Nonlinear resonance: a comparison between Landesman-Lazer and Ahmad-Lazer-Paul conditions

Alessandro Fonda and Maurizio Garrione

Abstract

We show that the Ahmad-Lazer-Paul condition for resonant problems is more general than the Landesman-Lazer one, discussing some relations with other existence conditions, as well. As a consequence, such a relation holds, for example, when considering resonant boundary value problems associated to linear elliptic operators, to the p-Laplacian and, in the scalar case, to an asymmetric oscillator.

1 Introduction

The aim of this paper is to establish some connections between classical existence conditions for nonlinear problems at resonance. In 1969, a paper by Lazer and Leach [29] for the periodic problem opened the way towards what today is usually called the *Landesman-Lazer condition*, introduced one year later in [26] for a semilinear elliptic problem. This type of condition has inspired several authors in the attempt of finding the right abstract formulation and providing different generalizations. Contributions in this direction were given, among others, by Brezis and Nirenberg [6], de Figueiredo [8], Fučik [20], Hess [23], Mawhin [32], Nečas [35], and Williams [41]. A significant alternative to the Landesman-Lazer condition was proposed by Ahmad, Lazer and Paul [1] in 1976. A variational formulation for it was given by Rabinowitz [36] in 1978, who introduced, for that purpose, his Saddle Point Theorem. More recent contributions in this direction can be found, e.g., in [9, 10, 13, 14, 15, 16, 17, 18, 24, 27, 28, 34].

For a general semilinear problem of the type

$$Lu = g(x, u),$$

where L is a linear differential operator with nontrivial kernel, there is a large literature dealing with different kinds of conditions to be imposed on the nonlinear function g in order to guarantee the existence of a solution. To this aim, different techniques have been exploited: Leray-Schauder degree theory, variational methods, and fixed point theorems in the phase plane.

Let us recall the nonresonance conditions proposed by Landesman-Lazer and Ahmad-Lazer-Paul. To fix the ideas, let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary, and $L : D(L) \subset L^2(\Omega) \to L^2(\Omega)$ a linear operator with compact resolvent. Assume $g : \Omega \times \mathbb{R} \to \mathbb{R}$ to be a L^2-bounded function, i.e., there exists $h \in L^2(\Omega)$ such that

$$|g(x, s)| \leq h(x),$$

for almost every $x \in \Omega$ and every $s \in \mathbb{R}$. Let G be a primitive of g in the second variable, i.e.,

$$G(x,s) = \int_0^s g(x,\tau)\,d\tau.$$

The *Landesman-Lazer condition* reads as follows:

(LL) for every $v \in \ker L \setminus \{0\}$,

$$\int_{\{v>0\}} \liminf_{s\to+\infty} g(x,s)v(x)\,dx + \int_{\{v<0\}} \limsup_{s\to-\infty} g(x,s)v(x)\,dx > 0,$$

while the *Ahmad-Lazer-Paul condition* can be written as

(ALP) as long as $v \in \ker L$,

$$\lim_{\|v\|_2\to+\infty} \int_\Omega G(x,v(x))\,dx = +\infty.$$

In the above, we used the notation

$$\{v>0\} = \{x\in\Omega \mid v(x)>0\}, \quad \{v<0\} = \{x\in\Omega \mid v(x)<0\}.$$

It is well known that (LL) and (ALP) are strongly related. Indeed, it was already proved in [1] (see also [33]) that, in some special cases, (LL) implies (ALP). For instance, this is surely true if the kernel of L has dimension equal to 1. However, even if it is commonly believed that (ALP) is more general than (LL), the implication in the general case, to our knowledge, has not been established yet.

In this paper, we will give a characterization of the Landesman-Lazer condition which allows to prove that (LL) implies (ALP) in a very general setting. Proposition 3 below provides this characterization, which can be of independent interest for a better understanding of (LL). No boundedness assumptions will be necessary on the set Ω - indeed, Ω can be an arbitrary σ-finite measure space - and more general nonlinearities and differential operators will be admitted in our framework. Moreover, we do not need a linear subspace as the kernel of L, but just a cone Σ, with some compactness properties. As possible examples, the nonlinear asymmetric oscillator equation and equations involving the p-Laplacian will be included in this setting. At the end of the paper, we will take into consideration another alternative condition related to both (LL) and (ALP), sometimes called *potential Landesman-Lazer* condition (see [39] and [40]).

2 Main result

Let (Ω,μ) be a σ-finite measure space (in the applications, Ω is usually an open subset of $\mathbb{R}^n$ with the standard Lebesgue measure). We will briefly write "measurable" in place of μ-measurable, and $L^q(\Omega)$ instead of $L^q(\Omega, d\mu)$. Let $g : \Omega\times\mathbb{R} \to \mathbb{R}$ be a L^1-Carathéodory function, i.e.,

- $x \mapsto g(x,s)$ is measurable for every $s \in \mathbb{R}$;
- $s \mapsto g(x,s)$ is continuous for almost every $x \in \Omega$;

- for every $R > 0$, there exists $\eta_R \in L^1(\Omega)$ such that, for almost every $x \in \Omega$, and every $s \in \mathbb{R}$ with $|s| \le R$,
$$|g(x,s)| \le \eta_R(x),$$

and let $p, q \in [1, +\infty]$ be conjugate exponents, i.e.,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We assume that there exist $d > 0$ and a nonnegative function $h \in L^q(\Omega)$ such that, for almost every $x \in \Omega$,

$$|s| \ge d \quad \Rightarrow \quad \operatorname{sgn}(s) g(x,s) \ge -h(x). \tag{1}$$

Let $\Sigma \subset L^p(\Omega)$ satisfy the following properties:

- if $u \in \Sigma$, and $\lambda > 0$, then $\lambda u \in \Sigma$;
- $\Sigma \cap S_1$ is compact in $L^p(\Omega)$, where $S_1 = \{u \in L^p(\Omega) \mid \|u\|_p = 1\}$.

Set $G(x,s) = \int_0^s g(x,\tau)\, d\tau$. We consider the following two conditions:

(LL) for every $v \in \Sigma \setminus \{0\}$,

$$\int_{\{v>0\}} \liminf_{s \to +\infty} g(x,s) v(x)\, d\mu + \int_{\{v<0\}} \limsup_{s \to -\infty} g(x,s) v(x)\, d\mu > 0;$$

(ALP) as long as $v \in \Sigma$,

$$\lim_{\|v\|_p \to +\infty} \int_\Omega G(x, v(x))\, d\mu = +\infty.$$

In this framework, we will prove the following statement:

Theorem 1. *(LL) implies (ALP).*

Several boundary value problems fit in the above setting. For example, the T-periodic problem associated to the differential equation $\ddot{u} + \lambda u = g(t,u)$, as first considered in [29], where λ is an eigenvalue of the differential operator, or to the more general asymmetric oscillator $\ddot{u} + \mu u^+ - \nu u^- = g(t,u)$, where the couple (μ, ν) belongs to the Dancer-Fučik spectrum. Indeed, concerning the homogeneous equation, if ϕ satisfies $\ddot{\phi} + \mu\phi^+ - \nu\phi^- = 0$, then every other solution is given by $\varphi(t) = \alpha\phi(t+\theta)$, with $\alpha \ge 0$ and $\theta \in [0,T]$.

Other examples are given by the Dirichlet (or Neumann) problem on a bounded domain associated to an elliptic equation like $\Delta u + \lambda u = g(x,u)$, where λ is an eigenvalue of the differential operator, or to a more general equation involving the p-Laplacian. However, some care could be recommended in this case, since the spectral properties of the p-Laplacian are not completely established yet (see, e.g., [4, 11, 12], and the references therein). On the other hand, if λ is the first eigenvalue, our assumptions are known to be fulfilled, since $\Sigma \cap S_1$ is a finite set.

In principle, boundary value problems associated to hyperbolic equations fit in our framework, as well. However, in this case we do not know about existence results under these general assumptions (see, however, [3, 36]).

It is worth underlining that, in the above statement, we do not need growth assumptions on g - which, however, are usually required to prove existence results - other than (1). And, moreover, in the applications, Ω has not necessarily to be bounded in $\mathbb{R}^n$. Problems on unbounded domains have been studied by several authors in the recent years, mainly with variational methods, yielding existence results by means of both Landesman-Lazer (see for instance [2, 31]) and Ahmad-Lazer-Paul conditions (see, e.g., [25, 30]). Our theorem could be useful in these cases, since it seems easier to check if (LL) holds, rather than (ALP).

Clearly, a symmetric result with respect to Theorem 1 can be stated if we take into account the following two conditions:

(LL$'$) for every $v \in \Sigma \setminus \{0\}$,

$$\int_{\{v>0\}} \limsup_{s\to+\infty} g(x,s)v(x)\,d\mu + \int_{\{v<0\}} \liminf_{s\to-\infty} g(x,s)v(x)\,d\mu < 0,$$

(ALP$'$) as long as $v \in \Sigma$,

$$\lim_{\|v\|_p\to+\infty} \int_\Omega G(x,v(x))\,d\mu = -\infty.$$

3 Proof of the main result

First of all, notice that condition (1) guarantees that the integrals appearing in (LL) are both well defined, with values in $\mathbb{R}\cup\{+\infty\}$. Along the proof of Theorem 1, we will show that, in this setting, the same is true for the integral appearing in (ALP).

Let us start the proof of Theorem 1. Since Ω is σ-finite, there is a family $\{K_m\}_{m\in\mathbb{N}}$ of measurable subsets of Ω such that

- $\mu(K_m) < +\infty$ for every $m \in \mathbb{N}$;
- $K_m \subset K_{m+1}$, for every $m \in \mathbb{N}$;
- $\cup_{m\in\mathbb{N}} K_m = \Omega$.

Define, for every $m \in \mathbb{N}$, the truncation function $\zeta_m : \Omega \to \mathbb{R}$ by

$$\zeta_m(x) = \begin{cases} m & \text{if } x \in K_m \\ 0 & \text{if } x \in \Omega \setminus K_m; \end{cases}$$

it is worth noticing that, for every $m \in \mathbb{N}$, ζ_m belongs to $L^q(\Omega)$, for every $q \geq 1$.
The following lemma says that, if the Landesman-Lazer condition is satisfied by g, then it is satisfied also by some suitable truncation of g.

Lemma 2. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy condition (LL). Then, setting*

$$g_m(x,s) = \begin{cases} \min\{g(x,s), \zeta_m(x)\} & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ \max\{g(x,s), -\zeta_m(x)\} & \text{if } s < 0, \end{cases}$$

there exists $\bar{m} \in \mathbb{N}$ such that, for every $m \geq \bar{m}$ and every $v \in \Sigma \setminus \{0\}$,

$$\int_{\{v>0\}} \liminf_{s\to+\infty} g_m(x,s)\, v(x)\, d\mu + \int_{\{v<0\}} \limsup_{s\to-\infty} g_m(x,s) v(x)\, d\mu > 0. \tag{2}$$

Proof. It suffices to prove the statement for every $v \in \Sigma \cap S_1$, since the left-hand side in (2) is positively homogeneous of degree 1 with respect to v. Consequently, we will assume $\|v\|_p = 1$.

Since $g(x,s) = \lim_{m\to+\infty} g_m(x,s)$ for almost every $x \in \Omega$ and every $s \in \mathbb{R} \setminus \{0\}$, this limit being monotone (increasing for $s > 0$, decreasing for $s < 0$), we can rewrite condition (LL) as

$$\int_{\{v>0\}} \Big(\liminf_{s\to+\infty} \lim_{m\to+\infty} g_m(x,s)\Big) v(x)\, d\mu + \int_{\{v<0\}} \Big(\limsup_{s\to-\infty} \lim_{m\to+\infty} g_m(x,s)\Big) v(x)\, d\mu > 0. \tag{3}$$

We show that it is possible to exchange the inferior limit and the limit under the first integral. First of all, since $g_m(x,s) \leq g(x,s)$ for almost every $x \in \Omega$ and every $s > 0$, it follows easily that

$$\lim_{m\to+\infty} (\liminf_{s\to+\infty} g_m(x,s)) \leq \liminf_{s\to+\infty} (\lim_{m\to+\infty} g_m(x,s)).$$

On the other hand, after having observed, in view of (1), that $\liminf_{s\to+\infty} g(x,s) > -\infty$ for almost every $x \in \Omega$, we have to consider the two following cases.

Assume that $x \in \Omega$ is such that $\liminf_{s\to+\infty} g(x,s) = +\infty$. Then, fixed $K > 0$ there exists s_K such that, if $s \geq s_K$, then $g(x,s) \geq K$. Moreover, there exists $m_x \in \mathbb{N}$ such that $x \in K_m$ for every $m \geq m_x$. For every $m \geq \max\{K, m_x\}$, then, it will be $g_m(x,s) \geq K$, for every $s \geq s_K$, from which $\liminf_{s\to+\infty} g_m(x,s) \geq K$, so that

$$\lim_{m\to+\infty} \liminf_{s\to+\infty} g_m(x,s) = +\infty.$$

Assume now that $x \in \Omega$ is such that $\liminf_{s\to+\infty} g(x,s) = l \in \mathbb{R}$. Then, fixed $\epsilon > 0$, there exists s_ϵ such that $g(x,s) \geq l - \epsilon$ for $s \geq s_\epsilon$. Moreover, there exists $m_x \in \mathbb{N}$ such that $x \in K_m$ for every $m \geq m_x$. For every $m \geq \max\{l, m_x\}$, then, we have $g_m(x,s) \geq l - \epsilon$, for every $s \geq s_\epsilon$, so that $\liminf_{s\to+\infty} g_m(x,s) \geq l - \epsilon$, from which we deduce that

$$\lim_{m\to+\infty} (\liminf_{s\to+\infty} g_m(x,s)) \geq l.$$

With the same computations, it is possible to show that the superior limit and the limit under the second integral can be exchanged.

According to (3), then,

$$\int_{\{v>0\}} \Big(\lim_{m\to+\infty} \liminf_{s\to+\infty} g_m(x,s)\Big) v(x)\, d\mu + \int_{\{v<0\}} \Big(\lim_{m\to+\infty} \limsup_{s\to-\infty} g_m(x,s)\Big) v(x)\, d\mu > 0.$$

The two sequences $(\liminf_{s\to+\infty} g_m(x,s)v(x))_m$ and $(\limsup_{s\to-\infty} g_m(x,s)v(x))_m$, considered on their domains of integration $\{v>0\}$ and $\{v<0\}$, respectively, are monotone increasing. Moreover, they are bounded from below by the L^1-functions $-h(x)v(x)$ and $h(x)v(x)$, respectively. By the monotone convergence theorem, then,

$$\lim_{m\to+\infty}\left(\int_{\{v>0\}}\liminf_{s\to+\infty} g_m(x,s)v(x)\,d\mu+\int_{\{v<0\}}\limsup_{s\to-\infty} g_m(x,s)v(x)\,d\mu\right)>0.$$

Hence, for every $v\in\Sigma\cap S_1$, there exists $M_v\in\mathbb{N}$ such that, for every $m\geq M_v$,

$$I_m(v):=\int_{\{v>0\}}\liminf_{s\to+\infty} g_m(x,s)v(x)\,d\mu+\int_{\{v<0\}}\limsup_{s\to-\infty} g_m(x,s)v(x)\,d\mu>0.$$

Choose $M\geq M_v$ and set

$$g_+(x)=\liminf_{s\to+\infty} g_M(x,s),\quad g_-(x)=\limsup_{s\to-\infty} g_M(x,s).$$

Observe that g_+ and g_- belong to $L^q(\Omega)$: similarly as before, indeed, for almost every $x\in\Omega$,

$$-h(x)\leq g_+(x)\leq\zeta_M(x),\quad -\zeta_M(x)\leq g_-(x)\leq h(x).$$

We now claim that $I_M:L^p(\Omega)\to\mathbb{R}$ is continuous at v. To show it, let $v_j\to v$ in $L^p(\Omega)$, and fix the following notations:

$$A_j^+=\{v_j\geq 0\},\quad A_j^-=\{v_j<0\},$$
$$A^+=\{v\geq 0\},\quad A^-=\{v<0\}.$$

We have

$$I_M(v_j)-I_M(v)=\Gamma_{1,j}+\Gamma_{2,j}+\Gamma_{3,j}+\Gamma_{4,j}\,,$$

where

$$\Gamma_{1,j}=\int_{A_j^+\cap A^+} g_+(x)(v_j(x)-v(x))\,d\mu,$$
$$\Gamma_{2,j}=\int_{A_j^-\cap A^-} g_-(x)(v_j(x)-v(x))\,d\mu,$$
$$\Gamma_{3,j}=\int_{A_j^-\cap A^+} (g_-(x)v_j(x)-g_+(x)v(x))\,d\mu,$$
$$\Gamma_{4,j}=\int_{A_j^+\cap A^-} (g_+(x)v_j(x)-g_-(x)v(x))\,d\mu.$$

As $j\to+\infty$, $\Gamma_{1,j}$ and $\Gamma_{2,j}$ vanish thanks to the Hölder inequality, since $v_j\to v$ in $L^p(\Omega)$. For what concerns $\Gamma_{3,j}$, notice first that, by the Lebesgue dominated convergence Theorem,

$$\int_{A_j^-\cap A^+} g_+(x)v(x)\,d\mu\to 0,$$

since $\chi_{A_j^-\cap A^+}(x)\to 0$ for almost every $x\in\Omega$. On the other hand, writing

$$\int_{A_j^-\cap A^+} g_-(x)v_j(x)\,d\mu=\int_{A_j^-\cap A^+} g_-(x)(v_j(x)-v(x))\,d\mu+\int_{A_j^-\cap A^+} g_-(x)v(x)\,d\mu,$$

arguing similarly we see that

$$\int_{A_j^- \cap A^+} g_-(x) v_j(x)\, d\mu \to 0.$$

This shows that $\Gamma_{3,j} \to 0$ as $j \to +\infty$. With the same reasonings, we see that $\Gamma_{4,j}$ vanishes, as well. The continuity of I_M is thus proved.
It follows that there exists $\delta > 0$ such that $I_M(w) > 0$ for $\|w - v\|_p < \delta$, namely

$$\int_{\{w>0\}} \liminf_{s\to+\infty} g_M(x,s) w(x)\, d\mu + \int_{\{w<0\}} \limsup_{s\to-\infty} g_M(x,s) w(x)\, d\mu > 0.$$

Since, thanks to our hypotheses, $\Sigma \cap S_1$ is compact in $L^p(\Omega)$, it will be possible to find $\bar{m} \in \mathbb{N}$ such that, for every $v \in \Sigma \cap S_1$,

$$I_{\bar{m}}(v) = \int_{\{v>0\}} \liminf_{s\to+\infty} g_{\bar{m}}(x,s) v(x)\, d\mu + \int_{\{v<0\}} \limsup_{s\to-\infty} g_{\bar{m}}(x,s) v(x)\, d\mu > 0.$$

The fact that (2) holds for every $m \geq \bar{m}$ is a simple consequence of the monotonicity of the integrands with respect to m. $\square$

We now give a characterization of the Landesman-Lazer condition.

Proposition 3. *The following conditions are equivalent:*

1) g satisfies (LL);

2) there exist $\bar{\eta} > 0$, $R \geq d$ and $\psi_-, \psi_+ \in L^q(\Omega)$ such that

- *$g(x,s) \geq \psi_+(x)$ for a.e. $x \in \Omega$, and every $s \geq R$;*
- *$g(x,s) \leq \psi_-(x)$ for a.e. $x \in \Omega$, and every $s \leq -R$;*
- *for every $v \in \Sigma$,*

$$\int_{\{v>0\}} \psi_+(x) v(x)\, d\mu + \int_{\{v<0\}} \psi_-(x) v(x)\, d\mu \geq \bar{\eta} \|v\|_p\,. \tag{4}$$

Moreover, there exists $M > 0$ such that

$$-h(x) \leq \psi_+(x) \leq M, \quad -M \leq \psi_-(x) \leq h(x),$$

for almost every $x \in \Omega$, and, if $x \in \Omega \setminus K_M$, then $\psi_+(x) \leq 0$ and $\psi_-(x) \geq 0$.

Proof. In view of the positive homogeneity of both sides of (4) with respect to v, it is not restrictive to assume $\|v\|_p = 1$. We will only prove that 1) implies 2), since the other implication is straightforward. Suppose that (LL) holds: by Lemma 2, using the same notations, there exists $\bar{m} \in \mathbb{N}$ such that, for every $m \geq \bar{m}$ and every $v \in \Sigma \setminus \{0\}$,

$$\int_{\{v>0\}} \liminf_{s\to+\infty} g_m(x,s) v(x)\, d\mu + \int_{\{v<0\}} \limsup_{s\to-\infty} g_m(x,s) v(x)\, d\mu > 0,$$

i.e.,

$$\int_{\{v>0\}} \Big(\lim_{n\to+\infty} \inf_{s\geq n} g_m(x,s) \Big) v(x)\, d\mu + \int_{\{v<0\}} \Big(\lim_{n\to+\infty} \sup_{s\leq -n} g_m(x,s) \Big) v(x)\, d\mu > 0.$$

Fix $M \geq \bar{m}$ and set

$$\gamma_n^+(x) = \inf_{s \geq n} g_M(x,s), \quad \gamma_n^-(x) = \sup_{s \leq -n} g_M(x,s).$$

Observe that, for every $n \geq d$, γ_n^+ and γ_n^- belong to $L^q(\Omega)$, since, for almost every $x \in \Omega$,

$$-h(x) \leq \gamma_n^+(x) \leq M, \quad -M \leq \gamma_n^-(x) \leq h(x).$$

On their domains of integration $\{v > 0\}$ and $\{v < 0\}$, respectively, the sequences of L^1-functions $\{\gamma_n^+ v\}_{n \geq d}$ and $\{\gamma_n^- v\}_{n \geq d}$ are both monotone increasing, and bounded from below by the L^1-functions $-hv$ and hv respectively. By the monotone convergence theorem,

$$\begin{aligned}
&\lim_{n \to +\infty} \left(\int_{\{v>0\}} \gamma_n^+(x) v(x)\, d\mu + \int_{\{v<0\}} \gamma_n^-(x) v(x)\, d\mu \right) = \\
&= \int_{\{v>0\}} \lim_{n \to +\infty} \gamma_n^+(x) v(x)\, d\mu + \int_{\{v<0\}} \lim_{n \to +\infty} \gamma_n^-(x) v(x)\, d\mu \\
&= \int_{\{v>0\}} \liminf_{s \to +\infty} g_M(x,s) v(x)\, d\mu + \int_{\{v<0\}} \limsup_{s \to -\infty} g_M(x,s) v(x)\, d\mu > 0.
\end{aligned}$$

For every $v \in \Sigma \cap S_1$, then, there exist $\eta_v > 0$ and $N_v \in \mathbb{N}$, with $N_v \geq d$, such that, for every $n \geq N_v$,

$$J_n(v) := \int_{\{v>0\}} \gamma_n^+(x) v(x)\, d\mu + \int_{\{v<0\}} \gamma_n^-(x) v(x)\, d\mu \geq \eta_v\,.$$

Choose $N \geq N_v$: with the same reasonings as in the proof of Lemma 2, we can show that $J_N : L^p(\Omega) \to \mathbb{R}$ is continuous. Hence, there exists $\delta > 0$ such that, if $\|w - v\|_p \leq \delta$,

$$\int_{\{w>0\}} \gamma_N^+(x) w(x)\, d\mu + \int_{\{w<0\}} \gamma_N^-(x) w(x)\, d\mu \geq \frac{\eta_v}{2}\,.$$

By the compactness of $\Sigma \cap S_1$ it is possible to find $\bar{n} \in \mathbb{N}$, with $\bar{n} \geq d$, and $\bar{\eta} > 0$ such that, for every $v \in \Sigma \cap S_1$,

$$\int_{\{v>0\}} \gamma_{\bar{n}}^+(x) v(x)\, d\mu + \int_{\{v<0\}} \gamma_{\bar{n}}^-(x) v(x)\, d\mu \geq \bar{\eta}.$$

Setting

$$\psi_+(x) = \gamma_{\bar{n}}^+(x), \quad \psi_-(x) = \gamma_{\bar{n}}^-(x),$$

the proof is easily completed, taking $R = \bar{n}$. □

Remark 4. In the study of an elliptic boundary value problem at resonance with the first eigenvalue, Gossez and Omari characterized the Landesman-Lazer condition, as well (see [21, Proposition 4.1]). In their particular case, the eigenspace is 1-dimensional and generated by a positive eigenfunction (see also, in a different context, [19, Lemma 1]).

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let $v \in \Sigma \setminus \{0\}$ and set

$$\Omega_v^+ = \{x \in \Omega \mid v(x) > R\},$$

$$\Omega_v^- = \{x \in \Omega \mid v(x) < -R\},$$

$$\Omega_v^0 = \{x \in \Omega \mid -R \leq v(x) \leq R\},$$

where $R > 0$ is given by Proposition 3. Writing

$$\int_\Omega G(x, v(x))\, d\mu = \int_{\Omega_v^+} G(x, v(x))\, d\mu + \int_{\Omega_v^-} G(x, v(x))\, d\mu + \int_{\Omega_v^0} G(x, v(x))\, d\mu,$$

we are led to consider each term separately. For what concerns the first one, notice that, using the notations of Proposition 3, we have $g(x, s) \geq \psi_+(x)$ for $s > R$, and $|g(x, s)| \leq \eta_R(x)$ for $|s| \leq R$, thanks to the Carathéodory assumptions. Moreover, recalling that $\psi_+(x) \leq M$ for almost every $x \in \Omega$ and $\psi_+(x) \leq 0$ for $x \in \Omega \setminus K_M$,

$$\begin{aligned}
G(x, v(x)) &= \int_0^R g(x, \tau)\, d\tau + \int_R^{v(x)} g(x, \tau)\, d\tau \\
&\geq -R\eta_R(x) + (v(x) - R)\psi_+(x) \\
&\geq -R\eta_R(x) + v(x)\psi_+(x) - R\psi_+(x)\chi_{K_M}(x) \\
&\geq -R\eta_R(x) + v(x)\psi_+(x) - RM\chi_{K_M}(x),
\end{aligned}$$

for almost every $x \in \Omega_v^+$. Hence,

$$\begin{aligned}
&\int_{\Omega_v^+} G(x, v(x))\, d\mu \geq -R\|\eta_R\|_1 + \int_{\Omega_v^+} \psi_+(x)v(x)\, d\mu - RM\mu(K_M) \\
&= -R(\|\eta_R\|_1 + M\mu(K_M)) + \int_{\{v>0\}} \psi_+(x)v(x)\, d\mu - \int_{\{0<v(x)\leq R\}} \psi_+(x)v(x)\, d\mu \\
&\geq -R(\|\eta_R\|_1 + M\mu(K_M)) + \int_{\{v>0\}} \psi_+(x)v(x)\, d\mu - \int_{\{0<v(x)\leq R\}\cap K_M} \psi_+(x)v(x)\, d\mu \\
&\geq -R(\|\eta_R\|_1 + M\mu(K_M)) + \int_{\{v>0\}} \psi_+(x)v(x)\, d\mu - \int_{\{0<v(x)\leq R\}\cap K_M} Mv(x)\, d\mu \\
&\geq -R(\|\eta_R\|_1 + 2M\mu(K_M)) + \int_{\{v>0\}} \psi_+(x)v(x)\, d\mu.
\end{aligned}$$

A similar computation yields

$$\int_{\Omega_v^-} G(x, v(x))\, d\mu \geq -R(\|\eta_R\|_1 + 2M\mu(K_M)) + \int_{\{v<0\}} \psi_-(x)v(x)\, d\mu,$$

while we have

$$\int_{\Omega_v^0} G(x, v(x))\, d\mu = \int_{\Omega_v^0} \int_0^{v(x)} g(x, \tau)\, d\tau\, d\mu \geq -R\|\eta_R\|_1\,. \tag{5}$$

Summing up, using (4), we have

$$\int_\Omega G(x, v(x))\, d\mu \geq \bar{\eta}\|v\|_p - R\big[3\|\eta_R\|_1 + 4M\mu(K_M)\big], \tag{6}$$

where $\bar{\eta} > 0$ is given by Proposition 3. This concludes the proof. $\square$

4 Some remarks and related conditions

In this section, we will make some remarks about Theorem 1 and take into account some variants of (LL) and (ALP), which have been considered in literature.

Remark 5. In view of Theorem 1, the Ahmad-Lazer-Paul condition is more general than the Landesman-Lazer one. However, in the setting of the theorem, adding some monotonicity assumption on g (with respect to s) makes the two conditions equivalent, as shown in the following proposition.

Proposition 6. *Assume that $g(x,s)$ is nondecreasing in s, for almost every $x \in \Omega$. Then (LL) and (ALP) are equivalent.*

Proof. Assume that (LL) is not satisfied. Hence, there exists $v \in \Sigma \setminus \{0\}$ such that

$$\int_{\{v>0\}} \liminf_{s\to+\infty} g(x,s)v(x)\,d\mu + \int_{\{v<0\}} \limsup_{s\to-\infty} g(x,s)v(x)\,d\mu \le 0;$$

setting $g_+(x) = \lim_{s\to+\infty} g(x,s)$, and $g_-(x) = \lim_{s\to-\infty} g(x,s)$, this reads

$$\int_{\{v>0\}} g_+(x)v(x)\,d\mu + \int_{\{v<0\}} g_-(x)v(x)\,d\mu \le 0.$$

Let us show that the function $F : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, defined by

$$F(\lambda) = \int_\Omega G(x, \lambda v(x))\,d\mu = \int_\Omega \int_0^{\lambda v(x)} g(x,\tau)\,d\tau,$$

is nonpositive for $\lambda > 0$. Indeed, since, for almost every $x \in \Omega$, and every $s \in \mathbb{R}$,

$$g_-(x) \le g(x,s) \le g_+(x),$$

we have

$$\begin{aligned} F(\lambda) &= \int_{\{v>0\}} \int_0^{\lambda v(x)} g(x,\tau)\,d\tau\,d\mu + \int_{\{v<0\}} \int_0^{\lambda v(x)} g(x,\tau)\,d\tau\,d\mu \\ &\le \lambda \int_{\{v>0\}} g_+(x)v(x)\,d\mu + \lambda \int_{\{v<0\}} g_-(x)v(x)\,d\mu \le 0. \end{aligned}$$

Consequently, $\limsup_{\lambda\to+\infty} F(\lambda) \le 0$, so that (ALP) does not hold. □

Remark 7. As shown in (6), condition (LL) implies that, for every $\beta \in [0,1[$,

$$\lim_{\|v\|_p\to+\infty} \frac{1}{\|v\|_p^\beta} \int_\Omega G(x, v(x))\,d\mu = +\infty,$$

as long as $v \in \Sigma$. Conditions of this type were considered, e.g., in [7, 22, 37, 38].

Remark 8. As already pointed out in [5], it is possible to compare conditions (LL) and (ALP) with another existence condition introduced by Tomiczek in [39] and [40], the so called *potential Landesman-Lazer condition.* In our abstract framework, with the same notations as before, such a condition can be written as follows:

(p-LL) for every $v \in \Sigma \setminus \{0\}$,

$$\int_{\{v>0\}} \liminf_{s\to+\infty} \frac{G(x,s)}{s} v(x)\, d\mu + \int_{\{v<0\}} \limsup_{s\to-\infty} \frac{G(x,s)}{s} v(x)\, d\mu > 0.$$

Let us first show that (LL) implies (p-LL). Using Proposition 3, with the notations therein, for almost every $x \in \Omega$ we have

$$\begin{aligned} \liminf_{s\to+\infty} \frac{G(x,s)}{s} &= \liminf_{s\to+\infty} \frac{G(x,s)-G(x,R)}{s} \\ &= \liminf_{s\to+\infty} \frac{1}{s}\int_R^s g(x,\tau)\, d\tau \\ &\geq \liminf_{s\to+\infty} \frac{s-R}{s}\psi_+(x) = \psi_+(x). \end{aligned}$$

By analogous computations, we see that

$$\limsup_{s\to-\infty} \frac{G(x,s)}{s} \leq \psi_-(x),$$

and the statement follows then from (4).

We now show that (p-LL) implies (ALP). To this aim, define the function

$$f(x,s) = \begin{cases} \dfrac{1}{s}(G(x,s)-G(x,d)) & \text{if } s \geq d \\ 0 & \text{if } -d < s < d \\ \dfrac{1}{s}(G(x,s)-G(x,-d)) & \text{if } s \leq -d \end{cases}$$

and notice that (p-LL) implies

$$\int_{\{v>0\}} \liminf_{s\to+\infty} f(x,s)v(x)\, d\mu + \int_{\{v<0\}} \limsup_{s\to-\infty} f(x,s)v(x)\, d\mu > 0,$$

i.e., f satisfies (LL). Moreover, it is easily seen that f satisfies the same L^1-Carathéodory conditions as g, and

$$|s| \geq d \quad \Rightarrow \quad \operatorname{sgn}(s) f(x,s) \geq -h(x),$$

as well. Consequently, assuming (p-LL), Lemma 2 and Proposition 3 apply, with g replaced by f, yielding the existence of $\bar\eta > 0$, $R \geq d$, and Ψ_+, Ψ_- belonging to $L^q(\Omega)$, such that $f(x,s) \geq \Psi_+(x)$ for $s \geq R$, $f(x,s) \leq \Psi_-(x)$ for $s \leq -R$, and

$$\int_{\{v>0\}} \Psi_+(x)v(x)\, d\mu + \int_{\{v<0\}} \Psi_-(x)v(x)\, d\mu \geq \bar\eta \|v\|_p\,, \tag{7}$$

for every $v \in \Sigma$. Moreover, there exists $M > 0$ such that

$$-h(x) \leq \Psi_+(x) \leq M, \quad -M \leq \Psi_-(x) \leq h(x),$$

for almost every $x \in \Omega$, and, if $x \in \Omega \setminus K_M$, then $\Psi_+(x) \leq 0$ and $\Psi_-(x) \geq 0$. Letting $v \in \Sigma \setminus \{0\}$ and

$$\Omega_v^+ = \{x \in \Omega \mid v(x) > R\},$$

$$\Omega_v^- = \{x \in \Omega \mid v(x) < -R\},$$
$$\Omega_v^0 = \{x \in \Omega \mid -R \le v(x) \le R\},$$

we write

$$\int_\Omega G(x, v(x))\, d\mu = \int_{\Omega_v^+} G(x, v(x))\, d\mu + \int_{\Omega_v^-} G(x, v(x))\, d\mu + \int_{\Omega_v^0} G(x, v(x))\, d\mu,$$

and consider each term separately. For what concerns the first one, since, for almost every $x \in \Omega_v^+$,

$$\begin{aligned} G(x, v(x)) &= \int_0^d g(x,\tau)\, d\tau + \int_d^{v(x)} g(x,\tau)\, d\tau \\ &\ge -d\eta_d(x) + f(x, v(x))v(x) \\ &\ge -d\eta_d(x) + \Psi_+(x)v(x), \end{aligned}$$

with similar computations as in the proof of Theorem 1 we obtain

$$\begin{aligned} \int_{\Omega_v^+} G(x, v(x))\, d\mu &\ge -d\|\eta_d\|_1 + \int_{\Omega_v^+} \Psi_+(x)v(x)\, d\mu \\ &\ge -d\|\eta_d\|_1 - RM\mu(K_M) + \int_{\{v>0\}} \Psi_+(x)v(x)\, d\mu. \end{aligned}$$

Similarly,

$$\int_{\Omega_v^-} G(x, v(x))\, d\mu \ge -d\|\eta_d\|_1 - RM\mu(K_M) + \int_{\{v<0\}} \Psi_-(x)v(x)\, d\mu,$$

while the integral on Ω_v^0 can be estimated as in (5). Hence, by (7),

$$\int_\Omega G(x, v(x))\, d\mu \ge \bar{\eta}\|v\|_p - \big[R\|\eta_R\|_1 + 2d\|\eta_d\|_1 + 2RM\mu(K_M)\big].$$

It follows that the Ahmad-Lazer-Paul condition is fulfilled.

Clearly, as a consequence of Proposition 6, condition (p-LL) is equivalent to both (LL) and (ALP) when g is nondecreasing with respect to its second variable.

References

[1] S. Ahmad, A. C. Lazer and J. L. Paul, Elementary critical point theory and perturbations of elliptic boundary value problems at resonance, Indiana Univ. Math. J. 25 (1976), 933–944.

[2] C. O. Alves, P. C. Carrião and O. H. Miyagaki, Existence and multiplicity results for a class of resonant quasilinear elliptic problems on $\mathbb{R}^N$, Nonlinear Anal. 39 (2000), 99–110.

[3] A. Bahri and H. Brezis, Periodic solutions of a nonlinear wave equation, Proc. Royal Soc. Edinburgh 85A (1980), 313–320.

[4] P. A. Binding and B. P. Rynne, The spectrum of the periodic p-Laplacian, J. Differential Equations 235 (2007), 199–218.

[5] D. Bonheure and C. Fabry, A variational approach to resonance for asymmetric oscillators, Commun. Pure Appl. Anal. 6 (2007), 163–181.

[6] H. Brezis and L. Nirenberg, Characterizations of the ranges of some nonlinear operators and applications to boundary value problems, Ann. Scuola Norm. Sup. Pisa 5 (1978), 225–326.

[7] A. Daouas and M. Timoumi, Subharmonics for not uniformly coercive Hamiltonian systems, Nonlinear Anal. 66 (2007), 571–581.

[8] D. de Figueiredo, The Dirichlet problem for nonlinear elliptic equations: a Hilbert space approach, Lecture Notes in Math., vol. 446, Springer, Berlin, 1975.

[9] P. Drábek, Landesman-Lazer type condition and nonlinearities with linear growth, Czechoslovak Math. J. 40 (1990), 70–86.

[10] P. Drábek, Landesman-Lazer condition for nonlinear problems with jumping nonlinearities, J. Differential Equations 85 (1990), 186–199.

[11] P. Drábek and S. B. Robinson, Resonance problems for the one-dimensional p-Laplacian, Proc. Amer. Math. Soc. 128 (1999), 755–765.

[12] P. Drábek and S. B. Robinson, Resonance problems for the p-Laplacian, J. Functional Analysis 169 (1999), 189–200.

[13] A. R. El Amrouss, Nontrivial solutions of semilinear equations at resonance, J. Math. Anal. Appl. 325 (2007), 19–35.

[14] C. Fabry, Landesman-Lazer conditions for periodic boundary value problems with asymmetric nonlinearities, J. Differential Equations 116 (1995), 405–418.

[15] C. Fabry and A. Fonda, Periodic solutions of nonlinear differential equations with double resonance, Ann. Mat. Pura Appl. 157 (1990), 99–116.

[16] C. Fabry and A. Fonda, Nonlinear equations at resonance and generalized eigenvalue problems, Nonlinear Anal. 18 (1992), 427–444.

[17] A. Fonda and M. Garrione, Double resonance with Landesman-Lazer conditions for planar systems of ordinary differential equations, preprint, available on-line at www.dmi.units.it/pubblicazioni/Quaderni_Matematici/600_2010.pdf

[18] A. Fonda and M. Ramos, Large-amplitude subharmonic oscillations for scalar second-order differential equations with asymmetric nonlinearities, J. Differential Equations 109 (1994), 354–372.

[19] A. Fonda and R. Toader, Periodic solutions of radially symmetric perturbations of Newtonian systems, preprint, available on-line at
www.dmi.units.it/pubblicazioni/Quaderni_Matematici/598_2010.pdf

[20] S. Fučik, Solvability of Nonlinear Equations and Boundary Value Problems, Reidel, Boston, 1980.

[21] J.-P. Gossez and P. Omari, Non-ordered lower and upper solutions in semilinear elliptic problems, Comm. Partial Differential Equations 19 (1994), 1163–1184.

[22] Z. Q. Han and Q. Li, Existence of critical point for abstract resonant problems with unbounded nonlinearities and applications to differential equations, Nonlinear Anal. 65 (2006), 1654–1668.

[23] P. Hess, On a theorem by Landesman and Lazer, Indiana Univ. Math. J. 23 (1974), 827–830.

[24] R. Iannacci and M. N. Nkashama, Nonlinear boundary value problems at resonance, Nonlinear Anal. 11 (1987), 455–473.

[25] G. Izquierdo Buenrostro and G. Lopez Garza, A resonance problem for the p-laplacian in $\mathbb{R}^N$, Electron. J. Differential Equations 112 (2005), 1–8.

[26] E. Landesman and A. C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech. 19 (1970), 609–623.

[27] E. Landesman, S. Robinson and A. Rumbos, Multiple solutions of semilinear elliptic problems at resonance, Nonlinear Anal. 24 (1995), 1049–1059.

[28] A. C. Lazer, A second look at the first result of Landesman-Lazer type, Electron. J. Differ. Equ. Conf. 5 (2000), 113–119.

[29] A. C. Lazer and D. E. Leach, Bounded perturbations of forced harmonic oscillators at resonance, Ann. Mat. Pura Appl. 82 (1969), 49–68.

[30] G. Lopez Garza and A. J. Rumbos, Resonance and strong resonance for semilinear elliptic equations in $\mathbb{R}^N$, Electron. J. Differential Equations 124 (2003), 1–22.

[31] G. Lopez Garza and A. J. Rumbos, Multiple solutions for resonant semilinear elliptic problems in $\mathbb{R}^N$, J. Math. Anal. Appl. 305 (2005), 367–379.

[32] J. Mawhin, Landesman-Lazer's type problems for nonlinear equations, Confer. Sem. Mat. Univ. Bari 147 (1977), 1–22.

[33] J. Mawhin, Problèmes de Dirichlet Variationnels Non-linéaires, Sémin. Math. Supérieures 104, Presses Univ. Montréal, Montréal, 1987.

[34] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, Springer, Berlin, 1989.

[35] J. Nečas, On the range of nonlinear operators with linear asymptotes which are not invertible, Comment. Math. Univ. Carolinae 14 (1973), 63–72.

[36] P. Rabinowitz, Some minimax theorems and applications to nonlinear partial differential equations, in: Nonlinear Analysis (collection of papers in honor of Erich H. Rothe), Academic Press, New York, 1978, pp. 161–177.

[37] C. L. Tang, Periodic solutions for nonautonomous second order systems with sublinear nonlinearity, Proc. Amer. Math. Soc. 126 (1998), 3263–3270.

[38] H. Tehrani, Existence result for an indefinite unbounded perturbation of a resonant Schrödinger equation, J. Differential Equations 236 (2007), 1–28.

[39] P. Tomiczek, A generalization of the Landesman-Lazer condition, Electron. J. Differential Equations 4 (2001), 1–11.

[40] P. Tomiczek, Potential Landesman-Lazer type conditions and the Fučik spectrum, Electron. J. Differential Equations 94 (2005), 1–12.

[41] S. Williams, A sharp sufficient condition for solution of a nonlinear elliptic boundary value problem, J. Differential Equations 8 (1970), 580–588.

Authors' address:

Alessandro Fonda
Dipartimento di Matematica e Informatica
Università di Trieste
P.le Europa 1
I-34127 Trieste
Italy
e-mail: a.fonda@units.it

Maurizio Garrione
SISSA - International School for Advanced Studies
Via Bonomea 265
I-34136 Trieste
Italy
e-mail: garrione@sissa.it

Mathematics Subject Classification: 34B15, 35J25.

Keywords: nonlinear boundary value problems; resonance; Landesman-Lazer condition.