Asymmetric Poincaré Inequalities and Solvability of Capillarity Problems

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1 Introduction

The aim of this work is twofold: on the one hand we establish an asymmetric version of the Poincaré inequality in the space of bounded variation functions, on the other hand, based on this result, we prove the existence of bounded variation solutions of a class of capillarity problems with asymmetric perturbations.

Let Ω be a bounded domain in $\mathbb{R}^N$, with a Lipschitz boundary $\partial\Omega$. The classical Poincaré inequality in $BV(\Omega)$ (see, e.g., [2, Theorem 3.44]) asserts that there exists a constant $c = c(\Omega) > 0$ such that every $u \in BV(\Omega)$, with $\int_\Omega u\,dx = 0$, satisfies

$$c \int_\Omega |u|\,dx \le \int_\Omega |Du|. \tag{1}$$

Here and in the sequel $\int_\Omega |Du|$ denotes the total variation of u. The largest constant $c = c(\Omega)$ for which (1) holds is called the Poincaré constant and is variationally characterized by

$$c = \min\Big\{ \int_\Omega |Dv| : v \in BV(\Omega), \int_\Omega v\,dx = 0, \int_\Omega |v|\,dx = 1 \Big\}. \tag{2}$$

Clearly, any minimizer yields the equality in (1).

We prove here an asymmetric counterpart of the Poincaré inequality (1), where u^+ and u^- weigh differently, i.e., the ratio $r = \frac{\int_\Omega u^+\,dx}{\int_\Omega u^-\,dx}$ is not necessarily 1. Namely, we show that for each $r > 0$ there exist constants $\mu = \mu(r,\Omega) > 0$ and $\nu = \nu(r,\Omega) > 0$, with $\nu = r\mu$, such that every $u \in BV(\Omega)$, with $\mu \int_\Omega u^+\,dx - \nu \int_\Omega u^-\,dx = 0$, satisfies

$$\mu \int_\Omega u^+\,dx + \nu \int_\Omega u^-\,dx \le \int_\Omega |Du|. \tag{3}$$

The constants μ and ν are variationally characterized by

$$\begin{aligned} \mu = \min\Big\{ \int_\Omega |Dv| : v \in BV(\Omega),\ & \int_\Omega v^+ dx - r\int_\Omega v^- dx = 0, \\ & \int_\Omega v^+ dx + r\int_\Omega v^- dx = 1 \Big\} \end{aligned} \tag{4}$$

and

$$\begin{aligned} \nu = \min\Big\{ \int_\Omega |Dv| : v \in BV(\Omega),\ & r^{-1}\int_\Omega v^+ dx - \int_\Omega v^- dx = 0, \\ & r^{-1}\int_\Omega v^+ dx + \int_\Omega v^- dx = 1 \Big\}, \end{aligned} \tag{5}$$

respectively. Clearly, any minimizer in (4), or (5), yields the equality in (3).

This construction allows us to single out in the plane a curve $\mathcal{C} = \mathcal{C}(\Omega)$ made up of all pairs $(\mu,\nu) = (\mu(r,\Omega), \nu(r,\Omega))$ defined by (4) and (5), by letting r vary in $\mathbb{R}_0^+$. From formulas (4) and (5) several properties of the curve $\mathcal{C}$ can be derived:

(i) $\mathcal{C}$ is symmetric with respect to the diagonal;

(ii) $\mathcal{C}$ is continuous, as both functions $\mu(r)$ and $\nu(r)$ are continuous;

(iii) $\mathcal{C}$ is strictly decreasing, in the sense that $\mu(r)$ is strictly decreasing and $\nu(r)$ is strictly increasing;

(iv) if $N \geq 2$, $\mathcal{C}$ is asymptotic to the lines $\mu = 0$ and $\nu = 0$, that is $\lim_{r\to 0^+} \mu(r) = +\infty$ and $\lim_{r\to 0^+} \nu(r) = 0$;

(v) if $N = 1$, we have that $(\mu, \nu) \in \mathcal{C}$ if and only if

$$\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \sqrt{2|\Omega|},$$

where $|\Omega|$ is the measure of the interval Ω; in particular, $\mathcal{C}$ is asymptotic to the lines $\mu = \frac{1}{2|\Omega|}$ and $\nu = \frac{1}{2|\Omega|}$.

The discrepancy occurring in the asymptotic behaviour of $\mathcal{C}$ between the cases $N = 1$ and $N \geq 2$ is due to the existence in higher dimension of functions having arbitrarily large oscillation and arbitrarily small variation.

We point out that the properties of the curve $\mathcal{C}$ resemble those of the first non-trivial curve of the Fučik spectrum of the p-Laplace operator, with $p > 1$, as described in [11, 10, 12, 9].

Once these results have been established – conclusions (iv) and (v) being the crucial ones – we use them to discuss the solvability of the capillarity problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1+|\nabla u|^2}\Big) = f(x,u) & \text{in } \Omega, \\ -\nabla u \cdot n/\sqrt{1+|\nabla u|^2} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

Here, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and n is the unit outer normal to $\partial\Omega$. Since our approach will be variational the natural context where this problem has to be settled is the space $BV(\Omega)$ of bounded variation functions (see, e.g., [13, 19, 17]). Solutions of (6) are defined as subcritical points of the action functional

$$\int_\Omega \sqrt{1+|Dv|^2} - \int_\Omega F(x,v)\,dx,$$

where

$$\int_\Omega \sqrt{1+|Dv|^2} = \int_\Omega \sqrt{1+|(Dv)^a|^2}\,dx + \int_\Omega |(Dv)^s|, \tag{7}$$

$Dv = (Dv)^a\,dx + (Dv)^s$ being the decomposition of the measure Dv in its absolutely continuous and singular parts with respect to the N-dimensional Lebesgue measure, and

$$F(x,s) = \int_0^s f(x,\xi)\,d\xi.$$

As in [33] we say that $u \in BV(\Omega)$ is a solution of (6) if

$$\int_\Omega \sqrt{1+|Dv|^2} \geq \int_\Omega \sqrt{1+|Du|^2} + \int_\Omega f(x,u)(v-u)\,dx, \tag{8}$$

for all $v \in BV(\Omega)$, that is, u minimizes in $BV(\Omega)$ the functional $v \mapsto \int_\Omega \sqrt{1+|Dv|^2} - \int_\Omega f(x,u)v\,dx$.

In order to start our discussion, let us consider for a while the simpler problem where the function f at the right-hand side of the equation in (6) does not depend on u, that is

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1+|\nabla u|^2}\Big) = f(x) & \text{in } \Omega, \\ -\nabla u \cdot n/\sqrt{1+|\nabla u|^2} = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

with $f \in L^\infty(\Omega)$ given. It is easy to see that (6) may have a solution only if $\int_\Omega f\,dx = 0$. A simple minimization argument based on the classical Poincaré inequality (1) shows that, assuming $\int_\Omega f\,dx = 0$, (9) has a solution if $\|f\|_{L^\infty} < c$ and it may have no solution if $\|f\|_{L^\infty} > c$. If we write $f = f^+ - f^-$, then the condition $\int_\Omega f\,dx = 0$ reads $\int_\Omega f^+\,dx = \int_\Omega f^-\,dx$ and the condition $\|f\|_{L^\infty} < c$ can be expressed by requiring that both $\operatorname*{ess\,sup}_\Omega f^+ < c$ and $\operatorname*{ess\,sup}_\Omega f^- < c$. Looking at elementary one-dimensional examples, where f is a piecewise constant function, one is led to guess that the existence of a solution of (9) can be still guaranteed even though $\operatorname*{ess\,sup}_\Omega f^+$ is large, provided that $\operatorname*{ess\,sup}_\Omega f^-$ is sufficiently small, allowing in this way asymmetric perturbations f. Indeed, let us denote by $\mathcal{A}$ the component of $(\mathbb{R}_0^+ \times \mathbb{R}_0^+) \setminus \mathcal{C}$ lying "below" $\mathcal{C}$ and by $\mathcal{B}$ the component lying "above" $\mathcal{C}$. Then, basically relying on the asymmetric Poincaré inequality (3), we prove that, assuming $\int_\Omega f\,dx = 0$ and setting $\operatorname*{ess\,sup}_\Omega f^+ = \mu$ and $\operatorname*{ess\,sup}_\Omega f^- = \nu$, problem (9) has a solution if $(\mu,\nu) \in \mathcal{A}$ and it may have no solution if $(\mu,\nu) \in \mathcal{B}$. Keeping in mind these facts we perform the study of the more general problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1+|\nabla u|^2}\Big) = f(x,u) + h(x) & \text{in } \Omega, \\ -\nabla u \cdot n/\sqrt{1+|\nabla u|^2} = \kappa(x) & \text{on } \partial\Omega. \end{cases} \tag{10}$$

We mention that this problem classically occurs in the study of capillary surfaces, of the equilibrium configurations of sessile or pendent drops [16]; more recently it has been observed that it plays some role in the theory of reaction-diffusion phenomena that exhibit saturation of the flux at large values of the gradient [37].

This paper is organized as follows. Section 2 is devoted to the construction and the study of the above mentioned properties of the curve $\mathcal{C}$. This curve is here defined for a more general functional than the total variation, namely for the functional $\mathcal{L} : BV(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{L}(u) = \int_\Omega |Du| - \int_\Omega hu\,dx + \int_{\partial\Omega} \kappa u\,d\mathcal{H}_{N-1},$$

which naturally arises in order to discuss the solvability of the problem (10). Similarly as above solutions of (10) are the subcritical points of the functional $\mathcal{I} : BV(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{I}(u) = \mathcal{L}(u) - \int_\Omega F(x,u)\,dx.$$

In Section 3 we discuss the solvability of problem (10). We first establish various technical tools that will be extensively used in the sequel. In particular, we prove some coercivity results for the functional $\mathcal{L}$ on suitable cones of $BV(\Omega)$ and we state a non-smooth version of the classical mountain pass lemma, without the Palais-Smale condition and adapted to the BV setting, inspired from [26, 22, 24, 32].

Second, we prove some simple non-existence results, which will justify the assumptions we are going to place later on the function f in order to achieve solvability of problem (10). In particular, we show that there exist functions $e \in L^\infty(\Omega)$, with $\int_\Omega e\,dx = 0$, and $g : \mathbb{R} \to \mathbb{R}$ continuous, bounded and strictly monotone, with

$$\lim_{s\to-\infty} g(s) < 0 < \lim_{s\to+\infty} g(s),$$

or

$$\lim_{s\to-\infty} g(s) > 0 > \lim_{s\to+\infty} g(s),$$

such that problem (6), with $f(x,s) = g(s) + e(x)$, has no solution. This means that results in the spirit of [23, 1] do not immediately carry over to this context.

Third, we prove several existence results for problem (10). To this aim we introduce in the study of problem (10) the conditions

$$\lim_{s\to\pm\infty} \int_\Omega F(x,s)\,dx = +\infty, \tag{11}$$

or

$$\lim_{s\to\pm\infty} \int_\Omega F(x,s)\,dx = -\infty. \tag{12}$$

In the frame of semilinear problems these assumptions are usually referred to as Ahmad-Lazer-Paul conditions after the seminal paper [1]. We couple these assumptions with the asymmetric two-sided restriction

$$\operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} f^+(x,s) < \mu \quad \text{and} \quad \operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} f^-(x,s) < \nu, \tag{13}$$

for some $(\mu,\nu) \in \mathcal{C}$. Assumptions (11) and (13) yield a mountain pass geometry for the functional $\mathcal{I}$; whereas assumptions (12) and (13) imply the boundedness from below and the coercivity of $\mathcal{I}$. Then elementary tools of non-smooth critical point theory ensure the existence of a solution of problem (10). It is worthwhile to observe that the above cited non-existence results show that, unlike in the semilinear case, condition (12), which requires that f lies in some sense to the left of the first eigenvalue $\lambda_1 = 0$, and the boundedness of f, with bounds unrelated to $\mathcal{C}$, do not guarantee solvability. Next

we replace the Ahmad-Lazer-Paul condition (12) with the stronger Hammerstein-type condition (cf. [21, 27]): there exists $\zeta \in L^1(\Omega)$, with $\zeta(x) \leq 0$ for a.e. $x \in \Omega$ and $\zeta(x) < 0$ on a set of positive measure, such that

$$\limsup_{s\to\pm\infty} \frac{F(x,s)}{|s|} \leq \zeta(x) \quad \text{uniformly a.e. in } \Omega.$$

Then we show that this condition yields the existence of a solution of (10), without any further assumption on f. We just notice that the Ahmad-Lazer-Paul condition (11) is implied by a Hammerstein-type condition assumed to the right of λ_1: there exists $\zeta \in L^1(\Omega)$, with $\zeta(x) \geq 0$ for a.e. $x \in \Omega$ and $\zeta(x) > 0$ on a set of positive measure, such that

$$\liminf_{s\to\pm\infty} \frac{F(x,s)}{|s|} \geq \zeta(x) \quad \text{uniformly a.e. in } \Omega.$$

However, in this case, assumption (13) cannot be dropped.

Fourth, we show that, in the case of dimension $N = 1$, we can replace the two-sided condition (13) with one of the following one-sided restrictions

$$\operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} f^+(x,s) < \tfrac{1}{2|\Omega|}, \quad \text{or} \quad \operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} f^-(x,s) < \tfrac{1}{2|\Omega|},$$

thus allowing f to be unbounded from below or from above, respectively. In Section 3 we also study the existence of multiple solutions of problem (10), when $\int_\Omega F(x,s)\,dx$ exhibits an oscillatory behaviour. In particular, we show that infinitely many solutions exist assuming, in addition to (13), the conditions

$$\liminf_{s\to\pm\infty} \int_\Omega F(x,s)\,dx = -\infty \quad \text{and} \quad \limsup_{s\to\pm\infty} \int_\Omega F(x,s)\,dx = +\infty.$$

Multiplicity results, under oscillatory conditions on the potential F, have been already considered for the periodic problem associated with the equation

$$-\Big(u'/\sqrt{1+u'^2}\Big)' = f(x,u),$$

as well as for other boundary value problems associated with the equation

$$-\Delta_p u = f(x,u),$$

where Δ_p is the p-Laplace operator, with $p > 1$ (see, e.g., [15, 36, 20, 29, 30] and the references therein).

We conclude this introduction mentioning that we cannot in general expect that bounded variation solutions of (10) are more regular; indeed, even simple one-dimensional examples can be constructed possessing only discontinuous solutions. For a discussion of this matter, that will be not faced here, we refer, e.g., to [13, 27, 19, 17, 35, 34].

Notations. We list a few notations that are used throughout this paper. We denote by $u \wedge v = \min\{u, v\}$, $u \vee v = \max\{u, v\}$ the pointwise minimum and the pointwise maximum of the functions u and v, respectively. We set $u^+ = u \vee 0$ and $u^- = -(u \wedge 0)$. We denote by $\mathcal{H}_k$, with $k \in \mathbb{N}$, the k-dimensional Hausdorff measure. We indicate by χ_E the characteristic function of the set E. If $E(\subseteq \mathbb{R}^N)$ is measurable, $|E|$ denotes the N-dimensional measure of E. If $\Omega(\subseteq \mathbb{R}^N)$ is an open set and $E(\subseteq \Omega)$ is a Caccioppoli set, $\mathrm{Per}(E)$ denotes the perimeter of E in Ω defined by $\mathrm{Per}(E) = \int_\Omega |D\chi_E|$. If $u : E(\subseteq \mathbb{R}^N) \to \mathbb{R}$, sometimes we write $\{u > 0\}$ (respectively, $\{u \geq 0\}$, $\{u = 0\}$) to denote the set $\{x \in E : u(x) > 0\}$ (respectively, $\{x \in E : u(x) \geq 0\}$, $\{x \in E : u(x) = 0\}$). For any $v \in BV(\Omega)$, $Du = (Du)^a dx + (Du)^s$ is the Lebesgue decomposition of the measure Du in its absolutely continuous part $(Du)^a dx$, with density function $(Du)^a$, and its singular part $(Du)^s$ with respect to the N-dimensional Lebesgue measure in $\mathbb{R}^N$, $|Du|$ denotes the total variation of the measure Du, $|Du| = |Du|^a dx + |Du|^s$ is the Lebesgue decomposition of $|Du|$, and $\frac{Du}{|Du|}$ is the density function of Du with respect to its total variation $|Du|$. Finally $\mathbb{R}_0^+$ and $\mathbb{R}^+$ denote the open interval $]0, +\infty[$ and the closed interval $[0, +\infty[$, respectively.

2 An asymmetric Poincaré inequality

Throughout this section we assume that

(h_0) *Ω is a bounded domain in $\mathbb{R}^N$ having a Lipschitz boundary $\partial\Omega$;*

(h_1) *$h \in L^p(\Omega)$, for some $p > N$, and $\kappa \in L^\infty(\partial\Omega)$;*

(h_2) *there exists a constant $\rho > 0$ such that*

$$\left| \int_B h \, dx - \int_{\partial\Omega} \kappa \chi_B \, d\mathcal{H}_{N-1} \right| \leq (1-\rho) \int_\Omega |D\chi_B|$$

for every Caccioppoli set $B \subseteq \Omega$.

In the case of dimension $N = 1$ condition (h_1) can be weakened in most cases by assuming

(h_1') *$h \in L^1(\Omega)$.*

We will generally leave the bother of the necessary changes to the interested reader.

Remark 2.1 Condition (h_2), introduced in [19], implies in particular that

$$\int_\Omega h \, dx - \int_{\partial\Omega} k \, d\mathcal{H}_{N-1} = 0. \tag{14}$$

We define a functional $\mathcal{L} : BV(\Omega) \to \mathbb{R}$ by setting

$$\mathcal{L}(v) = \int_\Omega |Dv| - \int_\Omega hv \, dx + \int_{\partial\Omega} \kappa v \, d\mathcal{H}_{N-1} \tag{15}$$

for every $v \in BV(\Omega)$.

Remark 2.2 $\mathcal{L}$ is continuous, due to the continuity of the trace map from $BV(\Omega)$ to $L^1(\partial\Omega)$ (see [2, Theorem 3.87]), is positively homogeneous of degree 1 and is invariant under constant shifts, i.e., $\mathcal{L}(v + r) = \mathcal{L}(v)$ for every $v \in BV(\Omega)$ and $r \in \mathbb{R}$. Thus, there exists a constant $\sigma > 0$ such that

$$\mathcal{L}(v) \leq \sigma \int_\Omega |Dv|$$

for every $v \in BV(\Omega)$.

In addition, $\mathcal{L}$ is coercive on the subspace of all $v \in BV(\Omega)$ with $\int_\Omega v \, dx = 0$ and is lower semicontinuous with respect to the L^q-convergence in $BV(\Omega)$ with $q = \frac{p}{p-1} \in [1, 1^*[$, where $1^* = \frac{N}{N-1}$ if $N \geq 2$ and $1^* = \infty$ if $N = 1$; this is the content of the next two lemmas.

Proposition 2.1. *Assume* (h_0), (h_1) *and* (h_2). *Then, for all* $v \in BV(\Omega)$, *we have*

$$\mathcal{L}(v) \geq \rho \int_\Omega |Dv|, \tag{16}$$

with ρ *defined in* (h_2).

Proof. The proof of this result closely follows the argument in [19, Lemma 2.1]. Fix any $v \in BV(\Omega)$. For each $t \in \mathbb{R}$, define the set

$$E_t = \{x \in \Omega : v(x) > t\}$$

and the function $\varphi_{E_t} \in BV(\Omega)$ by

$$\varphi_{E_t}(x) = \begin{cases} \chi_{E_t}(x) & \text{if } t > 0, \\ \chi_{E_t}(x) - 1 = -\chi_{\Omega \setminus E_t}(x) & \text{if } t \leq 0. \end{cases}$$

Then the representation

$$v(x) = \int_{-\infty}^{+\infty} \varphi_{E_t}(x)\, dt \tag{17}$$

holds for a.e. $x \in \Omega$. Hence we can write

$$\begin{aligned}
\int_{\partial\Omega} \kappa v \, d\mathcal{H}_{N-1} - \int_\Omega hv \, dx &= \int_{\partial\Omega} \kappa \int_{-\infty}^{+\infty} \varphi_{E_t}(x)\, dt \, d\mathcal{H}_{N-1} - \int_\Omega h \int_{-\infty}^{+\infty} \varphi_{E_t}(x)\, dt\, dx \\
&= \int_{-\infty}^{0} \left(\int_{\partial\Omega} \kappa \varphi_{E_t}(x)\, d\mathcal{H}_{N-1} - \int_\Omega h \varphi_{E_t}(x)\, dx \right) dt \\
&\quad + \int_{0}^{+\infty} \left(\int_{\partial\Omega} \kappa \varphi_{E_t}(x)\, d\mathcal{H}_{N-1} - \int_\Omega h \varphi_{E_t}(x)\, dx \right) dt \\
&= \int_{-\infty}^{0} \left(-\int_{\partial\Omega} \kappa \chi_{\Omega\setminus E_t}\, d\mathcal{H}_{N-1} + \int_{\Omega\setminus E_t} h\, dx \right) dt \\
&\quad + \int_{0}^{+\infty} \left(\int_{\partial\Omega} \kappa \chi_{E_t}(x)\, d\mathcal{H}_{N-1} - \int_{E_t} h\, dx \right) dt.
\end{aligned}$$

Using (h_2) and the coarea formula [6, Theorem 10.3], we get

$$\begin{aligned}
\int_{\partial\Omega} \kappa v \, d\mathcal{H}_{N-1} - \int_\Omega hv \, dx &\geq -(1-\rho) \left(\int_{-\infty}^{0} \int_\Omega |D\chi_{\Omega\setminus E_t}| \, dt + \int_0^{+\infty} \int_\Omega |D\chi_{E_t}| \, dt \right) \\
&= -(1-\rho) \left(\int_{-\infty}^{0} \int_\Omega |D\chi_{E_t}| \, dt + \int_0^{+\infty} \int_\Omega |D\chi_{E_t}| \, dt \right) \\
&= -(1-\rho) \left(\int_{-\infty}^{+\infty} \int_\Omega |D\chi_{E_t}| \, dt \right) = -(1-\rho) \int_\Omega |Dv|.
\end{aligned}$$

The last computation yields (16). □

Proposition 2.2. *Assume* (h_0), (h_1) *and* (h_2). *The functional* $\mathcal{L} : BV(\Omega) \to \mathbb{R}$ *is lower semicontinuous with respect to the* L^q*-convergence in* $BV(\Omega)$ *with* $q = \frac{p}{p-1}$, *i.e., if* $(v_n)_n$ *is a sequence in* $BV(\Omega)$ *converging in* $L^q(\Omega)$ *to a function* $v \in BV(\Omega)$, *then*

$$\mathcal{L}(v) \leq \liminf_{n\to+\infty} \mathcal{L}(v_n).$$

Proof. Let $(v_n)_n$ be a sequence in $BV(\Omega)$ converging in $L^q(\Omega)$ to a function $v \in BV(\Omega)$. It follows from [19, Lemma 2.2] that for every $\delta > 0$ there exists a constant C_δ such that, for all $w \in BV(\Omega)$,

$$\left|\int_{\partial\Omega} \kappa w \, d\mathcal{H}_{N-1}\right| \leq \left(1 - \tfrac{\rho}{2}\right) \int_{S_\delta} |Dw| + C_\delta \int_{S_\delta} |w| \, dx,$$

where

$$S_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}.$$

Fix $\delta > 0$. Then we can write

$$\begin{aligned}
\mathcal{L}(v) - \mathcal{L}(v_n) &= \int_\Omega |Dv| - \int_\Omega |Dv_n| \; - \int_\Omega h(v - v_n) \, dx + \int_{\partial\Omega} \kappa(v - v_n) \, d\mathcal{H}_{N-1} \\
&\leq \int_\Omega |Dv| - \int_\Omega |Dv_n| \; - \int_\Omega h(v - v_n) \, dx \\
&\quad + \int_{S_\delta} |D(v - v_n)| + C_\delta \int_{S_\delta} |v - v_n| \, dx.
\end{aligned}$$

Since $(v_n)_n$ converges to v in $L^q(\Omega)$ we get

$$\begin{aligned}
\mathcal{L}(v) - \liminf_{n\to+\infty} \mathcal{L}(v_n) &\leq \limsup_{n\to+\infty} \left(\int_\Omega |Dv| - \int_\Omega |Dv_n| \; + \int_{S_\delta} |D(v - v_n)| \right) \\
&\leq \limsup_{n\to+\infty} \left(\int_\Omega |Dv| - \int_\Omega |Dv_n| \; + \int_{S_\delta} |Dv| + \int_{S_\delta} |Dv_n| \right) \\
&\leq \int_{\Omega\setminus S_\delta} |Dv| - \liminf_{n\to+\infty} \int_{\Omega\setminus S_\delta} |Dv_n| \; + 2 \int_{S_\delta} |Dv|.
\end{aligned}$$

By the lower semicontinuity of the total variation with respect to the L^q-convergence in $BV(\Omega)$, we obtain

$$\mathcal{L}(v) \leq \liminf_{n\to+\infty} \mathcal{L}(v_n) + 2 \int_{S_\delta} |Dv|$$

for all $\delta > 0$. The conclusion follows letting $\delta \to 0$, as $\bigcap_{\delta>0} S_\delta = \emptyset$. □

A Poincaré-type inequality. We want to prove here an asymmetric version of the Poincaré inequality (1) which involves the functional $\mathcal{L}$. Namely, we will show that for each $r > 0$ there exist constants $\mu = \mu(r) > 0$ and $\nu = \nu(r) > 0$, which also depend on Ω, h and κ, such that every $u \in BV(\Omega)$, with

$$\mu \int_\Omega u^+ \, dx - \nu \int_\Omega u^- \, dx = 0, \tag{18}$$

satisfies

$$\mu \int_\Omega u^+ \, dx + \nu \int_\Omega u^- \, dx \leq \mathcal{L}(u). \tag{19}$$

For each $r > 0$ we define μ and ν through the variational formulas

$$\mu = \mu(r) = \inf \Big\{ \mathcal{L}(v) : v \in BV(\Omega), \int_\Omega v^+ dx - r \int_\Omega v^- dx = 0, \\ \int_\Omega v^+ dx + r \int_\Omega v^- dx = 1 \Big\} \tag{20}$$

and

$$\nu = \nu(r) = \inf \Big\{ \mathcal{L}(v) : v \in BV(\Omega), r^{-1} \int_\Omega v^+ dx - \int_\Omega v^- dx = 0, \\ r^{-1} \int_\Omega v^+ dx + \int_\Omega v^- dx = 1 \Big\}. \tag{21}$$

Proposition 2.1 implies that $\mu \geq 0$ and $\nu \geq 0$. For convenience we also set, for any $r > 0$,

$$C_r = \left\{ v \in BV(\Omega) : \int_\Omega v^+ \, dx = \tfrac{1}{2} \quad \text{and} \quad \int_\Omega v^- \, dx = \tfrac{1}{2r} \right\}.$$

Note that the constraints in (20) and in (21) can be equivalently expressed by requiring $v \in C_r$ and $r^{-1} v \in C_r$, respectively.

The curve $\mathcal{C}$ and its properties. We are now concerned with the study of the functions $r \mapsto \mu(r)$ and $r \mapsto \nu(r)$, and of the plane curve

$$\mathcal{C} = \{ (\mu(r), \nu(r)) : r \in \mathbb{R}_0^+ \}. \tag{22}$$

Proposition 2.3 (Minimum properties)**.** *Assume* (h_0), (h_1) *and* (h_2). *Then, for each* $r > 0$, *we have*

$$\mu(r) = \min\{ \mathcal{L}(v) : v \in C_r \} \quad \textit{and} \quad \nu(r) = \min\{ \mathcal{L}(v) : r^{-1} v \in C_r \}, \tag{23}$$

with $\mu(r) > 0$ *and* $\nu(r) = r\mu(r)$.

Proof. Fix $r > 0$. Let us show that the functional $\mathcal{L}$ has a minimum in the set C_r. Let $(v_n)_n$ be a minimizing sequence in C_r, that is,

$$\lim_{n\to+\infty} \mathcal{L}(v_n) = \mu(r) = \inf\{\mathcal{L}(v) : v \in C_r\} \geq 0.$$

By Proposition 2.1 the sequence $(v_n)_n$ is bounded in $BV(\Omega)$ and hence there exists a subsequence, that we still denote by $(v_n)_n$, which converges in $L^q(\Omega)$, with $q = \frac{p}{p-1}$, to some $v \in BV(\Omega)$. We have $v \in C_r$ and, by Proposition 2.2,

$$\mathcal{L}(v) \leq \liminf_{n\to+\infty} \mathcal{L}(v_n).$$

This implies that $\mathcal{L}(v) = \mu(r)$.

Moreover, we have $\mu(r) = \mathcal{L}(v) > 0$. Indeed, suppose by contradiction that $\mu(r) = 0$ and hence $\mathcal{L}(v) = 0$. Proposition 2.1 implies that $\int_\Omega |Dv| = 0$ and therefore, by [2, Proposition 3.2], v is constant, which is impossible as $v \in C_r$. Similar conclusions can be proved for $\nu(r)$.

Finally, let $v \in BV(\Omega)$ be such that $r^{-1}v \in C_r$ and $\mathcal{L}(v) = \nu(r)$. Setting $u = r^{-1}v$, we have $u \in C_r$ and $\mu(r) \leq \mathcal{L}(u) = r^{-1}\nu(r)$. Conversely, if $u \in C_r$ is such that $\mathcal{L}(u) = \mu(r)$, setting $v = ru$, we have $r^{-1}v \in C_r$ and $\nu(r) \leq \mathcal{L}(v) = r\mu(r)$. Thus the conclusion follows. □

Remark 2.3 It is clear that any minimizer u in (23) yields the equality in (19).

Proposition 2.4 (Asymmetric Poincaré inequality)**.** *Assume* (h_0), (h_1) *and* (h_2). *Take* $(\mu, \nu) \in \mathcal{C}$. *Then, every* $v \in BV(\Omega)$, *for which* (18) *holds, that is*

$$\mu \int_\Omega v^+ \, dx - \nu \int_\Omega v^- \, dx = 0,$$

also satisfies (19), *that is*

$$\mu \int_\Omega v^+ \, dx + \nu \int_\Omega v^- \, dx \leq \mathcal{L}(v).$$

Proof. The conclusion follows from (22) and Proposition 2.3. □

Remark 2.4 If $v \in BV(\Omega)$ satisfies (18) for some $(\mu, \nu) \in \mathcal{C}$, then we have, in particular,

$$\int_\Omega |v| \, dx \leq \frac{1}{2}\Big(\frac{1}{\mu} + \frac{1}{\nu}\Big)\mathcal{L}(v).$$

Proposition 2.5 (Symmetry)**.** *Assume* (h_0) *and suppose that* $h = 0$ *and* $\kappa = 0$. *Then, for each* $r > 0$, *we have* $\mu(r^{-1}) = \nu(r)$; *in particular,* $\mathcal{C}$ *is symmetric with respect to the diagonal.*

Proof. Let $v \in C_{r^{-1}}$ be such that $\int_\Omega |Dv| = \mu(r^{-1})$. Set $u = -v$. We have $r^{-1}u \in C_r$ and hence $\nu(r) \le \int_\Omega |Du| = \mu(r^{-1})$. Conversely, if $u \in BV(\Omega)$ is such that $r^{-1}u \in C_r$ and $\int_\Omega |Du| = \nu(r)$, setting $v = -u$, we have $v \in C_{r^{-1}}$ and hence $\mu(r^{-1}) \le \int_\Omega |Dv| = \nu(r)$. Thus we conclude that $\nu(r) = \mu(r^{-1})$. □

Proposition 2.6 (Continuity)**.** *Assume* (h_0), (h_1) *and* (h_2). *Then, the functions* $r \mapsto \mu(r)$ *and* $r \mapsto \nu(r)$ *are continuous.*

Proof. We first prove that the function $r \mapsto \mu(r)$ is lower semicontinuous. Fix $r \in \mathbb{R}_0^+$ and take any sequence $(r_n)_n$ in $\mathbb{R}_0^+$ with $\lim_{n\to+\infty} r_n = r$. We can suppose that

$$\liminf_{n\to+\infty} \mu(r_n) = \bar{\mu} \in [0, +\infty[,$$

because otherwise the conclusion is trivial. For each n, let $v_n \in C_{r_n}$ be such that $\mu(r_n) = \mathcal{L}(v_n)$. Since

$$\|v_n\|_{L^1} = \tfrac{1}{2}\big(1 + \tfrac{1}{r_n}\big)$$

for every n, the sequence $(v_n)_n$ is bounded in $L^1(\Omega)$. Moreover, we can extract a subsequence of $(v_n)_n$, still denoted by $(v_n)_n$, such that $(\mathcal{L}(v_n))_n$ converges to $\bar{\mu}$. Proposition 2.1 then implies that $(v_n)_n$ is bounded in $BV(\Omega)$. Hence, possibly passing to a further subsequence, we can suppose that $(v_n)_n$ converges in $L^q(\Omega)$, with $q = \frac{p}{p-1}$, to some function $v \in C_r$. Thus Proposition 2.2 yields

$$\mu(r) \le \mathcal{L}(v) \le \liminf_{n\to+\infty} \mathcal{L}(v_n) = \bar{\mu} = \liminf_{n\to+\infty} \mu(r_n).$$

Next we prove that the function $r \mapsto \mu(r)$ is upper semicontinuous. Fix $r \in \mathbb{R}_0^+$ and take any sequence $(r_n)_n$ in $\mathbb{R}_0^+$ with $\lim_{n\to+\infty} r_n = r$. Let $v \in C_r$ be such that $\mu(r) = \mathcal{L}(v)$. Define a sequence $(v_n)_n$ by setting for each n

$$v_n = v^+ - \tfrac{r}{r_n} v^-.$$

We have $v_n \in C_{r_n}$ and hence $\mu(r_n) \le \mathcal{L}(v_n)$. Since for all n

$$\|v_n - v\|_{BV} = |1 - \tfrac{r}{r_n}| \|v^-\|_{BV},$$

the sequence $(v_n)_n$ converges to v in $BV(\Omega)$. The continuity of the functional $\mathcal{L}$ in $BV(\Omega)$ finally yields

$$\mu(r) = \mathcal{L}(v) = \lim_{n\to+\infty} \mathcal{L}(v_n) \ge \limsup_{n\to+\infty} \mu(r_n).$$

Hence we conclude that the function $r \mapsto \mu(r)$ is continuous. The continuity of the map $r \mapsto \nu(r)$ follows from the relation $\nu(r) = r\mu(r)$. □

Proposition 2.7 (Monotonicity)**.** *Assume* (h_0), (h_1) *and* (h_2). *Then, the function* $r \mapsto \mu(r)$ *is strictly decreasing and the function* $r \mapsto \nu(r)$ *is strictly increasing.*

Proof. We only show that the map $r \mapsto \mu(r)$ is strictly decreasing; a similar argument allows us to prove that the function $r \mapsto \nu(r)$ is strictly increasing. Fix $r, s \in \mathbb{R}_0^+$, with $r < s$, and let $u \in C_r$ be such that $\mu(r) = \mathcal{L}(u)$. Let us define a function $\phi : \mathbb{R}^+ \to \mathbb{R}$ by setting

$$\phi(\xi) = \int_\Omega \left((u+\xi)^+ - s(u+\xi)^-\right) dx.$$

It is easy to see that ϕ is continuous. As $u \in C_r$, it follows that

$$\phi(0) = \int_\Omega \left(u^+ - su^-\right) dx < \int_\Omega \left(u^+ - ru^-\right) dx = 0.$$

Moreover, we have

$$\lim_{\xi \to +\infty} \phi(\xi) = +\infty,$$

as, on the one hand, Fatou's lemma implies that

$$\lim_{\xi \to +\infty} \int_\Omega (u+\xi)^+ \, dx = +\infty$$

and, on the other hand,

$$\sup_{\xi \in \mathbb{R}^+} \int_\Omega (u+\xi)^- \, dx \le \int_\Omega u^- \, dx.$$

As a consequence, there exists $\bar{\xi} \in \mathbb{R}_0^+$ such that $\phi(\bar{\xi}) = 0$, i.e.,

$$\int_\Omega (u+\bar{\xi})^+ \, dx = s \int_\Omega (u+\bar{\xi})^- \, dx.$$

Set

$$m = 2\int_\Omega (u+\bar{\xi})^+ \, dx > 2\int_\Omega u^+ \, dx = 1$$

and define a function $v \in BV(\Omega)$ by

$$v = \frac{u+\bar{\xi}}{m}.$$

A simple calculation shows that

$$\int_\Omega v^+ \, dx = \tfrac{1}{2} \quad \text{and} \quad \int_\Omega v^- \, dx = \tfrac{1}{2s},$$

that is $v \in C_s$. By (14), we get

$$\begin{aligned}
\mu(s) \le \mathcal{L}(v) &= \tfrac{1}{m}\left(\int_\Omega |D(u+\bar{\xi})| - \int_\Omega h(u+\bar{\xi}) \, dx + \int_{\partial\Omega} \kappa(u+\bar{\xi}) \, d\mathcal{H}_{N-1}\right) \\
&= \tfrac{1}{m}\left(\mathcal{L}(u) - \int_\Omega h\bar{\xi} \, dx + \int_{\partial\Omega} \kappa\bar{\xi} \, d\mathcal{H}_{N-1}\right) \\
&= \tfrac{1}{m}\mathcal{L}(u) = \tfrac{1}{m}\mu(r) < \mu(r),
\end{aligned}$$

which yields the conclusion. □

Proposition 2.8 (Asymptotic behaviour)**.** *Assume* (h_0), (h_1) *and* (h_2). *Then, we have*

$$\lim_{r\to 0^+} \mu(r) = +\infty \qquad \textit{and} \qquad \lim_{r\to +\infty} \nu(r) = +\infty.$$

Proof. We start with the following variant of the Poincaré inequality (1).
Claim. For each $\alpha \in]0,1]$ *there exists a constant* $c(\alpha)$ *such that every* $u \in BV(\Omega)$, *with*

$$|\{u = 0\}| \geq \alpha\, |\Omega|, \tag{24}$$

also satisfies

$$\int_\Omega |u|\, dx \leq c(\alpha) \int_\Omega |Du|. \tag{25}$$

Proof of the claim. Take $\alpha \in]0,1]$ and let $u \in BV(\Omega)$ satisfy (24). Set $\Omega_0 = \{u = 0\}$, $\tilde{\Omega} = \Omega \setminus \Omega_0$, and $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u$. The Poincaré inequality (1) yields

$$\begin{aligned}
\int_\Omega |u|\, dx = \int_{\tilde{\Omega}} |u|\, dx &= \int_{\tilde{\Omega}} |u - \bar{u} + \bar{u}|\, dx \\
&\leq \int_\Omega |u - \bar{u}|\, dx + \int_{\tilde{\Omega}} |\bar{u}|\, dx \\
&\leq \frac{1}{c}\int_\Omega |Du| \; + \int_{\tilde{\Omega}} |\bar{u}|\, dx,
\end{aligned}$$

where c is the Poincaré constant. We have

$$\int_{\tilde{\Omega}} |\bar{u}|\, dx = |\bar{u}||\tilde{\Omega}| = \frac{|\tilde{\Omega}|}{|\Omega|}\left|\int_\Omega u\, dx\right| \leq \frac{|\tilde{\Omega}|}{|\Omega|}\int_\Omega |u|\, dx$$

and then

$$\int_\Omega |u| \; \leq \frac{1}{c}\int_\Omega |Du| \; + \frac{|\tilde{\Omega}|}{|\Omega|}\int_\Omega |u|\, dx.$$

Since

$$\frac{|\tilde{\Omega}|}{|\Omega|} = \frac{|\Omega| - |\Omega_0|}{|\Omega|} \leq 1 - \alpha$$

we conclude that (25) holds, having set $c(\alpha) = \frac{1}{\alpha\, c}$. This concludes the proof of the claim.

The monotonicity of the function $r \mapsto \mu(r)$ implies that there exists

$$\lim_{r\to 0^+} \mu(r) = \bar{\mu} \in]0, +\infty].$$

Suppose by contradiction that $\bar{\mu} < +\infty$. Take any sequence $(r_n)_n$ in $\mathbb{R}_0^+$ such that $\lim_{n\to+\infty} r_n = 0$ and let $(v_n)_n$ be a sequence in $BV(\Omega)$ such that, for each n, $v_n \in C_{r_n}$ and

$$\mu(r_n) = \mathcal{L}(v_n).$$

We have

$$\int_\Omega v_n^+ \, dx = \tfrac{1}{2},$$

for all n, and

$$\lim_{n\to+\infty} \int_\Omega v_n^- \, dx = \lim_{n\to+\infty} \tfrac{1}{2r_n} = +\infty. \tag{26}$$

Moreover, as $\bar{\mu}$ is finite, Proposition 2.1 and the lattice property of $BV(\Omega)$ (see [2]) yields the existence of a constant M such that

$$\int_\Omega |Du_n^-| + \int_\Omega |Du_n^+| \le \int_\Omega |Du_n| \le M$$

for all n. Accordingly, the sequence $(u_n^+)_n$ is bounded in $BV(\Omega)$ and therefore, possibly passing to a subsequence still denoted by $(u_n^+)_n$, it converges in $L^q(\Omega)$, with $q = \frac{p}{p-1}$, and a.e. to a function $v \in BV(\Omega)$, with $v \ge 0$, such that

$$\int_\Omega v \, dx = \tfrac{1}{2}.$$

Let us prove that the measures of the essential supports of the functions u_n^+ remain bounded away from 0. Indeed, the pointwise convergence a.e. of $(u_n^+)_n$ to v implies that

$$\liminf_{n\to+\infty} \chi_{\{u_n>0\}}(x) \ge \chi_{\{v>0\}}(x)$$

for a.e. $x \in \Omega$. Then Fatou's lemma yields

$$\begin{aligned}\liminf_{n\to+\infty} |\{u_n > 0\}| &= \liminf_{n\to+\infty} \int_\Omega \chi_{\{u_n>0\}} \, dx \\ &\ge \int_\Omega \liminf_{n\to+\infty} \chi_{\{u_n>0\}} \, dx \ge \int_\Omega \chi_{\{v>0\}} \, dx = |\{v > 0\}|.\end{aligned}$$

Set

$$m = |\{v > 0\}| > 0.$$

As $\{u_n^+ > 0\} \subseteq \{u_n^- = 0\}$, we get, for all sufficiently large n,

$$|\{u_n^- = 0\}| \ge |\{u_n^+ > 0\}| \ge \tfrac{1}{2}m.$$

Applying the claim above with $\alpha = \frac{1}{2}\frac{m}{|\Omega|}$ to u_n^-, we get

$$\int_\Omega |u_n^-| \, dx \le c(\alpha) \int_\Omega |Du_n^-| \le c(\alpha)M,$$

for all sufficiently large n, thus contradicting (26).

By a similar argument we prove that $\lim_{r\to+\infty} \nu(r) = +\infty$. □

Proposition 2.9 (Asymptotic behaviour in dimension $N \geq 2$)**.** *Assume $N \geq 2$ and suppose that (h_0), (h_1) and (h_2) hold. Then, we have*

$$\lim_{r\to+\infty} \mu(r) = 0 \quad \textit{and} \quad \lim_{r\to 0^+} \nu(r) = 0.$$

Proof. We prove that $\lim_{r\to+\infty} \mu(r) = 0$; a similar argument shows that $\lim_{r\to 0^+} \nu(r) = 0$. The monotonicity of the function $r \mapsto \mu(r)$ implies that there exists

$$\lim_{r\to+\infty} \mu(r) = \inf_{r\in\mathbb{R}_0^+} \mu(r).$$

Let us prove that

$$\inf_{r\in\mathbb{R}_0^+} \mu(r) = 0, \tag{27}$$

that is, for every $\eta > 0$ there exist $r \in \mathbb{R}_0^+$ and $u \in C_r$ such that

$$\mathcal{L}(u) = \int_\Omega |Du| - \int_\Omega hu\, dx + \int_{\partial\Omega} \kappa u\, d\mathcal{H}_{N-1} < \eta. \tag{28}$$

Fix any $\eta > 0$. Pick $x_0 \in \Omega$ and denote by B_δ the closed ball centered in x_0 of radius $\delta > 0$. Take $\delta > 0$ so small that $B_\delta \subset \Omega$. Pick constants $a, b > 0$ and define a function $u \in BV(\Omega)$ by

$$u = -a\chi_{B_\delta} + b(1 - \chi_{B_\delta}) = b - (a + b)\chi_{B_\delta}.$$

We have

$$\int_\Omega |Du| = (a + b) \int_\Omega |D\chi_{B_\delta}|$$

and, by (14),

$$-\int_\Omega hu\, dx + \int_{\partial\Omega} \kappa u\, d\mathcal{H}_{N-1} = (a + b)\Big(\int_\Omega h\chi_{B_\delta}\, dx - \int_{\partial\Omega} \kappa\chi_{B_\delta}\, d\mathcal{H}_{N-1}\Big).$$

Then, using (h_2), we get

$$\begin{aligned}
\mathcal{L}(u) &= (a + b)\left(\int_\Omega |D\chi_{B_\delta}| + \int_\Omega h\chi_{B_\delta}\, dx - \int_{\partial\Omega} \kappa\chi_{B_\delta}\, d\mathcal{H}_{N-1}\right) \\
&\leq (a + b)\left(\int_\Omega |D\chi_{B_\delta}| + (1 - \rho)\int_\Omega |D\chi_{B_\delta}|\right) \\
&\leq 2(a + b)\int_\Omega |D\chi_{B_\delta}|.
\end{aligned}$$

From [14, Theorem 5.4.1], we have

$$\int_\Omega |D\chi_{B_\delta}| = N\delta^{N-1}\omega_N,$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$.

The function u belongs to C_r, for some $r \in \mathbb{R}_0^+$, if and only if

$$b(|\Omega| - \delta^N \omega_N) = \int_\Omega u^+ \, dx = \tfrac{1}{2} \quad \text{and} \quad a\delta^N \omega_N = \int_\Omega u^- \, dx = \tfrac{1}{2r}.$$

Moreover, u satisfies (28) if

$$2(a+b)N\delta^{N-1}\omega_N < \eta. \tag{29}$$

Take $a = 1$ and set

$$r = \frac{1}{2\delta^N \omega_N} \quad \text{and} \quad b = \frac{1}{2(|\Omega| - \delta^N \omega_N)}.$$

As $N \geq 2$ we have $\lim_{\delta \to 0} \delta^{N-1} = 0$. Hence plugging a and b in (29) yields

$$2N\omega_N \left(1 + \frac{1}{2(|\Omega| - \delta^N \omega_N)}\right) \delta^{N-1} \; < \eta,$$

where, as $N \geq 2$, the left-hand side goes to 0 letting $\delta \to 0$. Accordingly, a number $r > 0$ and a function $u \in C_r$ have been found such that (28) holds. Thus (27) follows. □

Proposition 2.10 (Asymptotic behaviour in dimension $N = 1$)**.** *Assume $N = 1$ and let $\Omega =]0, T[$. Suppose that (h_1') and (h_2) hold. Then, we have*

$$\lim_{r \to +\infty} \mu(r) > 0 \quad \textit{and} \quad \lim_{r \to 0^+} \nu(r) > 0.$$

Proof. We prove that $\lim_{r \to +\infty} \mu(r) > 0$; a similar argument shows that $\lim_{r \to 0^+} \nu(r) > 0$. The monotonicity of the function $r \mapsto \mu(r)$ implies that there exists

$$\lim_{r \to +\infty} \mu(r) = \bar{\mu} \in [0, +\infty[.$$

Assume by contradiction that $\bar{\mu} = 0$. Take any sequence $(r_n)_n$ in $\mathbb{R}_0^+$ such that $r_n \to +\infty$ and consider a corresponding sequence of function $(u_n)_n$ in $BV(0,T)$ such that

$$\int_0^T u_n^+ \, dx = \frac{1}{2} \quad \text{and} \quad \int_0^T u_n^- \, dx = \frac{1}{2r_n} \tag{30}$$

and

$$\mu(r_n) = \mathcal{L}(u_n).$$

From Proposition 2.1 we get

$$\lim_{n \to +\infty} \int_{]0,T[} |Du_n| \; = 0$$

and hence

$$\lim_{n\to+\infty}\big(\operatorname*{ess\,sup}_{]0,T[} u_n - \operatorname*{ess\,inf}_{]0,T[} u_n\big) = 0.$$

The conditions in (30) imply that

$$\operatorname*{ess\,inf}_{]0,T[} u_n \le 0 \le \operatorname*{ess\,sup}_{]0,T[} u_n$$

and therefore

$$\lim_{n\to+\infty} \operatorname*{ess\,sup}_{]0,T[} u_n = 0,$$

which yields a contradiction with the first condition in (30). □

The case $h = 0$ and $\kappa = 0$. It is clear that if $h = 0$ and $\kappa = 0$, then taking $r = 1$ we get

$$\mu(1) = \nu(1) = \min\Big\{\int_\Omega |Dv| : v \in BV(\Omega), \int_\Omega v\,dx = 0, \int_\Omega |v|\,dx = 1\Big\} = c. \tag{31}$$

We want to compare the Poincaré constant c with the second eigenvalue c_2 of the 1-Laplace operator with Neumann boundary conditions as defined in [8]. To this aim we recall the variational characterization of c_2 provided therein. Let A be a closed and symmetric subset of a Banach space and denote by $\gamma(A)$ its Krasnoselskii genus. We recall that $\gamma(A) \ge 2$ if no continuous odd function $g : A \to \mathbb{R}\setminus\{0\}$ exists. Set

$$\mathcal{F}_2 = \{A \subseteq L^1(\Omega) : A \text{ closed}, A = -A, \gamma(A) \ge 2\}.$$

Set $S = \{v \in L^1(\Omega) : \|v\|_{L^1} = 1\}$ and define a functional $\mathcal{E} : L^1(\Omega) \to \mathbb{R}$ by

$$\mathcal{E}(v) = \begin{cases} \displaystyle\int_\Omega |D(v)| & \text{if } v \in S \cap BV(\Omega), \\ +\infty & \text{if } v \in L^1(\Omega) \setminus \big(S \cap BV(\Omega)\big). \end{cases}$$

Then from [8] we have

$$c_2 = \inf_{A\in\mathcal{F}_2} \sup_{v\in A} \mathcal{E}(v). \tag{32}$$

Proposition 2.11. *Assume (h_0) and suppose that $h = 0$ and $\kappa = 0$. Then, we have $c \le c_2$.*

Proof. Pick any $A \in \mathcal{F}_2$. We want to prove that

$$c \le \sup_{v\in A} \mathcal{E}(v). \tag{33}$$

We may assume $A \subseteq S \cap BV(\Omega)$, otherwise the inequality is trivially satisfied. Observe that $\int_\Omega v_0\,dx = 0$ for some $v_0 \in A$. Indeed, otherwise, we would have $A = A^- \cup A^+$, with

$$A^- = \{v \in A : \int_\Omega v\,dx < 0\} \quad \text{and} \quad A^+ = \{v \in A : \int_\Omega v\,dx > 0\},$$

and we could define an odd continuous function $g : A \to \mathbb{R} \setminus \{0\}$ by setting $g(v) = \chi_{A^+} - \chi_{A^-}$, thus contradicting the assumption $\gamma(A) \geq 2$. Therefore we have $c \leq \int_\Omega |Dv_0| = \mathcal{E}(v_0)$ and thus (33) follows. Since (33) holds for all $A \in \mathcal{F}_2$, we conclude that $c \leq c_2$. $\square$

Remark 2.5 Let us set, for $p > N$,

$$X_p(\Omega) = \{z \in L^\infty(\Omega, \mathbb{R}^N) \mid \operatorname{div} z \in L^p(\Omega)\}.$$

For $u \in BV(\Omega)$ and $z \in X_p(\Omega)$, let $[z, n] \in L^\infty(\partial\Omega)$ be the weak trace on $\partial\Omega$ of the component of z along the outer normal n to $\partial\Omega$ and (Du, z) the Radon measure defined in [5]. Recall that Green's formula

$$\int_\Omega u \operatorname{div} z \, dx = \int_{\partial\Omega} [z, n] \, u \, d\mathcal{H}_{N-1} - \int_\Omega (Du, z) \tag{34}$$

holds [5, Theorem 1.9]. Using this formula and the method of Lagrange multipliers, it is proved in [8] that to the eigenvalue c_2 there correspond eigenfunctions $\varphi \in BV(\Omega)$, satisfying $\mathcal{E}(\varphi) = c_2$, for which there exists $z \in X_p(\Omega)$, with $\|z\|_\infty \leq 1$,

$$\begin{gathered} -\operatorname{div} z \in c_2 \operatorname{Sgn}(\varphi), \\ (D\varphi, z) = |D\varphi|, \\ [z, n] = 0 \quad \mathcal{H}_{N-1}\text{-a.e. on } \partial\Omega. \end{gathered}$$

A much more complete result holds in case of dimension $N = 1$.

Proposition 2.12 (Characterization of $\mathcal{C}$ in dimension $N = 1$)**.** *Assume $N = 1$ and let $\Omega =]0, T[$. Suppose that $h = 0$ and $\kappa = 0$. Then, we have*

$$\mathcal{C} = \Big\{(\mu, \nu) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \sqrt{2\,T}\Big\}.$$

In particular, for any fixed $(\mu, \nu) \in \mathcal{C}$, every $u \in BV(0, T)$ such that $\mu \int_0^T u^+ \, dx - \nu \int_0^T u^- \, dx = 0$ also satisfies

$$\mu \int_0^T u^+ \, dx + \nu \int_0^T u^- \, dx \leq \int_{]0,T[} |Du|.$$

Moreover, the equality is attained if and only if u is a positive multiple either of φ, or of $\varphi(T - \cdot)$, with

$$\varphi(x) = \begin{cases} \dfrac{1}{T} \dfrac{1}{2\mu} \dfrac{\sqrt{\mu} + \sqrt{\nu}}{\sqrt{\nu}} & \text{if } 0 < x < \dfrac{\sqrt{\nu}}{\sqrt{\mu} + \sqrt{\nu}} T, \\ -\dfrac{1}{T} \dfrac{1}{2\nu} \dfrac{\sqrt{\mu} + \sqrt{\nu}}{\sqrt{\mu}} & \text{if } \dfrac{\sqrt{\nu}}{\sqrt{\mu} + \sqrt{\nu}} T \leq x < T. \end{cases} \tag{35}$$

Remark 2.6 Under the assumptions of Proposition 2.12 we have that $(\frac{2}{T}, \frac{2}{T}) \in \mathcal{C}$, where $\frac{2}{T}$ is the second eigenvalue c_2 of the one-dimensional 1-Laplace operator with Neumann boundary conditions in $]0,T[$, defined by (32) and explicitly calculated in [8]. Moreover, $\mathcal{C}$ is asymptotic to the lines $\mu = \frac{1}{2T}$ and $\nu = \frac{1}{2T}$.

Proof. Part 1. We prove that, if $\mu, \nu \in \mathbb{R}_0^+$ are such that

$$\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \sqrt{2T},$$

then every $u \in BV(0,T)$ such that

$$\mu \int_0^T u^+ \, dx - \nu \int_0^T u^- \, dx = 0 \tag{36}$$

also satisfies

$$\mu \int_0^T u^+ \, dx + \nu \int_0^T u^- \, dx \leq \int_{]0,T[} |Du|, \tag{37}$$

the equality being attained if and only if u is a positive multiple either of φ, or of $\varphi(T-\cdot)$, with φ defined by (35).

The proof is divided into three steps.

Step 1. A monotone decreasing rearrangement. Take $u \in BV(0,T)$. Set, for each $t \in \mathbb{R}$,

$$E_t = \{x \in]0,T[: u(x) > t\}$$

and

$$E_t^* = [0, |E_t|] \cap]0,T[.$$

The monotone decreasing rearrangement of u (see, e.g., [28, Chaptre I]) is the function $u^* \in BV(0,T)$ defined, for a.e. $x \in]0,T[$, by

$$u^*(x) = \sup\{t : x \in E_t^*\}.$$

By [28, Theorem 1.1], we have

$$\int_0^T (u^*)^+ \, dx = \int_0^T u^+ \, dx \quad \text{and} \quad \int_0^T (u^*)^- \, dx = \int_0^T u^- \, dx. \tag{38}$$

Moreover, the Polya-Szegö inequality

$$\int_{]0,T[} |Du^*| \leq \int_{]0,T[} |Du| \tag{39}$$

holds. This follows by first observing that for each $t \in \mathbb{R}$

$$\mathrm{Per}(E_t^*) \leq \mathrm{Per}(E_t). \tag{40}$$

Indeed, we have $\mathrm{Per}(E_t^*) \le 1$ and, if $\mathrm{Per}(E_t) = 0$, then $\int_{]0,T[} |D\chi_{E_t}| = 0$ and χ_{E_t} is constant a.e. in $]0,T[$, up to a set of measure 0, either $E_t =]0,T[$ and hence $E_t^* =]0,T[$, or $E_t = \emptyset$ and hence $E_t^* = \emptyset$, and thus, in both cases, $\mathrm{Per}(E_t^*) = 0$. Then, by (40) the coarea formula (see [6, Theorem 10.3]) yields

$$\int_{]0,T[} |Du^*| = \int_{-\infty}^{+\infty} \mathrm{Per}(E_t^*)\, dt \le \int_{-\infty}^{+\infty} \mathrm{Per}(E_t)\, dt = \int_{]0,T[} |Du|.$$

Step 2. The inequality (37) *holds for all decreasing functions* $u :]0,T[\to \mathbb{R}$ *satisfying* (36). Let $u :]0,T[\to \mathbb{R}$ be a decreasing function satisfying (36) and

$$\mu \int_0^T u^+\, dx + \nu \int_0^T u^-\, dx = 1, \tag{41}$$

or, equivalently,

$$\int_0^T u^+\, dx = \frac{1}{2\mu} \quad \text{and} \quad \int_0^T u^-\, dx = \frac{1}{2\nu}. \tag{42}$$

Let $T_0 \in]0,T[$ be such that $u(x) \ge 0$ a.e. in $[0,T_0[$ and $u(x) \le 0$ a.e. in $[T_0,T]$. It is clear that $\operatorname*{ess\,sup}_{]0,T[} u \ge \frac{1}{2\mu T_0}$ and $\operatorname*{ess\,inf}_{]0,T[} u \le -\frac{1}{2\nu(T-T_0)}$ and therefore

$$\int_{]0,T[} |Du| = \operatorname*{ess\,sup}_{]0,T[} u - \operatorname*{ess\,inf}_{]0,T[} u \ge \frac{1}{2\mu T_0} + \frac{1}{2\nu(T-T_0)} \ge \frac{1}{2T}\Big(\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}}\Big)^2 = 1,$$

as the minimum of the function $\xi \mapsto \frac{1}{2\mu\xi} + \frac{1}{2\nu(T-\xi)}$ in $]0,T[$ is attained at $\frac{\sqrt{\nu}}{\sqrt{\mu}+\sqrt{\nu}}T$ and is equal to $\frac{1}{2T}\Big(\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}}\Big)^2$. Hence (37) holds, as u satisfies (41). The general conclusion, for all decreasing functions $u :]0,T[\to \mathbb{R}$ satisfying (36), follows by homogeneity.

Step 3. The inequality (37) *holds for all functions* $u \in BV(0,T)$ *satisfying* (36). Let $u \in BV(0,T)$ satisfy (36) and let u^* be the decreasing rearrangement of u as defined in Step 1, which satisfies (36) as well. By (39), (38) and Step 2 we have

$$\int_{]0,T[} |Du| \ge \int_{]0,T[} |Du^*| \ge \mu \int_0^T (u^*)^+\, dx + \nu \int_0^T (u^*)^-\, dx = \mu \int_0^T u^+\, dx + \nu \int_0^T u^-\, dx.$$

Step 4. The equality is attained both in (36) *and in* (37) *if and only if* u *is a positive multiple either of* φ*, or of* $\varphi(T-\cdot)$*, with* φ *defined by* (35). It is easily checked by a direct inspection that if u is a positive multiple of φ, or of $\varphi(T-\cdot)$, then it satisfies the equality both in (36) and in (37). Let us prove the converse implication. Assume that $u \in BV(0,T)$ satisfies the equality both in (36) and in (37). Possibly rescaling u, we can suppose that (42) holds too and hence

$$\int_{]0,T[} |Du| = 1 = \min\left\{ \int_{]0,T[} |Dv| : v \in BV(0,T), \int_0^T v^+\, dx = \frac{1}{2\mu}, \int_0^T v^-\, dx = \frac{1}{2\nu} \right\}.$$

Let u^* be the decreasing rearrangement of u, as defined in Step 1. We have

$$\int_0^T (u^*)^+\,dx = \frac{1}{2\mu} \quad \text{and} \quad \int_0^T (u^*)^-\,dx = \frac{1}{2\nu},$$

as well as

$$1 = \int_{]0,T[} |Du| \geq \int_{]0,T[} |Du^*| \geq 1.$$

Hence, by the coarea formula, we have

$$\int_{-\infty}^{+\infty} \mathrm{Per}(E_t)\,dt = \int_{]0,T[} |Du| = \int_{]0,T[} |Du^*| = \int_{-\infty}^{+\infty} \mathrm{Per}(E_t^*)\,dt$$

and, by (40), we conclude that, for a.e. $t \in \mathbb{R}$,

$$\mathrm{Per}(E_t) = \mathrm{Per}(E_t^*)$$

and therefore E_t is an interval having either 0 or T as one of its endpoints. Namely, we have that either $E_t = [0, |E_t|] \cap]0,T[$ for a.e. $t \in \mathbb{R}$, or $E_t = [T - |E_t|, T] \cap]0,T[$ for a.e. $t \in \mathbb{R}$. Then the representation formula (17) implies that either $u = u^*$, or $u = u^*(T - \cdot)$.

Suppose that $u = u^*$. Let $T_0 \in]0,T[$ be such that $u(x) \geq 0$ a.e. in $[0,T_0]$ and $u(x) \leq 0$ a.e. in $]T_0,T]$. As $\operatorname*{ess\,sup}_{]0,T[} u \geq \frac{1}{2\mu T_0}$ and $\operatorname*{ess\,inf}_{]0,T[} u \leq -\frac{1}{2\nu(T-T_0)}$, we have

$$1 = \int_0^T |Du|\,dx = \operatorname*{ess\,sup}_{]0,T[} u - \operatorname*{ess\,inf}_{]0,T[} u \geq \frac{1}{2\mu T_0} + \frac{1}{2\nu(T-T_0)} \geq \frac{1}{2T}\Big(\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}}\Big)^2 = 1,$$

and therefore $\operatorname*{ess\,sup}_{]0,T[} u = \frac{1}{2\mu T_0}$ and $\operatorname*{ess\,inf}_{]0,T[} u = -\frac{1}{2\nu(T-T_0)}$ and $T_0 = \frac{\sqrt{\nu}}{\sqrt{\mu}+\sqrt{\nu}}T$. Thus we conclude that $u(x) = \frac{1}{2\mu T_0}$ a.e. in $]0,T_0]$ and $u(x) = -\frac{1}{2\nu(T-T_0)}$ a.e. in $]T_0,T[$, i.e., $u = \varphi$. Similarly, we show that if $u = u^*(T - \cdot)$, then $u = \varphi(T - \cdot)$.

Part 2. We have

$$\mathcal{C} = \Big\{(\mu,\nu) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \sqrt{2\,T}\Big\}.$$

Assume that $\mu, \nu \in \mathbb{R}_0^+$ satisfy $\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \sqrt{2\,T}$. Then, setting $r = \frac{\nu}{\mu}$ we have that $\mu\varphi \in C_r$ and

$$\begin{aligned}\mu \int_{]0,T[} |D\varphi| &= \mu \min\Big\{ \int_{]0,T[} |Du| : u \in BV(0,T), \int_0^T u^+\,dx = \frac{1}{2\mu}, \int_0^T u^-\,dx = \frac{1}{2\nu}\Big\} \\ &= \min\Big\{ \int_{]0,T[} |Dv| : v \in C_r\Big\}.\end{aligned}$$

Thus we conclude that $(\mu,\nu) \in \mathcal{C}$.

Conversely, suppose that $(\mu, \nu) \in \mathcal{C}$. Take $\tilde{\mu}, \tilde{\nu} \in \mathbb{R}_0^+$ such that $\frac{1}{\sqrt{\tilde{\mu}}} + \frac{1}{\sqrt{\tilde{\nu}}} = \sqrt{2\,T}$ and $\frac{\tilde{\nu}}{\tilde{\mu}} = r = \frac{\nu}{\mu}$. We know from the previous step that

$$\tilde{\mu} = \min\Big\{ \int_{]0,T[} |Dv| : v \in C_r \Big\} = \mu.$$

Thus we conclude that $\mu = \tilde{\mu}$ and $\nu = \tilde{\nu}$. $\square$

3 Solvability of a capillarity problem

We collect in this section several statements concerning non-existence, existence and multiplicity of solutions of the problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1+|\nabla u|^2}\Big) = f(x,u)+h(x) & \text{in } \Omega, \\ -\nabla u\cdot n/\sqrt{1+|\nabla u|^2} = \kappa(x) & \text{on } \partial\Omega. \end{cases} \tag{43}$$

Hereafter we assume (h_0), (h_1), (h_2) and

(h_3) *$f:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies the Carathéodory conditions, i.e., for a.e. $x\in\Omega$, $f(x,\cdot):\mathbb{R}\to\mathbb{R}$ is continuous and, for every $s\in\mathbb{R}$, $f(\cdot,s):\Omega\to\mathbb{R}$ is measurable; moreover, there exist constants $a>0$ and $q\in\,]1,1^*[$ and a function $b\in L^p(\Omega)$, with $p>N$, such that*

$$|f(x,s)|\le a|s|^{q-1}+b(x)$$

for a.e. $x\in\Omega$ and every $s\in\mathbb{R}$.

Remark 3.1 Whenever (h_1) and (h_3) are assumed simultaneously, we suppose that $q=\frac{p}{p-1}$. We also notice that condition (h_3) obviously holds if f satisfies the Carathéodory conditions and

$$\operatorname*{ess\,sup}_{\Omega\times\mathbb{R}}|f(x,s)|<+\infty.$$

This is the situation that will occur for the most in the sequel. In the case of dimension $N=1$ condition (h_3) will be weakened by assuming that

(h_3') *$f:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies the L^1-Carathéodory conditions, i.e., f is a Carathéodory function such that, for every $r>0$, there exists a function $\gamma\in L^1(\Omega)$ such that $|f(x,s)|\le\gamma(x)$ for a.e. $x\in\Omega$ and $s\in\mathbb{R}$ with $|s|\le r$.*

As above, we set $F(x,s)=\displaystyle\int_0^s f(x,\xi)\,d\xi$ and we consider the functional $\mathcal{I}:BV(\Omega)\to\mathbb{R}$ defined by

$$\mathcal{I}(v)=\int_\Omega\sqrt{1+|Dv|^2}-\int_\Omega hv\,dx+\int_{\partial\Omega}\kappa v\,d\mathcal{H}_{N-1}-\int_\Omega F(x,v)\,dx, \tag{44}$$

where, for any $v\in BV(\Omega)$, $\displaystyle\int_\Omega\sqrt{1+|Dv|^2}$ is defined by (7), or equivalently by

$$\int_\Omega\sqrt{1+|Dv|^2}=\sup\Big\{\int_\Omega\Big(v\sum_{i=1}^N\frac{\partial w_i}{\partial x_i}+w_{N+1}\Big)\,dx : w_i\in C_0^1(\Omega)$$
$$\text{for } i=1,2,\dots,N+1 \text{ and } \Big\|\sum_{i=1}^{N+1}w_i^2\Big\|_{L^\infty}\le 1\Big\}.$$

Note that

$$\int_\Omega |Dv| \le \int_\Omega \sqrt{1+|Dv|^2} \le |\Omega| + \int_\Omega |Dv|.$$

We also set, for convenience,

$$\mathcal{J}(v) = \int_\Omega \sqrt{1+|Dv|^2} - \int_\Omega hv\,dx + \int_{\partial\Omega} \kappa v\,d\mathcal{H}_{N-1}.$$

The functional $\mathcal{J} : BV(\Omega) \to \mathbb{R}$ is convex and, also by the continuity of the trace map from $BV(\Omega)$ to $L^1(\partial\Omega)$ (see [2, Theorem 3.87]), is continuous. Moreover, the functional $\mathcal{F} : BV(\Omega) \to \mathbb{R}$ defined by $\mathcal{F}(v) = \int_\Omega F(x,v)\,dx$ is C^1 in $BV(\Omega)$. Accordingly, the following notion of solution is adopted.

Definition of solution. We say that a function $u \in BV(\Omega)$ is a solution of problem (43) if $\mathcal{F}'(u)$ is a subgradient at u of the functional $\mathcal{J}$, i.e., u satisfies

$$\mathcal{J}(v) - \mathcal{J}(u) \ge \int_\Omega f(x,u)(v-u)\,dx, \tag{45}$$

for every $v \in BV(\Omega)$.

Remark 3.2 Note that u is a solution of (43) if and only if u is a minimizer in $BV(\Omega)$ of the functional $\mathcal{K}_u : BV(\Omega) \to \mathbb{R}$ defined by $\mathcal{K}_u(v) = \mathcal{J}(v) - \int_\Omega f(x,u)v\,dx$.

Remark 3.3 It follows from [4] that $u \in BV(\Omega)$ satisfies the variational inequality (45), for every $v \in BV(\Omega)$, if and only if

$$\begin{aligned}\int_\Omega \frac{(Du)^a\,(D\phi)^a}{\sqrt{1+|(Du)^a|^2}}\,dx + \int_\Omega S\left(\frac{Du}{|Du|}\right)\frac{D\phi}{|D\phi|}\,|D\phi|^s \\ = \int_\Omega (f(x,u)+h)\phi\,dx - \int_{\partial\Omega} \kappa\phi\,d\mathcal{H}_{N-1}\end{aligned}$$

holds for every $\phi \in BV(\Omega)$ such that $|D\phi|^s$ is absolutely continuous with respect to $|Du|^s$. Here S is the N-dimensional sign function, i.e., $S(\xi) = |\xi|^{-1}\xi$ if $\xi \in \mathbb{R}^N \setminus \{0\}$ and $S(\xi) = 0$ if $\xi = 0$.

Remark 3.4 We point out that condition (h_2) has been introduced in [19], where it was shown to be necessary for the existence of a solution $u \in C^2(\bar\Omega)$ of the problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1+|\nabla u|^2}\Big) = h(x) & \text{in } \Omega, \\ -\nabla u \cdot n/\sqrt{1+|\nabla u|^2} = \kappa(x) & \text{on } \partial\Omega. \end{cases} \tag{46}$$

Let us verify that the weaker condition

$$\Big|\int_B h\,dx - \int_{\partial\Omega} \kappa\chi_B\,d\mathcal{H}_{N-1}\Big| \le \int_\Omega |D\chi_B|, \tag{47}$$

for every Caccioppoli set $B \subseteq \Omega$, is necessary for the existence of a solution $u \in BV(\Omega)$ of (46). Indeed, if $B \subseteq \Omega$ is any Caccioppoli set, taking $v = u + \chi_B$ as a test function in (45), we easily get

$$\int_\Omega h\chi_B\,dx - \int_{\partial\Omega} \kappa\chi_B\,d\mathcal{H}_{N-1} \\ \leq \int_\Omega \sqrt{1+|D(u+\chi_B)|^2} - \int_\Omega \sqrt{1+|Du|^2} \leq \int_\Omega |D\chi_B|.$$

Similarly, taking $v = u - \chi_B$ as a test function in (45), we obtain

$$-\int_\Omega h\chi_B\,dx + \int_{\partial\Omega} \kappa\chi_B\,d\mathcal{H}_{N-1} \leq \int_\Omega |D\chi_B|.$$

Hence (47) follows.

Conversely, assumptions (h_0), (h_1) and (h_2) are sufficient in order that problem (46) has a solution $u \in BV(\Omega)$.

Proposition 3.1. *Assume* (h_0), (h_1) *and* (h_2). *Then problem* (46) *has a solution* $w \in BV(\Omega)$ *with* $\int_\Omega w\,dx = 0$.

Proof. Define a closed subspace of $BV(\Omega)$ by setting $\mathcal{W} = \left\{w \in BV(\Omega) : \int_\Omega w\,dx = 0\right\}$. By Poincaré inequality, or Remark 2.4, $\mathcal{W}$ is a Banach space if endowed with the norm $\|w\|_{\mathcal{W}} = \int_\Omega |Dw|$. Moreover, the functional $\mathcal{J}$ restricted to $\mathcal{W}$ is bounded from below and coercive, by Proposition 2.1, and lower semicontinuous with respect to the L^q-convergence, with $q = \frac{p}{p-1}$, in $\mathcal{W}$ (see Proposition 3.5 below). Hence, $\mathcal{I} = \mathcal{J}$ has a minimum in $\mathcal{W}$ and any corresponding minimizer is a solution of (46). □

Our aim here is to find conditions on the perturbation f which preserve the solvability of problem (43).

We produce now some more explicit assumptions that imply the validity of (h_2). Let us set

$$\alpha = \inf\Big\{\int_\Omega |Dw| : w \in BV(\Omega),\ \int_\Omega w\,dx = 0,\ \|w\|_{L^{1^*}} = 1\Big\} \tag{48}$$

and

$$\beta = \inf\Big\{\int_\Omega |Dw| : w \in BV(\Omega),\ \int_\Omega w\,dx = 0,\ \int_{\partial\Omega} |w|\,d\mathcal{H}_{N-1} = 1\Big\}. \tag{49}$$

By the Sobolev-Poincaré inequality (see [2, Remark 3.50]) and the continuity of the trace embedding (see [2, Theorem 3.87]), we infer that $\alpha > 0$ and $\beta > 0$.

Proposition 3.2. *Assume* (h_0), (h_1),

(h_2') $\displaystyle\int_\Omega h\,dx = \int_{\partial\Omega} \kappa\,d\mathcal{H}_{N-1}$

and

(h_2'') $\beta^{-1}\|h\|_{L^N} + \alpha^{-1}\|\kappa\|_{L^\infty} < 1$.

Then (h_2) *holds.*

Proof. By (h_2'), Hölder inequality, (48) and (49) we get, for every $w \in BV(\Omega)$,

$$\begin{aligned}
&\int_\Omega hw\,dx - \int_{\partial\Omega} \kappa\, w\, d\mathcal{H}_{N-1} \\
&\quad = \int_\Omega h\Big(w - \frac{1}{|\Omega|}\int_\Omega w\,dx\Big)\,dx - \int_{\partial\Omega} \kappa\Big(w - \frac{1}{|\Omega|}\int_\Omega w\,dx\Big)\,d\mathcal{H}_{N-1} \\
&\quad \le \big(\alpha^{-1}\|h\|_{L^N} + \beta^{-1}\|\kappa\|_{L^\infty}\big)\int_\Omega |Dw|.
\end{aligned}$$

Hence, using (h_2''), we obtain (h_2) with $1-\rho = \alpha^{-1}\|h\|_{L^N} + \beta^{-1}\|\kappa\|_{L^\infty} < 1$. □

In the following statement we use the space $X_p(\Omega)$ introduced in Remark 2.5.

Proposition 3.3. *Assume* (h_0), (h_1) *and*

(h_2''') *there exists* $z \in X_p(\Omega)$, *with* $p > N$, *such that* $\operatorname{div} z = h$ *a.e. in* Ω, $[z,n] = \kappa$ $\mathcal{H}_{N-1}$*-a.e. on* $\partial\Omega$ *and* $\|z\|_{L^\infty} < 1$.

Then (h_2) *holds.*

Proof. Using Theorem 1.9 and Corollary 1.6 in [5], we get, for every $w \in BV(\Omega)$,

$$\begin{aligned}
\int_\Omega hw\,dx - \int_{\partial\Omega} \kappa\, w\, d\mathcal{H}_{N-1} &= \int_\Omega w \operatorname{div} z\,dx - \int_{\partial\Omega} [z,n]\, w\, d\mathcal{H}_{N-1} \\
&= -\int_\Omega (Dw, z) \le \|z\|_{L^\infty}\int_\Omega |Dw|.
\end{aligned}$$

Hence, by (h_2'''), we obtain (h_2) with $1-\rho = \|z\|_{L^\infty}$. □

Proposition 3.4. *Assume* $N = 1$ *and let* $\Omega =]0,T[$. *Assume further* (h_1'), (h_2') *and*

(h_2'''') $\|h\|_{L^1} < 1$ *and* $\|\kappa\|_{L^1} < 1$.

Then (h_2) *holds.*

Proof. Take $\rho > 0$ such that both $\|h\|_{L^1} < 1-\rho$ and $\|\kappa\|_{L^1} < 1-\rho$. Let $B \subseteq]0,T[$ be any Caccioppoli set. In case $B =]0,T[$ or $B = \emptyset$ the inequality in (h_2) is trivially satisfied. In case $0 \notin \bar{B}$ and $T \notin \bar{B}$ we have $\int_{]0,T[} |D\chi_B| \ge 2$ and $\int_{\partial]0,T[} \kappa\chi_B\, d\mathcal{H}_0 = 0$; hence the inequality in (h_2) is satisfied as

$$\Big|\int_B h\,dx - \int_{\partial]0,T[} \kappa\chi_B\, d\mathcal{H}_0\Big| \le \|h\|_{L^1} \le (1-\rho) \le (1-\rho)\int_{]0,T[} |D\chi_B|.$$

In case $0 \in \bar{B}$, $T \in \bar{B}$ and $B \neq]0,T[$ we have $\displaystyle\int_{]0,T[} |D\chi_B| \geq 2$ and $\int_{\partial]0,T[} \kappa\chi_B \, d\mathcal{H}_0 = \kappa(0) + \kappa(T)$; hence the inequality in (h_2) is satisfied as

$$\Big| \int_B h \, dx - \int_{\partial]0,T[} \kappa\chi_B \, d\mathcal{H}_0 \Big| = \Big| \int_B h \, dx - \int_0^T h \, dx \Big| \leq \|h\|_1 \leq (1-\rho) \leq (1-\rho) \int_{]0,T[} |D\chi_B|.$$

Suppose now that $0 \in \bar{B}$ and $T \notin \bar{B}$. Then we have $\displaystyle\int_{]0,T[} |D\chi_B| \geq 1$. Observe that, by (h_2'),

$$\begin{aligned} 2\Big(\int_0^T h^+ \, dx - \kappa(0) \Big) &= \int_0^T h^+ \, dx - \kappa(0) + \int_0^T h^- \, dx + \kappa(T) = \int_0^T |h| \, dx - \kappa(0) + \kappa(T) \\ &\leq 2(1-\rho) \end{aligned}$$

and hence

$$\int_0^T h^+ \, dx - \kappa(0) \leq 1 - \rho.$$

A similar computation yields

$$\int_0^T h^- \, dx + \kappa(0) \leq 1 - \rho.$$

Therefore we obtain

$$\begin{aligned} \Big| \int_B h \, dx - \int_{\partial]0,T[} \kappa\chi_B \, d\mathcal{H}_0 \Big| &= \Big| \int_B h^+ \, dx - \int_B h^- \, dx - \kappa(0) \Big| \\ &\leq \max\Big\{ \int_0^T h^+ \, dx - \kappa(0), \int_0^T h^- \, dx + \kappa(0) \Big\} \leq 1 - \rho. \end{aligned}$$

The case $0 \notin \bar{B}$ and $T \in \bar{B}$ is treated similarly. □

We now state some technical results which will be used in the sequel.

Proposition 3.5 (Lower semicontinuity)**.** *Assume* (h_0), (h_1) *and* (h_2). *The functional* $\mathcal{J} : BV(\Omega) \to \mathbb{R}$ *is lower semicontinuous with respect to the* L^q*-convergence in* $BV(\Omega)$ *with* $q = \frac{p}{p-1}$, *i.e., if* $(v_n)_n$ *is a sequence in* $BV(\Omega)$ *converging in* $L^q(\Omega)$ *to a function* $v \in BV(\Omega)$, *then*

$$\mathcal{J}(v) \leq \liminf_{n \to +\infty} \mathcal{J}(v_n).$$

Proof. The conclusion follows from [19, Proposition 2.1] taking $H = 0$ (cf. also Proposition 2.2). □

Proposition 3.6 (Lattice property)**.** *Assume* (h_0). *For every* $u, v \in BV(\Omega)$,

$$\mathcal{J}(u \vee v) + \mathcal{J}(u \wedge v) \leq \mathcal{J}(u) + \mathcal{J}(v).$$

Proof. We first recall that $BV(\Omega)$ is a lattice [3]. Then, also using [38, Theorem 1.56], we see that, for every $u, v \in W^{1,1}(\Omega)$,

$$\int_\Omega \sqrt{1+|\nabla(u\vee v)|^2}\,dx + \int_\Omega \sqrt{1+|\nabla(u\wedge v)|^2}\,dx \\ = \int_\Omega \sqrt{1+|\nabla u|^2}\,dx + \int_\Omega \sqrt{1+|\nabla v|^2}\,dx.$$

Take now $u, v \in BV(\Omega)$. The approximation property and the semicontinuity result, stated in [4, p. 491, p. 498] and Proposition 3.5, easily yield

$$\int_\Omega \sqrt{1+|D(u\vee v)|^2} + \int_\Omega \sqrt{1+|D(u\wedge v)|^2} \le \int_\Omega \sqrt{1+|Du|^2} + \int_\Omega \sqrt{1+|Dv|^2}.$$

As we have

$$\int_\Omega h(u\vee v)\,dx - \int_{\partial\Omega} \kappa(u\vee v)\,d\mathcal{H}_{N-1} + \int_\Omega h(u\wedge v)\,dx - \int_{\partial\Omega} \kappa(u\wedge v)\,d\mathcal{H}_{N-1} \\ = \int_\Omega hu\,dx - \int_{\partial\Omega} \kappa u\,d\mathcal{H}_{N-1} + \int_\Omega hv\,dx - \int_{\partial\Omega} \kappa v\,d\mathcal{H}_{N-1},$$

the conclusion follows. □

Proposition 3.7. *Assume $N = 1$ and let $\Omega =]0,T[$. Then, for every $u \in BV(0,T)$, we have*

$$\operatorname*{ess\,sup}_{]0,T[} u - \operatorname*{ess\,inf}_{]0,T[} u \le \int_{]0,T[} |Du|. \tag{50}$$

Proof. The proof is similar to that of [32, Proposition 2.9]. □

Remark 3.5 It is clear that equality in (50) is attained whenever u is monotone.

Proposition 3.8 (A continuous projection)**.** *Fix $\mu, \nu \in \mathbb{R}_0^+$. For each $v \in L^1(\Omega)$ there exists a unique $\mathcal{P}(v) \in \mathbb{R}$ such that*

$$\mu \int_\Omega (v - \mathcal{P}(v))^+\,dx - \nu \int_\Omega (v - \mathcal{P}(v))^-\,dx = 0. \tag{51}$$

The map $\mathcal{P} : L^1(\Omega) \to \mathbb{R}$ such that $v \mapsto \mathcal{P}(v)$ is a continuous projection.

Proof. Consider the continuous function $h : L^1(\Omega) \times \mathbb{R} \to \mathbb{R}$ defined by

$$h(v,s) = \mu \int_\Omega (v-s)^+\,dx - \nu \int_\Omega (v-s)^-\,dx.$$

Observe that, for each $v \in L^1(\Omega)$, the function $h(v,\cdot) : \mathbb{R} \to \mathbb{R}$ is strictly decreasing. Using Fatou's lemma we easily verify that

$$\lim_{s\to-\infty} h(v,s) = \lim_{s\to-\infty} \mu \int_\Omega (v-s)^+\,dx = +\infty$$

and

$$\lim_{s\to+\infty} h(v,s) = \lim_{s\to+\infty} \nu \int_\Omega (v-s)^- \, dx = -\infty.$$

Hence, by the strict monotonicity of $h(v,\cdot)$ and the intermediate value theorem, there exists a unique $\mathcal{P}(v) \in \mathbb{R}$ such that $h(v,\mathcal{P}(v)) = 0$, i.e. (51) holds. Moreover, we clearly have

$$\mathcal{P} \circ \mathcal{P} = \mathcal{P}.$$

Let us prove that $\mathcal{P}$ is continuous. Fix $v_0 \in L^1(\Omega)$ and pick $\varepsilon > 0$. Since $h(v_0,\cdot)$ is strictly decreasing and $h(v_0,\mathcal{P}(v_0)) = 0$ we have

$$h(v_0,\mathcal{P}(v_0)+\varepsilon) < 0 < h(v_0,\mathcal{P}(v_0)-\varepsilon).$$

By the continuity, for any fixed $s \in \mathbb{R}$, of the map $h(\cdot,s) : L^1(\Omega) \to \mathbb{R}$, we can find a neighbourhood V of v_0 such that

$$h(v,\mathcal{P}(v_0)+\varepsilon) < 0 < h(v,\mathcal{P}(v_0)-\varepsilon)$$

for all $v \in V$. Again by the strict monotonicity of the real function $h(v,\cdot)$ and the intermediate value theorem, we conclude that, for every $v \in V$, the unique point $\mathcal{P}(v)$ such that $h(v,\mathcal{P}(v)) = 0$ belongs to the interval $]\mathcal{P}(v_0)-\varepsilon, \mathcal{P}(v_0)+\varepsilon[$. This shows the continuity of $\mathcal{P}$ at v_0. □

Proposition 3.9 (A positive definite homogeneous form)**.** *Assume* (h_0), (h_1) *and* (h_2). *Let* $\zeta \in L^p(\Omega)$, *with* $p > N$, *be such that* $\zeta(x) \le 0$ *a.e. in* Ω *and* $\zeta(x) < 0$ *on a set of positive measure. Then there exists* $\delta > 0$ *such that*

$$\mathcal{L}(v) - \int_\Omega \zeta|v| \, dx \ge \delta \|v\|_{BV},$$

for all $v \in BV(\Omega)$.

Proof. Possibly replacing ζ with $-1 \vee \zeta$, we can assume $\zeta \in L^\infty(\Omega)$. Define $\mathcal{K} : BV(\Omega) \to \mathbb{R}$ by

$$\mathcal{K}(v) = \mathcal{L}(v) - \int_\Omega \zeta|v| \, dx.$$

Note that $\mathcal{K}(v) \ge 0$ for all $v \in BV(\Omega)$ and $\mathcal{K}(v) = 0$ if and only if $v = 0$. Indeed, if $\mathcal{K}(v) = 0$ then $\mathcal{L}(v) = 0$ and hence, by Proposition 2.1, $\int_\Omega |Dv| = 0$. By [2, Proposition 3.2], v is constant a.e. in Ω; therefore we easily conclude that $v = 0$. In order to prove the conclusion we suppose, by contradiction, that there exists a sequence $(w_n)_n$ in $BV(\Omega)$ such that, for each n,

$$0 \le \mathcal{K}(w_n) < \frac{1}{n}\|w_n\|_{BV}$$

and hence, setting $v_n = \dfrac{w_n}{\|w_n\|_{BV}}$,

$$0 \le \mathcal{K}(v_n) < \frac{1}{n}. \tag{52}$$

We also have, by Proposition 2.1,

$$\begin{aligned}\mathcal{K}(v_n) &= \mathcal{L}(v_n) + \int_\Omega |v_n|\,dx - \int_\Omega (\zeta+1)|v_n|\,dx \\ &\geq \rho\Big(\int_\Omega |Dv_n| + \int_\Omega |v_n|\,dx\Big) - \int_\Omega (\zeta+1)|v_n|\,dx \\ &= \rho - \int_\Omega (\zeta+1)|v_n|\,dx. \end{aligned} \tag{53}$$

Since the sequence $(v_n)_n$ is bounded in $BV(\Omega)$, there exists a subsequence, we still denote by $(v_n)_n$, which converges in $L^q(\Omega)$, with $q = \frac{p}{p-1}$, to some $v \in BV(\Omega)$. In particular we have

$$\lim_{n\to+\infty} \int_\Omega \zeta|v_n|\,dx = \int_\Omega \zeta|v|\,dx.$$

As we have, by (52), $\lim\limits_{n\to+\infty} \mathcal{K}(v_n) = 0$ and hence, by (53), $\lim\limits_{n\to+\infty} \int_\Omega (\zeta+1)|v_n|\,dx \geq \rho > 0$, we conclude that $v \neq 0$ and therefore $\mathcal{K}(v) > 0$. The lower semicontinuity of $\mathcal{K}$ with respect to the L^q-convergence finally yields

$$0 < \mathcal{K}(v) \leq \liminf_{n\to+\infty} \mathcal{K}(v_n) = 0,$$

which is a contradiction. □

Lemma 3.10 (A mountain pass theorem)**.** *Assume* (h_0), (h_1), (h_2) *and* (h_3). *Let* $x_0, x_1 \in BV(\Omega)$ *be given. Set*

$$\Gamma = \{\gamma \in C^0([0,1], BV(\Omega)) : \gamma(0) = x_0, \gamma(1) = x_1\}.$$

Suppose that

$$c_{\mathcal{I}} = \inf_{\gamma\in\Gamma} \max_{\xi\in[0,1]} \mathcal{I}(\gamma(\xi)) > \max\{\mathcal{I}(x_0), \mathcal{I}(x_1)\},$$

where $\mathcal{I}$ *is defined in* (44). *Then there exist sequences* $(\gamma_k)_k$, $(v_k)_k$, $(\varepsilon_k)_k$, *with* $\gamma_k \in \Gamma$, $v_k \in BV(\Omega)$ *and* $\varepsilon_k \in \mathbb{R}$ *such that* $\lim\limits_{k\to+\infty} \varepsilon_k = 0$, *satisfying for each* k

$$c_{\mathcal{I}} - \tfrac{1}{k} \leq \mathcal{I}(v_k) \leq \max_{\xi\in[0,1]} \mathcal{I}(\gamma_k(\xi)) \leq c_{\mathcal{I}} + \tfrac{1}{k}, \tag{54}$$

$$\min_{\xi\in[0,1]} \|v_k - \gamma_k(\xi)\|_{BV} \leq \tfrac{1}{k} \tag{55}$$

and, for all $v \in BV(\Omega)$,

$$\mathcal{J}(v) - \mathcal{J}(v_k) \geq \int_\Omega f(x, v_k)(v - v_k)\,dx + \varepsilon_k \|v - v_k\|_{BV}. \tag{56}$$

Proof. In [22, Theorem 5.1] a mountain pass theorem is proved for a continuous functional Φ on a complete metric space X, with distance d. In such a setting, a critical point of Φ is defined as a point $x \in X$ such that $\delta(\Phi, x) = 0$, where $\delta(\Phi, x)$ is the regularity constant of Φ at x. We recall (see [22, Definition 5.1]) that $x \in X$ is a δ-regular point of Φ if there is a neighbourhood U of x, $\alpha > 0$ and a continuous mapping $\Psi : U \times [0, \alpha] \to X$ such that, for all $(u, t) \in U \times [0, \alpha]$, $d\big(\Psi(u, t), u\big) \le t$ and $\Phi(u) - \Phi\big(\Psi(u, t)\big) \ge \delta t$. The regularity constant of Φ at x is

$$\delta(\Phi, x) = \sup\{\delta : \ \Phi \text{ is } \delta\text{-regular at } x\}.$$

If x is not a δ-regular point of Φ for any $\delta > 0$ we say that x is a critical point of Φ according to [22, Definition 5.1] and we set $\delta(\Phi, x) = 0$.

In our situation we have $X = BV(\Omega)$ and $\Phi = \mathcal{I}$.

Claim. $u \in BV(\Omega)$ is a critical point according to [22, Definition 5.1] if and only if 0 is a subgradient at u of $\mathcal{I}$, i.e., u satisfies (45) for all $v \in BV(\Omega)$. Suppose first that $u \in BV(\Omega)$ is not a critical point according to [22, Definition 5.1]. We prove that u does not satisfy (45) for some $v \in BV(\Omega)$. Indeed, u is a δ-regular point of $\mathcal{I}$ for some $\delta > 0$, i.e., there exist $\alpha > 0$, a neighbourhood U of u and a continuous mapping $\Psi : U \times [0, \alpha] \to BV(\Omega)$ such that, for all $(w, t) \in U \times [0, \alpha]$, $\|\Psi(w, t) - w\|_{BV} \le t$ and

$$\mathcal{I}(w) - \mathcal{I}\big(\Psi(w, t)\big) \ge \delta t.$$

Using the fact that the functional $\mathcal{F}$ is C^1 in $BV(\Omega)$, we have

$$\begin{aligned}\mathcal{J}\big(\Psi(u, t)\big) - \mathcal{J}(u) &= \mathcal{I}\big(\Psi(u, t)\big) - \mathcal{I}(u) + \int_\Omega F\big(x, \Psi(u, t)\big)\, dx - \int_\Omega F\big(x, u\big)\, dx \\ &\le -\delta t + \int_\Omega f(x, u)\big(\Psi(u, t) - u\big)\, dx + t\eta(t)\end{aligned}$$

where $\lim\limits_{t \to 0^+} \eta(t) = 0$, thus yielding, for t sufficiently small,

$$\mathcal{J}\big(\Psi(u, t)\big) - \mathcal{J}(u) < \int_\Omega f(x, u)\big(\Psi(u, t) - u\big)\, dx.$$

Therefore, u does not satisfy (45) for $v = \Psi(u, t)$.

Suppose now that it is false that $u_0 \in BV(\Omega)$ satisfies

$$\mathcal{J}(v) - \mathcal{J}(u_0) \ge \int_\Omega f(x, u_0)(v - u_0)\, dx$$

for all $v \in BV(\Omega)$. We shall prove that u_0 is not a critical point according to [22, Definition 5.1]. We are assuming that there exist $\delta > 0$ and $w \in BV(0, T)$ satisfying

$$\mathcal{J}\big(w\big) - \mathcal{J}(u_0) \le \int_\Omega f(x, u_0)(w - u_0)\, dx - 2\delta.$$

By the continuity in $BV(\Omega)$ of $\mathcal{J}$ and of the map $u \mapsto \int_\Omega f(x,u)(w-u)\,dx$, there exists a bounded neighbourhood U of u_0 in $BV(\Omega)$ such that

$$\mathcal{J}\big(w\big) - \mathcal{J}(u) \le \int_\Omega f(x,u)(w-u)\,dx - \delta$$

holds for all $u \in U$.

Take α, with $0 < \alpha < \inf\limits_{u\in U} \|w-u\|_{BV}$, and set $\tilde{\delta} = \inf_{u\in U} \frac{\delta}{\|w-u\|_{BV}} > 0$. Then we have

$$\begin{aligned}
\mathcal{J}\Big(u + t\tfrac{w-u}{\|w-u\|_{BV}}\Big) - \mathcal{J}(u) &\le \tfrac{t}{\|w-u\|_{BV}}\big(\mathcal{J}(w) - \mathcal{J}(u)\big) \\
&\le \int_\Omega f(x,u) t \tfrac{w-u}{\|w-u\|_{BV}}\,dx - \tfrac{t}{\|w-u\|_{BV}}\delta \\
&\le t\Big(\int_\Omega f(x,u)\tfrac{w-u}{\|w-u\|_{BV}}\,dx - \tilde{\delta}\Big)
\end{aligned}$$

for all $u \in U$ and all $t \in [0,\alpha]$. Define $\Psi : U \times [0,\alpha] \to BV(\Omega)$ by

$$\Psi(u,t) = u + t\frac{w-u}{\|w-u\|_{BV}}. \tag{57}$$

Then we have $\|\Psi(u,t) - u\|_{BV} = t$. Moreover, there exists a function η satisfying, possibly reducing U, $\lim\limits_{t\to 0^+} \eta(t) = 0$ uniformly with respect to $u \in U$, such that

$$\begin{aligned}
\mathcal{I}\big(\Psi(u,t)\big) - \mathcal{I}(u) &= \mathcal{J}\big(\Psi(u,t)\big) - \mathcal{J}(u) - \int_\Omega F\big(x,\Psi(u,t)\big)\,dx + \int_\Omega F\big(x,u\big)\,dx \\
&\le t\Big(\int_\Omega f(x,u)\tfrac{w-u}{\|w-u\|_{BV}}\,dx - \tilde{\delta}\Big) - \int_\Omega f(x,u) t \tfrac{w-u}{\|w-u\|_{BV}}\,dx + t\eta(t) \\
&= t\big(-\tilde{\delta} + \eta(t)\big),
\end{aligned} \tag{58}$$

for all $u \in U$ and all $t \in [0,\alpha]$. Possibly reducing α, we may assume that $\eta(t) \le \frac{1}{2}\tilde{\delta}$ for all $t \in [0,\alpha]$. Therefore, (58) yields $\mathcal{I}(u) - \mathcal{I}\big(\Psi(u,t)\big) \ge t\frac{1}{2}\tilde{\delta}$ for all $u \in U$ and all $t \in [0,\alpha]$, thus showing that u_0 is a $\frac{1}{2}\tilde{\delta}$-regular point of $\mathcal{I}$. This concludes the proof of the claim.

Fix $k \ge 1$ and pick any $\gamma_k \in \Gamma$ such that

$$\max_{\xi\in[0,1]} \mathcal{I}(\gamma_k(\xi)) \le c_{\mathcal{I}} + \tfrac{1}{k}.$$

According to Ekeland's variational principle (see, e.g., [26]), there is $\tilde{\gamma}_k \in \Gamma$ such that

$$\max_{\xi\in[0,1]} \|\tilde{\gamma}_k(\xi) - \gamma_k(\xi)\|_{BV} \le \tfrac{1}{\sqrt{k}},$$

$$\max_{\xi\in[0,1]} \mathcal{I}(\tilde{\gamma}_k(\xi)) \le \max_{\xi\in[0,1]} \mathcal{I}(\gamma_k(\xi)) \le c_{\mathcal{I}} + \tfrac{1}{k}$$

and

$$\max_{\xi\in[0,1]} \mathcal{I}(\gamma(\xi)) > \max_{\xi\in[0,1]} \mathcal{I}(\tilde{\gamma}_k(\xi)) - \tfrac{1}{\sqrt{k}} \max_{\xi\in[0,1]} \|\gamma(\xi) - \tilde{\gamma}_k(\xi)\|_{BV}$$

for all $\gamma \in \Gamma$ with $\gamma \neq \tilde{\gamma}_k$. In the proof of [22, Theorem 5.1], it is shown that there exists $\xi_k \in [0,1]$ such that

$$c_{\mathcal{I}} - \tfrac{1}{k} \leq \mathcal{I}(\tilde{\gamma}_k(\xi_k))$$

and

$$\delta(\mathcal{I}, \tilde{\gamma}_k(\xi_k)) \leq \tfrac{1}{\sqrt{k}}. \tag{59}$$

Set $v_k = \tilde{\gamma}_k(\xi_k)$. Then, (54) and

$$\min_{\xi\in[0,1]} \|v_k - \gamma_k(\xi)\|_{BV} \leq \tfrac{1}{\sqrt{k}} \tag{60}$$

are satisfied for all k. In particular,

$$\lim_{k\to+\infty} \mathcal{I}(v_k) = c_{\mathcal{I}}. \tag{61}$$

We claim that there exists a sequence $(\varepsilon_k)_k$ in $\mathbb{R}$, with $\lim_{k\to+\infty} \varepsilon_k = 0$, satisfying (56) for each k and all $v \in BV(\Omega)$. By contradiction, suppose that there exist k, $\varepsilon_k < -2\delta(\mathcal{I}, v_k)$, $\sigma > 0$ and $w \in BV(\Omega)$ satisfying

$$\mathcal{J}\big(w\big) - \mathcal{J}(v_k) \leq \int_\Omega f(x, v_k)(w - v_k)\, dx + \varepsilon_k \|w - v_k\|_{BV} - 2\sigma.$$

Arguing as in the claim above, we can actually assume that there exists a bounded neighbourhood U of v_k in $BV(\Omega)$ such that

$$\mathcal{J}\big(w\big) - \mathcal{J}(u) \leq \int_\Omega f(x, u)(w - u)\, dx + \varepsilon_k \|w - u\|_{BV} - \sigma$$

holds for all $u \in U$. Take α, with $0 < \alpha < \inf_{u\in U} \|w - u\|_{BV}$. Then, for all $u \in U$ and all $t \in [0, \alpha]$, we have

$$\mathcal{J}\Big(u + t\tfrac{w-u}{\|w-u\|_{BV}}\Big) - \mathcal{J}(u) \leq \int_\Omega f(x, u) t \tfrac{w-u}{\|w-u\|_{BV}}\, dx + t\varepsilon_k.$$

Define $\Psi : U \times [0, \alpha] \to BV(\Omega)$ as in (57). Then we have $\|\Psi(u, t) - u\|_{BV} = t$. Moreover, arguing as above, we can find a function η satisfying, possibly reducing U, $\lim_{t\to 0^+} \eta(t) = 0$ uniformly with respect to $u \in U$, and

$$\begin{aligned}\mathcal{I}\big(\Psi(u,t)\big) - \mathcal{I}(u) &= \mathcal{J}\big(\Psi(u,t)\big) - \mathcal{J}(u) - \int_\Omega F\big(x, \Psi(u,t)\big)\, dx + \int_\Omega F\big(x, u\big)\, dx \\ &\leq t\big(\varepsilon_k + \eta(t)\big),\end{aligned}$$

Possibly reducing α, we may assume that $\eta(t) \le -\frac{1}{2}\varepsilon_k$ for all $t \in [0,\alpha]$. Then $\mathcal{I}(u) - \mathcal{I}\big(\Psi(u,t)\big) \ge t\frac{1}{2}|\varepsilon_k|$ for all $t \in [0,\alpha]$, thus showing that v_k is a $\frac{1}{2}|\varepsilon_k|$-regular point of $\mathcal{I}$, contradicting the assumption $\varepsilon_k < -2\delta(\mathcal{I}, v_k)$.

We proved that the sequences $(\gamma_k)_k$, $(v_k)_k$ and $(\varepsilon_k)_k$, with $\lim_{k\to+\infty} \varepsilon_k = 0$, satisfy, for each k, (54), (60) and, for all $v \in BV(\Omega)$, (56). A relabelling of the subsequence with indexes k^2 finally satisfies all these properties and (55) as well. □

We conclude this section with two technical estimates on the functional $\mathcal{I}$, showing its coercivity on suitable cones.

Lemma 3.11 (A coercivity property)**.** *Assume* (h_0), (h_1), (h_2), (h_3') *and*

(h_4) *there exists* $(\mu,\nu) \in \mathcal{C}$ *such that*

$$\operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} f(x,s) < \mu \quad \text{and} \quad \operatorname*{ess\,inf}_{\Omega\times\mathbb{R}} f(x,s) > -\nu.$$

Then there exists $\eta > 0$ *such that*

$$\mathcal{I}(w+r) \ge \eta \int_\Omega |Dw| - \int_\Omega F(x,r)\,dx$$

for every $r \in \mathbb{R}$ *and* $w \in \mathcal{W}$*, where*

$$\mathcal{W} = \Big\{ w \in BV(\Omega) : \mu \int_\Omega w^+\,dx - \nu \int_\Omega w^-\,dx = 0 \Big\}. \tag{62}$$

Proof. By (h_4) there exists $\vartheta \in]0,1[$ such that

$$\operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} f(x,s) \le \vartheta\mu \quad \text{and} \quad \operatorname*{ess\,inf}_{\Omega\times\mathbb{R}} f(x,s) \ge -\vartheta\nu.$$

For any given $r \in \mathbb{R}$ and $w \in \mathcal{W}$ we have, by Proposition 2.4, Proposition 2.1 and (h_2),

$$\begin{aligned}
\mathcal{I}(w+r) &= \mathcal{J}(w) - \int_\Omega F(x,w+r)\,dx \\
&= \mathcal{J}(w) - \int_\Omega \big(F(x,w+r) - F(x,r)\big)\,dx - \int_\Omega F(x,r)\,dx \\
&= \mathcal{J}(w) - \int_\Omega \Big(\int_r^{r+w(x)} f(x,s)\,ds\Big)\operatorname{sgn}(w^+)\,dx \\
&\quad + \int_\Omega \Big(\int_{r+w(x)}^{r} f(x,s)\,ds\Big)\operatorname{sgn}(w^-)\,dx - \int_\Omega F(x,r)\,dx \\
&\ge \mathcal{L}(w) - \vartheta\mu \int_\Omega w^+\,dx - \vartheta\nu \int_\Omega w^-\,dx - \int_\Omega F(x,r)\,dx \\
&\ge (1-\vartheta)\mathcal{L}(w) - \int_\Omega F(x,r)\,dx \\
&\ge (1-\vartheta)\rho \int_\Omega |Dw| - \int_\Omega F(x,r)\,dx.
\end{aligned}$$

Hence the conclusion follows. □

Lemma 3.12 (A coercivity property in dimensiont $N = 1$)**.** *Suppose that $N = 1$ and let $\Omega =]0, T[$. Assume (h_1'), (h_2), (h_3') and*

(h_5) *there exists $g \in L^1(0, T)$, with $\|g^-\|_{L^1} < \rho$, with ρ defined in (h_2), such that $f(x, s) \geq g(x)$ for a.e. $x \in]0, T[$ and every $s \in \mathbb{R}$.*

Then there exists $\eta > 0$ such that

$$\mathcal{I}(v) \geq \eta \int_{]0,T[} |Dv| - \int_0^T F(x, \operatorname*{ess\,sup}_{]0,T[} v)\, dx$$

for every $v \in BV(0, T)$.

Proof. For any given $v \in BV(0, T)$ we have, by Proposition 2.1 and Proposition 3.7,

$$\begin{aligned}
\mathcal{I}(v) &= \mathcal{J}(v) + \int_0^T \Big(\int_{v(x)}^{\operatorname*{ess\,sup}_{]0,T[} v} f(x, s)\, ds \Big)\, dx - \int_0^T F(x, \operatorname*{ess\,sup}_{]0,T[} v)\, dx \\
&\geq \rho \int_{]0,T[} |Dv| - \int_0^T g^-(x)(\operatorname*{ess\,sup}_{]0,T[} v - v)\, dx - \int_0^T F(x, \operatorname*{ess\,sup}_{]0,T[} v)\, dx \\
&\geq \rho \int_{]0,T[} |Dv| - \|g^-\|_{L^1}(\operatorname*{ess\,sup}_{]0,T[} v - \operatorname*{ess\,inf}_{]0,T[} v) - \int_0^T F(x, \operatorname*{ess\,sup}_{]0,T[} v)\, dx \\
&\geq (\rho - \|g^-\|_{L^1}) \int_{]0,T[} |Dv| - \int_0^T F(x, \operatorname*{ess\,sup}_{]0,T[} v)\, dx.
\end{aligned}$$

Hence the conclusion follows. □

Remark 3.6 The conclusion of Lemma 3.12 still holds if we replace assumption (h_5) with

(h_5') *there exists $g \in L^1(0, T)$, with $\|g^+\|_{L^1} < \rho$, with ρ defined in (h_2), such that $f(x, s) \leq g(x)$ for a.e. $x \in]0, T[$ and every $s \in \mathbb{R}$.*

Remark 3.7 In dimension $N = 1$, if $h = 0$ and $\kappa = 0$, then we can take $\rho = 1$ in Proposition 3.12 and Remark 3.6.

3.1 Existence versus non-existence

In order to make more transparent our statements we assume in this subsection that $h = 0$ and $\kappa = 0$, so that the functional $\mathcal{L}$ is just the total variation. However, similar conclusions hold in the general case as well. We also write f in the form $f(x, s) = g(x, s) + e(x)$.

Our first result shows that the existence of solutions is guaranteed in the case where $g = 0$ and e lies, in some sense, "below" the curve $\mathcal{C}$ defined in (22).

Proposition 3.13. *Assume* (h_0). *Fix* $(\mu,\nu)\in\mathcal{C}$. *Then for every* $e\in L^\infty(\Omega)$, *with* $\int_\Omega e\,dx=0$, $\operatorname*{ess\,sup}_\Omega e<\mu$ *and* $\operatorname*{ess\,inf}_\Omega e>-\nu$, *the problem*

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1+|\nabla u|^2}\Big)=e(x) & \text{in } \Omega,\\ -\nabla u\cdot n/\sqrt{1+|\nabla u|^2}=0 & \text{on } \partial\Omega \end{cases} \tag{63}$$

has a solution $w\in BV(\Omega)$ *with* $\mu\int_\Omega w^+\,dx-\nu\int_\Omega w^-\,dx=0$.

Proof. For every $v\in BV(\Omega)$, let us write $v=w+\mathcal{P}(v)$, with $\mathcal{P}$ defined in Proposition 3.8; hence, we have

$$\mathcal{I}(v)=\mathcal{I}(w)=\int_\Omega\sqrt{1+|Dw|^2}-\int_\Omega ew\,dx.$$

By Lemma 3.11, the functional $\mathcal{I}$ is bounded from below. Let $(v_n)_n$ be a minimizing sequence; clearly, $(w_n)_n$ is a minimizing sequence too. Using again Lemma 3.11 we see that $(w_n)_n$ is bounded in $BV(\Omega)$ and hence it has a subsequence converging in $L^1(\Omega)$ to some $w\in\mathcal{W}$, where $\mathcal{W}$ is defined in (62). The usual semicontinuity argument shows that w is a minimizer and therefore a solution of (63). □

Our second result shows that the existence of solutions is not guaranteed if f lies, in some sense, "above" the curve $\mathcal{C}$ defined in (22).

Proposition 3.14. *Assume* (h_0). *Fix* $(\mu,\nu)\in\mathcal{C}$. *Then there exist functions* $e\in L^\infty(\Omega)$, *with* $\int_\Omega e\,dx=0$ *and either* $\operatorname*{ess\,sup}_\Omega e>\mu$ *and* $\operatorname*{ess\,inf}_\Omega e\le-\nu$, *or* $\operatorname*{ess\,sup}_\Omega e\ge\mu$ *and* $\operatorname*{ess\,inf}_\Omega e<-\nu$, *and a constant* $\gamma>0$ *such that, for any function* $g:\Omega\times\mathbb{R}\to\mathbb{R}$ *satisfying the Carathéodory conditions and* $\operatorname*{ess\,sup}_{\Omega\times\mathbb{R}}|g(x,s)|\le\gamma$, *the problem*

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1+|\nabla u|^2}\Big)=g(x,u)+e(x) & \text{in } \Omega,\\ -\nabla u\cdot n/\sqrt{1+|\nabla u|^2}=0 & \text{on } \partial\Omega \end{cases} \tag{64}$$

has no solution.

Proof. Let $\varphi\in BV(\Omega)\setminus\{0\}$ be such that

$$\mu\int_\Omega\varphi^+dx-\nu\int_\Omega\varphi^-dx=0\quad\text{and}\quad\int_\Omega|D\varphi|=\mu\int_\Omega\varphi^+dx+\nu\int_\Omega\varphi^-dx.$$

Pick $\rho,\sigma\in\mathbb{R}_0^+$ such that $\sigma\int_\Omega\mathrm{sgn}(\varphi^-)\,dx=\rho\int_\Omega\mathrm{sgn}(\varphi^+)\,dx$ and either $\rho>\mu$ and $\sigma\ge\nu$, or $\rho\ge\mu$ and $\sigma>\nu$. Define $e\in L^\infty(\Omega)$ by setting

$$e=\rho\,\mathrm{sgn}(\varphi^+)-\sigma\,\mathrm{sgn}(\varphi^-).$$

Clearly, we have $\int_\Omega e\,dx = 0$. Take $\gamma > 0$ and any function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the Carathéodory conditions and $\operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} |g(x,s)| \le \gamma$. Take any $u \in BV(0,T)$ and compute, for $k \in \mathbb{R}_0^+$,

$$\begin{aligned}
\mathcal{J}(k\varphi) - \int_\Omega (g(x,u)+e)k\varphi\,dx &\le |\Omega| + \int_\Omega |Dk\varphi| + \gamma\int_\Omega |k\varphi|\,dx - \int_\Omega ek\varphi\,dx \\
&= |\Omega| + k\int_\Omega |D\varphi| + k\gamma\int_\Omega |\varphi|\,dx - k\int_\Omega \big(\rho\,\mathrm{sgn}(\varphi^+) - \sigma\,\mathrm{sgn}(\varphi^-)\big)(\varphi^+ - \varphi^-)\,dx \\
&= |\Omega| + k\Big(\int_\Omega |D\varphi| - \mu\int_\Omega \varphi^+\,dx - \nu\int_\Omega \varphi^-\,dx\Big) \\
&\qquad - k(\rho-\mu)\int_\Omega \varphi^+\,dx - k(\sigma-\nu)\int_\Omega \varphi^-\,dx + k\gamma\int_\Omega |\varphi|\,dx \\
&= |\Omega| - k\Big((\rho-\mu-\gamma)\int_\Omega \varphi^+\,dx + (\sigma-\nu-\gamma)\int_\Omega \varphi^-\,dx\Big).
\end{aligned}$$

Hence we infer that

$$\inf_{v\in BV(\Omega)} \Big(\mathcal{J}(v) - \int_\Omega (g(x,u)+e)v\,dx\Big) = -\infty,$$

provided that $\gamma > 0$ is taken so small that $(\rho-\mu-\gamma)\int_\Omega \varphi^+\,dx + (\sigma-\nu-\gamma)\int_\Omega \varphi^-\,dx > 0$. Therefore problem (64) has no solution, according to Remark 3.2. □

Remark 3.8 We can replace the condition $\operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} |g(x,s)| \le \gamma$, for some small constant γ, with $|g(x,s)| \le \gamma(x)$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, for some $\gamma \in L^p(\Omega)$, with $p > N$, having a small L^p-norm.

In our third result we show that the conclusions of Proposition 3.14 can be somehow strengthened in the one-dimensional case.

Proposition 3.15. *Assume $N = 1$ and let $\Omega =]0,T[$. Fix $\rho, \sigma \in \mathbb{R}_0^+$ such that $\frac{1}{\rho} + \frac{1}{\sigma} < T$ and set $\tau = \frac{\sigma}{\rho+\sigma}T$. Then there exists $\gamma \in L^1(0,T)$ such that for every $e \in L^1(0,T)$, with $\frac{1}{\tau}\int_0^\tau e\,dx = -\rho$ and $\frac{1}{T-\tau}\int_\tau^T e\,dx = \sigma$ (and hence $\int_0^T e\,dx = 0$), and for every $g :]0,T[\times\mathbb{R} \to \mathbb{R}$ satisfying the Carathéodory conditions, with $|g(x,s)| \le \gamma(x)$ for a.e. $x \in]0,T[$ and every $s \in \mathbb{R}$, the problem*

$$\begin{cases} -\Big(u'/\sqrt{1+|u'|^2}\Big) = g(x,u) + e(x) & \text{in }]0,T[, \\ u'(0) = u'(T) = 0 \end{cases} \tag{65}$$

has no solution.

Proof. Let $\varphi \in BV(0,T)$ be given by $\varphi(x) = 1$, if $x \in [0,\tau[$, and $\varphi(x) = -1$, if $x \in]\tau, T]$. Take any $u \in BV(0,T)$ and compute, for $k \in \mathbb{R}_0^+$,

$$\begin{aligned}
&\mathcal{J}(k\varphi) - \int_0^T (g(x,u) - e)k\varphi\, dx \\
&\quad \le T + 2k + k\int_0^\tau e\, dx - k\int_\tau^T e\, dx + k\int_0^T |\gamma|\, dx \\
&\quad = T + 2k - k\rho\tau - k\sigma(T-\tau) + k\|\gamma\|_{L^1} \\
&\quad = T + 2k - 2k\tfrac{\rho\,\sigma}{\rho+\sigma}T + k\|\gamma\|_{L^1} \\
&\quad = T + 2k\left(1 - \tfrac{\rho\,\sigma}{\rho+\sigma}T + \tfrac{1}{2}\|\gamma\|_{L^1}\right).
\end{aligned}$$

Clearly, the last term tends to $-\infty$ as $k \to +\infty$, provided that $\frac{1}{2}\|\gamma\|_{L^1} \in \left]0, \frac{\rho\,\sigma}{\rho+\sigma}T - 1\right[$. Therefore u is not a solution of (65). □

Remark 3.9 Note that the curve

$$\Sigma_1 = \left\{(\rho,\sigma) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : \frac{1}{\rho} + \frac{1}{\sigma} = T\right\}$$

is the limit, as $p \to 1^+$, of the curves

$$\Sigma_p = \left\{(\rho,\sigma) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : \frac{1}{\rho^{\frac{1}{p}}} + \frac{1}{\sigma^{\frac{1}{p}}} = \frac{2T}{\pi_p}\right\},$$

where $\pi_p = \frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin(\frac{\pi}{p})}$. Note that $\pi_p \to 2$, as $p \to 1$.

Corollary 3.16. *Assume $N = 1$ and let $\Omega =]0,T[$. Fix $(\mu,\nu) \in \mathcal{C}$, i.e., $\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \sqrt{2T}$. Then there exist $\gamma \in L^1(0,T)$ and $e \in C^\infty([0,T])$, with $\int_\Omega e\, dx = 0$, $\operatorname*{ess\,inf}_{[0,T]} e < -\mu$ and $\operatorname*{ess\,sup}_{]0,T[} e > \nu$, such that for every $g :]0,T[\times \mathbb{R} \to \mathbb{R}$ satisfying the Carathéodory conditions, with $|g(x,s)| \le \gamma(x)$ for a.e. $x \in]0,T[$ and every $s \in \mathbb{R}$, problem (65) has no solution.*

Proof. Pick $\eta > 1$ such that $\frac{1}{\mu} + \frac{1}{\nu} < \eta T$ and set $\rho = \eta\mu$ and $\sigma = \eta\nu$. Then Proposition 3.15 easily yields the conclusion. □

Remark 3.10 Note that $\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \sqrt{2T}$ and $\frac{1}{\mu} + \frac{1}{\nu} = T$ if and only if $\mu = \nu = \frac{2}{T}$: in this case we can choose η as close to 1 as we want, and hence $\|e\|_\infty$ as close to the eigenvalue $\frac{2}{T}$ as we want.

Remark 3.11 From Proposition 3.14 it follows, in particular, that there exist functions $e \in L^\infty(\Omega)$, with $\int_\Omega e\, dx = 0$, and $g : \mathbb{R} \to \mathbb{R}$ continuous, bounded and strictly monotone, with

$$\lim_{s\to-\infty} g(s) < 0 < \lim_{s\to+\infty} g(s),$$

or

$$\lim_{s\to-\infty} g(s) > 0 > \lim_{s\to+\infty} g(s),$$

such that problem (64) has no solution. This will motivate the existence results we are going to present in the following subsection.

3.2 Existence results

We prove in this section the existence of solutions of problem (43). Let us observe that a necessary condition in order a solution u exists is that $\int_\Omega f(x,u)\,dx = 0$. This implies that f must change sign in $\Omega\times\mathbb{R}$. Here we will assume some stronger assumptions which guarantee such a condition.

3.2.1 Ahmad-Lazer-Paul conditions

We assume in this subsection the coercivity, or the anticoercivity, of the averaged potential map $s \mapsto \int_\Omega F(x,s)\,dx$ on $\mathbb{R}$. If $h = 0$ and $\kappa = 0$, and then $\mathcal{L}$ is the total variation, this assumption can be interpreted as a non-interference condition with the first eigenvalue $c_1 = 0$ of the 1-Laplace operator with Neumann boundary conditions as defined in [8]. This assumption will be coupled with a non-interference condition with the curve $\mathcal{C}$ defined in (22), as expressed by assumption (h_4). It is worth noting that, in the light of the non-existence results stated in Subsection 3.1, and in particular of Remark 3.11, assumption (h_4) cannot be omitted in this frame.

Our first result deals with the case where a coercivity condition on $s \mapsto \int_\Omega F(x,s)\,dx$ is assumed. In this case a solution is obtained by a minimax procedure.

Theorem 3.17. *Assume* (h_0), (h_1), (h_2), (h_3), (h_4) *and*

(h_7) $\displaystyle\lim_{s\to\pm\infty}\int_\Omega F(x,s)\,dx = +\infty.$

Then problem (43) *has at least one solution* u *such that*

$$-\min_{s\in\mathbb{R}}\int_\Omega F(x,s)\,dx \le \mathcal{I}(u) \le |\Omega| - \min_{s\in\mathbb{R}}\int_\Omega F(x,s)\,dx.$$

Proof. The proof is divided into three steps.
Step 1. Mountain pass geometry. By (h_3) the function $s \mapsto \int_\Omega F(x,s)\,dx$ is continuous on $\mathbb{R}$. Hence, using (h_7), we can find $a^-, a^+, b \in \mathbb{R}$, with $a^- < b < a^+$, such that

$$\int_\Omega F(x,b)\,dx = \min_{s\in\mathbb{R}}\int_\Omega F(x,s)\,dx,$$

and

$$\int_\Omega F(x,a^\pm)\,dx > |\Omega| + \int_\Omega F(x,b)\,dx.$$

Set

$$\mathcal{S} = \{v \in BV(\Omega) : \mathcal{P}(v) = b\} = \{w + b : w \in \mathcal{W}\},$$

where $\mathcal{P}$ and $\mathcal{W}$ are defined in Proposition 3.8 and by (62), respectively. By Lemma 3.11 we have

$$\inf_{v\in\mathcal{S}} \mathcal{I}(v) = \inf_{w\in\mathcal{W}} \mathcal{I}(w + b) \geq -\int_\Omega F(x, b)\, dx$$

and hence

$$\mathcal{I}(a^-) = |\Omega| - \int_\Omega F(x, a^-)\, dx < -\int_\Omega F(x, b)\, dx \leq \inf_{\mathcal{S}} \mathcal{I}(v), \tag{66}$$

$$\mathcal{I}(a^+) = |\Omega| - \int_\Omega F(x, a^-)\, dx < -\int_\Omega F(x, b)\, dx \leq \inf_{\mathcal{S}} \mathcal{I}(v). \tag{67}$$

Let us define

$$\Gamma = \{\gamma \in C^0\big([0, 1], BV(\Omega)\big) : \gamma(0) = a^-, \gamma(1) = a^+\}.$$

For any $\gamma \in \Gamma$ the function $\mathcal{P} \circ \gamma : [0, 1] \to \mathbb{R}$ is continuous and satisfies

$$\mathcal{P}(\gamma(0)) = a^- < b < a^+ = \mathcal{P}(\gamma(1)).$$

Therefore there exists $\bar{\xi} \in]0, 1[$ such that $\mathcal{P}(\gamma(\bar{\xi})) = b$, thus showing that $\gamma([0, 1]) \cap \mathcal{S} \neq \emptyset$, that is, the sets $\{a^-, a^+\}$ and $\mathcal{S}$ link. We set $x_0 = a^-$, $x_1 = a^+$ and

$$c_{\mathcal{I}} = \inf_{\gamma\in\Gamma} \max_{\xi\in[0,1]} \mathcal{I}(\gamma(\xi)).$$

By (66) and (67) we have

$$c_{\mathcal{I}} \geq \inf_{\mathcal{S}} \mathcal{I}(v) > \max\{\mathcal{I}(x_0), \mathcal{I}(x_1)\}.$$

Accordingly, Lemma 3.10 yields the existence of sequences $(v_k)_k$ and$(\varepsilon_k)_k$, with $v_k \in BV(\Omega)$ and $\varepsilon_k \in \mathbb{R}$, satisfying

$$\lim_{k\to+\infty} \varepsilon_k = 0,$$

(54), and in particular

$$\lim_{k\to+\infty} \mathcal{I}(v_k) = c_{\mathcal{I}},$$

and (56), that is

$$\mathcal{J}(v) - \mathcal{J}(v_k) \geq \int_\Omega f(x, v_k)(v - v_k)\, dx + \varepsilon_k \|v - v_k\|_{BV},$$

for each k and all $v \in BV(\Omega)$.

Step 2. The sequence $(v_k)_k$ is bounded in $BV(\Omega)$. By (h_4) there exists $\vartheta \in]0,1[$ such that

$$\operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} f(x,s) \le \vartheta\mu \quad \text{and} \quad \operatorname*{ess\,inf}_{\Omega\times\mathbb{R}} f(x,s) \ge -\vartheta\nu.$$

For each k we set $r_k = \mathcal{P}(v_k)$ and $w_k = v_k - r_k \in \mathcal{W}$. Taking $v = v_k - w_k^+$ in (56), we have by (h_4)

$$\begin{aligned}
\mathcal{J}(w_k) - \mathcal{J}(-w_k^-) &= \mathcal{J}(w_k) - \mathcal{J}(w_k - w_k^+) \\
&= \mathcal{J}(v_k) - \mathcal{J}(v_k - w_k^+) \\
&\le \int_\Omega f(x,v_k) w_k^+ \, dx - \varepsilon_k \|w_k^+\|_{BV} \\
&\le \vartheta\mu \int_\Omega w_k^+ \, dx - \varepsilon_k \|w_k^+\|_{BV}.
\end{aligned}$$

Similarly, taking $v = v_k + w_k^-$ in (56), we have by (h_4)

$$\begin{aligned}
\mathcal{J}(w_k) - \mathcal{J}(w_k^+) &= \mathcal{J}(w_k) - \mathcal{J}(w_k + w_k^-) \\
&= \mathcal{J}(v_k) - \mathcal{J}(v_k + w_k^-) \\
&\le -\int_\Omega f(x,v_k) w_k^- \, dx - \varepsilon_k \|w_k^-\|_{BV} \\
&\le \vartheta\nu \int_\Omega w_k^- \, dx - \varepsilon_k \|w_k^-\|_{BV}.
\end{aligned}$$

Summing up we obtain, by Proposition 3.6 and Proposition 2.4,

$$\begin{aligned}
\mathcal{L}(w_k) - |\Omega| &\le \mathcal{J}(w_k) - \mathcal{J}(0) \\
&\le \mathcal{J}(w_k) + \mathcal{J}(w_k) - \mathcal{J}(w_k^+) - \mathcal{J}(-w_k^-) \\
&\le \vartheta\mu \int_\Omega w_k^+ \, dx - \varepsilon_k \|w_k^+\|_{BV} + \vartheta\nu \int_\Omega w_k^- \, dx - \varepsilon_k \|w_k^-\|_{BV} \\
&\le \vartheta \mathcal{L}(w_k) + |\varepsilon_k| \|w_k\|_{BV}
\end{aligned}$$

and hence, by Proposition 2.1 and Remark 2.4,

$$\begin{aligned}
(1-\vartheta)\mathcal{L}(w_k) &\le |\varepsilon_k| \|w_k\|_{BV} + |\Omega| \\
&= |\varepsilon_k| \Big(\int_\Omega |Dw_k| + \int_\Omega |w_k| \, dx \Big) + |\Omega| \\
&\le |\varepsilon_k| \Big(\frac{1}{\rho} + \frac{1}{\mu} + \frac{1}{\nu} \Big) \mathcal{L}(w_k) + |\Omega|.
\end{aligned}$$

As $\lim_{k\to+\infty} \varepsilon_k = 0$, by Proposition 2.1, there is a constant $K > 0$ such that, for all k,

$$\mathcal{L}(w_k) \le K,$$

and hence,

$$\int_\Omega |Dw_k| \le K, \tag{68}$$

$$\int_\Omega |w_k|\,dx \le K, \tag{69}$$

$$\mathcal{J}(w_k) \le K. \tag{70}$$

From (54), (69), (70) we deduce, for all large k,

$$\begin{aligned} c_{\mathcal{I}} - 1 \le \mathcal{I}(v_k) &= \mathcal{I}(w_k + r_k) \\ &= \mathcal{J}(w_k) - \int_\Omega F(x, w_k + r_k)\,dx \\ &\le K - \int_\Omega \Big(\int_{r_k}^{r_k + w_k(x)} f(x,s)\,ds \Big)\,dx - \int_\Omega F(x, r_k)\,dx \\ &\le K + (\mu+\nu) \int_\Omega |w_k|\,dx - \int_\Omega F(x, r_k)\,dx \\ &\le K(1+\mu+\nu) - \int_\Omega F(x, r_k)\,dx. \end{aligned}$$

Using (h_7) we conclude that

$$\sup_k |r_k| < +\infty. \tag{71}$$

Combining (68), (69) and (71) finally yields

$$\sup_k \|v_k\|_{BV} < +\infty. \tag{72}$$

Step 3. Existence of a solution. Since the sequence $(v_k)_k$ is bounded in $BV(\Omega)$ there exist a subsequence, we still denote by $(v_k)_k$, and a function $u \in BV(\Omega)$, such that $\lim_{k\to+\infty} v_k = u$ in $L^q(\Omega)$, with $q = \frac{p}{p-1} \in [1, 1^*[$, and a.e. in Ω. Hence we have, by (h_3),

$$\lim_{k\to+\infty} f(x, v_k(x)) = f(x, u(x)),$$

for a.e. $x \in \Omega$, and, by the global boundedness of f implied by (h_4),

$$\lim_{k\to+\infty} f(\cdot, v_k) = f(\cdot, u)$$

in $L^p(\Omega)$. Similarly, we have

$$\lim_{k\to+\infty} F(x, v_k(x)) = F(x, u(x)),$$

for a.e. $x \in \Omega$, and, as $F(\cdot, s)$ grows at most linearly with respect to s uniformly a.e. in Ω,

$$\lim_{k\to+\infty} \int_\Omega F(x, v_k)\,dx = \int_\Omega F(x, u)\,dx.$$

Moreover, the lower semicontinuity of $\mathcal{J}$ with respect to the L^q-convergence in $BV(\Omega)$ implies

$$\liminf_{k\to+\infty} \mathcal{J}(v_k) \geq \mathcal{J}(u).$$

Finally, for any fixed $v \in BV(\Omega)$, we have

$$\lim_{k\to+\infty} \int_\Omega f(x, v_k)(v - v_k)\, dx = \int_\Omega f(x, u)(v - u)\, dx$$

and

$$\lim_{k\to+\infty} \varepsilon_k \|v - v_k\|_{BV} = 0.$$

Thus we get, passing to the inferior limit in (56),

$$\begin{aligned}\mathcal{J}(v) - \int_\Omega f(x, u)(v - u)\, dx &= \mathcal{J}(v) - \lim_{k\to+\infty} \int_\Omega f(x, v_k)(v - v_k)\, dx - \lim_{k\to+\infty} \varepsilon_k \|v - v_k\|_{BV} \\ &\geq \liminf_{k\to+\infty} \mathcal{J}(v_k) \geq \mathcal{J}(u),\end{aligned}$$

and hence

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \int_\Omega f(x, u)(v - u)\, dx$$

for all $v \in BV(\Omega)$, that is u is a solution of (43).

Step 4. A critical value estimate. Taking $v = u$ in (56), we get for all k

$$\mathcal{J}(u) - \int_\Omega f(x, v_k)(u - v_k)\, dx - \varepsilon_k \|u - v_k\|_{BV} \geq \mathcal{J}(v_k)$$

and hence

$$\mathcal{J}(u) = \lim_{k\to+\infty} \Big(\mathcal{J}(u) - \int_\Omega f(x, v_k)(u - v_k)\, dx - \varepsilon_k \|u - v_k\|_{BV} \Big) \geq \limsup_{k\to+\infty} \mathcal{J}(v_k).$$

Since, on the other hand,

$$\mathcal{J}(u) \leq \liminf_{k\to+\infty} \mathcal{J}(v_k),$$

we conclude that

$$\lim_{k\to+\infty} \mathcal{J}(v_k) = \mathcal{J}(u).$$

Thus we obtain

$$c_{\mathcal{I}} = \lim_{k\to+\infty} \mathcal{I}(v_k) = \lim_{k\to+\infty} \mathcal{J}(v_k) - \lim_{k\to+\infty} \int_\Omega F(x, v_k)\, dx = \mathcal{J}(u) - \int_\Omega F(x, u)\, dx = \mathcal{I}(u).$$

Taking as γ the segment joining a^- with a^+, we see that

$$c_{\mathcal{I}} \leq |\Omega| - \int_\Omega F(x, b)\, dx.$$

Since, on the other hand,

$$c_{\mathcal{I}} \geq \inf_{v\in\mathcal{S}} \mathcal{I}(v) \geq -\int_\Omega F(x,b)\,dx,$$

the conclusion follows. □

Now we find a solution by local minimization.

Proposition 3.18. *Assume* (h_0), (h_1), (h_2), (h_3), (h_4) *and*

(h_8) *there exist* $a^-, a^+ \in \mathbb{R}$, *with* $a^- < a^+$, *such that*

$$\max_{s\in[a^-,a^+]} \int_\Omega F(x,s)\,dx > |\Omega| + \max\Big\{\int_\Omega F(x,a^-)\,dx, \int_\Omega F(x,a^+)\,dx\Big\}.$$

Then problem (43) *has at least one solution* u *such that*

$$-\max_{s\in[a^-,a^+]} \int_\Omega F(x,s)\,dx \leq \mathcal{I}(u) \leq |\Omega| - \max_{s\in[a^-,a^+]} \int_\Omega F(x,s)\,dx$$

and

$$\mathcal{P}(u) \in \,]a^-, a^+[,$$

$\mathcal{P}$ *being the projection defined in Proposition* 3.8.

Proof. From (h_8) it follows that there exists $b \in \,]a^-, a^+[$ such that

$$\int_\Omega F(x,b)\,dx = \max_{s\in[a^-,a^+]} \int_\Omega F(x,s)\,dx$$

and

$$\mathcal{I}(b) = |\Omega| - \int_\Omega F(x,b)\,dx < -\max\Big\{\int_\Omega F(x,a^-)\,dx, \int_\Omega F(x,a^+)\,dx\Big\}.$$

Define the set

$$\mathcal{A} = \{v \in BV(\Omega) : \mathcal{P}(v) \in \,]a^-, a^+[\},$$

which is open in $BV(\Omega)$. By Lemma 3.11 we get

$$\mathcal{I}(b) \geq \inf_{v\in\mathcal{A}} \mathcal{I}(v) \geq \min_{r\in[a^-,a^+]} \Big(-\int_\Omega F(x,r)\,dx\Big) = -\int_\Omega F(x,b)\,dx.$$

Further, we have

$$\inf_{w\in\mathcal{W}} \mathcal{I}(w+a^-) \geq -\int_\Omega F(x,a^-)\,dx > \mathcal{I}(b) \geq \inf_{v\in\mathcal{A}} \mathcal{I}(v) \tag{73}$$

and

$$\inf_{w\in\mathcal{W}} \mathcal{I}(w+a^+) \geq -\int_\Omega F(x,a^+)\,dx > \mathcal{I}(b) \geq \inf_{v\in\mathcal{A}} \mathcal{I}(v). \tag{74}$$

Let $(v_k)_k$ be a sequence in $\mathcal{A}$ such that

$$\lim_{k\to+\infty} \mathcal{I}(v_k) = \inf_{v\in\mathcal{A}} \mathcal{I}(v).$$

We write, for each k, $v_k = w_k + r_k$, with $w_k \in \mathcal{W}$ and $r_k \in \,]a^-, a^+[$. Applying again Lemma 3.11 we get, for some $\eta > 0$ and all large k,

$$\eta \int_\Omega |Dw_k| \; - \int_\Omega F(x, r_k)\, dx \le \mathcal{I}(v_k) \le \mathcal{I}(b) + 1$$

and hence

$$\sup_k \int_\Omega |Dw_k| \; < +\infty.$$

Remark 2.4 and Remark 2.2 finally yield

$$\sup_k \|w_k\|_{BV} < +\infty.$$

Hence there exist subsequences of $(w_k)_k$ and $(r_k)_k$, that we still denote by $(w_k)_k$ and $(r_k)_k$ respectively, $w \in \mathcal{W}$ and $r \in [a^-, a^+]$ such that $\lim\limits_{k\to+\infty} w_k = w$ in $L^q(\Omega)$, with $q = \frac{p}{p-1}$, and a.e. in Ω, and $\lim\limits_{k\to+\infty} r_k = r$. Thus, setting $u = w + r$, we have

$$\lim_{k\to+\infty} \int_\Omega F(x, v_k)\, dx = \int_\Omega F(x, u)\, dx$$

and

$$\liminf_{k\to+\infty} \mathcal{J}(v_k) \ge \mathcal{J}(u).$$

Therefore we conclude that

$$\mathcal{I}(w + r) = \mathcal{I}(u) \le \lim_{k\to+\infty} \mathcal{I}(v_k) = \inf_{v\in\mathcal{A}} \mathcal{I}(v).$$

From (73) and (74) we infer that $r \in \,]a^-, a^+[$ and, hence, $u \in \mathcal{A}$ is a local minimizer of $\mathcal{I}$, with

$$-\int_\Omega F(x, b)\, dx \le \mathcal{I}(u) \le |\Omega| - \int_\Omega F(x, b)\, dx.$$

Finally, we can easily prove, using the convexity of $\mathcal{J}$ and the differentiability of the potential operator $\mathcal{F}$ in $BV(\Omega)$, that u satisfies (45), for all $v \in BV(\Omega)$, and hence it is a solution of problem (43). □

Remark 3.12 Condition (h_8) is clearly implied by

$$\liminf_{s\to\pm\infty} \int_\Omega F(x, s)\, dx = -\infty.$$

In the following result we assume an anticoercivity condition on the function $s \mapsto \int_\Omega F(x,s)\,dx$. In this case we find a solution which is a global minimizer.

Theorem 3.19. *Assume* (h_0), (h_1), (h_2), (h_3), (h_4) *and*

(h_9) $\displaystyle\lim_{s\to\pm\infty}\int_\Omega F(x,s)\,dx = -\infty.$

Then problem (43) *has at least one solution, which is a global minimizer of* $\mathcal{I}$ *in* $BV(\Omega)$.

Proof. The conclusion follows from Proposition 3.18 and Remark 3.12. Note however that conditions (h_4) and (h_9) imply, by Lemma 3.11, that the functional $\mathcal{I}$ is coercive and bounded from below. Hence, by a standard lower semicontinuity argument (see, e.g., the proof of Proposition 3.18), we conclude that a global minimizer does exist. □

Remark 3.13 It is known (see [1], or [18]) that the Ahmad-Lazer-Paul conditions (h_7) and (h_9) are respectively implied by the Landesman-Lazer conditions

(h_7') *there exists* $\ell \in L^1(0,T)$ *such that* $f(x,s)\,\mathrm{sgn}(s) \geq \ell(x)$ *for a.e.* $x \in]0,T[$ *and every* $s \in \mathbb{R}$ *and*

$$\int_\Omega \Big(\limsup_{s\to-\infty} f(x,s)\Big)\,dx < 0 < \int_\Omega \Big(\liminf_{s\to+\infty} f(x,s)\Big)\,dx$$

and

(h_9') *there exists* $\ell \in L^1(0,T)$ *such that* $f(x,s)\,\mathrm{sgn}(s) \leq \ell(x)$ *for a.e.* $x \in]0,T[$ *and every* $s \in \mathbb{R}$ *and*

$$\int_\Omega \Big(\liminf_{s\to-\infty} f(x,s)\Big)\,dx > 0 > \int_\Omega \Big(\limsup_{s\to+\infty} f(x,s)\Big)\,dx.$$

It is worthy to notice that, according to the non-existence results stated in Subsection 3.1 (see in particular Remark 3.11), assumption (h_4) cannot be omitted even if (h_7'), or (h_9'), is assumed in place of (h_7), or (h_9), respectively.

3.2.2 Hammerstein-type conditions

In this subsection we replace the Ahmad-Lazer-Paul condition (h_9) with the following Hammerstein-type condition to the left of the eigenvalue $c_1 = 0$, inspired from [21, 25]:

(h_9'') *there exists* $\zeta \in L^p(\Omega)$, *with* $p > N$, $\zeta(x) \leq 0$ *for a.e.* $x \in \Omega$ *and* $\zeta(x) < 0$ *on a set of positive measure, such that*

$$\limsup_{s\to\pm\infty} \frac{F(x,s)}{|s|} \leq \zeta(x)$$

uniformly a.e. in Ω.

Clearly, assumption (h_9'') implies (h_9). We point out that in this case condition (h_4) can be dropped. The following result is related to some classical results in [13, 27].

Theorem 3.20. *Assume* (h_0), (h_1), (h_2), (h_3) *and* (h_9''). *Then problem* (43) *has at least one solution, which is a global minimizer of* $\mathcal{I}$ *in* $BV(\Omega)$.

Proof. By (h_3) the functional $\mathcal{I}$ is well-defined and continuous in $BV(\Omega)$. Moreover, by (h_9'') Proposition 3.9 implies the existence of a constant $\delta > 0$ such that

$$\mathcal{L}(v) - \int_\Omega \zeta |v| \, dx \geq \delta \|v\|_{BV}$$

for all $v \in BV(\Omega)$. Fix $\varepsilon \in]0, \delta[$. By (h_3) and (h_9''), there is $\gamma \in L^1(\Omega)$ such that

$$F(x, s) \leq (\zeta(x) + \varepsilon) \, |s| + \gamma(x),$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$. Hence we have

$$\begin{aligned} \mathcal{I}(v) &\geq \mathcal{L}(v) - \int_\Omega (\zeta + \varepsilon) \, |v| \, dx - \|\gamma\|_{L^1} \\ &\geq \delta \|v\|_{BV} - \varepsilon \|v\|_{L^1} - \|\gamma\|_{L^1} \\ &\geq (\delta - \varepsilon) \|v\|_{BV} - \|\gamma\|_{L^1}, \end{aligned}$$

for all $v \in BV(\Omega)$. Therefore the functional $\mathcal{I}$ is bounded from below and coercive in $BV(\Omega)$. A standard lower semicontinuity argument (see, e.g., the proof of Proposition 3.18) shows that $\mathcal{I}$ has a global minimum. Since any minimizer u of $\mathcal{I}$ satisfies (45) for every $v \in BV(\Omega)$, we conclude that problem (43) has at least one solution. □

Remark 3.14 Condition (h_7) is implied by the counterpart of (h_9'') to the right of $\lambda_1 = 0$, i.e.

(h_7'') *there exists* $\zeta \in L^1(\Omega)$, *with* $\zeta(x) \geq 0$ *for a.e.* $x \in \Omega$ *and* $\zeta(x) > 0$ *on a set of positive measure, such that*

$$\liminf_{s \to \pm\infty} \frac{F(x, s)}{|s|} \geq \zeta(x)$$

uniformly a.e. in Ω.

Accordingly a version of Theorem 3.17, where (h_7) is substituted with (h_7''), holds.

3.2.3 One-sided conditions

In this section we show that in dimension $N = 1$ the two-sided condition (h_4) can be replaced by the one-sided condition (h_5), or (h_5'). This peculiarity is related to the asymptotic behaviour of the curve $\mathcal{C}$ which differs in the case $N = 1$ from the case $N \geq 2$. Condition (h_5) allows f to be unbounded from above and condition (h_5') allows f to be unbounded from below.

Theorem 3.21. *Assume $N = 1$ and let $\Omega =]0, T[$. Assume (h_1'), (h_2), (h_3') and (h_7). Suppose that either (h_5) or (h_5') holds. Then problem (43) has at least one solution.*

Proof. The proof resembles that of Theorem 3.17. Let us suppose that (h_5) holds.

Step 1. Mountain pass geometry. We set

$$\mathcal{S} = \{v \in BV(0,T) : \operatorname*{ess\,sup}_{]0,T[} v = 0\}.$$

Using (h_7) we can find $a^-, a^+ \in \mathbb{R}$, with $a^- < 0 < a^+$, such that

$$\min\Big\{\int_0^T F(x,a^-)\,dx, \int_0^T F(x,a^+)\,dx\Big\} > T = T + \int_0^T F(x,0)\,dx.$$

By Lemma 3.12 we have

$$\inf_{v\in\mathcal{S}} \mathcal{I}(v) \geq 0$$

and hence

$$\mathcal{I}(a^-) = T - \int_0^T F(x,a^-)\,dx < 0 \leq \inf_{\mathcal{S}} \mathcal{I}(v),$$
$$\mathcal{I}(a^+) = T - \int_0^T F(x,a^+)\,dx < 0 \leq \inf_{\mathcal{S}} \mathcal{I}(v).$$

We define

$$\Gamma = \{\gamma \in C^0\big([0,1], BV(0,T)\big) : \gamma(0) = a^-, \gamma(1) = a^+\}$$

and observe that $\gamma([0,1]) \cap \mathcal{S} \neq \emptyset$, for all $\gamma \in \Gamma$. We set $x_0 = a^-$, $x_1 = a^+$,

$$c_{\mathcal{I}} = \inf_{\gamma\in\Gamma} \max_{\xi\in[0,1]} \mathcal{I}(\gamma(\xi))$$

and note that

$$c_{\mathcal{I}} > \max\{\mathcal{I}(x_0), \mathcal{I}(x_1)\}.$$

Lemma 3.10 yields the existence of sequences $(v_k)_k$ and$(\varepsilon_k)_k$, with $v_k \in BV(\Omega)$ and $\varepsilon_k \in \mathbb{R}$, satisfying $\lim_{k\to+\infty} \varepsilon_k = 0$, (61) and (56) for each k and all $v \in BV(\Omega)$.

Step 2. The sequence $(v_k)_k$ is bounded in $BV(\Omega)$. Fix any k. By Proposition 3.7 we have

$$\begin{aligned}
\|\operatorname*{ess\,sup}_{]0,T[} v_k - v_k\|_{BV} &= \int_{]0,T[} |Dv_k| + \|\operatorname*{ess\,sup}_{]0,T[} v_k - v_k\|_{L^1} \\
&\leq \int_{]0,T[} |Dv_k| + T(\operatorname*{ess\,sup}_{]0,T[} v_k - \operatorname*{ess\,inf}_{]0,T[} v_k) \\
&\leq (1+T)\int_{]0,T[} |Dv_k|.
\end{aligned}$$

Hence, taking $v = \operatorname*{ess\,sup}_{]0,T[} v_k$ as a test function in (56) we obtain, using (h_5) too,

$$\begin{aligned}
\rho \int_{]0,T[} |Dv_k| &\leq \mathcal{L}(v_k) \leq \mathcal{J}(v_k) \\
&\leq T - \int_0^T f(x, v_k)(\operatorname*{ess\,sup}_{]0,T[} v_k - v_k)\, dx - \varepsilon_k \| \operatorname*{ess\,sup}_{]0,T[} v_k - v_k \|_{BV} \\
&\leq T + \int_0^T g^-(x)(\operatorname*{ess\,sup}_{]0,T[} v_k - \operatorname*{ess\,inf}_{]0,T[} v_k)\, dx + |\varepsilon_k|(1+T) \int_{]0,T[} |Dv_k| \\
&\leq T + \Big(\|g^-\|_{L^1} + |\varepsilon_k|(1+T) \Big) \int_{]0,T[} |Dv_k|.
\end{aligned}$$

This yields the existence of a constant $K > 0$ such that, for all k,

$$\int_{]0,T[} |Dv_k| \leq K.$$

and

$$\mathcal{J}(v_k) \leq K. \tag{75}$$

Let us verify that the sequence $(\operatorname*{ess\,sup}_{]0,T[} v_k)_k$ is bounded from below. By contradiction, assume this is false; then we have $\lim_{k\to+\infty} v_k(x) = -\infty$ uniformly a.e. in $]0,T[$. From (h_7) it follows that

$$\lim_{k\to+\infty} \int_0^T F(x, v_k)\, dx = +\infty. \tag{76}$$

Combining (75) and (76) yields a contradiction with

$$\lim_{k\to+\infty} \mathcal{I}(v_k) = c_{\mathcal{I}}.$$

Similarly we verify that the sequence $(\operatorname*{ess\,inf}_{]0,T[} v_k)_k$ is bounded from above. Therefore there exists $R > 0$ such that

$$\|v_k\|_{L^\infty} \leq \operatorname*{ess\,sup}_{]0,T[} v_k - \operatorname*{ess\,inf}_{]0,T[} v_k + R$$

holds for all k. Proposition 3.7 then yields $\sup_k \|v_k\|_{L^\infty} < +\infty$ and, hence, $\sup_k \|v_k\|_{BV} < +\infty$.

Step 3. Existence of a solution. The existence of a solution u of problem (43) is finally proved as in Step 3 in the proof of Theorem 3.17. □

3.3 Multiplicity results

In this subsection we discuss the existence of multiple solutions. Namely, under (h_4) the multiplicity of solutions can be proved, whenever the function $s \mapsto \int_\Omega F(x,s)\, dx$ exhibits an oscillatory behaviour at infinity.

Proposition 3.22. *(h_0), (h_1), (h_2), (h_3), (h_4) and*

(h_{10}) $\limsup_{s\to+\infty}\int_\Omega F(x,s)\,dx > |\Omega| + \liminf_{s\to+\infty}\int_\Omega F(x,s)\,dx.$

Then problem (43) has a sequence $(u_n)_n$ of solutions such that

$$-\limsup_{s\to+\infty}\int_\Omega F(x,s)\,dx \le \lim_{n\to+\infty}\mathcal{I}(u_n) \le |\Omega| - \limsup_{s\to+\infty}\int_\Omega F(x,s)\,dx$$

and

$$\lim_{n\to+\infty}\mathcal{P}(u_n) = +\infty.$$

Proof. Condition (h_{10}) implies the existence of sequences $(a_n^-)_n$, $(a_n^+)_n$ and $(b_n)_n$, with

$$\lim_{n\to+\infty} a_n^- = \lim_{n\to+\infty} a_n^+ = \lim_{n\to+\infty} b_n = +\infty$$

and

$$\lim_{n\to+\infty}\int_\Omega F(x,b_n)\,dx = \limsup_{s\to+\infty}\int_\Omega F(x,s)\,dx,$$

such that, for each n, $a_n^- < b_n < a_n^+$,

$$\int_\Omega F(x,b_n)\,dx = \max_{s\in[a_n^-,a_n^+]}\int_\Omega F(x,s)\,dx > |\Omega| + \int_\Omega F(x,a_n^-)\,dx,$$

and

$$\int_\Omega F(x,b_n)\,dx = \max_{s\in[a_n^-,a_n^+]}\int_\Omega F(x,s)\,dx > |\Omega| + \int_\Omega F(x,a_n^+)\,dx.$$

Hence Proposition 3.18 yields, for each n, the existence of a solution u_n of problem (43), satisfying $\mathcal{P}(u_n) \in\,]a_n^-, a_n^+[$ and

$$-\int_\Omega F(x,b_n)\,dx \le \mathcal{I}(u_n) \le |\Omega| - \int_\Omega F(x,b_n)\,dx.$$

Thus the conclusion follows. □

Remark 3.15 A result similar to Proposition 3.22 holds, where condition (h_{10}) is replaced by

$$\limsup_{s\to-\infty}\int_\Omega F(x,s)\,dx > |\Omega| + \liminf_{s\to-\infty}\int_\Omega F(x,s)\,dx.$$

In this case problem (43) has a sequence $(u_n)_n$ of solutions such that

$$-\limsup_{s\to-\infty}\int_\Omega F(x,s)\,dx \le \lim_{n\to+\infty}\mathcal{I}(u_n) \le |\Omega| - \limsup_{s\to-\infty}\int_\Omega F(x,s)\,dx$$

and

$$\lim_{n\to+\infty}\mathcal{P}(u_n) = -\infty.$$

From Proposition 3.22 we deduce the following statement.

Theorem 3.23. *Assume* (h_0), (h_1), (h_2), (h_3), (h_4) *and*

(h_{11}) $\limsup_{s\to+\infty}\int_\Omega F(x,s)\,dx = +\infty > \liminf_{s\to+\infty}\int_\Omega F(x,s)\,dx.$

Then problem (43) *has a sequence* $(u_n)_n$ *of solutions such that*

$$\lim_{n\to+\infty}\mathcal{I}(u_n) = -\infty \quad \text{and} \quad \lim_{n\to+\infty}\mathcal{P}(u_n) = +\infty.$$

Remark 3.16 From Remark 3.15 we deduce that a result similar to Theorem 3.23 holds, where condition (h_{11}) is replaced by

$$\limsup_{s\to-\infty}\int_\Omega F(x,s)\,dx = +\infty > \liminf_{s\to-\infty}\int_\Omega F(x,s)\,dx.$$

In this case problem (43) has a sequence $(u_n)_n$ of solutions such that

$$\lim_{n\to+\infty}\mathcal{I}(u_n) = -\infty \quad \text{and} \quad \lim_{n\to+\infty}\mathcal{P}(u_n) = -\infty.$$

The following local multiplicity result holds.

Proposition 3.24. *Assume* (h_0), (h_1), (h_2), (h_3), (h_4) *and*

(h_{12}) *there exist* a^-, a^+, b^-, b^+, *with* $b^- < a^- < a^+ < b^+$, *such that*

$$\begin{aligned}|\Omega| + \max\Big\{\int_\Omega F(x,b^-)\,dx, \int_\Omega F(x,b^+)\,dx\Big\} &< \int_\Omega F(x,b)\,dx\\ &< \min\Big\{\int_\Omega F(x,a^-)\,dx, \int_\Omega F(x,a^+)\,dx\Big\} - |\Omega|,\end{aligned}$$

where $b \in\,]a^-, a^+[$ *is such that*

$$\int_\Omega F(x,b)\,dx = \min_{s\in[a^-,a^+]}\int_\Omega F(x,s)\,dx.$$

Then problem (43) *has at least three solutions* $u^{(1)}, u^{(2)}, u^{(3)}$ *such that*

$$-\int_\Omega F(x,b)\,dx \le \mathcal{I}(u^{(1)}) \le |\Omega| - \int_\Omega F(x,b)\,dx,$$

$$\mathcal{I}(u^{(2)}) \le |\Omega| - \int_\Omega F(x,a^-)\,dx, \quad \mathcal{I}(u^{(3)}) \le |\Omega| - \int_\Omega F(x,a^+)\,dx,$$

$$\mathcal{P}(u^{(1)}) \in\,]b^-, b^+[, \quad \mathcal{P}(u^{(2)}) \in\,]b^-, b[, \quad \mathcal{P}(u^{(3)}) \in\,]b, b^+[.$$

Proof. The proof is divided into two parts.
Part 1. Existence of the first solution $u^{(1)}$. By assumption (h_{12}) we have

$$\int_\Omega F(x,b)\,dx + |\Omega| < \min\Big\{\int_\Omega F(x,a^-)\,dx, \int_\Omega F(x,a^+)\,dx\Big\}.$$

Step 1. Mountain pass geometry. As in Step 1 of the proof of Theorem 3.17, we set

$$\mathcal{S} = \{v \in BV(\Omega) : \mathcal{P}(v) = b\} = \{w + b : w \in \mathcal{W}\},$$

where $\mathcal{P}$ and $\mathcal{W}$ are defined in Proposition 3.8 and by (62), respectively,

$$\Gamma = \{\gamma \in C^0\big([0,1], BV(\Omega)\big) : \gamma(0) = a^-, \gamma(1) = a^+\}$$

and

$$c_{\mathcal{I}} = \inf_{\gamma\in\Gamma} \max_{\xi\in[0,1]} \mathcal{I}(\gamma(\xi)).$$

Since, by Lemma 3.11,

$$\max\{\mathcal{I}(a^-), \mathcal{I}(a^+)\} < \inf_{v\in\mathcal{S}} \mathcal{I}(v)$$

and $\gamma([0,1]) \cap \mathcal{S} \neq \emptyset$ for any $\gamma \in \Gamma$, Lemma 3.10 yields the existence of sequences $(\gamma_k)_k$, $(v_k)_k$ and$(\varepsilon_k)_k$, with $\gamma_k \in \Gamma$, $v_k \in BV(\Omega)$ and $\varepsilon_k \in \mathbb{R}$, satisfying $\lim_{k\to+\infty} \varepsilon_k = 0$, condition (54), that is

$$c_{\mathcal{I}} - \tfrac{1}{k} \le I(v_k) \le \max_{\xi\in[0,1]} I(\gamma_k(\xi)) \le c_{\mathcal{I}} + \tfrac{1}{k},$$

for each k, and condition (56), that is

$$\mathcal{J}(v) - \mathcal{J}(v_k) \ge \int_\Omega f(x,v_k)(v - v_k)\,dx + \varepsilon_k\|v - v_k\|_{BV},$$

for each k and all $v \in BV(\Omega)$. Notice that, by assumption (h_{12}), we have

$$\int_\Omega F(x,b)\,dx > |\Omega| + \max\Big\{\int_\Omega F(x,b^-)\,dx, \int_\Omega F(x,b^+)\,dx\Big\}.$$

Step 2. The sequence $(v_k)_k$ is bounded in $BV(\Omega)$. Let us set

$$\mathcal{B} = \{v \in BV(\Omega) : \mathcal{P}(v) \in \,]b^-, b^+[\}.$$

Observe that, for each k, $\gamma_k([0,1]) \cap \mathcal{B} \neq \emptyset$. Taking as γ the segment joining a^- with a^+, we see that

$$c_{\mathcal{I}} \le |\Omega| - \int_\Omega F(x,b)\,dx.$$

By (h_{12}) and Lemma 3.11 we deduce that

$$\inf_{w\in\mathcal{W}} \mathcal{I}(w + b^-) \ge -\int_\Omega F(x,b^-)\,dx > |\Omega| - \int_\Omega F(x,b)\,dx \ge c_{\mathcal{I}}$$

and

$$\inf_{w\in\mathcal{W}} \mathcal{I}(w+b^+) \geq -\int_\Omega F(x,b^+)\,dx > |\Omega| - \int_\Omega F(x,b)\,dx \geq c_{\mathcal{I}}.$$

Therefore, as $\lim_{k\to+\infty} \max_{\xi\in[0,1]} \mathcal{I}(\gamma_k(\xi)) = c_{\mathcal{I}}$, we obtain that, for large k, $\gamma_k([0,1]) \subset \mathcal{B}$. Set, for all k, $r_k = \mathcal{P}(v_k)$ and $w_k = v_k - r_k$, and recall (55), that is

$$\min_{\xi\in[0,1]} \|v_k - \gamma_k(\xi)\|_{BV} \leq \tfrac{1}{k}.$$

By the continuity of the projection $\mathcal{P}$, guaranteed by Proposition 3.8, we infer that there exists a decreasing sequence $(\eta_k)_k$, with $\lim_{k\to+\infty} \eta_k = 0$, such that, for each k, $r_k \in [b^- - \eta_k, b^+ + \eta_k]$. Then Lemma 3.11 yields

$$\begin{aligned}\eta \int_\Omega |Dw_k| &\leq \mathcal{I}(v_k) + \int_\Omega F(x,r_k)\,dx \\ &\leq c_{\mathcal{I}} + 1 + \max_{s\in[b^- -1, b^+ +1]} \int_\Omega F(x,s)\,dx\end{aligned}$$

for all large k. Thus we can conclude, by Remark 2.4, that $\sup_k \|w_k\|_{BV} < +\infty$ and, hence, $\sup_k \|v_k\|_{BV} < +\infty$ as well. Arguing as in Step 3 in the proof of Theorem 3.17, we prove the existence of a solution $u^{(1)}$ of problem (43) such that

$$-\min_{s\in[a^-,a^+]} \int_\Omega F(x,s)\,dx \leq \mathcal{I}(u^{(1)}) = c_{\mathcal{I}} \leq |\Omega| - \min_{s\in[a^-,a^+]} \int_\Omega F(x,s)\,dx.$$

By the continuity of $\mathcal{P}$ we have $\mathcal{P}(u^{(1)}) \in [b^-, b^+]$. By assumption (h_{12}) and Lemma 3.11 we actually see that $\mathcal{P}(u^{(1)}) \in]b^-, b^+[$.

Part 2. Existence of two further solutions $u^{(2)}$ and $u^{(3)}$. As we have

$$|\Omega| + \max\Big\{\int_\Omega F(x,b^-)\,dx, \int_\Omega F(x,b)\,dx\Big\} < \int_\Omega F(x,a^-)\,dx \leq \max_{s\in[b^-,b^+]} \int_\Omega F(x,s)\,dx,$$

Proposition 3.18 yields the existence of a solution $u^{(2)}$ of problem (43), satisfying

$$\mathcal{I}(u^{(2)}) \leq |\Omega| - \max_{s\in[b^-,b]} \int_\Omega F(x,s)\,dx \leq |\Omega| - \int_\Omega F(x,a^-)\,dx$$

and $\mathcal{P}(u^{(2)}) \in]b^-, b[$. Observe that $u^{(2)} \neq u^{(1)}$ because $\mathcal{I}(u^{(1)}) > |\Omega| - \int_\Omega F(x,a^-)\,dx$. Similarly we prove the existence of a solution $u^{(3)}$ of problem (43), satisfying

$$\mathcal{I}(u^{(3)}) \leq |\Omega| - \max_{s\in[b,b^+]} \int_\Omega F(x,s)\,dx \leq |\Omega| - \int_\Omega F(x,a^+)\,dx$$

and $\mathcal{P}(u^{(3)}) \in]b, b^+[$, which is different both from $u^{(1)}$ and from $u^{(2)}$. $\square$

From Proposition 3.24 we easily derive the following statement.

Theorem 3.25. *Assume* (h_0), (h_1), (h_2), (h_3), (h_4) *and*

(h_{13}) $\limsup_{s\to+\infty}\int_\Omega F(x,s)\,dx > -\infty = \liminf_{s\to\pm\infty}\int_\Omega F(x,s)\,dx.$

Then problem (43) *has two sequences* $(u_n^{(1)})_n$ *and* $(u_n^{(2)})_n$ *of solutions such that*

$$\lim_{n\to+\infty}\mathcal{I}(u_n^{(1)}) = +\infty, \qquad \limsup_{n\to+\infty}\mathcal{I}(u_n^{(2)}) < +\infty$$

and

$$\lim_{n\to+\infty}\mathcal{P}(u_n^{(2)}) = +\infty.$$

Proof. By assumption (h_{13}) we can find sequences $(b_n^-)_n$, with $\lim_{n\to+\infty} b_n^- = -\infty$, $(a_n^-)_n$, with $\lim_{n\to+\infty} a_n^- = +\infty$, $(a_n^+)_n$ and $(b_n^+)_n$, such that $b_n^- < a_n^- < a_n^+ < b_n^+$ and

$$\begin{aligned}|\Omega| + \max\Big\{\int_\Omega F(x,b_n^-)\,dx, \int_\Omega F(x,b_n^+)\,dx\Big\} &< \min_{s\in[a_n^-,a_n^+]}\int_\Omega F(x,s)\,dx\\ &< \min\Big\{\int_\Omega F(x,a_n^-)\,dx, \int_\Omega F(x,a_n^+)\,dx\Big\} - |\Omega|\end{aligned}$$

for every n. Hence, Proposition 3.24 yields the conclusion. □

Remark 3.17 A result similar to Theorem 3.25 holds, where condition (h_{13}) is replaced by

$$\limsup_{s\to-\infty}\int_\Omega F(x,s)\,dx > -\infty = \liminf_{s\to\pm\infty}\int_\Omega F(x,s)\,dx.$$

In this case problem (43) has two sequences $(u_n^{(1)})_n$ and $(u_n^{(2)})_n$ of solutions such that

$$\lim_{n\to+\infty}\mathcal{I}(u_n^{(1)}) = +\infty, \qquad \lim_{n\to+\infty}\mathcal{I}(u_n^{(2)}) < +\infty$$

and

$$\lim_{n\to+\infty}\mathcal{P}(u_n^{(2)}) = -\infty.$$

Finally notice that, if we assume

$$\limsup_{s\to\pm\infty}\int_\Omega F(x,s)\,dx > -\infty = \liminf_{s\to\pm\infty}\int_\Omega F(x,s)\,dx,$$

then problem (43) has three sequences $(u_n^{(1)})_n$, $(u_n^{(2)})_n$ and $(u_n^{(3)})_n$ of solutions such that

$$\lim_{n\to+\infty}\mathcal{I}(u_n^{(1)}) = +\infty, \quad \limsup_{n\to+\infty}\mathcal{I}(u_n^{(2)}) < +\infty, \quad \limsup_{n\to+\infty}\mathcal{I}(u_n^{(3)}) < +\infty$$

and

$$\lim_{n\to+\infty}\mathcal{P}(u_n^{(2)}) = +\infty \quad \text{and} \quad \lim_{n\to+\infty}\mathcal{P}(u_n^{(3)}) = -\infty.$$

References

[1] S. Ahmad, A.C. Lazer and J.L. Paul, Elementary critical point theory and perturbations of elliptic boundary value problems at resonance, Indiana Univ. Math. J. 25 (1976), 933–944.

[2] L. Ambrosio, N. Fusco and D. Pallara, "Functions of Bounded Variation and Free Discontinuity Problems", Clarendon Press, New York, 2000.

[3] L. Ambrosio, S. Mortola and V.M. Tortorelli, Functionals with linear growth defined on vector valued BV functions, J. Math. Pures Appl. 70 (1991), 269–323.

[4] G. Anzellotti, The Euler equation for functionals with linear growth, Trans. Amer. Math. Soc. 290 (1985), 483–501.

[5] G. Anzellotti, Pairings between measures and bounded functions and compensated compactness, Ann. Mat. Pura Appl. 135 (1983), 293–318.

[6] H. Attouch, G. Buttazzo and G. Michaille, "Variational Analysis in Sobolev and BV Spaces", Society for Industrial and Applied Mathematics, Philadelphia, 2006.

[7] G. Buttazzo, M. Giaquinta and S. Hildebrandt, "One-dimensional Variational Problems. An Introduction", Clarendon Press, Oxford, 1998.

[8] K.C. Chang, The spectrum of the 1-Laplace operator, Commun. Contemp. Math. 11 (2009), 865–894.

[9] M. Cuesta, D.G. de Figueiredo and J.P. Gossez, The beginning of the Fučik spectrum for the p-Laplacian, J. Differential Equations 159 (1999), 212–238.

[10] M. Cuesta and J.-P. Gossez, A variational approach to nonresonance with respect to the Fučík spectrum, Nonlinear Anal. 19 (1992), 487–500.

[11] M. d'Aujourd'hui, On the resonance set and the domain shape for a semilinear equation, Ann. Mat. Pura Appl. (4) 159 (1991), 89–100.

[12] D.G. de Figueiredo and J.P. Gossez, On the first curve of the Fučik spectrum of an elliptic operator, Differential Integral Equations 7 (1994), 1285–1302.

[13] M. Emmer, Esistenza, unicità e regolarità delle superfici di equilibrio nei capillari, Ann. Univ. Ferrara, 18 (1973), 79–94.

[14] L. Evans and R.F. Gariepy, "Measure theory and fine properties of functions", CRC Press, Boca Raton, FL, 1992.

[15] M.L.C. Fernandes, P. Omari and F. Zanolin, On the solvability of a semilinear two-point BVP around the first eigenvalue, Differential Integral Equations 2 (1989), 63–79.

[16] R. Finn, "Equilibrium capillary surfaces", Springer-Verlag, New York, 1986.

[17] R. Finn and C. Gerhardt, The internal sphere condition and the capillary problem, Ann. Mat. Pura Appl. (4) 112 (1977), 13–31.

[18] A. Fonda and M. Garrione, Nonlinear resonance: a comparison between Landesman-Lazer and Ahmad-Lazer-Paul conditions, Adv. Nonlinear Stud. 11 (2011), 391–404.

[19] E. Giusti, Boundary value problems for non-parametric surfaces of prescribed mean curvature, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 3 (1976), 501–548.

[20] P. Habets, R. Manásevich and F. Zanolin, A nonlinear boundary value problem with potential oscillating around the first eigenvalue, J. Differential Equations 117 (1995), 428–445.

[21] A. Hammerstein, Nichtlineare Integralgleichungen nebst Anwendungen, Acta Math. 54 (1930), 117–176.

[22] G. Katriel, Mountain pass theorems and global homeomorphism theorems, Ann. Inst. Henri Poincaré Analyse Non Linéaire 11 (1994), 189–209.

[23] E.M. Landesman and A.C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech. 19 (1969/1970), 609–623.

[24] V.K. Le, Some existence results on non-trivial solutions of the prescribed mean curvature equation, Adv. Nonlinear Stud. 5 (2005), 133–161.

[25] J. Mawhin, J.R. Jr. Ward and M. Willem, Variational methods and semilinear elliptic equations, Arch. Rational Mech. Anal. 95 (1986), 269–277.

[26] J. Mawhin and M. Willem, "Critical Point Theory and Hamiltonian Systems", Springer-Verlag, New York, 1989.

[27] M. Miranda, Dirichlet problem with L^1 data for the non-homogeneous minimal surface equation, Indiana Univ. Math. J. 24 (1974/75), 227–241.

[28] J. Mossino, "Inégalités isopérimétriques et applications en physique", Travaux en Cours, Hermann, Paris, 1984.

[29] F.I. Njoku, P. Omari and F. Zanolin, Multiplicity of positive radial solutions of a quasilinear elliptic problem in a ball, Adv. Differential Equations 5 (2000), 1545–1570.

[30] F. Obersnel and P. Omari, Positive solutions of elliptic problems with locally oscillating nonlinearities, J. Math. Anal. Appl. 323 (2006), 913–929.

[31] F. Obersnel and P. Omari, Existence and multiplicity results for the prescribed mean curvature equation via lower and upper solutions, Differential Integral Equations 22 (2009), 853–880.

[32] F. Obersnel and P. Omari, The periodic problem for curvature-like equations with asymmetric perturbations, J. Differential Equations 251 (2011), 1923–1971.

[33] F. Obersnel and P. Omari, Multiple bounded variation solutions of a capillarity problem, Discrete Contin. Dyn. Syst. 2011, Dynamical systems, differential equations and applications. 8th AIMS Conference. Suppl. Vol. II, 1129–1137.

[34] F. Obersnel and P. Omari, Existence, regularity and boundary behaviour of bounded variation solutions of a one-dimensional capillarity equation, Discrete Contin. Dyn. Syst. 33 (2013), 305–320.

[35] F. Obersnel, P. Omari and S. Rivetti, Existence, regularity and stability properties of periodic solutions of a capillarity equation in the presence of lower and upper solutions, Nonlinear Anal. Real World Appl. 13 (2012), 2830–2852.

[36] P. Omari and F. Zanolin, Infinitely many solutions of a quasilinear elliptic problem with an oscillatory potential, Comm. Partial Differential Equations 21 (1996), 721–733.

[37] P. Rosenau, Free energy functionals at the high-gradient limit, Phys. Rev. A 41 (1990), 2227–2230.

[38] G.M. Troianiello, "Elliptic Differential Equations and Obstacle Problems", Plenum Press, New York, 1987.