pm^n \to 0$. Thus, $c = \lim_n \varphi(e^n, v^n) = \lim_n \psi(e^n, v^n)$, contradicting assumption (15). The proof of Step 1 is thus concluded.

Step 2. There are numbers $a < \ell$ and $R > 0$ such that *(i).* $\varphi_a \subseteq \mathcal{N}(R)$; *(ii).* $\mathcal{N}(R)$ is invariant for the deformations in the class $\mathcal{D}$.

In view of Lemma 6.2, it is possible to choose the number $R > 0$ satisfying *(ii).* Since ψ is bounded, the complemented functional φ is bounded from below on $\left(\mathbb{B}_R^{E_-} + E_0 + E_+\right) \times \mathcal{V}$, and the result follows.

Step 3. $\sup\limits_{E_- \times \mathcal{V}} \varphi < \ell$.

Assume the contrary, and choose a sequence $\{(e_-^n, v^n)\}_n$ in $E_- \times \mathcal{V}$ such that

$$\lim_n \left[\tfrac{1}{2}\langle Le_-^n, e_-^n\rangle + \psi(e_-^n, v^n)\right] \geq \ell\,.$$

Since $\langle Le_-^n, e_-^n\rangle \leq -\varepsilon_0\|e_-^n\|^2$ and $\psi(e_-^n, v^n) < \ell$, we see that $e_-^n \to 0$ and $\psi(e_-^n, v^n) \to \ell$. From the boundedness of $\nabla_E\psi$ we conclude that $\psi(0, v^n) \to \ell$, which is impossible, since

$$\psi(0, v^n) \leq \max_{v\in\mathcal{V}} \psi(0, v) < \ell\,, \quad \text{for every } n\,.$$

The end of the proof. Pick numbers $a < \ell$ and $R > 0$ as provided by Step 2. Use now Step 3 to find some number b such that $\max\{a, \sup_{E_-\times\mathcal{V}} \varphi\} < b < \ell$. Then, $A(R) \subseteq \varphi_b$, and, by Lemma 6.5 ($\hbar$),

$$\mathrm{cat}^{\mathcal{D}}_{\mathcal{M},\varphi_a}(\varphi_b) \geq \mathrm{cat}^{\mathcal{D}}_{\mathcal{M},\varphi_a}(A(R))\,.$$

On the other hand, in view of Step 2 and Lemma 6.5 ($\hbar\hbar$), we see that

$$\mathrm{cat}^{\mathcal{D}}_{\mathcal{M},\varphi_a}(A(R)) \geq \mathrm{cat}^{\mathcal{D}}_{\mathcal{M},\mathcal{N}(R)}(A(R))\,,$$

and combining these inequalities with Lemma 6.7, we obtain

$$\mathrm{cat}^{\mathcal{D}}_{\mathcal{M},\varphi_a}(\varphi_b) \geq \mathrm{cl}(\mathcal{V}) + 1\,.$$

Since, by Step 1, the Palais–Smale condition $(\mathrm{PS})_c$ holds for every $c \in [a, b]$, Lemma 6.6 provides the existence of at least $\mathrm{cl}(\mathcal{V}) + 1$ critical points of φ. It concludes the proof. □

7 Morse theory and multiplicity of nondegenerate critical points

The aim of this section is to prove Theorem 5.1***(b)***: if ψ belongs to the class $\mathscr{A}^+$, then it has at least $\mathrm{sb}(\mathcal{V})$ critical points. Our approach will be divided into two steps. Firstly, we shall define a new class $\mathscr{A}^*$ of functionals ψ, which will be contained in $\mathscr{A}^+$, and we shall prove a version of Theorem 5.1***(b)*** for functionals ψ in this subclass. The second step will consist in showing that, given some functional ψ in the class $\mathscr{A}^+$, then there exists another functional ψ^* in the class $\mathscr{A}^*$ such that the associated complemented functionals φ and φ^* have exactly the same number of critical points. This will be shown provided only that φ has finitely many critical points, and after that, the proof of Theorem 5.1 will be complete.

7.1 A review on homology and Morse theory

In this subsection we collect a few basic elements from homology and Morse theory which will be needed in our proof of Theorem 5.1***(b)***. We do not claim any originality in these results, which are well-known to the specialists. However, we believe that having them all listed here can be helpful for some potential readers of this paper.

A topological pair is a couple (X, A), where X is a topological space and $A \subseteq X$ is a subset. Given such a topological pair (X, A) we shall denote by $H_*(X, A) = \{H_n(X, A)\}_{n \geq 0}$ the associated graded sequence of relative homology groups with real coefficients. Thus, $H_n(X, A)$ is, for each index n, a real vector space whose elements are equivalence classes of singular chains having zero boundary. Moreover, this is done in such a way that $H_*(X, \emptyset) = H_*(X)$. A map between topological pairs $f : (X, A) \to (Y, B)$ is a continuous mapping $f : X \to Y$ such that $f(A) \subseteq B$. Such a mapping induces a corresponding sequence of linear transformations $f_* : H_*(X, A) \to H_*(Y, B)$, and this correspondence is functorial, in the sense that $\left[\mathrm{Id}_{(X,A)}\right]_* = \mathrm{Id}_{H_*(X,A)}$ and, for any maps $f : (X, A) \to (Y, B)$, $g : (Y, B) \to (Z, C)$, one has that $(g \circ f)_* = g_* \circ f_*$. See, e.g., [34, Ch. 4] for more details.

We recall that a homotopy between $f, g : (X, A) \to (Y, B)$ is a continuous map $h : [0, 1] \times X \to Y$ such that $h(0, \cdot) = f$, $h(1, \cdot) = g$, and $h([0, 1] \times A) \subseteq B$. If such a homotopy exists, f and g are called homotopic. The following lemma, which states the homotopy invariance of the homology, is completely standard.

Lemma 7.1 (Deformation)**.** *If $f, g : (X, A) \to (Y, B)$ are homotopic, then the induced linear maps $f_*, g_* : H_*(X, A) \to H_*(Y, B)$ are equal.*

A second important fact is the so-called Künneth formula. It relates the relative homology groups of a product of spaces with those of each factor, see, e.g. [12, p. 5], [34, p. 235]. We shall be interested only in the following particularly simple case ($\otimes$ denotes the usual tensor product):

Lemma 7.2 (Künneth formula)**.** *Let (X, A) be a topological pair, and Y a topological space. Then,*

$$H_n(X \times Y, A \times Y) \cong \bigoplus_{i+j=n} \left[H_i(X, A) \otimes H_j(Y)\right],$$

for every integer $n \geq 0$.

It will be convenient to have at hand the relative homology groups of the pairs $(\mathbb{R}^N, \mathbb{S}^{N-1})$ or, what is the same (by Lemma 7.1), of the pairs $(\overline{\mathbb{B}^N}, \mathbb{S}^{N-1})$. Again, this is a completely standard result.

Lemma 7.3. $H_n(\mathbb{R}^N, \mathbb{S}^{N-1}) \cong H_n(\overline{\mathbb{B}^N}, \mathbb{S}^{N-1}) \cong \begin{cases} 0\,, & \text{if } n \neq N\,, \\ \mathbb{R}\,, & \text{if } n = N\,. \end{cases}$

Finally, we shall need the elementary result given below. Since we could not find it in the literature, this time we include the short proof.

Lemma 7.4. *Let X be a topological space, and let $Z \subseteq Y$ be subsets of X. Assume that there exists a continuous map $h : [0, 1] \times X \to X$, with*

- *$h(0, x) = x$ for every $x \in X$;*

- *$h(t,Y) \subseteq Y$ and $h(t,Z) \subseteq Z$, for every $t \in [0,1]$;*
- *$h(1,Y) \subseteq Z$.*

Then, the map $i_* : H_*(X,Z) \to H_*(X,Y)$, induced by the inclusion $i : (X,Z) \to (X,Y)$, is an isomorphism.

Proof. We consider the mapping $r : (X,Y) \to (X,Z)$, defined by $r(x) := h(1,x)$, for $x \in X$. Then,

$$i_* \circ r_* = (i \circ r)_* = \mathrm{Id}_{H_*(X,Y)} \,,$$

since $h : [0,1] \times X \to X$ is a continuous deformation of (X,Y), connecting the identity with $i \circ r$. Moreover,

$$r_* \circ i_* = (r \circ i)_* = \mathrm{Id}_{H_*(X,Z)} \,,$$

since $h : [0,1] \times X \to X$ is also a continuous deformation of (X,Z), connecting the identity with $r \circ i$. Hence, i_* is an isomorphism, with inverse $(i_*)^{-1} = r_*$. $\square$

The dimensions $\dim H_n(X,A)$ of the homology vector spaces associated with a given topological pair (X,A) are called *Betti numbers*. These numbers play an important role in classical Morse theory, since, under compactness conditions, they can be used to estimate the number of critical points of a Morse function (i.e., one having only nondegenerate critical points). This is done by letting X, A be sublevel sets of the given functional. More precisely, let ϕ be a C^1-smooth functional defined on some C^2-smooth, finite-dimensional manifold without boundary. Assume further that ϕ is C^2-smooth on some open set containing all its critical points, and that it is a Morse function. Assume finally that $a < b$ are real numbers such that the Palais – Smale condition $(PS)_c$ holds at every level $c \in [a,b]$. The result below is a classical consequence of the so-called weak Morse inequalities (see, e.g. [13, Example 3 (p. 328) and Corollary 5.1.28 - Theorem 5.1.29 (p. 339)] or [26, Theorem 8.2 (p. 182) and Remark 2 (p. 189)]).

Lemma 7.5. *Under these conditions, the functional ϕ has at least $\mathrm{sb}(\phi_b, \phi_a) = \sum_{n=0}^{\infty} \dim H_n(\phi_b, \phi_a)$ critical points in $\phi^{-1}([a,b])$.*

7.2 The class $\mathscr{A}^*$ and a weak version of Theorem 5.1*(b)*

We keep the general setting and the notation from Subsection 5.2 (including (18)). Let the functional $\psi : \mathcal{M} \to \mathbb{R}$ belong to the class $\mathscr{A}^+$; since $\nabla_E \psi$ is bounded, we may find some $R_1 > 0$ such that[2]

$$\begin{cases} \|e_-\| \geq R_1 & \Rightarrow \quad \langle Le + \nabla_E \psi(e,v), e_- \rangle < 0 \,, \\ \|e_+\| \geq R_1 & \Rightarrow \quad \langle Le + \nabla_E \psi(e,v), e_+ \rangle > 0 \,. \end{cases} \tag{31}$$

[2] As in page 19 we denote by $e_\pm$, e_0, the orthogonal projections on $E_\pm$, E_0 of the vector $e \in E$. Also, we will write $\tilde{e}$ to denote $e_+ + e_-$.

After possibly replacing ψ, L, ℓ by $-\psi, -L, -\ell$, we see that, in order to prove Theorem 5.1***(b)***, there is no loss of generality in assuming that

$$\psi(e,v) < \ell\,, \quad \text{for every } (e,v) \in E \times \mathcal{V}\,.$$

It will be said that ψ belongs to the class $\mathscr{A}^*$ provided that the following additional three conditions hold: *(a)* the space E is finite-dimensional; *(b)* there exists a compact set $K \subseteq E_0$ such that, denoting $\widetilde{E} := E_- \oplus E_+$,

$[\mathbf{K_1}]$ all critical points of φ belong to $(K + \widetilde{E}) \times \mathcal{V}$,

$[\mathbf{K_2}]$ $\displaystyle\sup_{(K+\widetilde{E})\times\mathcal{V}} \psi < \ell - R_1^2/2,$
for some positive constants ℓ, R_1 satisfying $[\boldsymbol{\psi}_1]$ and (31), respectively;

and *(c):* there exists a continuous map $m : [0,1] \times E_0 \to E_0$ such that

$[\mathbf{m_1}]$ $m(0, e_0) = e_0, \qquad e_0 \in E_0,$

$[\mathbf{m_2}]$ $m(t, e_0) = e_0, \qquad t \in [0,1],\ e_0 \in K,$

$[\mathbf{m_3}]$ $m(1, E_0) = K,$

$[\mathbf{m_4}]$ $\psi(e_- + m(t, e_0) + e_+, v) \le \psi(e, v), \qquad t \in [0,1],\ e \in E,\ v \in \mathcal{V}.$

Summarizing, assumptions $[\mathbf{m_{1-4}}]$ require K to be a strong deformation retract of E_0; moreover, the sublevel sets of $\psi(\cdot + \tilde{e}, v)$ are kept invariant by m. In rough terms, membership to the class $\mathscr{A}^*$ can be described by saying that, in addition to the Hilbert space E being finite-dimensional, the saddle geometry of the functional is stressed. The main result of this subsection will be the following:

Proposition 7.6. *Let $\psi : \mathcal{M} \to \mathbb{R}$ be in the class $\mathscr{A}^*$; then the associated complemented functional φ has at least* $\mathrm{sb}(\mathcal{V})$ *critical points.*

Let us start the proof. Recalling assumption $[\mathbf{K_2}]$, it is possible to find a real number $b < \ell$ with

$$\sup\Big\{\psi(e,v) : (e,v) \in (K + \widetilde{E}) \times \mathcal{V}\Big\} + \frac{R_1^2}{2} < b\,.$$

In this way we have, by (18) and (31),

$$\sup\Big\{\varphi(e,v) : (e,v) \in (E_- + K + B_+) \times \mathcal{V}\Big\} < b\,, \tag{32}$$

where $B_+ := \overline{\mathbb{B}_{R_1}^{E_+}}$. We shall use this inequality later.

On the other hand, we recall (from [$\boldsymbol{\psi}_1$]) that ψ is bounded; hence we may find a constant $a \in \mathbb{R}$ such that

$$\varphi(e,v) > a\,, \quad \text{for every } e \in E \text{ with } \|e_-\| \le R_1 + 1\,, \text{ and } v \in \mathcal{V}\,. \tag{33}$$

Observe that $a < b < \ell$. On the other hand, φ satisfies the Palais–Smale condition $(\mathrm{PS})_c$, for every $c < \ell$ (remember Step 1 in Section 6); consequently (in view of Lemma 7.5), Proposition 7.6 will be proved if we show the following

Lemma 7.7. *Under the above,* $\mathrm{sb}(\varphi_b, \varphi_a) \ge \mathrm{sb}(\mathcal{V})$.

Proof. Using again the boundedness of ψ, we may find a constant $R_2 > R_1 + 1$, such that, setting $B_- := \overline{\mathbb{B}^{E_-}_{R_2}}$, one has

$$\varphi(e,v) < a\,, \quad \text{if } e \in (E_- \setminus B_-) + E_0 + B_+\,. \tag{34}$$

Let the homotopy $m_+ : [0,1] \times E_+ \to E_+$ be defined by

$$m_+(t, e_+) = \begin{cases} e_+\,, & \text{if } e_+ \in B_+\,, \\ (1-t)e_+ + tR_1 \dfrac{e_+}{\|e_+\|}\,, & \text{if } e_+ \in E_+ \setminus B_+\,, \end{cases}$$

and let the deformation $h_+ : [0,1] \times \mathcal{M} \to \mathcal{M}$ be given as

$$h_+\big(t, (e,v)\big) := \big(e_- + e_0 + m_+(t, e_+), v\big)\,.$$

We observe that, by (31),

$$\varphi\big(h_+(t,(e,v))\big) \le \varphi(e,v)\,, \quad \text{for every } (e,v) \in \mathcal{M} \text{ and } t \in [0,1]\,,$$

and hence the sublevels φ_a, φ_b are kept invariant by the deformation. Consequently, by Lemma 7.1,

$$H_*(\varphi_b, \varphi_a) \cong H_*\Big[\varphi_b \cap \big((E_- + E_0 + B_+) \times \mathcal{V}\big), \varphi_a \cap \big((E_- + E_0 + B_+) \times \mathcal{V}\big)\Big]\,. \tag{35}$$

On the other hand, remembering (33) and (34), we see that

$$\Big((\overline{E_- \setminus B_-}) + E_0 + B_+\Big) \times \mathcal{V} \subseteq \varphi_a \subseteq \Big((E_- \setminus \overline{\mathbb{B}^{E_-}_{R_1+1}}) + E_0 + E_+\Big) \times \mathcal{V}\,. \tag{36}$$

This time we consider the homotopy $m_- : [0,1] \times E_- \to E_-$ defined by

$$m_-(t, e_-) = \begin{cases} e_-\,, & \text{if } e_- \in B_-\,, \\ (1-t)e_- + tR_2 \dfrac{e_-}{\|e_-\|}\,, & \text{if } e_- \in E_- \setminus B_-\,. \end{cases}$$

In view of the first inclusion of (36) we see that the deformation

$$h_- : [0,1] \times \mathcal{M} \to \mathcal{M}\,, \qquad \big(t, (e,v)\big) \mapsto \big(m_-(t, e_-) + e_0 + e_+, v\big)\,,$$

keeps invariant both sets $\varphi_b \cap \big((E_- + E_0 + B_+) \times \mathcal{V}\big)$ and $\varphi_a \cap \big((E_- + E_0 + B_+) \times \mathcal{V}\big)$. Hence, by Lemma 7.1, expression (35) can be simplified to

$$H_*(\varphi_b, \varphi_a) \cong H_*(X, Y)\,, \tag{37}$$

where $X = \varphi_b \cap \big((B_- + E_0 + B_+) \times \mathcal{V}\big)$, and $Y = \varphi_a \cap \big((B_- + E_0 + B_+) \times \mathcal{V}\big)$.

Let us consider now the piecewise linear function $u :]0,+\infty[\to]0,+\infty[$ given by

$$u(\rho) := \begin{cases} \rho\,, & \text{if } \rho \in]0,R_1] \cup [R_2,+\infty[\,, \\ R_1(R_1+1-\rho) + R_2(\rho - R_1)\,, & \text{if } \rho \in [R_1, R_1+1]\,, \\ R_2\,, & \text{if } \rho \in [R_1+1, R_2]\,, \end{cases}$$

and the homotopy $n : [0,1] \times E_- \to E_-$ given by

$$n(t,e_-) := \begin{cases} (1-t)e_- + t\,\dfrac{u(\|e_-\|)}{\|e_-\|}\,e_-\,, & \text{if } e_- \neq 0\,, \\ 0\,, & \text{if } e_- = 0\,. \end{cases}$$

We use this map to construct the deformation

$$\eta : [0,1] \times \mathcal{M} \to \mathcal{M}\,, \qquad \eta(t,(e,v)) = (n(t,e_-) + e_0 + e_+, v)\,,$$

and observe that, by (31), both sets X, Y are kept invariant by η. Moreover, the second inclusion of (36) gives

$$\eta(1,Y) = ((\partial B_-) + E_0 + B_+) \times \mathcal{V} =: Z\,,$$

and the set Z is also kept invariant by the deformation η. Remembering (37) and applying Lemma 7.4, we obtain that

$$H_*(\varphi_b, \varphi_a) \cong H_*(X,Z)\,.$$

Finally, let $m : [0,1] \times E_0 \to E_0$ be given by the fact that ψ belongs to the class $\mathscr{A}^*$. We consider the associated homotopy

$$h : [0,1] \times \mathcal{M} \to \mathcal{M}\,, \qquad \big(t,(e,v)\big) \mapsto \big(e_- + m(t,e_0) + e_+, v\big)\,.$$

It follows from [$\mathbf{m_4}$] that both sets X, Z are kept invariant by h. Moreover, (32) implies that $X \supseteq (B_- + K + B_+) \times \mathcal{V}$, and hence

$$h(1,X) = (B_- + K + B_+) \times \mathcal{V}\,, \qquad h(1,Y) = ((\partial B_-) + K + B_+) \times \mathcal{V}\,,$$

and we finally deduce that

$$H_*(\varphi_b, \varphi_a) \cong H_*((B_- + K + B_+) \times \mathcal{V}\,, ((\partial B_-) + K + B_+) \times \mathcal{V})\,.$$

Since B_+ is contractible, a deformation argument gives

$$H_*(\varphi_b, \varphi_a) \cong H_*((B_- + K) \times \mathcal{V}\,, ((\partial B_-) + K) \times \mathcal{V})\,.$$

By the Künneth formula (Lemma 7.2), for $n = 0,1,2,\ldots$ we have

$$H_n(B_- \times K \times \mathcal{V}, (\partial B_-) \times K \times \mathcal{V}) \cong \bigoplus_{i+j=n} [H_i(B_- \times \mathcal{V}, (\partial B_-) \times \mathcal{V}) \otimes H_j(K)]\,,$$

and hence, the corresponding Betti numbers satisfy

$$\dim H_n((B_- + K) \times \mathcal{V}\,, ((\partial B_-) + K) \times \mathcal{V}) \geq \dim H_n(B_- \times \mathcal{V}, (\partial B_-) \times \mathcal{V}).$$

Using again the Künneth formula (and combining it with Lemma 7.3), we get

$$\begin{aligned} H_n(B_- \times \mathcal{V}, (\partial B_-) \times \mathcal{V}) &\cong \bigoplus_{i+j=n} [H_i(B_-, \partial B_-) \otimes H_j(\mathcal{V})] \\ &\cong \mathbb{R} \otimes H_{n-r}(\mathcal{V}) \cong H_{n-r}(\mathcal{V})\,, \end{aligned}$$

where $r = \dim E_-$. We conclude that $\dim H_n(\varphi_b, \varphi_a) \geq \dim H_{n-r}(\mathcal{V})$, for any $n = r, r+1, r+2, \ldots$, and the result follows. □

7.3 From the class $\mathscr{A}^+$ to the class $\mathscr{A}^*$

The aim of this subsection is to obtain Theorem 5.1***(b)*** from Proposition 7.6. Our argument will be divided in two steps. Firstly, we shall use a Liapunov–Schmidt reduction procedure to replace φ by a new functional $\widehat{\varphi}$. It will still be the complemented of a functional in the class $\mathscr{A}^+$, and will also have the same number of critical points, but its domain will be $\mathcal{M}^* := F \times \mathcal{V}$, the subspace $F \subseteq E$ being finite-dimensional. In the second step we shall modify our functional to the complemented of a functional in the class $\mathscr{A}^*$ (so that Proposition 7.6 applies). Again, while performing this procedure we shall be careful to keep the number of critical points.

*First Step: It suffices to check Theorem 5.1**(b)** when the space E is finite-dimensional.*

To show this statement let us go back to the framework of Theorem 5.1***(b)*** and assume that $\psi : \mathcal{M} \to \mathbb{R}$ is in the class $\mathscr{A}^+$. We shall start with the following remark:

Lemma 7.8. *The set*

$$\mathfrak{K} := \Big\{ \mathrm{Hess}_E \psi(e, v)\mathfrak{e} : (e, v) \in \mathcal{M},\ \mathfrak{e} \in E,\ \|\mathfrak{e}\| \leq 1 \Big\}\,,$$

is relatively compact in E.

Proof. Using a contradiction argument, assume instead the existence of $\epsilon_0 > 0$ and sequences $\{(e^n, v^n)\}_n \subseteq \mathcal{M}$, $\{\mathfrak{e}^n\}_n \subseteq E$, such that $\|\mathfrak{e}^n\| \leq 1$ for every n, and

$$\big\| \mathrm{Hess}_E \psi(e^n, v^n)\mathfrak{e}^n - \mathrm{Hess}_E \psi(e^m, v^m)\mathfrak{e}^m \big\| \geq \epsilon_0\,, \quad \text{for } m \neq n\,.$$

Combining assumption $[\boldsymbol{\psi}_2]$ with the fact that the differential at any point of a completely continuous map is compact [35, Theorem 1.40, p. 27], we see that the linear map $\mathrm{Hess}_E \psi(e^n, v^n) \in \mathscr{L}(E)$ is compact, for every n. On the other hand, $[\boldsymbol{\psi}_4]$ states that, after possibly passing to a subsequence, we

may assume that $\{\mathrm{Hess}_E\psi(e^n, v^n)\}_n$ converges in $\mathscr{L}(E)$, and well-known arguments show that its limit T must again be a compact operator, see, e.g. [15, Proposition 4.2(b)]. However, for $n \neq m$ big enough one has:

$$\|T\mathfrak{e}^n - T\mathfrak{e}^m\| \geq \epsilon_0/2\,,$$

and hence the sequence $\{T\mathfrak{e}^n\}_n$ has no convergent subsequences. This is a contradiction, and concludes the proof. □

We begin by observing that the critical points of φ are exactly the solutions $(e, v) \in E \times \mathcal{V}$ of the system

$$(S)\quad \begin{cases} Le + \nabla_E\psi(e, v) = 0\,, \\ \nabla_{\mathcal{V}}\psi(e, v) = 0\,. \end{cases}$$

The Hilbert space E was assumed separable, and hence, there are increasing sequences $\{F^n_\pm\}_n \subseteq E_\pm$ of finite-dimensional subspaces with dense union in $E_\pm$. Choose some index $n \in \mathbb{N}$, to be fixed later, and set $F_\pm := F^n_\pm$, $F := F_+ \oplus E_0 \oplus F_-$. In view of (18), $L(F) \subseteq F$. Rewriting the points of E in the form $e = f + g$, where $f \in F$ and $g \in F^\perp$, and denoting by $\pi : E \to F$, $f + g \mapsto f$ the orthogonal projection, the first equation of system (S) splits as

$$\begin{cases} Lf + \pi[\nabla_E\psi(f + g, v)] = 0\,, \\ Lg + (\mathrm{Id} - \pi)[\nabla_E\psi(f + g, v)] = 0\,. \end{cases} \tag{38}$$

If n is big, the last equation above can be solved in the variable g. We check this below:

Lemma 7.9. *Assume that n has been chosen large enough. Then, there exists a C^1-smooth map $G : \mathcal{M}^* := F \times \mathcal{V} \to F^\perp$such that, if $f \in F$, $v \in \mathcal{V}$, and $g \in F^\perp$,*

$$Lg + (\mathrm{Id} - \pi)[\nabla_E\psi(f + g, v)] = 0 \quad \Leftrightarrow \quad g = G(f, v)\,. \tag{39}$$

Moreover, $G(e_0, v) = 0$ for any $e_0 \in E_0$ with $\|e_0\| \geq R$, and both G and its partial differential $G'_F : \mathcal{M}^* \to \mathscr{L}(F, F^\perp)$ are globally bounded, and satisfy

$$\lim_{\substack{\|e_0\|\to\infty \\ e_0\in E_0}} G(e_0 + b, v) = 0\,, \qquad \lim_{\substack{\|e_0\|\to\infty \\ e_0\in E_0}} G'_F(e_0 + b, v) = 0\,, \tag{40}$$

uniformly with respect to b belonging to bounded subsets of F, and $v \in \mathcal{V}$.

Proof. By assumption $[\boldsymbol{\psi}_4]$, the second-order derivative of ψ is bounded, and hence, $\nabla_E\psi : \mathcal{M} \to E$ is Lipschitz continuous in its first variable. Let $\alpha > 0$ be an associated Lipschitz constant. In view of Lemma 7.8, the set $\mathfrak{K}$ defined there

is relatively compact, and hence there exists a finite set of points $e_1, \dots, e_p$ in E such that $\bigcup_{j=1}^p \mathbb{B}^E_{1/(4\alpha)}(e_j) \supseteq \mathfrak{K}$. On the other hand, the set

$$\bigcup_{n=1}^{\infty} F^n = \Big(\bigcup_{n=1}^{\infty} F^n_-\Big) \oplus E_0 \oplus \Big(\bigcup_{n=1}^{\infty} F^n_+\Big)$$

is dense in E; therefore, we can find some $n_0 \in \mathbb{N}$ and points $f_1, \dots, f_p$ in F^{n_0} such that $\|f_i - e_i\| < 1/(4\alpha)$. It follows that $\bigcup_{j=1}^p \mathbb{B}^E_{1/(2\alpha)}(f_j) \supseteq \mathfrak{K}$.

Fix now some $n \geq n_0$. Then, $\|(\mathrm{Id} - \pi)e\| \leq 1/(2\alpha)$, for every $e \in \mathfrak{K}$ and, consequently, for any $f \in F$ and $v \in \mathcal{V}$, the map $g \mapsto (\mathrm{Id} - \pi)[\nabla_E \psi(f+g, v)]$ is Lipschitz continuous on $F^\perp$, with associated Lipschitz constant $\alpha/(2\alpha) = 1/2$. Since, on the other hand, $L^{-1} : \widetilde{E} \to \widetilde{E}$ is Lipschitz continuous with Lipschitz constant 1 (by (18)), we see that, for any $f \in F^\perp$ and $v \in \mathcal{V}$, the map

$$g \mapsto L^{-1}(\mathrm{Id} - \pi)[\nabla_E \psi(f + g, v)]$$

is a contraction on $F^\perp$. Thus, the Banach Contraction Theorem ensures that it has a unique fixed point $g = G(f, v)$ on $F^\perp$, which is the unique solution of the second equation in (38). And the implicit function theorem ensures that G is C^1-smooth. The remaining statements on G follow easily from its definition and assumptions $[\boldsymbol{\psi}_{2-4}]$. □

Let us consider the functionals $\widehat{\psi}, \widehat{\varphi} : \mathcal{M}^* \to \mathbb{R}$ defined by

$$\widehat{\psi}(f, v) := \tfrac{1}{2}\big\langle LG(f, v), G(f, v)\big\rangle + \psi(f + G(f, v), v)\,,$$
$$\widehat{\varphi}(f, v) := \tfrac{1}{2}\big\langle Lf, f\big\rangle + \widehat{\psi}(f, v) = \varphi\big(f + G(f, v), v\big)\,.$$

We consider the map $\Upsilon : \mathcal{M}^* \to \mathcal{M}$ defined by $\Upsilon(f, v) := \big(f + G(f, v), v\big)$. Straightforward computations, combined with (39), give

$$\nabla_F \widehat{\psi} = \pi \circ (\nabla_E \psi) \circ \Upsilon\,, \qquad \nabla_{\mathcal{V}} \widehat{\psi} = (\nabla_{\mathcal{V}} \psi) \circ \Upsilon\,,$$

so that $\widehat{\psi}$ (and consequently, also $\widehat{\varphi}$) is actually a C^2-smooth functional. Furthermore,

$$\begin{cases} \nabla_{\mathcal{M}^*} \widehat{\varphi}(f, v) = \nabla_{\mathcal{M}} \varphi\big(\Upsilon(f, v)\big), \\ \mathrm{Hess}_F \widehat{\varphi}(f, v) = \mathrm{Hess}_E \varphi\big(\Upsilon(f, v)\big) \circ \big[\mathrm{Id}_F + G'_F(f, v)\big]\,, \end{cases}$$

for every $(f, v) \in \mathcal{M}^*$. It is now easy to check that $\widehat{\psi}$ satisfies assumptions $[\boldsymbol{\psi}_{1-4}]$ and the critical points of $\widehat{\varphi}$ are in a one-to-one correspondence (given by Υ) with the critical points of φ. Finally,

$$\mathrm{Hess}_{\mathcal{M}^*} \widehat{\varphi}(f, v) = \mathrm{Hess}_{\mathcal{M}} \varphi\big(\Upsilon(f, v)\big) \circ \Upsilon'(f, v)\,,$$

at every critical point (f, v) of $\widehat{\varphi}$. Thus, $\widehat{\varphi}$ satisfies also $[\boldsymbol{\psi}_5]$, concluding the discussion of the first step.

Second Step: Intensifying the saddle geometry.

Having in mind the First Step, in order to conclude the proof of Theorem 5.1 we shall assume that $\dim E < +\infty$, and the functional $\psi : \mathcal{M} = E \times \mathcal{V} \to \mathbb{R}$ belongs to the class $\mathscr{A}^+$. As we already said before, without loss of generality we shall only consider the case $\psi(e,v) < \ell$ for every $(e,v) \in \mathcal{M}$. Moreover, we may assume that (18) holds, and the complemented functional $\varphi : \mathcal{M} \to \mathbb{R}$ has finitely many critical points, otherwise there is nothing to prove.

Lemma 7.10. *Under the above, there exists a functional $\psi^* : \mathcal{M} \to \mathbb{R}$ in the class $\mathscr{A}^*$ such that all critical points of the complemented functional φ^* are critical points of φ, and vice-versa.*

Proof. By (31), there is a number $R_1 > 0$ such that

$$\nabla_{\widetilde{E}}\varphi(e,v) = Le + \nabla_{\widetilde{E}}\psi(e,v) \neq 0\,, \quad \text{if } \max\{\|e_-\|, \|e_+\|\} \geq R_1\,. \tag{41}$$

On the other hand, in view of assumption $[\boldsymbol{\psi}_3]$ and the comments preceding this lemma, there is some $R > 0$ such that, for $e_0 \in E_0$,

$$0 \neq \nabla_{\mathcal{M}}\varphi(e_0,v) = \nabla_{\mathcal{M}}\psi(e_0,v) \in E_0\,, \quad \text{if } \|e_0\| \geq R\,. \tag{42}$$

In particular, $\nabla_{\mathcal{V}}\psi(e_0,v) = 0$, and $\psi(e_0,v) = h(e_0)$ does not depend on $v \in \mathcal{V}$ if $\|e_0\| \geq R$. The function $h : E_0 \setminus \mathbb{B}_R^{E_0} \to \mathbb{R}$ defined in this way is C^2-smooth and, by $[\boldsymbol{\psi}_{1,2,4}]$ and (42), satisfies

$$\begin{cases} h(e_0) < \ell, \quad \nabla h(e_0) \neq 0\,, \quad \text{for any } e_0 \in E_0 \setminus \mathbb{B}_R^{E_0}\,, \\ \lim\limits_{\|e_0\|\to\infty} h(e_0) = \ell\,, \quad \lim\limits_{\|e_0\|\to\infty} \nabla h(e_0) = 0\,, \quad \lim\limits_{\|e_0\|\to\infty} \operatorname{Hess} h(e_0) = 0\,. \end{cases} \tag{43}$$

Since, in view of (42), one has that $\nabla_{\widetilde{E}}\psi(e_0,v) = 0$ for any $e_0 \in E_0$ with $\|e_0\| \geq R$, the triangle inequality gives

$$\begin{aligned} \|\nabla_{\widetilde{E}}\varphi(e,v)\| &= \|Le + \nabla_{\widetilde{E}}\psi(e,v) - \nabla_{\widetilde{E}}\psi(e_0,v)\| \\ &\geq \|\tilde{e}\| - \|\nabla_{\widetilde{E}}\psi(e_0+\tilde{e},v) - \nabla_{\widetilde{E}}\psi(e_0,v)\|\,. \end{aligned} \tag{44}$$

(We used (18) to obtain $\|Le\| = \|\tilde{e}\|$.) Remembering $[\boldsymbol{\psi}_4]$, we may replace R by a bigger constant so that

$$\|\mathrm{Hess}_E\psi(e,v)\| < \tfrac{1}{2}\,, \quad \text{if } \|e_0\| \geq R \text{ and } \|e_\pm\| \leq R_1\,.$$

Hence, for $\|e_0\| \geq R$ the map $\tilde{e} \mapsto \nabla_{\widetilde{E}}\psi(e_0+\tilde{e},v)$ is contractive on $\overline{\mathbb{B}_{R_1}^{E_-}} + \overline{\mathbb{B}_{R_1}^{E_+}}$, and (44) gives

$$\nabla_{\widetilde{E}}\varphi(e,v) \neq 0\,, \text{ if } \|e_0\| \geq R \text{ and } 0 < \max\{\|e_-\|, \|e_+\|\} \leq R_1\,,$$

which, when combined with (41) means that

$$\nabla_{\widetilde{E}}\varphi(e,v) \neq 0\,, \text{ if } \|e_0\| \geq R \text{ and } \tilde{e} \neq 0\,. \tag{45}$$

In particular, (42) and (45) imply that φ has no critical points (e,v) with $\|e_0\| \geq R$.

Choose numbers $\alpha < \beta < \ell$, $R' > R$ such that

$$h(e_0) < \alpha\,, \ \text{ if } \|e_0\| = R\,, \qquad h(e_0) > \alpha\,, \ \text{ if } \|e_0\| \geq R'\,,$$

and

$$h(e_0) < \beta\,, \ \text{ if } \|e_0\| \leq R'\,.$$

Pick also some C^2-smooth functions $n, \omega : \mathbb{R} \to \mathbb{R}$, with

$$n(\rho) = \begin{cases} 1\,, & \text{if } \rho \leq R'\,, \\ 0\,, & \text{if } \rho \geq R'+1\,, \end{cases} \qquad n'(\rho) < 0\,, \ \text{ if } R' < \rho < R'+1\,,$$

and

$$\omega(t) = 0\,, \ \text{ if } t < \alpha\,, \qquad \omega'(t) > 0\,, \ \text{ if } t > \alpha\,.$$

Define, for $\lambda > 0$,

$$\psi^*_\lambda(e,v) := \begin{cases} \psi(e,v)\,, & \text{if } \|e_0\| < R\,, \\ n(\|e_0\|)\,\psi(e,v) + \lambda\omega(h(e_0))\,, & \text{if } \|e_0\| \geq R\,. \end{cases}$$

For λ big enough, one has

$$\ell + \lambda \max\{\omega(h(e_0)) : R \leq \|e_0\| \leq R'+1\} < \lambda\omega(\ell)\,,$$

and one easily checks that ψ^*_λ then satisfies assumptions $[\boldsymbol{\psi}_{1-4}]$, the new limit along E_0 being $\lambda\omega(\ell)$. The associated complemented functional φ^*_λ coincides with φ as long as $\|e_0\| \leq R$, and satisfies

$$\begin{cases} \nabla_{E_0}\varphi^*_\lambda(e_0,v) = \big(1 + \lambda\omega'(h(e_0))\big)\nabla h(e_0) \neq 0\,, & \text{if } R \leq \|e_0\| \leq R'\,, \\ \nabla_{\widetilde{E}}\varphi^*_\lambda(e,v) = \nabla_{\widetilde{E}}\varphi(e,v) \neq 0\,, & \text{if } R \leq \|e_0\| \leq R'\,,\ \tilde{e} \neq 0\,, \end{cases}$$

by (43) and (45), respectively. Hence,

$$\nabla_{\mathcal{M}}\varphi^*_\lambda(e,v) \neq 0\,, \ \text{ if } R \leq \|e_0\| \leq R'\,. \tag{46}$$

Since n and ψ are bounded with bounded differentials, while ∇h is bounded away from zero on $\overline{\mathbb{B}^{E_0}_{R'+1}} \setminus \mathbb{B}^{E_0}_{R'}$, we can take a larger constant λ so that

$$\langle \nabla_{E_0}\varphi^*_\lambda(e,v), \nabla h(e_0)\rangle = \langle \nabla_{E_0}\psi^*_\lambda(e,v), \nabla h(e_0)\rangle > 0\,, \ \text{ if } \|e_0\| \geq R'\,.$$

We set $\psi^* := \psi^*_\lambda$. Combining the inequality above with (46), we see that $\varphi^* := \varphi^*_\lambda$ has no critical points (e,v) with $\|e_0\| \geq R$.

Two important consequences follow: on the one hand, all critical points of φ^* are indeed critical points of φ, and vice-versa; hence, $[\boldsymbol{\psi}_5]$ also holds for φ^*, so that ψ^* belongs to the class $\mathscr{A}^+$; on the other hand, setting

$$K := \mathbb{B}^{E_0}_R \cup \{e_0 \in E_0 \setminus \mathbb{B}^{E_0}_R : h(e_0) \leq \beta\}\,,$$

we see that $[\mathbf{K_1}]$ holds for φ^*. Moreover, after possibly replacing λ by a bigger number, we see that

$$\sup_{(K+\widetilde{E})\times\mathcal{V}} \psi^*_\lambda < \lambda\omega(\ell) - \frac{R_1^2}{2}\,,$$

and hence, also assumption $[\mathbf{K_2}]$ holds for ψ^*. Now, the homotopy $m : [0,1] \times E_0 \to E_0$ satisfying $[\mathbf{m_{1-4}}]$ can be built as follows: we keep all the points of K fixed, while, if $e_0 \in E_0 \setminus K$, then $m(\cdot, e_0)$ is the curve, starting from $m(0, e_0) = e_0$, which follows backwards the flow lines of ∇h, and arrives at the point $m(1, e_0)$ where the flow first meets K. (Notice that this flow is transversal to the boundary of K.) Hence, ψ^* belongs to the class $\mathscr{A}^*$. It completes the proof. $\square$

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