Periodic solutions of perturbed Hamiltonian systems in the plane by the use of the Poincaré–Birkhoff Theorem

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In the 100th anniversary of Poincaré's last geometric theorem

Abstract

We prove the existence of periodic solutions for a planar non-autonomous Hamiltonian system which is a small perturbation of an autonomous system, in the presence of a non-isochronous period annulus. To this aim we use the Poincaré–Birkhoff fixed point theorem, even if the boundaries of the annulus are neither assumed to be invariant for the Poincaré map, nor to be star-shaped. As a consequence, we show how to deal with the problem of bifurcation of subharmonic solutions near a given nondegenerate periodic solution. In this framework, we only need little regularity assumptions, and we do not need to introduce any Melnikov type functions.

1 Introduction

The Poincaré–Birkhoff fixed point theorem, named also the "twist theorem" or the "Poincaré's last geometric theorem", in the original formulation [62], asserts the existence of at least two fixed points for an area-preserving homeomorphism Ψ of a planar circular annulus $\mathcal{A} = \overline{B}(0, R_2) \setminus B(0, R_1)$ onto itself, such that the points of the inner boundary Γ_1 are advanced along Γ_1 in the clockwise sense and the points of the outer boundary Γ_2 are advanced along Γ_2 in the counter-clockwise sense.

This remarkable result, conjectured by Henri Poincaré, was published by him with some reluctance in 1912, the year of his death. As Poincaré says:

> *Je n'ai jamais presenté au public un travail aussi inachevé; je crois donc nécessaire d'expliquer en quelques mots les raisons qui m'ont déterminé à le publier, et d'abord celles qui m'avaient engagé à l'entreprendre. [...] J'ai donc été amené à rechercher si ce théorème est vrai ou faux, mais j'ai rencontré des difficultés auxquelles je ne m'attendais pas. [...] Il semble que dans ces conditions, je devrais m'abstenir de toute publication tant que je n'aurai pas résolu la question; mais après les inutiles efforts que j'ai faits pendant de longs mois, il m'a paru que le plus sage était de laisser le problème mûrir, en m'en reposant durant quelques années; cela serait très bien si j'étais sûr de pouvoir le reprendre un jour; mais à mon âge je ne puis en répondre. D'un autre côté, l'importance du sujet est trop grande et l'ensemble des résultats obtenus trop considérable déjà, pour que je me résigne à les laisser définitivement infructueux. [...] Je pense que ces considérations suffisent à me justifier.* [1]

Poincaré also checked the validity of his conjecture in various special cases, but a complete proof was only provided by George D. Birkhoff [5], in 1913, with respect to the existence of at least one fixed point. For the existence of a second fixed point, Birkhoff refers to a remark by Poincaré, according to which [62, p. 377]

> *Il y en aura au moins deux puisque l'Analysis situs (et en particulier le théorème de Kronecker) nous montre immédiatement qu'elle doit en avoir un nombre pair.* [2]

[1] I have never presented such an incomplete work to the public; therefore, I think it necessary to briefly explain the reasons which convinced me to publish it, and, above all, those which drove me to start it. [...] So, I was led to research the veracity of this theorem, but I met some unexpected difficulties. [...] It seems that, in such a situation, I should refrain from any publication until I have solved the problem; but, after all the pointless efforts made over many months, I thought that the wiser choice was to leave the problem to mature, while resting for some years; this would have been fine if I had been sure to be able to take it up again one day; but at my age I cannot be so sure. On the other hand, the importance of the subject is too great and the quantity of results so far obtained too considerable, to resign myself to let them definitively unfruitful. [...] I think that these considerations are sufficient to justify me. [Our translation.]

[2] There will be at least two of them, since the Analysis situs (and in particular Kronecker theorem) shows us immediately that their number must be even. [Our translation.]

In modern terms, since the homeomorphism Ψ is homotopic to the identity, its fixed point index on $\mathcal{A}$ is equal to zero (which is the Euler characteristic of the annulus). So, the existence of a fixed point *with nonzero index* implies the existence of a second fixed point. It was not so clear, however, how to prevent the existence of only one fixed point with zero index.

Hence, as observed by Birkhoff himself [7, p. 299], the proof for a second fixed point needed to be made more precise. Therefore, in 1926, Birkhoff provided a proof of the existence of a second fixed point in any case, and also replaced the condition about the preservation of the areas with a more general assumption of topological nature. In 1977, due to the skepticism of some mathematicians on the validity of Birkhoff's original proof, M. Brown and W. D. Neumann [14] were led to a very careful and detailed checking of Birkhoff's proof, showing in a very reasonable manner its correctedness, up to the present standards of rigorousness (see also [17, 39, 58, 69] for different approaches to the proof of the existence of the second fixed point). A modern approach to the proof of the twist theorem also uses a suggestion of de Kérékjartó [47], showing a connection with the Brouwer plane translation theorem [13], see e.g. [38, 40, 71], and the references therein.

Applications of the twist theorem to dynamical systems problems coming from nonlinear mechanics and geometry were already suggested by Poincaré in [62] and studied by Birkhoff in [6, 8].

In the case of planar non-autonomous ordinary differential equations, when one looks for the existence of periodic solutions or subharmonic solutions via the search of the fixed points of the Poincaré map or of its iterates, respectively, a major difficulty in the application of the Poincaré–Birkhoff theorem in the version stated above is the construction of annular regions which are invariant under these transformations. Hence, a modification of this fixed point theorem in which the invariance conditions for the annulus and its inner and outer boundaries are not assumed became necessary for the applications. Birkhoff himself, motivated by different dynamical applications, was interested in proving some extensions of the theorem along these directions. In particular, in [6, 7], he showed that his proof worked also when the annulus is not necessarily invariant under Ψ but its inner boundary is still rotated onto itself.

In 1976, H. Jacobowitz [45, 46], following a suggestion of J. Moser [56], proposed a modified version of the Poincaré–Birkhoff theorem for a topolog-

ical pointed disc, showing how to apply it to the search of periodic solutions to some superlinear second order differential equations. Applications in this direction were also given by P. Hartman [42] and G. J. Butler [15]. In view of these results, W.-Y. Ding [24, 25], J. Franks [34, 35], and C. Rebelo [63] offered new versions of the Poincaré–Birkhoff theorem where the boundary invariance assumption is removed. See e.g. [19] for a review on this subject.

Let us now state a modern version of the Poincaré–Birkhoff theorem (see [63, Corollary 2 and Remark 2]). For a simple closed curve Γ in $\mathbb{R}^2$, we denote by $I(\Gamma)$ the open domain bounded by Γ, and by $\bar{I}(\Gamma)$ its closure.

Theorem 1 *Let $\mathcal{A} = \bar{I}(\Gamma_2) \setminus I(\Gamma_1)$, with $0 \in I(\Gamma_1)$, be an annular region bounded by two curves Γ_1 and Γ_2, which are strictly star-shaped with respect to the origin. Let $\Psi : \bar{I}(\Gamma_2) \to \Psi(\bar{I}(\Gamma_2))$ be an area-preserving homeomorphism such that $0 \in \Psi(I(\Gamma_1))$. On the universal covering space $\{(\theta, \rho) : \theta \in \mathbb{R},\ \rho > 0\}$, with the standard covering projection $\Pi : (\theta, \rho) \mapsto (\rho \cos\theta, \rho \sin\theta)$, consider a lifting of $\Psi|_{\mathcal{A}}$ of the form*

$$h(\theta, \rho) = (\theta + \gamma(\theta, \rho), \eta(\theta, \rho))\,,$$

where $\gamma(\theta, \rho)$ and $\eta(\theta, \rho)$ are continuous, and 2π-periodic in their first variable. Correspondingly, for $\tilde{\Gamma}_1 = \Pi^{-1}(\Gamma_1)$ and $\tilde{\Gamma}_2 = \Pi^{-1}(\Gamma_2)$, assume the twist condition

$$\gamma(\theta, \rho) > 0 \quad \text{on} \quad \tilde{\Gamma}_1\,, \qquad \gamma(\theta, \rho) < 0 \quad \text{on} \quad \tilde{\Gamma}_2\,.$$

Then, Ψ has two fixed points z_1, z_2 in the interior of $\mathcal{A}$, such that

$$\gamma(\Pi^{-1}(z_1)) = \gamma(\Pi^{-1}(z_2)) = 0\,.$$

The star-shapedness assumption on the boundaries of the annulus is a delicate hypothesis. In [25], Ding assumed only Γ_1 to be star-shaped. He considered this as a technical assumption, although crucial for his proof, based on the Jacobowitz version of the twist theorem, but probably unnecessary. However, it has been shown by R. Martins and A. J. Ureña in [54] that the assumption of having star-shaped boundaries is not eliminable. And, more recently, P. Le Calvez and J. Wang in [51] provided an example showing that *a star-shapedness assumption on the outer boundary* is also needed. In the same paper, corrections to some previous proofs were also provided.

For the proof of Theorem 1 we refer to [63], where a direct reduction to the classical Poincaré–Birkhoff theorem for the standard annulus, already settled in [14], is done. In our opinion, this seems to be a safer approach, without invoking the Jacobowitz result, which probably hides some difficulty, in view of the counter-examples given in [54, 51].

The above theorem has been used by many authors to prove existence and multiplicity of periodic solutions of some non-autonomous Hamiltonian systems in a variety of situations, see, e.g., [10, 11, 12, 15, 16, 19, 20, 22, 23, 27, 28, 29, 30, 31, 32, 33, 43, 45, 53, 63, 64, 65, 66, 72, 73, 74]. Most of these papers rely on Ding's version of the Poincaré–Birkhoff theorem, safely applying it to annuli for which both boundaries are star-shaped. For related results, see also [49, 50]. It can be noticed how, in recent years, this powerful tool has been attracting a rapidly growing interest in the applications.

An interesting feature of Theorem 1 and some of its variants, when applied to the Poincaré map associated to a planar system, is that we can take

$$\gamma(\theta, \rho) = \Theta(\theta, \rho) - 2\pi j \,,$$

where $\Theta(\theta, \rho)$ is the angular displacement of a solution in the phase–plane, and j is an integer. In this case, the fixed points correspond to periodic solutions making exactly j rotations around the origin (in the sense of the winding number). This additional information has often been used to distinguish among different type of solutions, thus providing multiplicity results in a variety of different situations. It also represents a crucial argument in some proofs of the minimality of the period, when trying to detect subharmonic solutions.

In this paper, we focus our attention on periodic solutions of non-autonomous planar Hamiltonian systems

$$J\dot{u} = \nabla H(t, u; \varepsilon) \,, \tag{1}$$

which are small T-periodic perturbations of an autonomous system

$$J\dot{u} = \nabla \mathcal{H}(u) \,, \tag{2}$$

in the sense that $H(t, u; 0) = \mathcal{H}(u)$. Such periodic solutions are obtained as fixed points of the Poincaré map $\mathcal{P}_\varepsilon$ associated to (1), or of some of its iterates, which in turn is a small perturbation of the Poincaré map $\mathcal{P}_0$ associated to (2), or the corresponding iterates, respectively.

We assume the existence of a non-isochronous period annulus $\mathcal{A}$ for (2), *whose boundaries are not necessarily star-shaped*. Notice that, even if $\mathcal{A}$ is invariant for $\mathcal{P}_0$, in general it will not be invariant for $\mathcal{P}_\varepsilon$. We then prove that, for ε sufficiently small, there are a large number of periodic solutions for the non-autonomous system (1), lying in $\mathcal{A}$.

In this setting, Theorem 1 cannot be applied directly. We will need to transform our problem to an equivalent one, in some suitable new coordinates, by the use of a canonical transformation for system (2). In this new setting, $\mathcal{A}$ becomes a classical annular region, and $\mathcal{P}_\varepsilon$ is modified accordingly. The non-isochronicity of the annulus will then guarantee a twist condition for the Poincaré map in the new coordinates, and Theorem 1 will provide the fixed points we need.

The existence of subharmonic solutions for Hamiltonian systems like (1) is often claimed as one of the by-products of *KAM theory*, which provides the existence of a whole family of invariant curves for the Poincaré map in a given annulus. For this subject, we refer to the classical book of Moser [57], since a review of the manyfold recent developments of the theory is beyond the aim of this paper. However, notice that, in order to apply KAM theory, one needs rather severe differentiability conditions, together with a strong twist monotonicity assumption. On the contrary, our aim is to reduce to the minimum the regularity assumptions on the Poincaré map, and we do not need any monotonicity on the twist.

As a corollary of our main theorem, we directly obtain a bifurcation type result, both for harmonic and subharmonic solutions, by only assuming the existence of a nondegenerate periodic orbit for the autonomous system (2). Such kind of results have been obtained in the literature by the use of different techniques. For instance, using variational methods combined with bifurcation type arguments, M. Willem [70] proved a general existence theorem for harmonic solutions of perturbed Hamiltonian systems in $\mathbb{R}^{2M}$, following some earlier results by A. Ambrosetti, V. Coti Zelati and I. Ekeland [2] (see also [26]). The existence of subharmonic solutions near an equilibrium, in the spirit of Birkhoff and Lewis [9], was also studied by many authors, see e.g. [1] and the references therein. Our corollary deals only with the case $M = 1$, but provides the existence of subharmonic solutions with precise information on their periods, as well.

Another approach used to deal with (not necessarily Hamiltonian) per-

turbations of Hamiltonian systems is based on the so-called *subharmonic Melnikov function*, for which we refer to [37]. Let us only quote the pioneering work of Loud [52] and Lazer [48], without entering in the details of the large literature dealing with this topic. However, we emphasize that, in the case of Hamiltonian perturbations, our approach does not need assumptions related to any type of Melnikov functions.

2 The main result

We consider a *period annulus* $\mathcal{A} \subseteq \mathbb{R}^2$ for an autonomous Hamiltonian system

$$J\dot{u} = \nabla\mathcal{H}(u)\,. \tag{3}$$

The Hamiltonian function $\mathcal{H}$ is only defined on $\mathcal{A}$, and we want it to be twice continuously differentiable there. Needless to say, we denote by J the standard 2×2 symplectic matrix, namely

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The inner and the outer components of the boundary of $\mathcal{A}$ are the Jordan curves γ_i and γ_e, respectively. We are therefore assuming that all the solutions with initial point in $\mathcal{A}$ are periodic and their orbits are not contractible in $\mathcal{A}$.

More precisely, we assume that $\mathcal{A}$ is *not isochronous*, i.e., the periods of the solutions in $\mathcal{A}$ cover an interval $[\mathcal{T}_{\min}\,, \mathcal{T}_{\max}]$, with $\mathcal{T}_{\min} < \mathcal{T}_{\max}$. Moreover, without loss of generality, we assume that $\mathcal{A}$ surrounds the origin.

We then consider the Hamiltonian system

$$J\dot{u} = \nabla H(t, u)\,, \tag{4}$$

with Hamiltonian function $H : \mathbb{R} \times \mathcal{A} \to \mathbb{R}$, whose gradient with respect to its second variable, denoted by $\nabla H(t, u)$, is continuous in (t, u), and also locally Lipschitz continuous in u, and is T-periodic in its first variable, for some $T > 0$.

Theorem 2 *Given two positive integers m and n such that*

$$\mathcal{T}_{\min} < \frac{mT}{n} < \mathcal{T}_{\max}\,, \tag{5}$$

there is an $\bar{\varepsilon} > 0$ such that, if

$$|\nabla H(t,u) - \nabla \mathcal{H}(u)| \leq \bar{\varepsilon}\,, \qquad \textit{for every } t \in [0,T] \textit{ and } u \in \mathcal{A}\,,$$

then system (4) has at least two mT-periodic solutions, whose orbits are contained in $\mathcal{A}$, which make exactly n rotations around the origin in the period time mT.

The above statement generalizes [16, Theorem 1], where the case of small perturbations of planar systems generated by the second order equation $x'' + g(x) = 0$ was considered, assuming the energy level lines to be star-shaped. A similar type of result can also be found in [44, Theorem 2] in the case of non-Hamiltonian perturbations. However, in [44], a Melnikov type assumption is required, while this is not needed in our Hamiltonian setting.

The proof of Theorem 2 will be given in Section 4. Let us state one of its immediate consequences.

Corollary 1 *For any positive integer N there is a $\bar{\varepsilon}_N > 0$ such that, if*

$$|\nabla H(t,u) - \nabla \mathcal{H}(u)| \leq \bar{\varepsilon}_N\,, \qquad \textit{for every } t \in [0,T] \textit{ and } u \in \mathcal{A}\,,$$

then system (4) has at least N periodic solutions, whose orbits are contained in $\mathcal{A}$.

<u>Proof</u> It is sufficient to take N couples of positive relatively prime integers m and n satisfying (5). Since, for each of them, there is a suitable $\bar{\varepsilon}_{m,n} > 0$ for which the perturbed equation has two periodic solutions, taking as $\bar{\varepsilon}_N$ the smallest of those $\bar{\varepsilon}_{m,n}$ we obtain $2N$ periodic solutions. To be sure that such solutions are distinct it will be sufficient that the quotients m/n be pairwise distinct. ■

As a particular case, we will now show that such a situation arises when one can find a nondegenerate periodic solution of (3).

Definition 1 *We say that a T_0-periodic solution u_0 of (3) is* nondegenerate *if every T_0-periodic solution of the linearized system*

$$J\dot{v} = \mathcal{H}''(u_0(t))v \tag{6}$$

is of the type $v(t) = c\,\dot{u}_0(t)$, for some $c \in \mathbb{R}$.

Proposition 1 *If u_0 is a nondegenerate and nonconstant periodic solution of (3), with minimal period T_0, then there is a period annulus, containing the orbit of u_0 in its interior, whose inner and outer boundaries are the orbits of two solutions with periods $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, such that*

$$\textit{either} \quad \mathcal{T}_1 < T_0 < \mathcal{T}_2\,, \quad \textit{or} \quad \mathcal{T}_2 < T_0 < \mathcal{T}_1\,.$$

The proof of Proposition 1 is more or less known (see, e.g., [44, 50]), and is provided, for the reader's convenience, in the Appendix. As an immediate consequence, we have the following.

Corollary 2 *Assume that (3) has a nondegenerate and nonconstant T_0-periodic solution u_0, and let $\mathcal{U}_0$ be a given neighborhood of its orbit. Then, there is a $\bar{\delta} > 0$ with the following property: given two integers m and n for which*

$$\left|\frac{mT}{n} - T_0\right| \leq \bar{\delta}\,, \tag{7}$$

there is a $\bar{\varepsilon} > 0$ such that, if

$$|\nabla H(t,u) - \nabla \mathcal{H}(u)| \leq \bar{\varepsilon}\,, \qquad \textit{for every } t \in [0,T] \textit{ and } u \in \mathcal{U}_0\,,$$

then system (4) has at least two mT-periodic solutions, whose orbits are contained in $\mathcal{U}_0$, which make exactly n rotations around the origin in the period time mT.

<u>Proof</u> Proposition 1 guarantees the existence of a period annulus $\mathcal{A}$, containing in its interior the orbit of u_0, such that the periods of the solutions in $\mathcal{A}$ cover an interval $[\alpha, \beta]$, with $\alpha < T_0 < \beta$. Taking $\bar{\delta} = \min\{T_0 - \alpha, \beta - T_0\}$, Theorem 2 directly applies. ■

Using the same argument as for Corollary 1, we have the following direct consequence of Corollary 2.

Corollary 3 *Assume that (3) has a nondegenerate and nonconstant T_0-periodic solution u_0, and let $\mathcal{U}_0$ be a given neighborhood of its orbit. Then, for any positive integer N there is a $\bar{\varepsilon}_N > 0$ such that, if*

$$|\nabla H(t,u) - \nabla \mathcal{H}(u)| \leq \bar{\varepsilon}_N\,, \qquad \textit{for every } t \in [0,T] \textit{ and } u \in \mathcal{U}_0\,,$$

then system (4) has at least N periodic solutions, whose orbits are contained in $\mathcal{U}_0$.

Notice that, in order to apply Theorem 2, one could also consider situations where there is a "degenerate" and nonconstant periodic solution u_0 of (3), provided that some conditions on higher order derivatives of the time-map are imposed in order to determine an annulus, near the orbit of u_0, whose boundary is made by orbits with different periods. Conditions of this type have been provided, e.g., by Hale and Táboas [41], and Rothe [67].

As a particular case, we can apply the above results to a system like

$$J\dot{u} = \nabla H(t, u; \varepsilon)\,, \tag{8}$$

with Hamiltonian function $H : \mathbb{R} \times \mathcal{A} \times [-1, 1] \to \mathbb{R}$, whose gradient with respect to its second variable, denoted by $\nabla H(t, u; \varepsilon)$, is continuous in $(t, u; \varepsilon)$, and also locally Lipschitz continuous in u, and is T-periodic in its first variable, for some $T > 0$. We assume that

$$H(t, u; 0) = \mathcal{H}(u)\,, \qquad \text{for every } \ t \in [0, T] \text{ and } \ u \in \mathcal{A}\,,$$

where $\mathcal{H}$ is a twice continuously differentiable function on the period annulus $\mathcal{A}$. As above, we assume that the periods of the solutions of (3) in $\mathcal{A}$ cover an interval $[\mathcal{T}_{\min}\,, \mathcal{T}_{\max}]$, with $\mathcal{T}_{\min} < \mathcal{T}_{\max}$.

Recalling that $\mathcal{A}$ is a compact set, and therefore $|\nabla H(t, u; \varepsilon) - \nabla \mathcal{H}(u)| \to 0$ as $\varepsilon \to 0$, uniformly for $(t, u) \in [0, T] \times \mathcal{A}$, we immediately get the following statements.

Theorem 3 *Given two positive integers m and n satisfying (5), there is an $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, then system (8) has at least two mT-periodic solutions, whose orbits are contained in $\mathcal{A}$, which make exactly n rotations around the origin in the period time mT.*

Corollary 4 *For any positive integer N there is a $\bar{\varepsilon}_N > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}_N$, then system (8) has at least N periodic solutions, whose orbits are contained in $\mathcal{A}$.*

Analogous statements hold in the case when a nondegenerate periodic solution of (4) exists.

Corollary 5 *Assume that (3) has a nondegenerate and nonconstant T_0-periodic solution u_0, and let $\mathcal{U}_0$ be a given neighborhood of its orbit. Then, there is a $\bar{\delta} > 0$ with the following property: given two integers m and n for*

which (7) holds, there is a $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, then system (8) has at least two mT-periodic solutions, whose orbits are contained in $\mathcal{U}_0$, which make exactly n rotations around the origin in the period time mT.

Corollary 6 *Assume that (3) has a nondegenerate and nonconstant T_0-periodic solution u_0, and let $\mathcal{U}_0$ be a given neighborhood of its orbit. Then, for any positive integer N there is a $\bar{\varepsilon}_N > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}_N$, then system (8) has at least N periodic solutions, whose orbits are contained in $\mathcal{U}_0$.*

These last two corollaries may also be interpreted as a bifurcation type result from a nondegenerate periodic orbit, and should be compared with [70, Theorem 1], where the case $T = T_0$ is considered, for a system in $\mathbb{R}^{2M}$, with $M \geq 1$, and the existence of two T_0-periodic solutions is proved. See also [2, 26]. Analogous results are available in the non-Hamiltonian setting, with further assumptions on some Melnikov type functions. Notice also that Theorem 3 could be used to provide an alternative proof to some multiplicity results concerning T-periodic solutions, like, e.g., Theorems 6 and 8 in [28].

3 Some examples

In this section, we want to produce some examples of autonomous Hamiltonian systems in the plane whose orbits are not star-shaped. A direct inspection of the geometry of such systems permits to detect several annuli over which our results can be applied.

As a first example, we deal with a classical problem in constrained mechanics, the "bead on a wire" system. Let us consider a rigid wire contained in a vertical plane. We assume the wire to lie in a region where a constant force acts. Without loss of generality, we may assume such a force to be represented by a vertical vector $(0, -\kappa)$, with $\kappa > 0$ (for instance, it could be the gravitational force, which is approximatively constant in a small region of space). A ball of mass m is constrained to move, without friction, along the wire. We describe the wire with a C^1-curve $w : [s^-, s^+] \to \mathbb{R}^2$, with $w(s) = (w_1(s), w_2(s))$, parametrized by its arc length. We may identify the ball's position on the wire by means of the function $s(t)$, where t is the time and s denotes the arc length with respect to one of the end points of the wire. According to the discussion in [3, Chapter 3], the ball's motion is described by the solution to the differential equation $\ddot{s} = f(s)$, where

$f(s) = \frac{1}{m}F^{\tan}$, $F^{\tan}$ being the tangential component of the applied force, i.e., $F^{\tan} = (0, -\kappa) \cdot (w_1'(s), w_2'(s)) = -\kappa w_2'(s)$. Hence, the ball's motion is described by the solutions of the differential equation

$$\ddot{s} + \frac{\kappa}{m} w_2'(s) = 0\,,$$

equivalent to the planar Hamiltonian system

$$\dot{s} = q, \qquad -\dot{q} = \frac{\kappa}{m} w_2'(s)\,,$$

with Hamiltonian function

$$\mathcal{H}(s, q) = \frac{q^2}{2} + \frac{\kappa}{m} w_2(s)\,.$$

Choosing appropriately the function $w_2(s)$, we can construct Hamiltonian systems with non-star-shaped orbits. For instance we may choose

$$\mathcal{H}(s, q) = \frac{q^2}{2} + s^2(1 - s^2)(4 - s^2)\,,$$

which has three minima on the s-axis, surrounded by closed level sets. The region covered by such level sets has a bounded boundary, the union of the homoclinic and the heteroclinic connections of the two saddle points. Outside those separatrices, all level sets are closed. Infinitely many of them are non-star-shaped, as shown in Figure 1.

We can easily construct period annuli with star-shaped boundaries in the regions around the three minima. Moreover, there is a large choice of period annuli with non-star-shaped boundaries in the region outside the separatrices.

Using a different approach, we will now construct some other examples of Hamiltonian functions, with an isolated minimum at the origin O, having no star-shaped level sets in a neighbourhood of O.

Let $\varphi :]0, +\infty[\to]0, +\infty[$ be a twice continuously differentiable function and, for every positive $r = \sqrt{x^2 + y^2}$, consider the linear map associated to the matrix

$$A_\varphi(r) = \begin{pmatrix} \cos\varphi(r) & \sin\varphi(r) \\ -\sin\varphi(r) & \cos\varphi(r) \end{pmatrix}$$

which rotates clockwise the vector (x, y) of an angle $\varphi(r)$. We call *circular shear* the transformation $\Phi : \mathbb{R}^2 \backslash \{O\} \to \mathbb{R}^2 \backslash \{O\}$, defined by

$$\Phi(x, y) = A_{\varphi(r)}(x, y)\,.$$

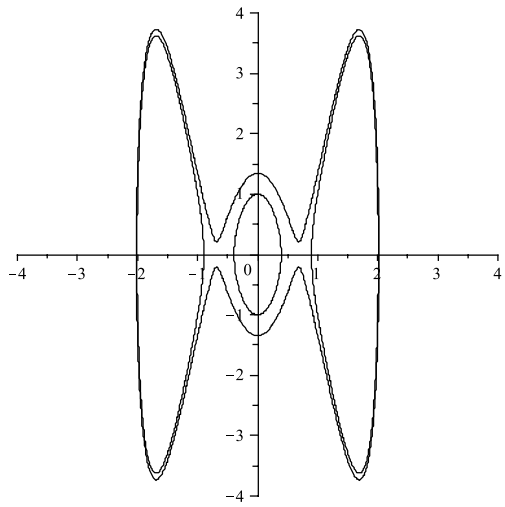

Figure 1: The levels 0.5, 0.9 of $\mathcal{H}(s,q) = \frac{q^2}{2} + s^2(1-s^2)(4-s^2)$.

Such a map leaves invariant the circles centered at the origin, operating on each one of them as a clockwise rotation of an angle $\varphi(\sqrt{x^2+y^2})$. Its inverse is the map $A_{-\varphi(r)}(x,y)$. We can extend Φ to the whole plane by setting $\Phi(O) = O$.

The function $\Phi(x,y)$ is an area-preserving transformation. In fact, denoting by Γ the transformation from rectangular to polar coordinates,

$$\Gamma(x,y) = (r\cos\theta, r\sin\theta)\,,$$

and $\mathbb{T}$ the "triangular" map

$$\mathbb{T}(r,\theta) = (r, \theta + \varphi(r))\,,$$

we have that

$$\Phi = \Gamma^{-1} \circ \mathbb{T} \circ \Gamma\,.$$

Circles centered at the origin are invariant for the transformation Γ, hence the Jacobian determinant J_Φ of Φ can be computed as follows

$$J_\Phi = J_{\Gamma^{-1}} \cdot J_{\mathbb{T}} \cdot J_\Gamma = \frac{1}{r} \cdot 1 \cdot r = 1\,.$$

Even if φ is not even defined at $r = 0$, the map $\Phi(x,y)$ is area-preserving, since a single point is not relevant in area's measure.

Consider the function

$$G(X,Y) = \frac{X^2}{a^2} + \frac{Y^2}{b^2}$$

with $a, b \in \mathbb{R}$ such that $0 < a < b$. The level sets $G = l^2$, with $l > 0$, are ellipses centered at O, that meet the X- and Y-axis at the points $(\pm la, 0)$ and $(0, \pm lb)$, respectively. We define the function

$$\mathcal{H}(x,y) = G(\Phi(x,y))\,.$$

The level sets of $\mathcal{H}$ are obtained from the ellipses by means of the transformation $A_{-\varphi(r)}(x,y)$. Let us choose φ as follows:

$$\varphi(r) = \frac{1}{r(b-a)}\,.$$

It is easily seen that

$$\varphi(la) - \varphi(lb) > \frac{1}{lb^2}\,.$$

As l goes to zero, the difference $\varphi(la) - \varphi(lb)$ goes to $+\infty$. This fact is responsible for a distorsion in the orbits, so that, as shown in Figure 2, where we have chosen $a^2 = \frac{1}{10}$, $b^2 = 1$, the orbits cannot be star-shaped. Lower levels display higher distortions.

In order to construct a Hamiltonian function of class C^∞, we can apply a smoothing procedure developed in [55] in order to prove the existence of smooth first integrals of planar centers. First, we observe that the function $\mathcal{H}$ is C^∞ on $B(O,\epsilon) \setminus \{O\}$ and continuous on $B(O,\epsilon)$, for some $\epsilon > 0$. Then, following [55, Section 1], it is possible to find a scalar function F such that $F \circ \mathcal{H}$ is C^∞ on $B(O,\epsilon)$, and such that the composed function $F \circ \mathcal{H}$ has the same level sets as $\mathcal{H}$.

It is even possible to construct a function defined on all of $\mathbb{R}^2$, which reflects the above properties. The transformation

$$\zeta(x,y) = \left(x \tan \frac{\pi\sqrt{x^2+y^2}}{2\epsilon}\,,\; y \tan \frac{\pi\sqrt{x^2+y^2}}{2\epsilon} \right)$$

is a diffeomorphism taking the disk $B(O,\epsilon)$ onto all of $\mathbb{R}^2$. The map ζ leaves all rays invariant, hence, taking $\mathcal{H}$ as above, the function $\mathcal{K}(x,y) = \mathcal{H}(\zeta^{-1}(x,y))$ is defined on all of $\mathbb{R}^2$ and has non-star-shaped level sets.

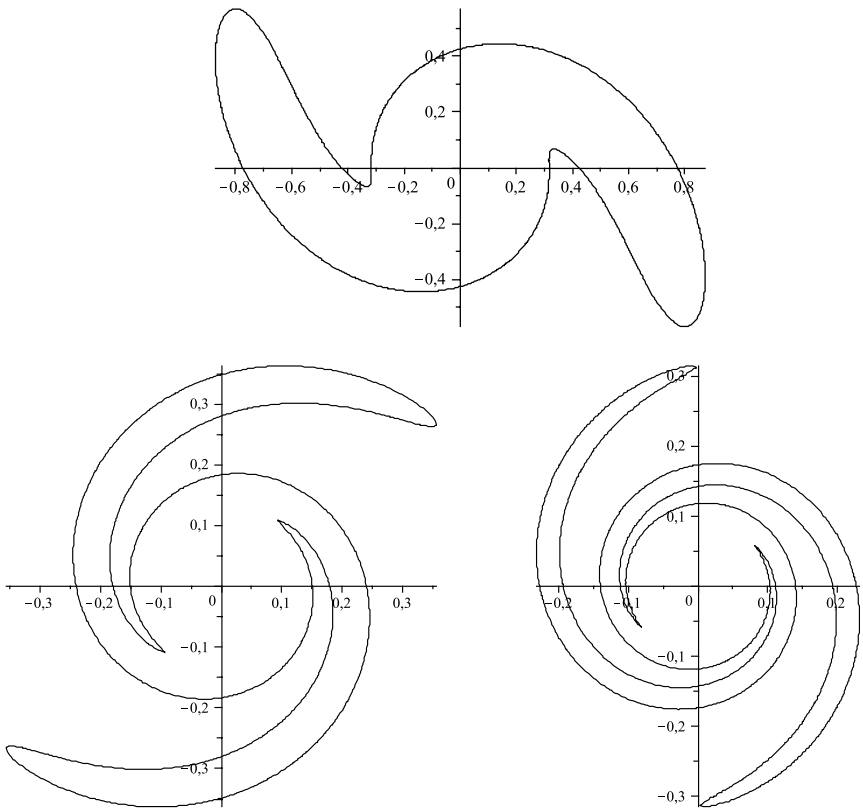

Figure 2: The level sets $\mathcal{H} = 1$, $\mathcal{H} = 0.2$ and $\mathcal{H} = 0.1$

4 The proof of Theorem 2

We begin by selecting from $\mathcal{A}$ two solutions with minimal periods $\mathcal{T}_{\min}$ and $\mathcal{T}_{\max}$, respectively. These solutions determine a smaller period annulus, over which the same assumptions of Theorem 2 hold. Therefore, we may assume without loss of generality that the periods of the inner boundary orbit γ_i and of the outer boundary orbit γ_e are the extremals of the interval $[\mathcal{T}_{\min}, \mathcal{T}_{\max}]$. Just to fix the ideas, we assume that $\mathcal{T}_{\min}$ is the period of γ_i, and $\mathcal{T}_{\max}$ is the period of γ_e. The other case is analogous.

Set $h_i = \mathcal{H}(\gamma_i)$ and $h_e = \mathcal{H}(\gamma_e)$. Noticing that

$$\nabla\mathcal{H}(u) \neq 0\,, \qquad \text{for every } u \in \mathcal{A}\,,$$

we have that $h_i \neq h_e$. We will assume, for instance,

$$h_i < h_e\,,$$

the other case being treated similarly. We thus have

$$h_i = \min\{\mathcal{H}(u) : u \in \mathcal{A}\}\,, \qquad h_e = \max\{\mathcal{H}(u) : u \in \mathcal{A}\}\,.$$

Clearly, nothing will change by adding a constant to the Hamiltonian function $\mathcal{H}$. Since $\mathcal{A}$ is compact, we can then assume, without loss of generality, that $h_i > 0$. Notice that, under the above assumptions, all the solutions in $\mathcal{A}$ rotate clockwise with respect to the origin.

We then see that the orbits of the gradient system

$$\dot{u} = \nabla\mathcal{H}(u)\,, \tag{9}$$

starting at a point of the inner boundary γ_i, meet exactly once all the cycles in $\mathcal{A}$, eventually leaving $\mathcal{A}$ at a point of the outer boundary γ_e. Let us fix a reference orbit δ^* of (9), such that $\delta^*(0) \in \gamma_i$, and $\delta^*(\sigma^*) \in \gamma_e$, for some $\sigma^* > 0$.

In the following, we will denote by $V_{\mathcal{H}}$ the vector field associated to the differential system (3). For $u \in \mathcal{A}$, let us denote by $\mathcal{T}(u)$ the minimal period of the cycle passing through u. By the implicit function theorem, it can be seen that the function $\mathcal{T}(u)$ is twice continuously differentiable (see, e.g., [59, Theorem 3.4.1]).

Lemma 1 *There is a twice continuously differentiable function $\nu : [h_i, h_e] \to \mathbb{R}$ such that $\mathcal{T}(u) = \nu(\mathcal{H}(u))$, for every $u \in \mathcal{A}$.*

<u>Proof</u> Consider the reference curve $\delta^*(s)$, solution of (9). The function $h(s) = \mathcal{H}(\delta^*(s))$ has positive derivative, hence it is invertible. Since h is C^2, its inverse is also C^2, and, setting

$$\nu(S) = \mathcal{T}(\delta^*(h^{-1}(S)))\,,$$

for all $S = h(s)$, we have that $\nu(\mathcal{H}(\delta^*(s))) = \mathcal{T}(\delta^*(s))$. Since both $\mathcal{T}$ and $\mathcal{H}$ are constant along the solutions of (3), the proof is easily concluded. ■

We now consider the system

$$J\dot{u} = \frac{\mathcal{T}(u)}{2\pi}\,\nabla\mathcal{H}(u)\,. \tag{10}$$

The period annulus $\mathcal{A}$ is isochronous for (10), with period 2π. Setting

$$\mathcal{K}(u) = \frac{1}{2\pi}\int_0^{\mathcal{H}(u)} \nu(s)\,ds\,,$$

we have that

$$\nabla\mathcal{K}(u) = \frac{1}{2\pi}\nu(\mathcal{H}(u))\nabla\mathcal{H}(u) = \frac{\mathcal{T}(u)}{2\pi}\,\nabla\mathcal{H}(u)\,.$$

So, (10) is itself a Hamiltonian system, generated by $\mathcal{K}$. Let us denote by $V_{\mathcal{K}}$ the vector field associated to the differential system (10), and by $\phi_{\mathcal{K}}(t,u)$ the flow defined by (10) on $\mathcal{A}$. One has $V_{\mathcal{K}} = \mathcal{T}V_{\mathcal{H}}/2\pi$.

For $u \in \mathcal{A}$, let us denote by $\tau(u) \in [0, 2\pi[$ the minimum time for which $\phi_{\mathcal{K}}(-\tau(u), u) \in \delta^*$. The regularity of the map $\tau(u)$ comes from the following lemma.

Lemma 2 *The function $\tau : \mathcal{A} \to \mathbb{R}$ is twice continuously differentiable.*

Proof. In order to prove the regularity of τ at the point $u_0 \in \mathcal{A}$, let u^* be the unique point of δ^* such that $\phi_{\mathcal{K}}(-\tau(u_0), u_0) = u^*$, or, equivalently, $\phi_{\mathcal{K}}(\tau(u_0), u^*) = u_0$. Since u^* is a non-singular point of $\nabla\mathcal{H}$, there exists a neighbourhood U^* of u^* such that $U^* \cap \delta^*$ is the graph of a C^2 single-variable function $y = \zeta(x)$ (or $x = \zeta(y)$; in this case the proof proceeds in a similar way). In other words, $U^* \cap \delta^*$ is the zero-level set of the two-variables function $Z(x,y) = \zeta(x) - y$, whose gradient is $\nabla Z(x,y) = (\zeta'(x), 1) \neq (0,0)$. Such a gradient is orthogonal to δ^*, hence it is parallel to $V_{\mathcal{H}}$, and to $V_{\mathcal{K}} = \mathcal{T}V_{\mathcal{H}}/2\pi$, at every point of $U^* \cap \delta^*$. Let us set $U(u_0) = \phi_{\mathcal{K}}(\tau(u_0), U^*)$. The map $\phi_{\mathcal{K}}(\tau(u_0), \cdot)$ is a diffeomorphism, hence $U(u_0)$ is a neighbourhood of u_0. Let us consider the function $Z(\phi_{\mathcal{K}}(t,u))$, defined on $U(u_0)$. Differentiating such a function with respect to t one has

$$\begin{aligned}\frac{\partial}{\partial t}Z(\phi_{\mathcal{K}}(t,u)) &= \nabla Z(\phi_{\mathcal{K}}(t,u)) \cdot \frac{\partial\phi_{\mathcal{K}}(t,u)}{\partial t}\\ &= \nabla Z(\phi_{\mathcal{K}}(t,u)) \cdot V_{\mathcal{K}}(\phi_{\mathcal{K}}(t,u)).\end{aligned}$$

Such a scalar product does not vanish if $\phi_{\mathcal{K}}(t,u) \in U^* \cap \delta^*$, since on $U^* \cap \delta^*$ one has $\nabla Z(u) \neq 0$ and $\nabla Z(u)$ is parallel to $V_{\mathcal{K}}(u)$. Hence, the partial derivative of $Z(\phi_{\mathcal{K}}(t,u))$ with respect to t does not vanish at the point $(-\tau(u_0), u_0)$. By the implicit function theorem, the relationship $Z(\phi_{\mathcal{K}}(t,u)) = 0$ defines locally, at $(-\tau(u_0), u_0)$, a C^2-function $t(u)$. By construction, one has $t(u) = -\tau(u)$, which gives the regularity of $\tau(u)$ at u_0. ■

Following [68], we define the function $\Lambda : \mathcal{A} \to \mathbb{R}^2$ as

$$\Lambda(u) = \left(\sqrt{2\mathcal{K}(u)} \cos(-\tau(u)) \, , \sqrt{2\mathcal{K}(u)} \sin(-\tau(u)) \right) .$$

Let us denote by k_i, k_e, the values of $\mathcal{K}(u)$ corresponding to the level sets $\mathcal{H}(u) = h_i$, $\mathcal{H}(u) = h_e$, respectively. The map Λ transforms each level set $\{u : \mathcal{K}(u) = k\}$ into the circle $\{v : |v|^2 = 2k\}$. We denote by J_Λ the Jacobian determinant of Λ. For the sake of conciseness, we sometimes write $c(u)$ for $\cos(\tau(u))$, and $s(u)$ for $\sin(\tau(u))$. Moreover, we set

$$r_i = \sqrt{2k_i} \, , \qquad r_e = \sqrt{2k_e} \, .$$

Lemma 3 *The function Λ is an area-preserving diffeomorphism that takes the set $\mathcal{A}$ onto the annulus*

$$\mathcal{B} = \{ v \in \mathbb{R}^2 : r_i \leq |v| \leq r_e \} \, .$$

The system (3) is transformed by Λ into the system

$$J\dot{v} = \frac{2\pi}{\mathcal{T}(\Lambda^{-1}(v))} \, v \, . \tag{11}$$

<u>Proof</u> The Jacobian matrix of Λ is

$$J_\Lambda = \begin{bmatrix} \dfrac{\mathcal{K}_x}{\sqrt{2\mathcal{K}}} c - \sqrt{2\mathcal{K}} s \tau_x & \dfrac{\mathcal{K}_y}{\sqrt{2\mathcal{K}}} c - \sqrt{2\mathcal{K}} s \tau_y \\ -\dfrac{\mathcal{K}_x}{\sqrt{2\mathcal{K}}} s - \sqrt{2\mathcal{K}} c \tau_x & -\dfrac{\mathcal{K}_y}{\sqrt{2\mathcal{K}}} s - \sqrt{2\mathcal{K}} c \tau_y \end{bmatrix} .$$

Hence, the Jacobian determinant of Λ is

$$\det J_\Lambda = \tau_x \mathcal{K}_y - \tau_y \mathcal{K}_x \, .$$

Since $\mathcal{K}_x \tau_y - \tau_x \mathcal{K}_y$ is the derivative of $\tau(x,y)$ along the solutions of (10), i.e., the derivative of the time of (10) with respect to itself, one has $\det J_\Lambda = 1$. This shows that Λ is a local diffeomorphism.

Let us set $v = \Lambda(u)$, with $v = (w, z) \in \mathbb{R}^2$. Then,

$$\begin{aligned}\dot{w} &= \left(\frac{\mathcal{K}_x c}{\sqrt{2\mathcal{K}}} - \sqrt{2\mathcal{K}} s \tau_x\right)\mathcal{K}_y - \left(\frac{\mathcal{K}_y c}{\sqrt{2\mathcal{K}}} - \sqrt{2\mathcal{K}} s \tau_y\right)\mathcal{K}_x \\ &= -\sqrt{2\mathcal{K}} s(\tau_x \mathcal{K}_y - \tau_y \mathcal{K}_x) = z\,,\end{aligned}$$

and

$$\begin{aligned}\dot{z} &= -\left(\frac{\mathcal{K}_x s}{\sqrt{2\mathcal{K}}} + \sqrt{2\mathcal{K}} c \tau_x\right)\mathcal{K}_y + \left(\frac{\mathcal{K}_y s}{\sqrt{2\mathcal{K}}} + \sqrt{2\mathcal{K}} c \tau_y\right)\mathcal{K}_x \\ &= -\sqrt{2\mathcal{K}} c(\tau_x \mathcal{K}_y - \tau_y \mathcal{K}_x) = -w\,.\end{aligned}$$

Hence, Λ transforms (10) into

$$J\dot{v} = v\,. \tag{12}$$

Consequently, Λ transforms (3) into (11).

An orbit γ of (10) is taken into the circle of radius $\sqrt{2\mathcal{K}(\gamma)}$ centered at the origin of the v-plane. Let us denote by $\theta = \arg(v)$ the angle between by the vector $v = (w, z)$ and the positive w-semiaxis. A point of an orbit of (12) is uniquely identified by the corresponding value of θ. Similarly, a point of an orbit of (10) is uniquely identified by the corresponding value of $\tau(u) \in [0, 2\pi[$. Since $\dot{\theta} = \dot{\tau} = 1$, we have that Λ is injective on every orbit, and the proof of the lemma is thus completed. ■

Since $\det \Lambda'(u) = 1$ for every $u \in \mathcal{A}$, the function $\Lambda : \mathcal{A} \to \mathcal{B}$ is symplectic (cf. [36]), i.e.,

$$\Lambda'(u)^T J \Lambda'(u) = J\,, \tag{13}$$

for every $u \in \mathcal{A}$. (Here $\mathbb{M}^T$ denotes the transposed of a matrix $\mathbb{M}$.) Then, using (13), one easily sees that the function Λ transforms the Hamiltonian system (4) into

$$J\dot{v} = \nabla L(t, v)\,, \tag{14}$$

where

$$L(t, v) = H(t, \Lambda^{-1}(v))\,.$$

(As usual, we denote by ∇L the gradient with respect to the second variable.) Moreover, we can write equivalently (11) as

$$J\dot{v} = \nabla \mathcal{L}(v)\,, \tag{15}$$

where

$$\mathcal{L}(v) = \mathcal{H}(\Lambda^{-1}(v))\,.$$

We now need to extend our Hamiltonian system (14) from $\mathcal{B}$ to the whole plane $\mathbb{R}^2$. Let m, n be two positive integers verifying (5). Then, there is a (mT/n)-periodic solution of (11), whose orbit lies in the interior of $\mathcal{B}$. Its orbit is indeed a circle, with radius $\bar{r} \in]r_i, r_e[$. Let $\delta > 0$ be such that

$$2\delta < \min\{\bar{r} - r_i, r_e - \bar{r}\}\,,$$

and define the annuli

$$\begin{aligned}\mathcal{B}' &= \{v \in \mathbb{R}^2 : r_i + \delta \le |v| \le r_e - \delta\}\,,\\ \mathcal{B}'' &= \{v \in \mathbb{R}^2 : r_i + 2\delta \le |v| \le r_e - 2\delta\}\,.\end{aligned}$$

Let us denote by $\mathcal{T}_{i,2\delta}$ and $\mathcal{T}_{e,2\delta}$ the periods of the circular orbits of (11) with radius $r_i + 2\delta$ and $r_e - 2\delta$, respectively. If δ is small enough, we can assume that

$$\mathcal{T}_{i,2\delta} < \frac{mT}{n} < \mathcal{T}_{e,2\delta}\,. \tag{16}$$

Consider now a C^2-function $\chi : [0, +\infty[\to [0, +\infty[$, such that

$$\chi(r) = \begin{cases} 1 & \text{if } r \in [r_i + \delta, r_e - \delta]\,,\\ 0 & \text{if } r \notin [r_i, r_e]\,.\end{cases}$$

Let $\tilde{L} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$\tilde{L}(t, v) = \begin{cases} \chi(|v|)L(t, v) & \text{if } v \in \mathcal{B}\,,\\ 0 & \text{if } v \in \mathbb{R}^2 \setminus \mathcal{B}\,.\end{cases}$$

This is a continuous function, T-periodic in its first variable, and twice continuously differentiable in its second variable. We can then consider the Hamiltonian system

$$J\dot{v} = \nabla\tilde{L}(t, v)\,, \tag{17}$$

which extends (14) to the whole plane $\mathbb{R}^2$. Notice that

$$v \in \mathcal{B}' \quad \Rightarrow \quad \tilde{L}(t, v) = L(t, v)\,.$$

Moreover, all the points in $\mathbb{R}^2 \setminus \mathcal{B}$ are equilibria for (17).

Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be the Poincaré map for the period T, associated to system (17). It is well-known that this map is an area-preserving homeomorphism. Notice that $\mathcal{P}(0) = 0$. Let $\mathcal{P}_0 : \mathcal{B}'' \to \mathcal{B}''$ be the Poincaré map for the period T, associated to system (11) which, we recall, is also written as (15). By (16), the twist condition is verified on $\mathcal{B}''$ by the map obtained as the composition of the m-th iterate of $\mathcal{P}_0$, and a counter-clockwise rotation of angle $2\pi n$. By compactness and continuous dependence, there is a $\bar{\varepsilon}_1 > 0$ such that, if

$$\left|\nabla \tilde{L}(t,v) - \nabla \mathcal{L}(v)\right| < \bar{\varepsilon}_1 \,, \qquad \text{for every } \ t \in [0,T] \text{ and } \ v \in \mathcal{B}'' \,, \tag{18}$$

the same twist condition on $\mathcal{B}''$ is satisfied by the map obtained as the composition of the m-th iterate of $\mathcal{P}$, and a counter-clockwise rotation of angle $2\pi n$.

Hence, if (18) is satisfied, we can apply Theorem 1, and we get two mT-periodic solutions $v_1(t)$, $v_2(t)$ of (17), with initial point in $\mathcal{B}''$, which make exactly n rotations around the origin in the period time mT. Moreover, if $\bar{\varepsilon}_1$ is small enough, all the solutions of (17) starting inside $\mathcal{B}''$ at the time $t = 0$ will remain in $\mathcal{B}'$ for all times $t \in [0, mT]$. Therefore, $v_i(t) \in \mathcal{B}'$ for every $t \in \mathbb{R}$, and hence $v_i(t)$ is a mT-periodic solution of (14), for $i = 1, 2$.

Once $\bar{\varepsilon}_1 > 0$ has been found, there is a $\bar{\varepsilon} > 0$ such that, if

$$|\nabla H(t,u) - \nabla \mathcal{H}(u)| \le \bar{\varepsilon} \,, \qquad \text{for every } \ t \in [0,T] \text{ and } \ u \in \mathcal{A} \,,$$

then (18) holds, so that, by the above argument, there are two mT-periodic solutions $v_1(t)$, $v_2(t)$ of (14) which lie in $\mathcal{B}'$, and make exactly n rotations around the origin in the period time mT. Setting $u_i(t) = \Lambda^{-1} v_i(t)$, for $i = 1, 2$, we have found two mT-periodic solutions of (4), which lie in $\mathcal{A}$, and make exactly n rotations around the origin in the period time mT. The proof of Theorem 2 is thus completed.

Appendix: Proof of Proposition 1

For the reader's convenience, we provide here a proof of Proposition 1. We follow essentially the arguments in [50].

Assume that u_0 is a nonconstant periodic solution of (3), with minimal period T_0. Let $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$\Psi(\theta, r) = u_0(\theta) + rJ\dot{u}_0(\theta)\,.$$

Notice that $\Psi(\theta, r)$ is T_0-periodic in its first variable. We denote by $u(t; \theta, r)$ the solution of the Cauchy problem

$$\begin{cases} J\dot{u} = \nabla\mathcal{H}(u) \\ u(0) = \Psi(\theta, r)\,. \end{cases} \tag{19}$$

If $|r|$ is small, such a solution is periodic in its first variable, with a minimal period which will be denoted by $T(\theta, r)$. Notice that, since $\Psi(\theta, 0) = u_0(\theta)$, one has that

$$T(\theta, 0) = T_0\,.$$

By standard arguments (cf. [18]), it is possible to determine two functions $\vartheta, \rho : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ such that

$$u(t; \theta, r) = \Psi(\vartheta(t; \theta, r), \rho(t; \theta, r))\,.$$

These functions are continuously differentiable, at least as far as $|r|$ is small. Since $u(0; \theta, r) = \Psi(\theta, r)$ and $u(t; \theta, 0) = u_0(t + \theta)$, we have that

$$\vartheta(0; \theta, r) = \theta\,, \qquad \rho(0; \theta, r) = r\,, \tag{20}$$

and

$$\vartheta(t; \theta, 0) = t + \theta\,, \qquad \rho(t; \theta, 0) = 0\,. \tag{21}$$

We can then see that

$$\frac{\partial u}{\partial r}(t; \theta, 0) = \frac{\partial \vartheta}{\partial r}(t; \theta, 0)\dot{u}_0(t + \theta) + \frac{\partial \rho}{\partial r}(t; \theta, 0)J\dot{u}_0(t + \theta)\,. \tag{22}$$

Moreover, by the periodicity, if $|r|$ is small enough,

$$\vartheta(T(\theta, r); \theta, r) = \theta + T_0\,, \qquad \rho(T(\theta, r); \theta, r) = r\,;$$

so, since $r \mapsto \vartheta(T(\theta, r); \theta, r)$ is constant, we have that

$$\frac{\partial \vartheta}{\partial t}(T(\theta, r); \theta, r)\frac{\partial T}{\partial r}(\theta, r) + \frac{\partial \vartheta}{\partial r}(T(\theta, r); \theta, r) = 0\,.$$

In particular, taking $r = 0$, and recalling (21),

$$\frac{\partial T}{\partial r}(\theta, 0) + \frac{\partial \vartheta}{\partial r}(T_0; \theta, 0) = 0\,. \tag{23}$$

The following lemma immediately leads to the conclusion of the proof of Proposition 1.

Lemma 4 *The solution u_0 of (3) is nondegenerate if and only if*

$$\frac{\partial T}{\partial r}(\theta, 0) \neq 0\,, \qquad \textit{for every } \theta \in \mathbb{R}\,. \tag{24}$$

<u>Proof</u> First, assume that u_0 is nondegenerate. By contradiction, recalling (23), assume that there is a θ_0 for which

$$\frac{\partial \vartheta}{\partial r}(T_0; \theta_0, 0) = 0\,. \tag{25}$$

Let us define

$$V(t) = \frac{\partial u}{\partial \theta}(t; \theta_0, 0)\,, \qquad W(t) = \frac{\partial u}{\partial r}(t; \theta_0, 0)\,.$$

Notice that $V(t) = \dot{u}_0(t + \theta_0)$, by (21). It is then easy to see that both these functions $V(t)$ and $W(t)$ solve the linear differential equation

$$J\dot{u} = \mathcal{H}''(u_0(t + \theta_0))u\,. \tag{26}$$

Since, recalling (20) and (21), the two vectors $V(0) = \dot{u}_0(\theta_0)$ and $W(0) = J\dot{u}_0(\theta_0)$ are orthogonal, these two solutions $V(t)$ and $W(t)$ are linearly independent.

Since, $V(t) = \dot{u}_0(t+\theta_0)$, we have that $V(t)$ is T_0-periodic, so that $V(T_0) = V(0)$. On the other hand, by (22) and (25),

$$\begin{aligned} W(T_0) &= \frac{\partial \vartheta}{\partial r}(T_0; \theta_0, 0)\dot{u}_0(T_0 + \theta_0) + \frac{\partial \rho}{\partial r}(T_0; \theta_0, 0)J\dot{u}_0(T_0 + \theta_0) \\ &= \frac{\partial \rho}{\partial r}(T_0; \theta_0, 0)J\dot{u}_0(\theta_0)\,, \end{aligned}$$

so that $W(T_0) = \alpha_0 W(0)$, with $\alpha_0 = \frac{\partial \rho}{\partial r}(T_0; \theta_0, 0)$. Let $X(t) = (V(t), W(t))$ be the 2×2 matrix whose columns coincide with $V(t)$ and $W(t)$. By Liouville

Theorem, the function $t \mapsto \det X(t)$ is constant, hence $\det X(0) = \det X(T_0)$, i.e.,

$$\det(\dot{u}_0(\theta_0), J\dot{u}_0(\theta_0)) = \det(\dot{u}_0(\theta_0), \alpha_0 J\dot{u}_0(\theta_0)) \,.$$

We then have that $\alpha_0 = 1$, so that $W(t)$ is T_0-periodic, as well as $V(t)$. Then, all the solutions of (26) are T_0-periodic. By a simple change of variable, we then conclude that all the solutions of (6) are T_0-periodic, in contradiction with the assumption that u_0 is nondegenerate.

Now, assume that (24) holds. Following the above construction for $\theta_0 = 0$, we define $V(t) = \frac{\partial u}{\partial \theta}(t; 0, 0)$ and $W(t) = \frac{\partial u}{\partial r}(t; 0, 0)$. Clearly, $V(t) = \dot{u}_0(t)$ is T_0-periodic. On the other hand, since $\frac{\partial \vartheta}{\partial r}(T_0; 0, 0) \neq 0$, we have that $W(T_0) \neq W(0)$, so that $W(t)$ is not T_0-periodic. Therefore, the set of T_0-periodic solutions of (6) is only made of the multiples of $V(t)$. Hence, u_0 is nondegenerate, and the lemma is thus proved. ■

Remark 1 In the above proof, in order to define the local coordinates, besides taking the position of the function $u_0(t)$, we have chosen to use the direction of the normal vector to its orbit $J\dot{u}_0(t)$. An equivalent approach could be made using the gradient lines of the Hamiltonian function, i.e., the orbits of the gradient system (9).

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