A NOTE ON SECANTS OF GRASSMANNIANS

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Abstract. Let $\mathbb{G}(k, n)$ be the Grassmannian of k-subspaces in an n-dimensional complex vector space, $k \geq 3$. For a projective variety X, let $\sigma_s(X)$ denote its s-secant variety, namely the closure of the union of linear spans of all the s-tuples of independent points lying on X. We classify all defective $\sigma_s(\mathbb{G}(k, n))$ for $s \leq 12$.

1. Introduction

Let $X \subset \mathbb{P}^N$ be a non-degenerate projective variety. The *s-th secant variety* $\sigma_s(X)$ is defined to be the closure of the union of linear spans of all the s-tuples of independent points lying on X.

Let $\mathbb{G}(k, n)$ denote the *Grassmannian* parametrizing k-subspaces in an n-dimensional complex vector space. It is embedded in $\mathbb{P}^N = \mathbb{P}(\Lambda^k \mathbb{C}^n)$ via the Plücker map, where $N = \binom{n}{k} - 1$. Remark that if we identify points in $\mathbb{P}^N$ with general skew-symmetric tensors, then points in $\mathbb{G}(k, V)$ correspond to decomposable skew-symmetric tensors.

Consider the secant variety $\sigma_s(\mathbb{G}(k, n))$. An immediate computation shows that:

$$\dim \sigma_s(\mathbb{G}(k, n)) \leq \min\{sk(n-k) + s - 1, N\}. \tag{1.1}$$

We say that $\sigma_s(\mathbb{G}(k, n))$ *has expected dimension* if equality holds. Otherwise $\sigma_s(\mathbb{G}(k, n))$ is called *defective* and its *defect* is the difference between the right and left hand side in (1.1).

The aim of this paper is to give a contribution in the classification of all defective $\sigma_s(\mathbb{G}(k, n))$. Such a classification is a highly non-trivial and in many cases wide open problem. There is an extensive related literature—not only on Grassmannians, but on many other homogeneous varieties, such as Segre products [AOP09a], Segre-Veronese varieties [AB09], Lagrangian Grassmannians [BB10], Spinor varieties [Ang11] and Veronese varieties [AH95]. The latter is the only case where the classification is complete. For a recent survey on the subject and its applications we refer the reader to [Lan].

If $k = 2$ then $\mathbb{G}(k, n)$ is a Grassmannian of (projective) lines and $\sigma_s(\mathbb{G}(k, n))$ is almost always defective—it corresponds to the locus of skew-symmetric matrices of rank at most $2s$. Thus throughout the paper we assume $k \geq 3$. Only four defective cases are known then, and we believe that they are the only ones.

Conjecture 1.1. [BDdG07, Conjecture 4.1] *Let $k \geq 3$. Then $\sigma_s(\mathbb{G}(k, n))$ has the expected dimension except for the cases $(k, n; s) = (3, 7; 3), (4, 8; 3), (4, 8; 4)$ and $(3, 9; 4)$.*

Conjecture 1.1 can be implicitly found in previous works, for example in [CGG05]. In [BDdG07] the authors use a computational technique to check that the conjecture holds true for $n \leq 15$. (The same result for $n \leq 14$ can be found in [McG06].)

In [CGG05] explicit bounds on $(k, n; s)$ were found for $\sigma_s(\mathbb{G}(k, n))$ to have expected dimension. Improving these bounds and using the monomial technique Abo, Ottaviani and Peterson showed that the conjecture is true for $s \leq 6$ [AOP09b]. The main result of this paper is the following:

Theorem 1.2. *If $k \geq 3$ then $\sigma_s(\mathbb{G}(k, n))$ has the expected dimension for every $s \leq 12$, except for the cases $(k, n; s) = (3, 7; 3), (4, 8; 3), (4, 8; 4)$ and $(3, 9; 4)$.*

Exploiting the bounds given in [AOP09b] and computational results of [BDdG07] Theorem 1.2 can be strengthened:

Theorem 1.3. *If $k \geq 3$, $k \leq \frac{n}{2}$ then $\sigma_s(\mathbb{G}(k, n))$ has the expected dimension:*

(1) for $n \leq 15$, all k and s, except $(k, n; s) = (3, 7; 3), (4, 8; 3), (4, 8; 4)$ and $(3, 9; 4)$

(2) for $n > 15$, $k \geq 7$, $s \leq \max\{111, \frac{n-k+3}{3}\}$

(3) for $n > 15$, $3 \leq k \leq 6$, s as follows:

(a) $k = 3$, $s \leq \max\{12, \frac{n}{3}\}$

(b) $k = 4$, $s \leq \max\{30, \frac{n-1}{3}\}$

(c) $k = 5$, $s \leq \max\{59, \frac{n-2}{3}\}$

(d) $k = 6$, $s \leq \max\{90, \frac{n-3}{3}\}$

2. A lemma on tangent spaces

Let $V \simeq \mathbb{C}^n$ be a complex vector space of dimension n. The Grassmannian $\mathbb{G}(k, V) = \mathbb{G}(k, n)$ is the variety parametrizing k-subspaces in V. The Grassmannian $\mathbb{G}(k, V)$ embeds in $\mathbb{P}(\bigwedge^k V)$ via Plücker map.

We start by introducing a useful description of the affine tangent space to the Grassmannian. (Recall that the affine tangent space is the tangent space to the affine cone of the variety.) Its proof is an immediate consequence of Leibniz rule.

Lemma 2.1. *Let $E = e_1 \wedge \ldots \wedge e_k$ be a point of $\mathbb{G}(k, V)$, where $e_i \in V$. The affine tangent space to $\mathbb{G}(k, V)$ at E is:*

$$\hat{T}_E \mathbb{G}(k, V) = \sum_{j=1}^{k} e_1 \wedge \ldots \wedge e_{j-1} \wedge V \wedge e_{j+1} \wedge \ldots \wedge e_k.$$

Using compact notation we can write:

$$\hat{T}_E \mathbb{G}(k, V) = \bigwedge^{k-1} E \wedge V.$$

Using the description above we can prove the following linear algebra lemma.

Lemma 2.2. *For $i = 1 \dots s$, let $E_i = e_{i,1} \wedge \dots \wedge e_{i,k}$ be points of $\mathbb{G}(k,V)$ such that the spaces $\hat{T}_{E_i}\mathbb{G}(k,V)$ are linearly independent in $\bigwedge^k V$. (Where $(e_{i,j})_{j=1\dots k}$ are elements of V.)*
Let W be a complex vector space of dimension $m > n$, and consider $V \hookrightarrow W$ any immersion. Then the spaces $\hat{T}_{E_i}\mathbb{G}(k,W)$ are linearly independent in $\bigwedge^k W$.
(We keep the notation E_i for the image of the subspaces E_i inside W.)

Proof. The spaces:

$$\begin{aligned}
\hat{T}_{E_i}\mathbb{G}(k,W) &= \textstyle\bigwedge^{k-1} E_i \wedge W \\
&= \textstyle\bigwedge^{k-1} E_i \wedge (V \oplus W/V) \\
&= \left(\textstyle\bigwedge^{k-1} E_i \wedge V\right) \oplus \left(\textstyle\bigwedge^{k-1} E_i \wedge W/V\right)
\end{aligned}$$

live inside:

$$\textstyle\bigwedge^k W = \bigwedge^k (V \oplus W/V) = \displaystyle\bigoplus_{h=0}^{k} \textstyle\bigwedge^{k-h} V \otimes \bigwedge^h (W/V)$$

and more precisely the situation is:

$$\begin{array}{ccccc}
\hat{T}_{E_i}\mathbb{G}(k,V) & = & \bigwedge^{k-1} E_i \wedge V & \oplus & \bigwedge^{k-1} E_i \wedge W/V \\
\cap & & \cap & & \cap \\
\bigwedge^k W & \subseteq & \bigwedge^k V & \oplus & \bigwedge^{k-1} V \otimes (W/V)
\end{array}$$

The pieces $\bigwedge^{k-1} E_i \wedge V$ in the first summand are linearly independent by our assumption, and since the sum is direct, the result follows if we prove the linear independence of the pieces $\bigwedge^{k-1} E_i \wedge W/V$ in the second summand above. Elements of $\bigwedge^{k-1} E_i \wedge W/V$ are of the form:

$$\sum_{j=1}^{k} a_{i,j}(e_{i,1} \wedge \dots \wedge e_{i,j-1} \wedge w \wedge e_{i,j+1} \wedge \dots \wedge e_{i,k}),$$

for some coefficients $a_{i,j}$ and some nonzero element $w \in W/V$. Without loss of generality we ignore these coefficients in what follows. Linear dependence would mean that there exist $\alpha_1, \dots, \alpha_s$ such that:

$$\begin{aligned}
0 &= \sum_{i=1}^{s} \alpha_i \left(\sum_{j=1}^{k} e_{i,1} \wedge \dots \wedge e_{i,j-1} \wedge w \wedge e_{i,j+1} \wedge \dots \wedge e_{i,k} \right) \\
&= \left(\sum_{i=1}^{s} \sum_{j=1}^{k} (-1)^{\epsilon} \alpha_i (e_{i,1} \wedge \dots \wedge e_{i,j-1} \wedge e_{i,j+1} \wedge \dots \wedge e_{i,k}) \right) \wedge w
\end{aligned}$$

where we use $(-1)^\epsilon$ as a reminder that there might be a sign change. (These we can also ignore without losing any generality.) Since w is nonzero we get that:

$$\sum_{i=1}^{s}\sum_{j=1}^{k}(-1)^\epsilon\alpha_i(e_{i,1}\wedge\ldots\wedge e_{i,j-1}\wedge e_{i,j+1}\wedge\ldots\wedge e_{i,k})=0$$

in $\bigwedge^{k-1}V$. Now let $\mu\in V$ be any vector and consider:

$$\sum_{i=1}^{s}\sum_{j=1}^{k}\alpha_i(e_{i,1}\wedge\ldots\wedge e_{i,j-1}\wedge\mu\wedge e_{i,j+1}\wedge\ldots\wedge e_{i,k})=0$$

in $\bigwedge^{k}V$, which is a contradiction and concludes the proof. □

3. Results

Recall from the introduction that given $X\subset\mathbb{P}^N$ a non-degenerate projective variety, its s-th secant variety $\sigma_s(X)$ is defined to be the closure of the union of linear spans of all the s-tuples of independent points lying on X:

$$\sigma_r(X)=\overline{\bigcup_{p_1,\ldots,p_r\in X}\langle p_1,\ldots,p_r\rangle}.$$

If X is non-degenerate and $\dim X=d$, then

$$\dim\sigma_s(X)\le\min\{sd+s-1,N\}. \tag{3.1}$$

If equality holds we say that $\sigma_s(X)$ has the expected dimension, otherwise we call $\sigma_s(X)$ defective, and define its defect to be the difference between the two numbers. If $\dim\sigma_s(X)=N$ we say that $\sigma_s(X)$ fills the ambient space.

We want to classify all defective $\sigma_s(\mathbb{G}(k,n))$. Since $\dim\mathbb{G}(k,n)=k(n-k)$ note that (3.1) reduces to (1.1).

We recall the main tool to compute the dimension of secant varieties, Terracini Lemma. (For a proof see [Zak93, Proposition 1.10].)

Lemma 3.1 (Terracini Lemma)**.** *Let $p_1,\ldots,p_r$ be general points in X and let z be a general point of $\langle p_1,\ldots,p_r\rangle$. Then the affine tangent space to $\sigma_r(X)$ at z is given by*

$$\hat{\mathrm{T}}_z\sigma_r(X)=\hat{\mathrm{T}}_{p_1}X+\cdots+\hat{\mathrm{T}}_{p_r}X$$

where $\hat{\mathrm{T}}_{p_i}X$ denotes the affine tangent space to X at p_i.

Lemma 3.2. *If $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension and does not fill the ambient space, then $\sigma_s(\mathbb{G}(k,m))$ has the expected dimension for every $m\ge n$.*

Proof. The statement is an immediate consequence of Lemma 2.2 together with Terracini Lemma 3.1. □

Theorem 3.3. *If $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension and does not fill the ambient space, then $\sigma_s(\mathbb{G}(k+t,n+t))$ has the expected dimension for every $t\ge 0$.*

Proof. This follows from the duality of Grassmannians: $\mathbb{G}(k, V) \simeq \mathbb{G}(n-k, V^*)$. If $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension, so does $\sigma_s(\mathbb{G}(n-k,n))$. Then using Lemma 3.2 also $\sigma_s(\mathbb{G}(n-k,n+t))$ has the expected dimension for every $t \geq 0$. Since $\mathbb{G}(n-k,n+t) \simeq \mathbb{G}(n+t-(n-k),n+t) = \mathbb{G}(k+t,n+t)$, the statement follows. $\square$

Theorem 1.2. *If $k \geq 3$ then $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension for every $s \leq 12$, except for the cases $(k,n;s) = (3,7;3),(4,8;3),(4,8;4)$ and $(3,9;4)$.*

Proof. The proof of Theorem 1.2 is an easy consequence of Theorem 3.3 together with the computational evidence provided in [BDdG07]. Duality of Grassmannians allows us to assume that $k \leq \frac{n}{2}$. The case $n \leq 15$ has been checked in [BDdG07]. Now take $\sigma_s(\mathbb{G}(k,n))$, with k, s as required and $n > 15$. Since for the given values of s the secant variety $\sigma_s(\mathbb{G}(3,15))$ has the expected dimension and does not fill the ambient space, using Lemma 3.2 we can conclude that the statement is true for $\sigma_s(\mathbb{G}(3,n-(k-3))$. For our choice of range of s, k and n we can also claim that $\sigma_s(\mathbb{G}(3,n-(k-3))$ does not fill the ambient space. Theorem 3.3 with $t = k-3$ then implies that the statement is true for $\sigma_s(\mathbb{G}(3+(k-3),n-(k-3)+(k-3)) = \sigma_s(\mathbb{G}(k,n))$. $\square$

Remark 3.1. Theorem 1.2 can be restated in terms of Conjecture 1.1. Remark that all defective cases mentioned in the conjecture have $\sigma_s(\mathbb{G}(k-1,n-1))$ that is either defective or fills the ambient space, so Theorem 3.3 is no contradiction to the conjecture.

To the detriment of its clean statement, Theorem 1.2 can be strengthened using all of values of k in the computational results of [BDdG07] on $\mathbb{G}(k,15)$. For a more complete statement, we also include bounds on $(k,n;s)$ proved in [AOP09b] using the monomial technique. (Their result is in fact an extension of [CGG05, Theorem 2.1].)

Theorem 3.4. [AOP09b, Theorem 3.3] *If $3(s-1) \leq n-k$ and $k \geq 3$ then $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension and does not fill the ambient space.*

We conclude with this stronger statement. Its proof is immediate from the proof of Theorem 1.2, Theorem 3.4 and an explicit computation of the maximal $s = s(k)$ such that the secant $\sigma_s(\mathbb{G}(k,15))$ does not fill the ambient space.

Theorem 1.3. *If $k \geq 3$, $k \leq \frac{n}{2}$ then $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension:*

(1) for $n \leq 15$, all k and s, except $(k,n;s) = (3,7;3),(4,8;3),(4,8;4)$ and $(3,9;4)$
(2) for $n > 15$, $k \geq 7$, $s \leq \max\{111, \frac{n-k+3}{3}\}$
(3) for $n > 15$, $3 \leq k \leq 6$, s as follows:
   (a) $k = 3$, $s \leq \max\{12, \frac{n}{3}\}$
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   (c) $k = 5$, $s \leq \max\{59, \frac{n-2}{3}\}$
   (d) $k = 6$, $s \leq \max\{90, \frac{n-3}{3}\}$

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