Periodic, subharmonic, and quasi-periodic oscillations under the action of a central force

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Abstract

We consider planar systems driven by a central force which depends periodically on time. If the force is sublinear and attractive, then there is a connected set of subharmonic and quasi-periodic solutions rotating around the origin at different speeds; moreover, this connected set stretches from zero to infinity. The result still holds allowing the force to be attractive only in average provided that an uniformity condition is satisfied and there are no periodic oscillations with zero angular momentum. We provide examples showing that these assumptions cannot be skipped.

1 Introduction

The motion of a particle subjected to the influence of an (autonomous) central force field in the plane may be mathematically modelled as a system of differential equations:

$$\ddot{x} = -f(|x|)\frac{x}{|x|}, \qquad x \in \mathbb{R}^2\backslash\{0\}. \tag{1}$$

Many phenomena of the nature obey to laws of this type. For instance, the newtonian equation for the motion of a particle subjected to the gravitational attraction of a sun which lies at the origin

$$\ddot{x} = -\frac{c\,x}{|x|^3}, \qquad x \in \mathbb{R}^2\backslash\{0\}, \tag{2}$$

corresponds to the choice $f(r) := c/r^2$ for some positive constant $c > 0$. If, on the contrary, $c < 0$, equation (2) still has a relevant physical meaning, as it may be used to model Rutherford's scattering of α particles by heavy atomic nuclei.

If the force field (1) is attractive, i.e., $f > 0$, then there is a collection of solutions of (1) which rotate around the origin at constant angular speeds. Indeed, direct computations show that $x(t) = r(\cos(\omega t), \sin(\omega t))$ satisfies (1) if and only if $|\omega| = \sqrt{f(r)/r}$. These solutions, which we shall call *copernican* in what follows, are all of them periodic; however, excepting the case in which $f(r) = c\,r$ is linear, the period $2\pi/|\omega| = 2\pi\sqrt{r/f(r)}$ will depend on the solution.

On the other hand, already Newton [12], in his study of Kepler's Second Law considered the problem of a central force divided into instantaneous impulses which take place periodically in time. Leaving aside the issue of the discreteness (which would require to work with measure-type forces), this motivates us to consider the following question:

What is left from the copernican orbits when the central force field depends periodically on time?

More specifically, in this paper we shall study systems of the form

$$\ddot{x} = -f(t, |x|)\frac{x}{|x|}, \qquad x \in \mathbb{R}^2 \backslash \{0\}, \qquad (3)$$

where the L^1-Carathéodory function

$$f : (\mathbb{R}/T\mathbb{Z}) \times]0, +\infty[\to \mathbb{R}, \qquad (t, r) \mapsto f(t, r),$$

is $T-$periodic in the time variable t for some $T > 0$. It may have a singularity at $r = 0$; consequently, the solutions of our equation (3) 'live' on the punctured plane $\mathbb{R}^2 \backslash \{0\}$.

Throughout this paper, continuous functions $x : \mathbb{R} \to \mathbb{R}^2 \backslash \{0\}$ will be routinely decomposed in polar coordinates, $x(t) = r_x(t)(\cos \theta_x(t), \sin \theta_x(t))$. Such a function will be called *T-radially periodic* if the modulus r_x is T-periodic and there exists some number $\omega \in \mathbb{R}$ such that $\theta_x(t) - \omega t$ is T-periodic. In this case, $\omega = (\theta_x(T) - \theta_x(0))/T$, so that this number may be interpreted as the average angular speed of x. It will be called the *rotation number* of x, and denoted by $\omega = \text{rot}(x)$.

For instance, $x_\omega(t) = (2 + \sin t)(\cos \omega t, \sin \omega t)$ is 2π-radially periodic, independently of the value of the parameter ω, which is the rotation number of x. Already in this first example we observe that radially periodic curves may not be periodic of any period; this is indeed the case of x_ω if $\omega \notin \mathbb{Q}$. Actually, an arbitrary T-radially periodic curve is T-periodic if and only if its rotation number is an integer multiple of $2\pi/T$. If the rotation number belongs instead to $(2\pi/T)\mathbb{Q}$, then the curve will not be T-periodic, but a subharmonic. On the other hand, if the modulus r_x is not constant and $\text{rot}(x) \notin (2\pi/T)\mathbb{Q}$, then the T- radially-periodic curve x will not be periodic of any period; instead, it will be quasi-periodic on the two frequencies $\omega_1 = 2\pi/T$ and $\omega_2 = \text{rot}(x)$. This is easy to check as, in complex notation, $x(t) = r(t)e^{i\theta(t)}$ may be decomposed as the product of the $T-$periodic function $r(t)e^{i\theta(t) - i\,\text{rot}(x)t}$ and the $2\pi/\text{rot}(x)-$periodic function $e^{i\,\text{rot}(x)t}$.

Without further assumptions, our equation (3) may not have any bounded solution at all; this is actually the case, if, for instance, the force field is repulsive. This intuitive statement may be checked by introducing polar coordinates $x(t) = r(t)(\cos \theta(t), \sin \theta(t))$ in (3), thus obtaining the system

$$\ddot{r} = \mu^2/r^3 - f(t, r), \qquad \dot{\theta} = \mu/r^2, \qquad (4)$$

where $\mu = r^2\dot{\theta}$ is the angular momentum (which remains constant along solutions, cf. [3]). Now, if $f < 0$, the first equation above implies that r is strictly convex, and, consequently, it cannot be globally bounded. Thus, we shall assume, in a first approach, that our force is *attractive*:

(H_1) $\qquad\qquad\qquad\qquad f(t, r) > 0 \text{ on } \mathbb{R} \times]0, +\infty[.$

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Even under this assumption, our equation may not have bounded orbits if f is allowed to grow linearly on r. To check this fact, it suffices to consider forcing terms of the form $f(t,r) := h(t)r$, giving rise to the following Hill's type equation on the punctured plane:

$$\ddot{x} = -h(t)x, \qquad x \in \mathbb{R}^2\backslash\{0\}.$$

Observe that $x = (x_1, x_2) : \mathbb{R} \to \mathbb{R}^2\backslash\{0\}$ is a solution if and only if both components x_i solve the corresponding one-dimensional Hill's equation

$$\ddot{x}_i = -h(t)x_i, \qquad x_i \in \mathbb{R}.$$

Assumption (H_1) holds provided that $h > 0$. But the T-periodic function $h > 0$ may be chosen so that the equation is hyperbolic, preventing the existence of nontrivial bounded solutions, see [11]. We owe this observation to R. Ortega.

To avoid this second pathology, we shall restrict our study to forcing terms f which are *sublinear near infinity*, i.e., there exists some function $h \in L^1_{loc}(\mathbb{R})$ and some number $r_0 > 0$ such that

(H_2) $\qquad |f(t,r)| \leq h(t)r$ on $\mathbb{R} \times [r_0, +\infty[, \qquad \lim_{r\to\infty} f(t,r)/r = 0$ for a.e. $t \in \mathbb{R}$.

Let us consider the set of all T-radially periodic curves in the punctured plane $\mathbb{R}^2\backslash\{0\}$. It becomes a topological space after being endowed with the topology of the uniform convergence on compact intervals of time.

Theorem 1.1. *Assume (H_1) and (H_2). Then, there exists a connected set $\mathcal{C}$ of T-radially periodic solutions of (3) which goes from zero to infinity, meaning that*

$$\{\min |x| : x \in \mathcal{C}\} =]0, +\infty[. \tag{5}$$

Assumption (H_1) may be criticized on the grounds that it is probably too restrictive. One might consider the possibility of extending this theorem to forcing terms f which are positive only in average, i.e.,

(W_1) $\qquad \displaystyle\int_0^T f(t,r)\,dt > 0$ for every $r > 0$.

However, this is not enough, and we shall give a counterexample in Section 6. It will exhibit $\mathcal{C}^\infty$ regularity (not just Carathéodory's) and will be bounded (not just sublinear).

Proposition 1.2. *There exists a bounded and $\mathcal{C}^\infty$ function $f : (\mathbb{R}/2\pi\mathbb{Z})\times\,]0, +\infty[\to \mathbb{R}$ verifying (W_1), but such that, for any connected set $\mathcal{C}$ of 2π-radially periodic solutions of (3), the interval $\{|x(t)| : t \in \mathbb{R}, \ x \in \mathcal{C}\}$ is bounded.*

Condition (W_1) turned out to be too weak. But the idea of formulating a generalization of Theorem 1.1 in which f might change sign provided that it is, in some sense, positive in average, is not completely erroneous. We define, for any $\rho \geq 1$, the function

$$f_\rho(t, \lambda) := \min_{\lambda/\rho \leq r \leq \lambda\rho} f(t,r), \qquad (t, \lambda) \in \mathbb{R}\times\,]0, +\infty[. \tag{6}$$

Observe that $f_1 = f$. For $\rho > 1$ one still has the inequality $f_\rho \leq f$.

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Theorem 1.3. *Assume that*

(M_1) *there are numbers $\rho_* > 1$ and $\lambda_1 > 0$ such that $\int_0^T f_{\rho_*}(t, \lambda)dt > 0$ if $\lambda \geq \lambda_1$.*

Assume also (H_2). Then, there is a connected set $\mathcal{C}$ of T-radially periodic solutions to (3) coming from infinity in the sense that

$$\text{the interval } \{\min |x| : x \in \mathcal{C}\} \text{ is unbounded from above.} \tag{7}$$

Assumption (M_1) is certainly weaker than (H_1), and thus, Theorem 1.3 may be seen as a generalization of Theorem 1.1 (actually, only a *partial* generalization, since also the thesis (7) is weaker than than (5)). The assumptions of Theorem 1.3 apply, for instance, when f has the form

$$f(t, r) := f_*(t, r) + e(t), \tag{8}$$

provided that the L^1-Carathéodory function $f_* : \mathbb{R} \times]0, +\infty[\to \mathbb{R}$ verifies assumptions (H_1) and (H_2), and the $L^1(\mathbb{R}/T\mathbb{Z})$-function e has nonnegative mean:

$$\frac{1}{T} \int_0^T e(t)\, dt \geq 0. \tag{9}$$

Notice however that, under suitable additional conditions, assumption (W_1) may imply (M_1). This is for instance the case if f is *monotone in r*, meaning that $f(t, \cdot)$ is either increasing for a.e. $t \in \mathbb{R}$ of decreasing for a.e. $t \in \mathbb{R}$ (we remark that the definition does not include those functions which are sometimes increasing and sometimes decreasing depending on the value of t). Consequently, if f is monotone in r and verifies (W_1) and (H_2), Theorem 1.3 holds for equation (3).

A third class of equations where Theorem 1.3 applies corresponds to forces f of the type

$$f(t, r) := c(t)r^\gamma + e(t), \qquad x \in \mathbb{R}^2 \backslash \{0\}, \tag{10}$$

where $-\infty < \gamma < 1$ and $c, e \in L^1(\mathbb{R}/T\mathbb{Z})$ are T-periodic and verify

$$\int_0^T c(t)\, dt > 0, \qquad \int_0^T e(t)\, dt \geq 0,$$

as it may be easily checked. Nonlinearities of these forms (8,10) were previously considered in [6], under an extra assumption at infinity, which was needed to parameterize a family of large-amplitude solutions with the angular momentum. In this paper we avoid that assumption by using as parameter the distance from our solution to the origin at time zero.

As we already observed, Theorem 1.3 is *not* a true generalization of Theorem 1.1 because it does not state anymore that the connected set $\mathcal{C}$ of radially periodic solutions approaches the origin. It motivates the question of whether one may find actual examples where $\mathcal{C}$ cannot be continued up to the origin. Somehow, this is expectable, since the assumptions (M_1) and (H_2) only refer to the behavior of f in a neighborhood of infinity. We have studied in detail equations with the form (10) and $e \equiv 0$,

$$\ddot{x} = -c(t)|x|^\gamma \frac{x}{|x|}, \tag{11}$$

assuming that $c \in C(\mathbb{R}/T\mathbb{Z})$ and $\int_0^T c(t)dt > 0$. We get the following result:

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Proposition 1.4. *If $0 \leq \gamma < 1$, then (11) has a connected set $\mathcal{C}$ of T-radially periodic solutions going from zero to infinity in the sense of (5).*

On the other hand, for every $\gamma < 0$ there exists some continuous and 1-periodic function c_γ with positive mean and such that (11) does not have 1-radially periodic solutions x with $\min_{\mathbb{R}} |x| = 1$.

Then, what happens with the connected set $\mathcal{C}$? Why, in some cases, it does not continue up to the origin? We may obtain a further insight on the situation by going back to the equivalent formulation (4). Observe that if $x = r(\cos\theta, \sin\theta)$ is a solution with angular momentum μ, then

$$\tilde{x} = r(\cos(-\theta), \sin(-\theta))$$

is another one, this time with angular momentum $-\mu$. Thus, together with $\mathcal{C}$, there is a second connected set of solutions coming from infinity:

$$\tilde{\mathcal{C}} = \left\{ \tilde{x} : x \in \mathcal{C} \right\}.$$

It may happen that both connected sets $\mathcal{C}$ and $\tilde{\mathcal{C}}$ coincide. This will be the situation if $\mathcal{C}$ contains some solutions with zero angular momentum $\mu = 0$. Looking back to the second equation of (4) we see that such a solution must have zero rotation number, as it lives in a ray emanating from the origin. Precisely, it will have the form

$$x(t) = r(t)\,v\,,$$

where $|v| = 1$ and r is a T-periodic solution of the one-dimensional equation

$$\ddot{r} = -f(t, r)\,, \qquad\qquad r > 0\,. \tag{12}$$

Thus, if we do not want to allow the connected set $\mathcal{C}$ go back to infinity through $\tilde{\mathcal{C}}$, we have to prevent it to contain such solutions:

Theorem 1.5. *Assume (M_1) and (H_2), the assumptions of Theorem 1.3. Then, there exists a connected set $\mathcal{C}$ of T-radially periodic solutions of (3) coming from infinity, i.e. (7). Furthermore, $\mathcal{C}$ may be chosen so that either it approaches the origin in the sense of (5), or it contains some solutions with zero angular momentum.*

Theorem 1.5 may be considered as a generalization of Theorem 1.1, since f being positive implies that (12) does not have T-periodic solutions. This statement can be easily checked by integrating both sides of the equation, to get

$$\dot{r}(T) - \dot{r}(0) = \int_0^T \ddot{r}(t)dt = -\int_0^T f(t, r(t))\,dt\,,$$

so that $\dot{r}(T) < \dot{r}(0)$ and r cannot be T-periodic. The argument may be repeated to show that (12) does not have T-periodic solutions if f has the form (8) for some L^1-Carathéodory function f_* with (H_1) and (H_2), and some T-periodic function $e \in L^1_{loc}(\mathbb{R})$ verifying (9). And a similar situation occurs if f is monotone in r and verifies (W_1) and (H_2). Thus, in all these

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cases there is a connected set $\mathcal{C}$ of T-radially periodic solutions going from zero to infinity in the sense of (5). We shall also rely on Theorem 1.5 to prove the first part of Proposition 1.4.

Remark that Theorem 1.5 includes Theorem 1.3. This result implies the existence of T-radially periodic solutions of (3) with big amplitudes, motivating the study the appearance of these solutions. It turns out that they look like copernican, meaning that the ratio between their maximum and minimum distances to the origin is close to one. Moreover, their angular speeds get small. This result, which is related to Lemmas 1,2 and 3 of [6], holds under the common requirements of Theorems 1.3 and 1.5: assumptions (M_1) and (H_2).

Proposition 1.6. *Assume (M_1) and (H_2). Then, for each $\epsilon > 0$ there exists some number $\lambda_2 > 0$ such that, for any T-radially periodic solution $x = r(\cos\theta, \sin\theta)$ of (3) with $\max r \geq \lambda_2$, one has*

$$\frac{\max r}{\min r} < 1 + \epsilon, \qquad\qquad 0 < |\dot\theta(t)| < \epsilon \text{ on } \mathbb{R}. \qquad (13)$$

The combination of some elements taken from (the proof of) Theorem 1.5 and Proposition 1.6 will lead us to:

Theorem 1.7. *Assume (M_1) and (H_2). Then, there exists some number $\bar\omega > 0$ with the property that for every real number $0 < \omega < \bar\omega$ there is a $T-$radially periodic solution $x_\omega = r_\omega(\cos\theta_\omega, \sin\theta_\omega)$ of (3) such that $\mathrm{rot}(x_\omega) = \omega$. Moreover,*

$$\lim_{\omega\to 0}(\min r_\omega) = +\infty, \qquad\qquad \lim_{\omega\to 0}\left[\frac{\min r_\omega}{\max r_\omega}\right] = 1, \qquad (14)$$

and

$$\dot\theta_\omega > 0 \text{ for every } \omega \in \,]0,\bar\omega[, \qquad\qquad \lim_{\omega\to 0}\dot\theta_\omega(t) = 0 \text{ uniformly w.r.t. } t \in \mathbb{R}. \qquad (15)$$

Choose some integer $k \in \mathbb{N}$ big enough so that $\omega_k := 2\pi/(kT) \in \,]0,\bar\omega[$ and observe that the T-radially periodic solution having rotation number ω_k, which we shall now call x_k, is actually kT-periodic, winding once around the origin on each period. We arrive to the following result, which may be seen as a generalization of Theorem 4 of [6]:

Corollary 1.8. *Assume (M_1) and (H_2). Then, there exists some $k_1 \geq 1$ such that, for any integer $k \geq k_1$, equation (3) has some subharmonic solution $x_k = r_k(\cos\theta_k, \sin\theta_k)$ with minimal period kT, which makes exactly one revolution around the origin in the period time kT. These solutions verify*

$$\lim_{k\to\infty}(\min r_k) = +\infty, \qquad\qquad \lim_{k\to\infty}\left[\frac{\min r_k}{\max r_k}\right] = 1,$$

and

$$\dot\theta_k > 0 \text{ for every integer } k \geq k_1, \qquad\qquad \lim_{k\to\infty}\dot\theta_k(t) = 0 \text{ uniformly w.r.t. } t \in \mathbb{R}.$$

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When $\omega \notin (2\pi/T)\mathbb{Q}$, the radially-periodic solution x_ω given by Theorem 1.7 is quasiperiodic of the frequencies $\omega_1 = 2\pi/T$, $\omega_2 = \omega$, this was already observed. We deduce:

Corollary 1.9. *Assume (M_1) and (H_2). Then, there exists some $\bar{\omega} > 0$ such that, for any number $0 < \omega < \bar{\omega}$ not commensurable with $2\pi/T$, equation (3) has some quasiperiodic solution $x_\omega = r_\omega(\cos\theta_\omega, \sin\theta_\omega)$ of the frequencies $\omega_1 = 2\pi/T$, $\omega_2 = \omega$. These solutions verify (14) and (15).*

For a systematic treatment of non-radially symmetric systems with a singularity, by the use of variational methods, the reader can consult [1] and the references therein. See also [7] for some results obtained by the use of degree theory.

2 Near infinity, radially periodic solutions are close to copernican

In this section we exploit the sublinearity of f to obtain some insight on the solutions of (3) with big amplitude. Our main goal will consist in showing Proposition 1.6, the result stating that, as the amplitude grows to infinity, radially periodic solutions become similar to copernican, while spinning slower and slower.

Along this Section we shall assume (M_1) and (H_2). It will be convenient to use polar coordinates $x = r(\cos\theta, \sin\theta)$, and consequently, we go back to the equivalent system (4). If x is T-radially periodic, then r must be T-periodic. In combination with the first equation of (4), it leads us to the boundary value problem

$$\begin{cases} \ddot{r} = \mu^2/r^3 - f(t,r), & r > 0, \\ r(0) = r(T), \ \dot{r}(0) = \dot{r}(T). \end{cases} \tag{16}$$

Observe that the parameter μ (the angular momentum of the solution) has not a prefixed value, and thus, solutions are couples (r, μ). For such a solution one has

$$-\ddot{r} \le f(t,r), \tag{17}$$

i.e., r is a lower solution of (12). And for these lower solutions, the first part of Proposition 1.6 holds (we let $\rho = 1 + \epsilon$):

Lemma 2.1. *For each $\rho > 1$ there exists some number $\lambda_2 > 0$ such that every T-periodic solution of (17) with $\max r \ge \lambda_2$ verifies*

$$\frac{\max r}{\min r} < \rho.$$

Proof. Using a contradiction argument, assume that the result were not true. Then, it would be possible to find a sequence $\{r_n\}_n$ of lower solutions of (17) with $\max r_n \to +\infty$ and

$$\frac{\max r_n}{\min r_n} \ge \rho_0 \qquad \text{for some } \rho_0 > 1.$$
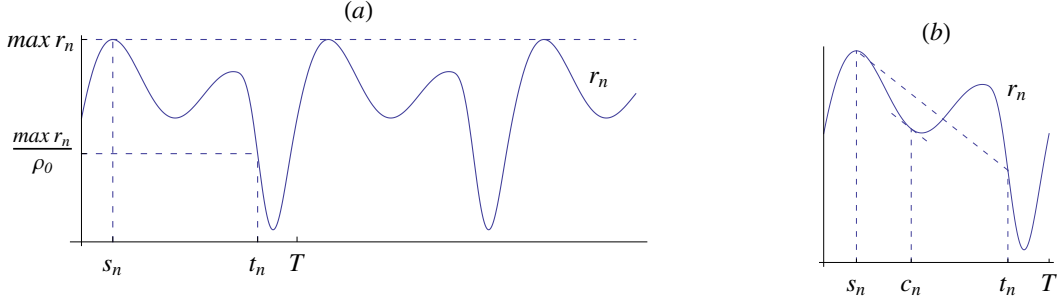
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Figure 1: The graph of r_n and the choices of s_n, t_n, c_n.

Then, $\min r_n \leq (\max r_n)/\rho_0$, and we may find instants of time $0 \leq s_n < t_n \leq T$ such that

$$r_n(s_n) = \max r_n\,, \qquad r_n(t_n) = (\max r_n)/\rho_0\,, \qquad r_n(t_n) \leq r_n(t) \leq r_n(s_n) \text{ if } t \in [s_n, t_n]\,,$$

see Figure 1(a). We use now Lagrange's Mean value Theorem and find some time $c_n \in\,]s_n, t_n[$ such that

$$\dot{r}_n(c_n) = \frac{r_n(t_n) - r_n(s_n)}{t_n - s_n} = \frac{\frac{\max r_n}{\rho_0} - \max r_n}{t_n - s_n} = -\frac{(\rho_0 - 1)(\max r_n)}{\rho_0(t_n - s_n)} \leq -\frac{(\rho_0 - 1)(\max r_n)}{\rho_0 T}\,,$$

see Figure 1(b). We denote $K_0 := (\rho_0 - 1)/(\rho_0 T)$, which is a positive constant and verifies

$$\dot{r}_n(c_n) \leq -K_0(\max r_n)\,, \qquad n \in \mathbb{N}\,. \tag{18}$$

On the other hand, $\dot{r}(s_n) = 0$, and again by the Mean Value Theorem (this time in its integral form),

$$\dot{r}_n(c_n) = \dot{r}_n(c_n) - \dot{r}_n(s_n) = \int_{s_n}^{c_n} \ddot{r}_n(t)dt \geq -\int_{s_n}^{c_n} f(t, r_n(t))dt\,,$$

which, in combination with (18), gives

$$\int_{s_n}^{c_n} f(t, r_n(t))dt \geq K_0(\max r_n)\,, \qquad n \in \mathbb{N}\,. \tag{19}$$

We define now, for each $n \in \mathbb{N}$,

$$\tilde{r}_n(t) := \max\left\{r_n(t), r_n(t_n)\right\}\,, \qquad t \in \mathbb{R}\,,$$

which is again continuous and $T-$periodic. Moreover, on $[s_n, c_n]$, r_n and $\tilde{r}_n$ coincide, and $\max \tilde{r}_n = \max r_n$. Using (19) we deduce that

$$K_0(\max \tilde{r}_n) = K_0(\max r_n) \leq \int_{s_n}^{c_n} f(t, r_n(t))dt = \int_{s_n}^{c_n} f(t, \tilde{r}_n(t))dt \leq$$

$$\leq \int_{s_n}^{c_n} |f(t, \tilde{r}_n(t))|dt \leq \int_0^T |f(t, \tilde{r}_n(t))|dt\,,$$

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and then
$$K_0 \leq \int_0^T \frac{|f(t, \tilde{r}_n(t))|}{\max \tilde{r}_n} \, dt \leq \int_0^T \left| \frac{f(t, \tilde{r}_n(t))}{\tilde{r}_n(t)} \right| \, dt \,. \tag{20}$$

However, assumption (H_2) implies that the sequence $f(t, \tilde{r}_n(t))/\tilde{r}_n(t)$ converges to zero pointwise on $[0, T]$, and is dominated by the integrable function h. Lebesgue's Convergence Theorem then implies that
$$\lim_{n \to \infty} \int_0^T \left| \frac{f(t, \tilde{r}_n(t))}{\tilde{r}_n(t)} \right| \, dt = 0 \,,$$

contradicting (20). $\qquad \square$

To continue, we remember assumption (M_1) and choose some number $\rho_* > 1$ as given there. We define the function
$$\mathcal{M}(\lambda) := 1 + \sqrt{\frac{\rho_*^3 \lambda^3}{T} \int_0^T \left(\max_{\lambda/\rho_* \leq r \leq \rho_* \lambda} |f(t, r)| \right) dt} \,, \qquad \lambda > 0 \,. \tag{21}$$

Observe that $\mathcal{M}$ is continuous. Moreover,
$$\frac{\mathcal{M}(\lambda)}{\lambda^2} = \frac{1}{\lambda^2} + \sqrt{\frac{\rho_*^3}{\lambda T} \int_0^T \left(\max_{\lambda/\rho_* \leq r \leq \rho_* \lambda} |f(t, r)| \right) dt} \leq \frac{1}{\lambda^2} + \sqrt{\frac{\rho_*^4}{T} \int_0^T \left(\max_{\lambda/\rho_* \leq r \leq \rho_* \lambda} \frac{|f(t, r)|}{r} \right) dt} \,,$$

and the combination of assumption (H_2) and Lebesgue's Theorem implies that
$$\frac{\mathcal{M}(\lambda)}{\lambda^2} \to 0 \qquad \text{as } \lambda \to +\infty \,,$$

i.e., $\mathcal{M}$ is subquadratic. The following result collects some properties of our equation and the function $\mathcal{M}$ which will be needed later. As before, $\rho_* > 1$ is given by assumption (M_1):

Lemma 2.2. *There exists some $\lambda_0 > 0$ with the following properties:*

(i) *For any solution (r, μ) of (16) with $\max r \geq \lambda_0$, one has that $r(t_0)/\rho_* < r(t) < \rho_* \, r(t_0)$ for any $t_0, t \in \mathbb{R}$.*

(ii) *If $r = r(t)$ is continuous and T-periodic, and verifies, for some $\lambda \geq \lambda_0/\rho_*$,*
$$\lambda/\rho_* < r(t) < \rho_* \lambda \text{ for every } t \in \mathbb{R} \,, \tag{22}$$

then
$$0 < \int_0^T f(t, r(t)) dt < \mathcal{M}(\lambda)^2 \int_0^T \frac{1}{r(t)^3} dt \,.$$

Proof. Remembering (M_1), there is some number $\lambda_1 > 0$ such that, if $\lambda \geq \lambda_1$, then
$$\int_0^T f(t, r(t)) dt > 0 \text{ for any } r \in C(\mathbb{R}/T\mathbb{Z}) \text{ verifying } (22) \,.$$

Choose next $\lambda_2 > 0$ as given by Lemma 2.1 for $\rho = \rho_*$, and let $\lambda_0 := \max\{\rho_* \lambda_1, \lambda_2\}$.

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(i): If (r, μ) is a solution of (16) with $\max r \geq \lambda_0$, then $\max r \geq \lambda_2$ and for any time $t_0 \in \mathbb{R}$ one has

$$\frac{\max r}{r(t_0)} < \rho_* , \qquad \frac{r(t_0)}{\min r} < \rho_* ,$$

or, what is the same,

$$\frac{r(t_0)}{\rho_*} < \min r \leq \max r < \rho_* r(t_0) ,$$

and *(i)* follows.

(ii): Let now $r \in C(\mathbb{R}/T\mathbb{Z})$ verify (22) for some $\lambda \geq \lambda_0/\rho_*$. Then, $\lambda \geq \lambda_1$, and the first part of *(ii)* follows from assumption (M_1). Concerning the second part, we recall (21) and observe that

$$\mathcal{M}(\lambda)^2 \int_0^T \frac{1}{r(t)^3} dt \geq \mathcal{M}(\lambda)^2 \frac{T}{\rho_*^3 \lambda^3} > \int_0^T \left(\max_{\lambda/\rho_* \leq r \leq \rho_* \lambda} |f(t,r)| \right) dt \geq \int_0^T f(t, r(t)) dt ,$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

The following result states the boundedness of the set of solutions (r, μ) of (16) for which the minimum of r lies between two given bounds. It will follow from Lemma 2.2*(i)*:

Corollary 2.3. *Given constants $0 < k < K$, there exists some $M > 0$ such that, whenever (r, μ) is a $T-$periodic solution of (16) with $k \leq \min r \leq K$, then*

$$|\mu| \leq M, \qquad\qquad \max r \leq M .$$

Proof. After possibly replacing K by a bigger quantity, it is not restrictive to assume that $K \geq \lambda_0$, the constant appearing in Lemma 2.2. Then, any solution (r, μ) of (16) with $k \leq \min r \leq K$ verifies

$$\max r < M_1 := \rho_* K . \tag{23}$$

Next, we remember that f is L^1-Carathéodory and find some function $h \in L^1(\mathbb{R}/T\mathbb{Z})$ such that $|f(t, x)| \leq h(t)$ for a.e. $t \in \mathbb{R}$ and all $x \in [k, M_1]$. We let

$$M_2 := \sqrt{\frac{M_1^3}{T} \int_0^T h(t) dt} .$$

Choose now a solution (r, μ) of (16) with $k \leq \min r \leq K$. In view of (23), $\max r \leq M_1$, and integrating both sides of the equation of (16) we get

$$0 = \mu^2 \int_0^T \frac{1}{r(t)^3} dt - \int_0^T f(t, r(t)) dt \geq \frac{\mu^2 T}{M_1^3} - \int_0^T h(t) dt ,$$

and we deduce that $|\mu| \leq M_2$. It completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Our next step will consist in showing that the angular momentum of radially periodic solutions with big amplitude may be bounded by the subquadratic function $\mathcal{M}$ applied to the distance from our solution to the origin.

Lemma 2.4. *There exists some $\lambda_0 > 0$ such that every solution (r, μ) of (16) with $\max r \geq \lambda_0$ verifies*

$$0 < |\mu| < \mathcal{M}(r(t_0)) \quad \text{for any } t_0 \in \mathbb{R}. \tag{24}$$

Proof. Let $\lambda_0 > 0$ be as in Lemma 2.2 and let (r, μ) be a solution of (16) with $\max r \geq \lambda_0$. Then Lemma 2.2*(i)* implies that (22) holds for $\lambda = r(t_0)$ and any $t_0 \in \mathbb{R}$. In particular, $r(t_0) \geq \lambda_0/\rho_*$, and Lemma 2.2*(ii)* gives

$$0 < \int_0^T f(t, r(t))dt < \mathcal{M}(r(t_0))^2 \int_0^T \frac{1}{r(t)^3} dt.$$

On the other hand, integrating on the equation of (16) we observe that

$$0 = \int_0^T \ddot{r}(t)dt = \mu^2 \int_0^T \frac{1}{r(t)^3} dt - \int_0^T f(t, r(t))\, dt,$$

and the result follows.

$\square$

To close this Section, we prove Proposition 1.6. We have only to combine Lemmas 2.1 and 2.4.

Proof of Proposition 1.6. Fix $\lambda_0 > 0$ as in Lemma 2.4. Choose now some positive number $\epsilon > 0$ and find $\lambda_1 \geq \lambda_0$ such that $\mathcal{M}(\lambda)/\lambda^2 < \epsilon$ for any $\lambda \geq \lambda_1$. Now, let $\lambda_2 > 0$ be as given by Lemma 2.1 for $\rho = 1 + \epsilon$. We may replace, if we want, λ_2 by a greater number, so that it is not restrictive to assume that $\lambda_2 \geq \lambda_1$.

Finally, pick some solution $x = r(\cos\theta, \sin\theta)$ of (3) with $\max r \geq \lambda_2$. We consider the angular momentum $\mu = r(t)^2 \dot{\theta}(t)$, which does not depend on time. Thus, (r, μ) is now a solution of (16), and Lemma 2.1 implies that $\max r/\min r < 1 + \epsilon$. On the other hand, since $\max r \geq \lambda_1 \geq \lambda_0$, we have

$$r(t)^2 |\dot{\theta}(t)| = |\mu| \leq \mathcal{M}(r(t)) < \epsilon\, r(t)^2, \qquad t \in \mathbb{R},$$

so that $|\dot{\theta}(t)| < \epsilon$ for any $t \in \mathbb{R}$. This concludes the proof.

$\square$

3   Radially periodic solutions in the plane versus periodic solutions of a one-dimensional equation

In this Section we provide an equivalent formulation of Theorem 1.5 using polar coordinates. With this aim, let $x = r(\cos\theta, \sin\theta)$ be a $T-$radially periodic solution of (3); as observed in

the previous Section, $r = r_x$ must be a solution of the periodic boundary value problem (16), the parameter $\mu = \mu_x = r^2 \dot\theta$ being the angular momentum of the solution.

Conversely, let now (r, μ) be a solution of (16). Then

$$\theta_{r,\mu}(t) = \int_0^t \frac{\mu}{r(\tau)^2}\, d\tau\,, \qquad\qquad t \in \mathbb{R}\,,$$

solves the second equation of system (4). It further verifies that $\theta_{r,\mu}(t+T) = \theta_{r,\mu}(t) + \theta_{r,\mu}(T)$ for any $t \in \mathbb{R}$, and, consequently,

$$x_{r,\mu}(t) = r(t)(\cos\theta_{r,\mu}(t), \sin\theta_{r,\mu}(t))\,, \qquad\qquad t \in \mathbb{R}\,,$$

is a T−radially periodic solution of (3), its rotation number being $\theta_{r,\mu}(T)/T$. Since $\theta_{r,\mu}(0) = 0$, this solution verifies

$$x_{r,\mu}(0) = (r(0), 0) \in {]0, +\infty[} \times \mathbb{R}\,,$$

i.e., at the initial time our solution crosses the horizontal axis on its positive side. This discussion leads us to consider the mappings

$$\Phi : \mathcal{R} \to \mathcal{X}\,,\ \ (r, \mu) \mapsto x_{r,\mu}\,, \qquad\qquad \Psi : \mathcal{X} \to \mathcal{R}\,,\ \ x \mapsto (r_x, \mu_x)\,,$$

the sets $\mathcal{R}$ and $\mathcal{X}$ being defined by

$$\mathcal{R} := \Big\{\text{sols. } (r, \mu) \text{ of } (16)\Big\}\,, \quad \mathcal{X} := \Big\{T\text{-rad. per. sols. } x \text{ of } (3) \text{ with } x(0) \in {]0, +\infty[} \times \mathbb{R}\Big\}\,.$$

Observe that Φ and Ψ are mutually inverse bijections. Moreover, if $\mathcal{R}$ and $\mathcal{X}$ are endowed, respectively, with the topologies inherited from $\mathbb{R}$ times the space $C(\mathbb{R}/T\mathbb{Z})$ of continuous and T−periodic functions on the real line, and the topology of uniform convergence on compact sets, then Φ and Ψ become mutually inverse homeomorphisms.

What relevance has this discussion with respect to Theorem 1.5? To answer this question we recall that, roughly speaking, this result states the existence of a 'large' connected set $\mathcal{C}$ of T-radially periodic solutions of (3). In principle, $\mathcal{C}$ may be not contained into $\mathcal{X}$, as the functions of $\mathcal{C}$ are not obliged to cross the positive part of the horizontal axis at time zero. Observe, however, that our equation (3) is rotation-invariant. By this we mean that, if $x = r_x(\cos\theta_x, \sin\theta_x)$ is a solution, and R is the rotation of some angle θ_0 around the origin, then $Rx = r_x(\cos(\theta_x + \theta_0), \sin(\theta_x + \theta_0))$ is again a solution. In this way, we may rotate the elements of $\mathcal{C}$ to build a second connected set $\mathcal{C}'$ of solutions of (3) which is now contained inside $\mathcal{X}$:

$$\mathcal{C}' := \Big\{r_x\Big(\cos\big(\theta_x - \theta_x(0)\big), \sin\big(\theta_x - \theta_x(0)\big)\Big) : x = r_x(\cos\theta_x, \sin\theta_x) \in \mathcal{C}\Big\}\,.$$

Notice that $\{\min|x| : x \in \mathcal{C}'\} = \{\min|x| : x \in \mathcal{C}\}$, so that there is no restriction in looking for the connected set $\mathcal{C}$ in the smaller space $\mathcal{X}$. The homeomorphism Ψ will then send it into another connected set $\mathcal{K} \subset \mathcal{R}$, which will verify

$(\mathcal{K}_1)$ the interval $\Big\{\min r : (r, \mu) \in \mathcal{K}\Big\}$ is unbounded (from above),

and one of the following:

$(\mathcal{K}_{2a})$ $\Big\{ \min r : (r, \mu) \in \mathcal{K} \Big\} =]0, +\infty[$

$(\mathcal{K}_{2b})$ $\Big\{ (r, \mu) \in \mathcal{K} : \mu = 0 \Big\} \neq \emptyset.$

In this way, the following result may be seen as a corollary of Theorem 1.5:

Lemma 3.1. *Assume (M_1) and (H_2). Then, there exists a connected set $\mathcal{K} \subset \mathcal{R}$ verifying $(\mathcal{K}_1)$, and either $(\mathcal{K}_{2a})$ or $(\mathcal{K}_{2b})$ above.*

Actually, Lemma 3.1 is equivalent to Theorem 1.5, because Φ and Ψ are homeomorphisms. And this will be the spirit of our approach; we shall prove Theorem 1.5 in the form given by Lemma 3.1. Before, it will be convenient to establish a functional analysis framework for our problem.

4 Two-sided continuation for equations depending on a parameter

Let Y be a Banach space, $\Omega \subset \mathbb{R} \times Y$ an open set, and let

$$H : \Omega \to Y, \quad (\lambda, y) \mapsto H(\lambda, y),$$

be a continuous operator verifying the standard compactness assumption:

(C) *Subsets of $\mathbb{R} \times Y$ which are bounded, closed, and contained inside Ω, are mapped by H into relatively compact sets.*

This property plays an important role, as it will allow us to use the Leray-Schauder degree arguments. We look for solutions of the fixed-point equation

$$y = H(\lambda, y), \qquad (\lambda, y) \in \Omega. \qquad (25)$$

For this, we further assume the existence of some $\lambda_0 \in \mathbb{R}$ and some open and bounded set $\Upsilon_{\lambda_0} \subset Y$ such that $\{\lambda_0\} \times \overline{\Upsilon}_{\lambda_0} \subset \Omega$ and

(i) Equation (25) has no solutions $(\lambda, y) \in \Omega$ such that $\lambda = \lambda_0$ and $y \in Y \backslash \Upsilon_{\lambda_0}$.

(ii) $\deg_{LS}(\mathrm{Id}_Y - H(\lambda_0, \cdot), \Upsilon_{\lambda_0}) \neq 0$.

To state the main result of this section it will be convenient to introduce some notation. Given any set $A \subset \Omega$, we divide it into its (not necessarily disjoint) left and right pieces $A_\pm$. The right one A_+ is defined by

$$A_+ := \Big\{ (\lambda, y) \in A : \lambda \geq \lambda_0 \Big\},$$

while A_- is given analogously after reversing the inequality. We also consider the set Σ of solutions of (25),

$$\Sigma := \Big\{ (\lambda, y) \in \Omega : y = H(\lambda, y) \Big\}. \qquad (26)$$

Proposition 4.1. *Assume* **(C)***, (i), and (ii). Then, there exists a connected subset* $\mathcal{C} \subset \Sigma$ *such that each piece* $\mathcal{C}_\pm$ *lies under the following disjunctive: either*

$(a_\pm)$ $\mathcal{C}_\pm$ *is unbounded in* $\mathbb{R} \times Y$,

$(b_\pm)$ *or* $\mathcal{C}_\pm$ *goes up to the boundary of* Ω, *i.e. its distance to the boundary of* Ω *is zero.*
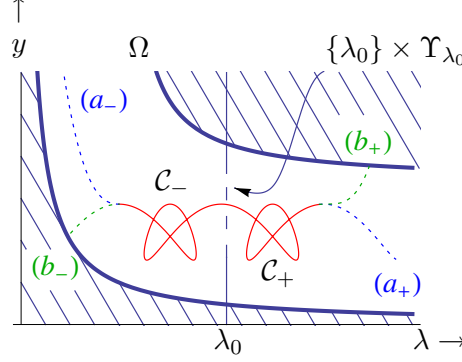

Figure 2: The four possibilities for the connected set $\mathcal{C}$

Remark. *The two possibilities* $(a_-), (b_-)$ *are to be combined with* $(a_+), (b_+)$, *giving then rise to four different situations:* (a_-)-(a_+), (a_-)-(b_+), (b_-)-(a_+), *and* (b_-)-(b_+).

This result is well known, and even though we could not find it in the literature in the form presented here, related results may be found, for instance, in [2, 4, 8, 9, 10]. We shall actually need only a corollary of this result, which we describe next. It will be convenient to introduce first the following notation: given any subset $\Gamma \subset \Omega$ and any $\lambda \in]0, +\infty[$ we denote by Γ_λ the vertical section $\Gamma_\lambda := \{y \in Y : (\lambda, y) \in \Gamma\}$.

In the case we are interested in, Ω will be a cylinder

$$\Omega =]0, +\infty[\times V,$$

where V is an open (and possibly unbounded) subset of Y. With other words, we are assuming that $\Omega_\lambda = V$ is always the same set for every $\lambda \in]0, +\infty[$. As before, we shall also need **(C)**, we fix some value $\lambda_0 > 0$ and we denote by $\Omega_\pm$ the sides of Ω to the left and the right of λ_0.

But this time we shall further assume the existence of some open[1] subset $\Upsilon \subset \Omega_+$ verifying

(1.) the adherence $\overline{\Upsilon}$ is still contained in Ω_+,

(2.) Υ is bounded over bounded intervals of λ, i.e. $\Upsilon \cap ([\lambda_0, \lambda] \times Y)$ is bounded for every $\lambda > \lambda_0$,

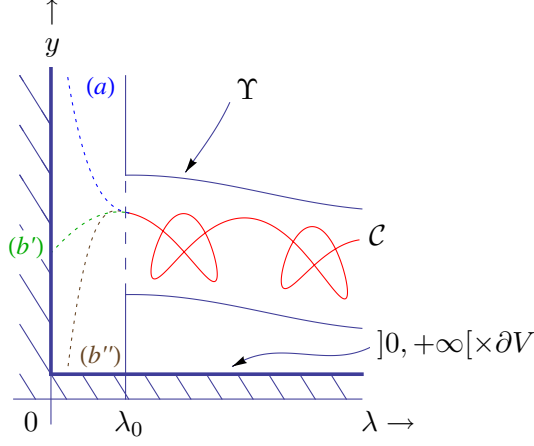(3.) Equation (25) has no solutions $(\lambda, y) \in \Omega_+$ such that $(\lambda, y) \notin \Upsilon$.

Figure 3: The three possibilities for the connected set $\mathcal{C}$

Observe that assumption (2.) implies in particular that Υ_{λ_0} is bounded, while, by (1.), $\{\lambda_0\} \times \overline{\Upsilon}_{\lambda_0} \subset \Omega$. Both things were required in Proposition 4.1. On the other hand, observe that (3.) implies assumption (i) there. As in (26), we denote by Σ to the set of solutions of our equation (25):

Corollary 4.2. *Assume (1.), (2.), and (3.) above. Assume further that (ii) holds[2]. Then, there exists a connected subset $\mathcal{C} \subset \Sigma$ such that $\mathcal{C}_\lambda \neq \emptyset$ for any $\lambda \geq \lambda_0$. Moreover, either*

(a) $\mathcal{C} \cap (]0, \lambda_0[\times Y)$ *is unbounded,*

(b') *or $\mathcal{C}$ goes all the way up to $\lambda = 0$, i.e., $\mathcal{C}_\lambda \neq \emptyset$ for any $\lambda > 0$,*

(b'') *or $\mathcal{C} \cap (]0, \lambda_0[\times Y)$ goes up to the boundary of V, i.e. its distance to $]0, \lambda_0[\times \partial V$ is zero.*

Proof. Applying Proposition 4.1 we deduce the existence of a connected set $\mathcal{C}$ for which each piece $\mathcal{C}_\pm$ lies under the disjunctive $(a_\pm)$-$(b_\pm)$. For instance, $\mathcal{C}_+$ must verify either (a_+) or (b_+). But this time, our assumptions (1.) and (3.) prevent (b_+) from happening, so that we must have (a_+), and in view of (2.), we deduce that $\mathcal{C}_\lambda \neq \emptyset$ for any $\lambda \geq \lambda_0$, as claimed.

We still have the disjunctive (a_-)-(b_-). Possibility (a_-) is now called (a). Concerning to possibility (b_-), one observes that the boundary of our cylindrical domain Ω may be decomposed into two parts: $\{0\} \times \bar{V}$ and $]0, +\infty[\times \partial V$. Thus, if the distance from $\mathcal{C}_-$ to $\partial \Omega$ is zero, it is because either the distance to $\{0\} \times \bar{V}$ is zero or the distance to $]0, +\infty[\times \partial V$ is zero. The first possibility is what we called (b') and the second one is (b''). The proof is complete. $\qquad\square$

Finally, we state a one-sided variant of Corollary 4.2 which we shall need later. It is also well-known and the proof is skipped:

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[1]Here, the adjective 'open' is referred to Ω_+.

[2]In particular, $\Upsilon_{\lambda_0} \neq \emptyset$.

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Corollary 4.3. *Under the framework of Corollary 4.2, for any $\lambda^* \geq \lambda_0$, there exists a connected subset $\mathcal{C}^* \subset \Sigma$ such that*

$$\{\lambda > 0 : \mathcal{C}^*_\lambda \neq \emptyset\} = [\lambda^*, +\infty[.$$

5   A connected set of solutions coming from infinity

Throughout this Section we assume (M_1) and (H_2). In order to apply the abstract framework developed in the previous section to find connected sets of solutions (r, μ) of (16), we have to reformulate this problem as a fixed point equation in a Banach space. The equation will depend on a one-dimensional parameter; however, this parameter will not be μ, but the value λ of our T-periodic unknown r at the integer multiples of the period. For this reason, our first step will be to rewrite r as

$$r = \lambda(1 + \tilde{r}),$$

where $\lambda = r(0)$ and $\tilde{r} = r/\lambda - 1$. It follows now that $\tilde{r}(0) = 0$, motivating us to consider the space

$$C_0(\mathbb{R}/T\mathbb{Z}) := \left\{ \tilde{r} \in C(\mathbb{R}/T\mathbb{Z}) \text{ such that } \tilde{r}(0) = 0 \right\},$$

which is endowed with the uniform norm $\|\cdot\|_\infty$. This space has the following property: given any integrable function $h \in L^1(\mathbb{R}/T\mathbb{Z})$ there exists a unique function $Kh \in C_0(\mathbb{R}/T\mathbb{Z})$ which has $W^{2,1}$ regularity on $[0, T]$ and whose second derivative there is h. Moreover, the mapping $K : L^1(\mathbb{R}/T\mathbb{Z}) \to C_0(\mathbb{R}/T\mathbb{Z})$ defined in this way is linear and compact. Observe that $\tilde{r} = Kh$ is continuous and T-periodic by the way it has been defined; however, the left and right derivatives at the integer multiples of T will coincide if and only if the mean-value projection of h on the space of constant functions (naturally identified with $\mathbb{R}$),

$$Ph := \frac{1}{T} \int_0^T h(t)dt,$$

vanishes.

On the other hand, since r was taken positive, $\lambda \in]0, +\infty[$ and $\tilde{r} > -1$ on $\mathbb{R}$. Thus, we shall look for our solutions $(\lambda, \tilde{r}, \tilde{\mu})$ on the set

$$\Omega :=]0, +\infty[\times V,$$

where $V := \left\{ (\tilde{r}, \tilde{\mu}) \in C_0(\mathbb{R}/T\mathbb{Z}) \times \mathbb{R} : \tilde{r} > -1 \right\}$ is an open subset of the Banach space

$$Y := C_0(\mathbb{R}/T\mathbb{Z}) \times \mathbb{R},$$

(the elements of this space are pairs of the form $y = (\tilde{r}, \tilde{\mu})$).

Observe that, instead of the angular momentum μ we have used here a new letter $\tilde{\mu}$. Well, problem (16) has the property that, if (r, μ) is a solution, then so is $(r, -\mu)$. We do not want this situation to be translated to our abstract framework, and that is the reason why

we have introduced the new variable $\tilde{\mu}$. When $\tilde{\mu} \geq 0$, then $\tilde{\mu} = \mu$ will be just the angular momentum, but if $\tilde{\mu} < 0$ then we will be looking for solutions with zero angular momentum $\mu = 0$. It means that we are going to loose the solutions with negative angular momentum; on the other hand, if there is some solution with $\mu = 0$, then it will be repeated for each value of $\tilde{\mu} \in \,]-\infty, 0]$. Having this in mind, the 'modified Nemitski functional' for our problem (16) is defined by

$$N : \Omega \to L^1(\mathbb{R}/T\mathbb{Z}), \qquad (\lambda, \tilde{r}, \tilde{\mu}) \mapsto \begin{cases} \dfrac{-f(\cdot, \lambda(1+\tilde{r}))}{\lambda} & \text{if } \tilde{\mu} < 0 \\[3mm] \dfrac{\tilde{\mu}^2}{\lambda^4(1+\tilde{r})^3} - \dfrac{f(\cdot, \lambda(1+\tilde{r}))}{\lambda} & \text{if } \tilde{\mu} \geq 0 \, . \end{cases}$$

We rewrite (16) as a Lyapunov-Schmidt-type system on Ω:

$$\tilde{r} = K(\mathrm{Id} - P)N(\lambda, \tilde{r}, \tilde{\mu}), \qquad\qquad PN(\lambda, \tilde{r}, \tilde{\mu}) = 0 \, . \qquad (27)$$

This system may be seen as a fixed point equation on V depending on the parameter λ:

$$\begin{pmatrix} \tilde{r} \\ \tilde{\mu} \end{pmatrix} = H\left(\lambda, \begin{pmatrix} \tilde{r} \\ \tilde{\mu} \end{pmatrix}\right) := \begin{pmatrix} K(\mathrm{Id} - P)N(\lambda, \tilde{r}, \tilde{\mu}) \\ \tilde{\mu} + PN(\lambda, \tilde{r}, \tilde{\mu}) \end{pmatrix}, \qquad \lambda \in \,]0, +\infty[, \quad \begin{pmatrix} \tilde{r} \\ \tilde{\mu} \end{pmatrix} \in V. \quad (28)$$

Observe that $H :]0, +\infty[\times V \to Y$ is continuous; moreover, it verifies the compactness assumption which we labeled **(C)** in the previous Section. We want to apply Corollary 4.2, but before, we have to place ourselves inside the framework of this result. Thus, we fix some number $\rho_* > 1$ as given by assumption (M_1), we recall the function $\mathcal{M} :]0, +\infty[\to \mathbb{R}$ defined in (21), and we pick some number $\lambda_0 > 0$ verifying simultaneously the statements of Lemmas 2.2 and 2.4. Finally, we define

$$\Upsilon := \left\{ \left(\lambda, \begin{pmatrix} \tilde{r} \\ \tilde{\mu} \end{pmatrix}\right) \in \,]0, +\infty[\times Y : \lambda \geq \lambda_0, \, \frac{1}{\rho_*} - 1 < \tilde{r}(t) < \rho_* - 1 \, \forall t \in \mathbb{R}, \, 0 < \tilde{\mu} < \mathcal{M}(\lambda) \right\}.$$

Observe that Υ is an open subset of $\Omega_+ = \left\{ \left(\lambda, \begin{pmatrix} \tilde{r} \\ \tilde{\mu} \end{pmatrix}\right) \in \Omega : \lambda \geq \lambda_0 \right\}$. The following result states that the other assumptions of Corollary 4.2 are also satisfied.

Lemma 5.1. *Statements (1.), (2.), (3.), and (ii) from the previous section hold.*

Proof. (1.) and (2.) follow immediately from the definition of Υ. To see (3.), choose some $\lambda \geq \lambda_0$ and some fixed point $(\tilde{r}, \tilde{\mu})$ of $H(\lambda, \cdot)$. We let $r := \lambda(1 + \tilde{r})$ and $\mu = \max\{\tilde{\mu}, 0\}$, so that (r, μ) is a solution of (16). Moreover, $\max r \geq r(0) = \lambda \geq \lambda_0$, and, for $t_0 = 0$, Lemma 2.2 *(i)* implies that $\lambda/\rho_* < r(t) < \rho_* \lambda$, or, equivalently,

$$\frac{1}{\rho_*} - 1 < \tilde{r}(t) < \rho_* - 1, \qquad\qquad t \in \mathbb{R}.$$

On the other hand, Lemma 2.4 states that $0 < |\mu| < \mathcal{M}(\lambda)$. It follows that $\mu = \tilde{\mu} > 0$ and $(\lambda, \tilde{r}, \tilde{\mu}) \in \Upsilon$, proving (3.).

Before going into the proof of *(ii)* we first notice that, as a consequence of our sublinearity assumption (H_2) and the function $\mathcal{M}$ being subquadratic, on the set $\overline{\Upsilon}$ we have:

$$\lim_{\lambda \to +\infty} \left\| N(\lambda, \tilde{r}, \tilde{\mu}) \right\|_{L^1} = 0 \quad \text{as } \lambda \to +\infty, \text{ uniformly with respect to } (\tilde{r}, \tilde{\mu}).$$

But the linear operators $I - P : L^1(\mathbb{R}/T\mathbb{Z}) \to L^1(\mathbb{R}/T\mathbb{Z})$ and $K : L^1(\mathbb{R}/T\mathbb{Z}) \to C_0(\mathbb{R}/T\mathbb{Z})$ are continuous, and, consequently,

$$\lim_{\lambda \to +\infty} \left\| K(\mathrm{Id} - P)N(\lambda, \tilde{r}, \tilde{\mu}) \right\|_{\infty} = 0 \quad \text{as } \lambda \to +\infty, \text{ uniformly with respect to } (\tilde{r}, \tilde{\mu}),$$

on $\overline{\Upsilon}$. Thus, we may find some number $\lambda_* \geq \lambda_0$ such that, if $\lambda \geq \lambda_*$ and $(\tilde{r}, \tilde{\mu}) \in \overline{\Upsilon}_\lambda$, then

$$\frac{1}{\rho_*} - 1 < K(\mathrm{Id} - P)N(\lambda, \tilde{r}, \tilde{\mu}) < \rho_* - 1. \tag{29}$$

Let us show now *(ii)*. In view of (3.), $\deg_{LS}\left(\mathrm{Id}_Y - H(\lambda, \cdot), \Upsilon_\lambda\right)$ does not depend on $\lambda \geq \lambda_0$, and we shall actually show that this degree is equal to -1. Our strategy will consist in using a homotopy to link $H(\lambda_*, \cdot)$ and the mapping

$$F : \overline{\Upsilon}_{\lambda_*} \to Y, \qquad \begin{pmatrix} \tilde{r} \\ \tilde{\mu} \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ \tilde{\mu} + PN(\lambda_*, 0, \tilde{\mu}) \end{pmatrix}.$$

Before constructing the homotopy let us observe that

$$\Upsilon_{\lambda_*} = \mathcal{O} \times \left]0, \mathcal{M}(\lambda_*)\right[, \tag{30}$$

where $\mathcal{O} = \left\{ \tilde{r} \in C_0(\mathbb{R}/T\mathbb{Z}) : 1/\rho_* - 1 < \tilde{r} < \rho_* - 1 \right\}$, which is open and contains $\tilde{r} \equiv 0$. Since

$$\mathrm{Id}_Y - F : \mathcal{O} \times \left]0, \mathcal{M}(\lambda_*)\right[\to Y, \qquad \begin{pmatrix} \tilde{r} \\ \tilde{\mu} \end{pmatrix} \mapsto \begin{pmatrix} \tilde{r} \\ -PN(\lambda_*, 0, \tilde{\mu}) \end{pmatrix},$$

then, the product formula (see, for instance, [5]) gives

$$\deg_{LS}(\mathrm{Id}_Y - F, \Upsilon_{\lambda_*}) = \deg\left(-PN(\lambda_*, 0, \cdot) \left]0, \mathcal{M}(\lambda_*)\right[\right).$$

On the other hand, Lemma 2.2 *(ii)* implies, for $r(t) \equiv \lambda_*$, that

$$PN(\lambda_*, 0, 0) < 0 < PN(\lambda_*, 0, \mathcal{M}(\lambda_*)),$$

and consequently, $\deg_{LS}(\mathrm{Id}_Y - F, \Upsilon_{\lambda_*}) = -1$.

We define now our homotopy $M : [0, 1] \times \overline{\Upsilon}_{\lambda_*} \to Y$ by

$$M\left(s, \begin{pmatrix} \tilde{r} \\ \tilde{\mu} \end{pmatrix}\right) := \begin{pmatrix} s\, K(\mathrm{Id} - P)N(\lambda_*, \tilde{r}, \tilde{\mu}) \\ \tilde{\mu} + PN(\lambda_*, s\, \tilde{r}, \tilde{\mu}) \end{pmatrix}, \qquad \left(s, \begin{pmatrix} \tilde{r} \\ \tilde{\mu} \end{pmatrix}\right) \in [0, 1] \times \overline{\Upsilon}_{\lambda_*}.$$

In this way, M is completely continuous, $M(1, \cdot) = H(\lambda_*, \cdot)$ and $M(0, \cdot) = F$. To complete the argument we still have to show that $M(s, \cdot)$ has no fixed points in $\partial \Upsilon_{\lambda_*}$ for any $s \in [0, 1]$. But, in view of (30),

$$\partial \Upsilon_{\lambda_*} = \left(\partial \mathcal{O} \times \left[0, \mathcal{M}(\lambda_*)\right] \right) \cup \left(\mathcal{O} \times \left\{0, \mathcal{M}(\lambda_*)\right\} \right).$$

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In the set $\partial\mathcal{O} \times [0, \mathcal{M}(\lambda_*)]$ there are no fixed points of $M(s, \cdot)$ since, by (29),

$$s\, K(\mathrm{Id} - P)N(\lambda_*, \tilde{r}, \tilde{\mu}) \in \mathcal{O}, \qquad (\tilde{r}, \tilde{\mu}) \in \overline{\Upsilon}_{\lambda_*}.$$

On the other hand, Lemma 2.2 *(ii)* implies

$$PN(\lambda_*, s\,\tilde{r}, 0) < 0 < PN(\lambda_*, s\,\tilde{r}, \mathcal{M}(\lambda_*)), \qquad \tilde{r} \in \mathcal{O},$$

and then, $\mathcal{O} \times \{0, \mathcal{M}(\lambda_*)\}$ does not contain fixed points of $M(s, \cdot)$ either. It concludes the proof.

$\square$

We are now ready to prove the main result of the paper:

Proof of Theorem 1.5. As observed at the end of Section 3, Theorem 1.5 is equivalent to Lemma 3.1; thus, we shall prove this result instead. Combining Lemma 5.1 and Corollary 4.2 we find some connected set $\mathcal{C} \subset]0, +\infty[\times Y$ of solutions $(\lambda, \tilde{r}, \tilde{\mu})$ of our Lyapunov-Schmidt system (27) with

$$\{\lambda : (\lambda, \tilde{r}, \tilde{\mu}) \in \mathcal{C}\} \supset [\lambda_0, +\infty[\,, \tag{31}$$

and, either

(a) $\{(\tilde{r}, \tilde{\mu}) : (\lambda, \tilde{r}, \tilde{\mu}) \in \mathcal{C}, \ 0 < \lambda < \lambda_0\}$ is unbounded,

(b′) or $\{\lambda : (\lambda, \tilde{r}, \tilde{\mu}) \in \mathcal{C}\} =\,]0, +\infty[$,

(b″) or $\inf \{\, \min \tilde{r} : (\lambda, \tilde{r}, \tilde{\mu}) \in \mathcal{C}, \ 0 < \lambda < \lambda_0\} = -1$.

We define

$$\mathcal{K} := \Big\{ \big(\lambda(1 + \tilde{r}), \, \max\{\tilde{\mu}, 0\}\big) : (\lambda, \tilde{r}, \tilde{\mu}) \in \mathcal{C} \Big\},$$

which is a connected set of solutions (r, μ) of (16). Moreover, since $\tilde{r}(0) = 0$ for any $(\lambda, \tilde{r}, \tilde{\mu}) \in \mathcal{C}$, then (31) implies that the interval $\{\max r : (r, \mu) \in \mathcal{K}\}$ is unbounded, and, by Lemma 2.1, also the interval $\{\min r : (r, \mu) \in \mathcal{K}\}$ is unbounded, showing our assertion $(\mathcal{K}_1)$.

It remains to show the alternative $(\mathcal{K}_{2a})$-$(\mathcal{K}_{2b})$. We shall assume that none of these possibilities hold to arrive to a contradiction with the disjunctive (a)-(b')-(b'').

Indeed, if $(\mathcal{K}_{2a})$ does not hold, there must exist some lower bound $k > 0$ such that

$$\min r \geq k, \qquad (r, \mu) \in \mathcal{K}, \tag{32}$$

and in particular

$$\lambda \geq k \qquad (\lambda, \tilde{r}, \tilde{\mu}) \in \mathcal{C}, \tag{33}$$

contradicting (b'). Another consequence of (32) is that

$$\tilde{r} \geq \frac{k}{\lambda} - 1 > \frac{k}{\lambda_0} - 1 > -1, \qquad (\lambda, \tilde{r}, \tilde{\mu}) \in \mathcal{C}, \ \ 0 < \lambda < \lambda_0,$$

contradicting (b'').

On the other hand, if $(\mathcal{K}_{2b})$ does not hold either, then $\mu > 0$ for all $(r, \mu) \in \mathcal{K}$, and

$$\mathcal{K} = \left\{ \left(\lambda(1 + \tilde{r}), \mu \right) : (\lambda, \tilde{r}, \mu) \in \mathcal{C} \right\}. \tag{34}$$

Now, choose some element $(\lambda, \tilde{r}, \mu) \in \mathcal{C}$ with $0 < \lambda < \lambda_0$. We let $r := \lambda(1 + \tilde{r})$; by (34), $(r, \mu) \in \mathcal{K}$ and (r, μ) is a solution of (16) with $\min r < \lambda_0$. But, by (32), also $\min r \geq k$, and Corollary 2.3 implies the existence of some $M > 0$ (depending on k and λ_0 but not on r or μ), such that

$$\lambda(1 + \max \tilde{r}) \leq M \text{ and } |\mu| \leq M,$$

and, in view of (33),

$$k(1 + \max \tilde{r}) \leq M \text{ and } |\mu| \leq M,$$

contradicting (a). This contradiction concludes the proof.

$\square$

We are going to prove now Theorem 1.7. In order to do so, we first establish an elementary result on a priori bounds for the first derivative of T-radially periodic functions whose second derivative is controlled:

Lemma 5.2. *Let the $W_{loc}^{2,1}(\mathbb{R})$ function $x : \mathbb{R} \to \mathbb{R}^2$ be T-radially periodic. If $c \in L^1(\mathbb{R}/T\mathbb{Z})$ verifies $|\ddot{x}(t)| \leq c(t)$ for a.e. $t \in \mathbb{R}$, then*

$$\max |\dot{x}| \leq 2 \min |x| + T\|c\|_{L^1}, \qquad \max |x| - \min |x| \leq \left(2 \min |x| + T\|c\|_{L^1} \right) T.$$

Proof. Since both $|x|$ and $|\dot{x}|$ are T-periodic, we may find points $t_0, s_0 \in \mathbb{R}$ such that

$$|\dot{x}(t_0)| = \max |\dot{x}|, \qquad |x(s_0)| = \min |x|, \qquad s_0 \leq t_0 \leq s_0 + T.$$

On the other hand, remembering that $|\ddot{x}| \leq c$,

$$|\dot{x}(s) - \dot{x}(t_0)| \leq \|c\|_{L^1}, \qquad s \in [s_0, s_0 + T]. \tag{35}$$

We integrate in both sides of this inequality, to obtain

$$\left| x(s_0 + T) - x(s_0) - \dot{x}(t_0) \right| = \left| \int_{s_0}^{s_0 + T} \dot{x}(s) ds - \dot{x}(t_0) \right| \leq \int_{s_0}^{s_0 + T} \left| \dot{x}(s) - \dot{x}(t_0) \right| dt \leq T\|c\|_{L^1},$$

and thus, $\max |\dot{x}| = |\dot{x}(t_0)| \leq |x(s_0 + T)| + |x(s_0)| + T\|c\|_{L^1} = 2 \min |x| + T\|c\|_{L^1}$ the first half of the statement. We choose now $s_1 \in [s_0, s_0 + T]$ such that $|x(s_1)| = \max |x|$ and observe that

$$\max |x| - \min |x| \leq \left| x(s_1) - x(s_0) \right| = \left| \int_{s_0}^{s_1} \dot{x}(s) dt \right| \leq \int_{s_0}^{s_1} |\dot{x}(s)| dt \leq \left(2 \min |x| + T\|c\|_{L^1} \right) T,$$

concluding the proof.

$\square$

Proof of Theorem 1.7. Using Proposition 1.6, choose some $\lambda_2 > 0$ such that any T-radially periodic solution $x = r(\cos\theta, \sin\theta)$ of (3) with $\max r \geq \lambda_2$ verifies

$$\dot\theta \neq 0 \text{ on } \mathbb{R}, \qquad\qquad \min r \geq 1. \tag{36}$$

Combine now Lemma 5.1 with Corollary 4.3 and use similar arguments to those carried out in the proof of Theorem 1.5 to obtain the existence of a connected set $\mathcal{C}^*$ of T-radially periodic solutions $x = r(\cos\theta, \sin\theta)$ of (3) such that

$$\{r(0) : r(\cos\theta, \sin\theta) \in \mathcal{C}^*\} = [\lambda^*, +\infty[, \tag{37}$$

for some $\lambda^* \geq \lambda_2$. Then, $\dot\theta \neq 0$ for any $x = r(\cos\theta, \sin\theta) \in \mathcal{C}^*$, and, after possibly replacing $\mathcal{C}^*$ by

$$\tilde{\mathcal{C}}^* = \left\{ r\big(\cos(-\theta), \sin(-\theta)\big) : r(\cos\theta, \sin\theta) \in \mathcal{C}^* \right\},$$

we may assume that

$$\dot\theta > 0 \text{ for any } r(\cos\theta, \sin\theta) \in \mathcal{C}^*. \tag{38}$$

We consider now the set

$$I := \{\operatorname{rot}(x) : x \in \mathcal{C}^*\},$$

which is an interval, because $\mathcal{C}$ is connected and rot is continuous. From (38) we see that $I \subset \,]0, +\infty[$. On the other hand, (37) and Proposition 1.6 imply that

$$I \supset \,]0, \bar\omega[$$

for some $\bar\omega > 0$. With other words, for any $0 < \omega < \bar\omega$ there exists some element $x_\omega \in \mathcal{C}^*$ such that $\operatorname{rot}(x_\omega) = \omega$. The first part of (15) follows now from (38).

To conclude the proof, it will suffice to show that $\lim_{\omega \to 0} \max |x_\omega| = +\infty$, since (14) and the second part of (15) will then follow from Proposition 1.6. Thus, we use a contradiction argument and assume that there is some sequence $\omega_n \to 0$ such that $\max |x_{\omega_n}|$ is bounded, say, by some constant $K > 0$. Taking into account the second part of (36) we observe that

$$1 \leq |x_{\omega_n}| \leq K, \qquad\qquad n \in \mathbb{N}. \tag{39}$$

But f was assumed to be L^1-Carathéodory, and then, there is some function $c \in L^1(\mathbb{R}/\mathbb{Z})$ such that $|f(t, r)| \leq c(t)$ for a.e. $t \in \mathbb{R}/\mathbb{Z}$ and every $r \in [1, K]$. From the equation (3) we conclude that

$$|\ddot{x}_{\omega_n}(t)| \leq c(t) \qquad \text{for a.e. } t \in \mathbb{R} \text{ and all } n \in \mathbb{N}. \tag{40}$$

Inequality (39) states that the sequence $\{x_{\omega_n}\}$ is uniformly bounded, and Lemma 5.2 states that also $\{\dot{x}_{\omega_n}\}$ is uniformly bounded. In particular, $\{x_{\omega_n}\}$ is equicontinuous, and, in view of (40), also $\{\dot{x}_{\omega_n}\}$ is equicontinuous. Thus, Ascoli-Arzelà's Theorem applies and states that, after possibly passing to a subsequence, one may assume that there exists some C^1 function $x_* : \mathbb{R} \to \mathbb{R}^2$ such that $\{x_{\omega_n}\} \to x_*$ and $\{\dot{x}_{\omega_n}\} \to \dot{x}_*$ uniformly on compact sets. It immediately follows that x_* is T-radially periodic. On the other hand, $\max |x_{\omega_n}| \geq \lambda_2$ for

any n, and we deduce that $\max |x_*| \geq \lambda_2$. Finally rewriting our equation (3) in its integral form (for $x = x_{\omega_n}$)

$$\dot{x}_{\omega_n}(t) = \dot{x}_{\omega_n}(0) + \int_0^t f(s, |x_{\omega_n}(s)|) \frac{x_{\omega_n}(s)}{|x_{\omega_n}(s)|} ds, \qquad t \in \mathbb{R},$$

and taking limits as $n \to \infty$, we see that also x_* is a solution of (3). But then, (36) implies that $\mathrm{rot}\, x_* \neq 0$, contradicting the continuity of rot. It concludes the proof.

$\square$

6   Examples, counterexamples...

In this last Section we construct the examples announced in Propositions 1.2 and 1.4. We start with Proposition 1.2, and, with this aim, we consider the following increasing sequence of positive, 2π-periodic functions:

$$r_n(t) := n + \frac{1}{4}\sin t, \qquad\qquad n \geq 1, \ \ t \in \mathbb{R}.$$

Our proof for this result will be based on the following elementary result of real analysis:

Lemma 6.1. *There exists a bounded and $\mathcal{C}^\infty$ function $f : (\mathbb{R}/2\pi\mathbb{Z}) \times\,]0, +\infty[\to \mathbb{R}$ such that $f(t, r_n(t)) = -1$ for any $n \in \mathbb{N}$ and*

$$\int_0^{2\pi} f(t, r)dt > 0, \qquad\qquad r \in\,]0, +\infty[\,.$$

We shall not give a detailed proof of Lemma 6.1 here, as it may be considered an exercise; we just point out that f may be chosen with the form $f(t, r) = u(r - (1/4)\sin t)$ for some suitable function $u : \mathbb{R} \to \mathbb{R}$.

We remark that the condition above on the positivity of the integrals over horizontal lines may be improved. Indeed, one easily checks that given some constant $M > 0$, the function f can be chosen so that $\int_0^{2\pi} f(t, r)dt \geq M$ for any $r > 0$. In this way, also Proposition 1.2 could be sharpened; given $M > 0$ the function f may be chosen so that $\int_0^T f(t, r)dt \geq M$ for any $r > 0$.

Lemma 6.1 will lead us to Proposition 1.2. We see it below:

Proof of Proposition 1.2. We choose f as given by Lemma 6.1 and observe that the sequence $\{r_n\}_n$ is made of strict upper solutions for the equation $-\ddot{r} = f(t, r)$. In other words,

$$-\ddot{r}_n(t) > f(t, r_n(t)), \qquad t \in \mathbb{R}, \ n \in \mathbb{N}. \tag{41}$$

Let now $\mathcal{C}$ be a connected set of 2π-radially periodic solutions of (3). Using the results of Section 3 we see that $\mathcal{K} := \Psi(\mathcal{C})$ is a connected set of solutions (r, μ) of (16). We recall that our aim is to show that the interval

$$\Big\{ |x(t)| : t \in \mathbb{R}, \ x \in \mathcal{C} \Big\} = \Big\{ r(t) : t \in \mathbb{R}, \ (r, \mu) \in \mathcal{K} \Big\}$$

is bounded. With this aim we choose some element $(r_\blacktriangle, \mu_\blacktriangle) \in \mathcal{K}$ and fix some $n_0 \in \mathbb{N}$ such that $r_{n_0}(t) > r_\blacktriangle(t)$ for any $t \in [0, T]$. We consider the sets

$$A := \left\{ (r, \mu) \in \mathcal{K} : r(t) < r_{n_0}(t) \text{ for all } t \in \mathbb{R} \right\},$$

$$B := \left\{ (r, \mu) \in \mathcal{K} : r(t_*) > r_{n_0}(t_*) \text{ for some } t_* \in \mathbb{R} \right\}.$$

Observe that A and B are open in $\mathcal{K}$. Moreover, we claim that $A \cup B = \mathcal{K}$. Indeed, the contrary would mean the existence of some element $(r, \mu) \in \mathcal{K}$ such that

$$r \le r_{n_0} \text{ on } \mathbb{R}, \qquad\qquad r(t_0) = r_{n_0}(t_0) \text{ for some } t_0 \in \mathbb{R},$$

which is not possible because, in view of (17), r is a lower solution of $-\ddot{r} = f(t, r)$ while, as observed in (41), r_{n_0} is an strict upper solution.

Since $\mathcal{K}$ is connected, one of these sets must be empty. But $(r_\blacktriangle, \mu_\blacktriangle) \in A$, and we deduce that $B = \emptyset$. It concludes the proof. $\qquad\square$

We undertake now the proof of Proposition 1.4. This result is divided in two parts, the first one concerning the case $0 \le \gamma < 1$, and the second about the case $\gamma < 0$, and they will be treated separately.

The first part will follow from Theorem 1.5. We only have to check that the one-dimensional equation

$$-\ddot{r} = c(t)r^\gamma, \qquad r > 0, \tag{42}$$

does not have T-periodic solutions if $0 < \gamma < 1$ and $c \in C(\mathbb{R}/T\mathbb{Z})$ has positive mean. Our strategy will consist in building a continuous family of lower solutions for this equation. Well, not all of them exactly lower solutions, but something close to that:

Lemma 6.2. *Assume that $0 \le \gamma < 1$ and $\int_0^T c(t)dt > 0$. Then, there is a continuous function $\psi : (\mathbb{R}/T\mathbb{Z}) \times [0, +\infty[\to \mathbb{R}, \ (t, p) \mapsto \psi(t, p)$, verifying*

(i) $\psi(t, 0) = 0$ and $\lim_{p \to +\infty} \psi(t, p) = +\infty$ uniformly with respect to $t \in \mathbb{R}/T\mathbb{Z}$.

(ii) ψ is a C^2 mapping on the set $\Omega := \{(t, p) \in (\mathbb{R}/T\mathbb{Z}) \times [0, +\infty[: \psi(t, p) > 0\}$; moreover,

$$-\partial_{tt}\psi(t, p) - c(t)\psi(t, p)^\gamma < 0, \qquad (t, p) \in \Omega.$$

Proof. Let $\bar{c} = (1/T) \int_0^T c(t)dt > 0$ and $\tilde{c}(t) = c(t) - \bar{c}$. Then, $\tilde{c}$ has zero mean value and it is possible to find some T-periodic function $H : \mathbb{R} \to \mathbb{R}$ of class C^2 such that $\ddot{H} = \tilde{c}$. This function H is determined up to constants, and then, we may choose it so that $H(t) > 0$ for any $t \in \mathbb{R}$. We define

$$\psi(t, p) := \left[\max \left\{ 0, p - (1 - \gamma)H(t) \right\} \right]^{\frac{1}{1-\gamma}}, \qquad (t, p) \in (\mathbb{R}/T\mathbb{Z}) \times [0, +\infty[.$$

Now, *(i)* is immediate. On the other hand, one easily checks that ψ solves the first-order equation

$$-\partial_t\psi(t, p) - \dot{H}(t)\psi(t, p)^\gamma = 0, \qquad (t, p) \in \Omega,$$

from where *(ii)* follows easily. $\qquad\square$

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Proof of Proposition 1.4 for $0 \leq \gamma < 1$. Using a contradiction argument, assume that $r = r(t)$ were a T-periodic solution of (42) and define

$$p_* := \min \left\{ p > 0 \text{ such that the graphs of } \psi(\cdot, p) \text{ and } r \text{ intersect} \right\}.$$

Then, $\psi(t, p_*) \leq r(t)$ for every $t \in \mathbb{R}$ and, for some t_0, one has the equality $\psi(t_0, p_*) = r(t_0)$, which is contradictory with the fact that r and $\psi(\cdot, p_*)$ are, respectively, a solution and a strict lower solution of (42). It completes the proof. $\square$

We show now the second part of Proposition 1.4. With this aim, we start by choosing some $\gamma < 0$ which will be henceforth fixed and observe that any solution x of the equation

$$\ddot{x} = -c(t)|x|^\gamma \frac{x}{|x|} \,, \tag{43}$$

with $\min |x| = 1$ verifies

$$|\ddot{x}(t)| \leq |c(t)| \qquad \text{for a.e. } t \in \mathbb{R} \,.$$

But then, Lemma 5.2 gives a priori bounds for x and $|\dot{x}|$, which, in our case, read:

$$|x(t)|, \ |\dot{x}(t)| \leq 3 + \|c\|_{L^1} \,, \qquad t \in \mathbb{R} \,. \tag{44}$$

At this point we introduce a contradiction argument, and, from now on, we shall assume that

[**A**] for any $c \in C(\mathbb{R}/\mathbb{Z})$ with $\int_0^1 c(t)dt > 0$, equation (43) has some 1-radially periodic solution x with $\min |x| = 1$.

In this situation, the bounds (44) may be combined with an approximation argument to deduce that, in fact, (43) must have 1-radially periodic solutions for a wider class of functions $c = c(t)$:

Lemma 6.3. *Assume* [**A**]. *Then, equation (43) has some* 1-*radially periodic solution x with* $\min |x| = 1$ *for any $c \in L^1(\mathbb{R}/\mathbb{Z})$ with $\int_0^1 c(t)dt \geq 0$, .*

Proof. Fix $c \in L^1(\mathbb{R}/\mathbb{Z})$ with $\int_0^1 c(t)dt \geq 0$ and choose some sequence $\{c_n\} \subset C(\mathbb{R}/\mathbb{Z})$ with $\int_0^1 c_n(t)dt > 0$ such that $\{c_n\} \to c$ in the L^1 sense. Correspondingly, for each $n \in \mathbb{N}$ we find an associated 1-radially periodic solution x_n of (43) with $c = c_n$ such that $\min |x_n| = 1$. But $\{c_n\}$ is bounded in $L^1(\mathbb{R}/\mathbb{Z})$, and (44) states that $\{x_n\}$ and $\{\dot{x}_n\}$ are uniformly bounded on $\mathbb{R}$. In particular, $\{x_n\}$ is equicontinuous, and using Ascoli-Arzelà Theorem, we may assume, after possibly passing to a subsequence, that $\{x_n\}$ converges uniformly on compact sets to some continuous function $x_* : \mathbb{R} \to \mathbb{R}^2$ with $\min |x_*| = 1$. It is not restrictive to assume that, moreover, $\{\dot{x}_n(0)\}$ converges to some number $\beta \in \mathbb{R}$. Since x_n is 1-radially periodic for each n, we conclude that x_* is also 1-radially periodic. Finally, from the integral form of our equation

$$\dot{x}_n(t) = \dot{x}_n(0) - \int_0^t c_n(t)|x_n(t)|^\gamma \frac{x_n(t)}{|x_n(t)|} \, dt \,, \qquad n \in \mathbb{N} \,,$$

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it follows that $\dot{x}_n$ converges uniformly on compact sets to the function

$$y(t) := \beta - \int_0^t c(t)|x_*(t)|^\gamma \frac{x_*(t)}{|x_*(t)|}\, dt\,, \qquad t \in \mathbb{R}\,,$$

and then, $y = \dot{x}_*$. In particular, x_* solves (43) and the proof is complete.

$\square$

To prepare the next step we define, for each $0 < \delta < 1/2$, the function $c_\delta \in L^1(\mathbb{R}/\mathbb{Z})$ by

$$c_\delta(t) := \begin{cases} \dfrac{1 - 2\delta}{2\delta} & \text{if } -\delta \le t \le \delta\,, \\[2mm] -1 & \text{if } \delta < t < 1 - \delta\,, \end{cases}$$

and extended periodically. Observe that $\int_0^1 c_\delta(t)dt = 0$ and consequently, Lemma 6.3 states the existence of some 1-radially periodic solution x_δ of the equation

$$\ddot{x}_\delta = -c_\delta(t)|x_\delta|^\gamma \frac{x_\delta}{|x_\delta|}\,,$$

with $\min |x_\delta| = 1$. We define $r_\delta := |x_\delta|$, which, remembering (17), verifies

$$\ddot{r}_\delta(t) \ge -c_\delta(t)r_\delta(t)^\gamma\,.$$

and, in particular, we have the inequality

$$\ddot{r}_\delta(t) \ge r_\delta(t)^\gamma > 0\,, \qquad\qquad t \in]\delta, 1 - \delta[\,, \tag{45}$$

so that r_δ is convex on $[\delta, 1 - \delta]$. The next step is devoted to show that, if δ is small, then r_δ has a 'U-like shape' on this interval.

Lemma 6.4. *If $\delta \in]0, 1/2[$ is small enough, then $\dot{r}_\delta(\delta) < 0 < \dot{r}_\delta(1 - \delta)$.*

Proof. Choose $0 < \delta < 1/2$ small enough so that $20\,\delta < 5^\gamma(1 - 2\delta)^2$. We shall see that this δ satisfies the conditions above.

With this aim, we start by observing that, for any $0 < \delta < 1/2$, $\|c_\delta\|_{L^1} < 2$, and, by Lemma 5.2, we know that both $r_\delta = |x_\delta|$ and $|\dot{x}_\delta|$ are bounded by 5. When combined with (45), the first fact gives

$$\ddot{r}_\delta(t) \ge 5^\gamma\,, \qquad\qquad t \in]\delta, 1 - \delta[\,, \tag{46}$$

On the other hand, the second fact implies that $|\dot{r}_\delta(t)| \le |\dot{x}_\delta(t)| \le 5$ and we get

$$|r_\delta(1 - \delta) - r_\delta(\delta)| = |r_\delta(-\delta) - r_\delta(\delta)| \le 10\,\delta\,. \tag{47}$$

We use now a contradiction argument, and assume that, for instance, $\dot{r}_\delta(\delta) \ge 0$. Then, by (46), $\dot{r}(t) \ge 5^\gamma(t - \delta)$ for any $t \in [\delta, 1 - \delta]$, and we obtain that

$$r(1 - \delta) - r(\delta) \ge 5^\gamma(1 - 2\delta)^2/2\,,$$

and, in view of (47), $10\delta \ge 5^\gamma(1-2\delta)^2/2$. This contradicts our choice of δ and thus, $\dot{r}_\delta(\delta) < 0$. Similarly, $\dot{r}_\delta(1 - \delta) < 0$.

$\square$

The end of the proof of Proposition 1.4. The combination of (45) and Lemma 6.4 implies that, if δ is small enough, then r_δ has an unique critical point $t_0 \in [\delta, 1-\delta]$, which is, by the way, a local minimum. But r_δ is 1-periodic, and thus, it must attain its maximum somewhere on $]1-\delta, 1+\delta[$. In particular, the set

$$\{t \in\,]1-\delta, 1+\delta[:\, \dot{r}_\delta(t) = 0\}$$

is nonempty, and we call $t_\pm$ to the minimum and the maximum of this set. Consequently,

$$1 - \delta < t_- \leq t_+ < 1 + \delta\,, \qquad\qquad \dot{r}_\delta(t_-) = \dot{r}_\delta(t_+) = 0\,.$$
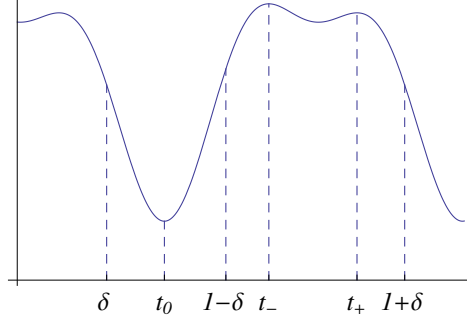

Figure 4: The graph of r_δ

Now,

$$0 = \int_{t_0}^{t_-} \ddot{r}_\delta(s)ds = \int_{t_0}^{1-\delta} \ddot{r}_\delta(s)ds + \int_{1-\delta}^{t_-} \ddot{r}_\delta(s)ds \geq \int_{t_0}^{1-\delta} r_\delta(s)^\gamma ds - \int_{1-\delta}^{t_-} \left(\frac{1-2\delta}{2\delta} \right) r_\delta(s)^\gamma ds\,.$$

Observe now that, if $s \in [t_0, 1-\delta[$, then $r(s)^\gamma > r(1-\delta)^\gamma$, while, if $s \in\,]1-\delta, t_-]$, then $r(s)^\gamma < r(1-\delta)^\gamma$. It follows that

$$0 > r(1-\delta)^\gamma(1-\delta-t_0) - \left(\frac{1-2\delta}{2\delta} \right) r(1-\delta)^\gamma(t_- - 1 + \delta)\,,$$

or, what is the same,

$$\frac{t_- - (1-\delta)}{2\delta} > \frac{1-\delta-t_0}{1-2\delta}\,. \tag{48}$$

On the other hand, a similar reasoning starting from the equality

$$0 = \int_{t_+}^{1+t_0} \ddot{r}_\delta(s)ds = \int_{t_+}^{1+\delta} \ddot{r}_\delta(s)ds + \int_{1+\delta}^{1+t_0} \ddot{r}_\delta(s)ds\,,$$

leads us to the inequality

$$\frac{1+\delta-t_+}{2\delta} > \frac{t_0-\delta}{1-2\delta}\,,$$

which, when added to (48) yields $t_- > t_+$, a contradiction. The contradiction comes from having assumed [**A**] and concludes the proof.

$\square$

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