

Generalized Lenard chains and Separation of Variables

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It is known that integrability properties of soliton equations follow from the existence of Lenard chains of symmetries, constructed by means of a Nijenhuis (i.e. hereditary) operator. In this paper, we will discuss the role of such an operator in the framework of separation of variables (SoV) for the classical Hamilton-Jacobi equation. We will show that the existence of generalized Lenard chains, for a given finite-dimensional Hamiltonian system, assures that the corresponding Hamilton-Jacobi equation can be solved by SoV in a set of special coordinates, naturally defined by the Nijenhuis operator. Finally, we will discuss some examples of generalized Lenard chains and, in particular, of the so-called quasi-bi-Hamiltonian systems.

Keywords: Bi-Hamiltonian structures, Quasi-bi-Hamiltonian systems, Separation of Variables

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1. Introduction

One of the most effective method to solve Hamiltonian systems is to find a complete integral of the corresponding Hamilton-Jacobi equation. Let us briefly recall the geometric setting of Hamiltonian dynamics. Let (M, ω) be a symplectic manifold, i.e. a $2n$ -dimensional manifold endowed with a non degenerate closed 2-form ω , said to be a symplectic form. Such a geometrical structure selects a privileged dynamics on M , the one given by Hamiltonian vector fields defined by

$$i_{X_H}\omega = -dH \tag{1.1}$$

(i_{X_H} denotes the contraction operator w.r.t. the vector field X_H) or, equivalently,

$$X_H = (\omega^b)^{-1}dH, \quad (1.2)$$

where $\omega^b : TM \rightarrow T^*M$ denotes the fiber bundles isomorphism induced by ω . The function H is said to be the Hamiltonian function of the vector field X_H . A symplectic form acting on vector fields is equivalent to a non degenerate Poisson bracket defined as

$$\{f, g\} := \omega(X_f, X_g) = \langle df, X_g \rangle, \quad (1.3)$$

i.e. as a skew-symmetric composition law on the ring $C^\infty(M)$ satisfying

$$\{f, gh\} = \{f, g\}h + \{f, h\}g \quad (\text{Leibniz rule}) \quad (1.4)$$

$$0 = \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \quad (\text{Jacobi identity}) \quad (1.5)$$

$$\{f, g\} = 0 \quad \forall f \implies dg = 0 \quad (\text{nondegenerate}) \quad (1.6)$$

A system of coordinates $(\mathbf{q}, \mathbf{p}) := (q_1, \dots, q_n, p_1, \dots, p_n)$ satisfying $\{q_i, p_j\} = \delta_{ij}$, $\{q_i, q_j\} = \{p_i, p_j\} = 0$ is said to be a system of canonical or Darboux coordinates. In such coordinates the (time-independent) Hamilton-Jacobi equation corresponding to a Hamiltonian vector field X_H reads

$$H(q_1, \dots, q_n, \frac{\partial W}{\partial q_1}, \dots, \frac{\partial W}{\partial q_n}) = E. \quad (1.7)$$

A solution $W(\mathbf{q}; a_1, \dots, a_n)$, depending on \mathbf{q} and on n parameters $\{a_i\}_{1 \leq i \leq n}$ in such a way that $\det[\frac{\partial^2 W}{\partial q_i \partial a_j}] \neq 0$, is said to be a complete integral of (1.7). Most of the cases in which a complete integral is found, occur when W is an additively separated function of the coordinates q_i

$$W(\mathbf{q}; a_1, \dots, a_n) = \sum_{i=1}^n W_i(q_i; a_1, \dots, a_n). \quad (1.8)$$

In such a case H is said to be separable and the coordinates (\mathbf{q}, \mathbf{p}) are said to be separated coordinates for H , in order to stress that the possibility to find a separated complete integral of (1.7) depends on the choice of the coordinates.

One of the main issues in the classical theory of separation of variables (SoV) is to find criteria to decide if a given Hamiltonian function H is separable in an assigned system of canonical coordinates and in the affirmative case to find a separated complete integral of the Hamilton-Jacobi equation. In this regard, we recall the test by Levi-Civita (1904). It states that H is separable if and only if the following conditions

$$0 = \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial q_j} \frac{\partial^2 H}{\partial p_i \partial p_j} + \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial q_i \partial q_j} - \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial H}{\partial p_i} \frac{\partial^2 H}{\partial p_j \partial q_i} \quad (1.9)$$

($1 \leq i < j \leq n$), are satisfied.

As a separable Hamiltonian is also integrable after Liouville, one can start with an integrable system, i.e. with a set of n independent Hamiltonian functions in involution w.r.t. the Poisson bracket (1.3). In this framework, we have to quote the following theorem in the tradition of the *Italian school*.

Theorem 1.1. (Benenti, 1980) *The Hamiltonian functions $\{H_i\}_{1 \leq i \leq n}$ are separable in a set of canonical coordinates (\mathbf{q}, \mathbf{p}) if and only if they are in separable involution, i.e. if they satisfy*

$$\{H_i, H_j\}_{|k} = \frac{\partial H_i}{\partial q_k} \frac{\partial H_j}{\partial p_k} - \frac{\partial H_i}{\partial p_k} \frac{\partial H_j}{\partial q_k} = 0, \quad (1.10)$$

($1 \leq k \leq n$), where no summation over k is understood.

However, such a theorem as well as the Levi–Civita test are not constructive, since they do not help to find a complete integral of the Hamilton–Jacobi equation (1.7). In contrast, a constructive definition of SoV was given by Jacobi and later rediscovered by Sklyanin (1995) in the framework of Lax systems.

Definition 1.2. *The Hamiltonian functions $\{H_i\}_{1 \leq i \leq n}$ are separable in a set of canonical coordinates (\mathbf{q}, \mathbf{p}) if there exist n equations*

$$\Phi_i(q_i, p_i; H_1, \dots, H_n) = 0 \quad i = 1, \dots, n \quad (1.11)$$

such that $\det\left[\frac{\partial \Phi_i}{\partial H_j}\right] \neq 0$. They are said to be Jacobi–Sklyanin separation equations for $\{H_i\}_{1 \leq i \leq n}$.

However, the three above–mentioned criteria of separability are not intrinsic since they require to know explicitly the coordinates (\mathbf{q}, \mathbf{p}) in order to be applied. Recently, a new geometric approach to SoV has been developed, based on the bi–Hamiltonian theory (Morosi & Tondo 1997, Falqui & Pedroni 2003). It has succeeded in giving intrinsic and constructive criteria of separability and has connected the classical theory of SoV with the modern theory by Sklyanin. This survey article is settled on the bi–Hamiltonian approach and can be seen as complementary to the article of Falqui (this volume). It is organized as follows. In §2 we will introduce the phase spaces where the bi–Hamiltonian theory of SoV takes place. They are manifolds M endowed with two geometric structures satisfying two suitable compatibility conditions. Such structures are a symplectic form ω and a Nijenuis (or hereditary) operator N acting on the tangent bundle of M . For this reason such manifolds have been called ωN manifolds. Whilst the symplectic form defines the algebra of Hamiltonian vector fields, the Nijenuis operator defines sets of distinguished coordinates that are separated coordinates for a special class of Hamiltonian vector fields, those belonging to generalized Lenard chains. We will introduce these chains step by step, starting in §3, with classical Lenard chains and with bi–Hamiltonian (BH) systems, widely known in soliton theory. Then, in §4 we will pass to Fröbenius chains and to quasi–bi–Hamiltonian systems (QBH), illustrated with the new and relevant examples of the Jacobi–Rosochatius system and of the Kowalesvski top. Finally, we will present the concept of (true) generalized Lenard chains with the examples of the Lagrange top and of the t_5 -Boussinesq stationary flow.

2. Bi–Hamiltonian manifolds and ωN manifolds

Generally, a Poisson bracket (see, e.g., Vaisman 1996) is defined as a skew–symmetric composition law on $C^\infty(M)$ which satisfies only (1.4), (1.5). Equivalently, it is defined by a Poisson bi-vector field, i.e. a skew–symmetric linear map $P : T^*M \mapsto TM$

with vanishing Schouten bracket

$$0 = [P, P]_S(\alpha, \beta) := \mathcal{L}_{P\beta}(P)\alpha + P(i_{P\alpha}d\beta) \quad \alpha, \beta \in T^*M, \quad (2.1)$$

by means of

$$\{f, g\} := \langle df, Pdg \rangle \quad (2.2)$$

(\mathcal{L} denotes the Lie derivatives). In the special case of symplectic manifolds, $P := (\omega^\flat)^{-1}$ is a Poisson bi-vector. Generalizing (1.2) the vector field $X_g := Pdg$ is said to be the Hamiltonian vector field with Hamiltonian function g . It is an immediate consequence of (2.1) that each Hamiltonian vector field is a symmetry of P

$$\mathcal{L}_{X_g}P = 0 \quad \forall g \in C^\infty(M). \quad (2.3)$$

Bi-Hamiltonian manifolds were introduced by Magri (1978) as models of phase space for soliton equations.

Definition 2.1. *A bi-Hamiltonian manifold (M, P_0, P_1) is a manifold M endowed with two Poisson bi-vectors such that*

$$0 = 2[P_0, P_1]_S(\alpha, \beta) := \mathcal{L}_{P_0\beta}(P_1)\alpha + P_1(i_{P_0\alpha}d\beta) + \mathcal{L}_{P_1\beta}(P_0)\alpha + P_0(i_{P_1\alpha}d\beta) \quad (2.4)$$

Such a condition assures that the linear combination $P_1 - \lambda P_0$ is a Poisson pencil, i.e. it is a Poisson bi-vector for each $\lambda \in \mathbb{C}$, and therefore the corresponding bracket $\{\cdot, \cdot\}_{P_1 - \lambda P_0}$ is a pencil of Poisson bracket. Condition (2.4) is known as the compatibility condition between P_0 and P_1 .

What happens if one of the Poisson tensor, say P_0 , is invertible, and its inverse is a symplectic operator $\omega^\flat := P_0^{-1}$? In this case, it is remarkable that the bi-Hamiltonian manifold M turns out to be an ωN manifold (see Magri & Morosi 1984) because the composed operator $N := P_1 P_0^{-1}$ results a Nijenhuis (or hereditary) operator compatible with the symplectic form ω .

Definition 2.2. *A linear map $N : TM \mapsto TM$ which satisfy the following quadratic differential condition $\forall X, Y \in TM$*

$$[NX, NY] - N([X, NY] + [NX, Y]) + N^2[X, Y] = 0 \quad (2.5)$$

is said to be a Nijenhuis (or hereditary) operator.

Such a condition implies that the distribution of the eigenvectors of N is integrable. Moreover, it allows to introduce a new derivation d_N (Kosmann-Schwarzbach, Y. & Magri, F. 1990) on the Grassman algebra of k -forms satisfying

$$dd_N = -d_N d \quad (2.6)$$

$$d_N^2 = 0. \quad (2.7)$$

In a local chart (x^1, \dots, x^{2n}) such a derivation acts on a function f and on a 1-form $\alpha = \sum_{i=1}^{2n} \alpha_i dx^i$ respectively as

$$d_N f = \sum_{i=1}^{2n} \frac{\partial f}{\partial x^i} (N^*)^i_j dx^j \quad (2.8)$$

$$d_N \alpha = \sum_{i=1}^{2n} d_N \alpha_i \wedge dx^i - \sum_{i=1}^{2n} \alpha_i dd_N x^i \quad (2.9)$$

(see Magri 2003).

Proposition 2.3. (Magri & Morosi 1984) *Let (M, ω, N) be an ωN manifold. The Poisson bivector $P_0 := (\omega^b)^{-1}$ and the Nijenhuis operator N satisfy the following compatibility condition*

$$\mathcal{L}_{P_0\alpha}(N)Y = P_0(i_Y d_N\alpha - i_{NY}d\alpha) \quad \forall \alpha \in T^*M, \forall Y \in TM \quad (2.10)$$

From now on, we make some assumption on the spectral properties of N . Actually, we suppose that at each point (or in a dense open subset) of M , the Nijenhuis tensor field N admits n distinct eigenvalues λ_i ($i = 1, \dots, n$) of algebraic multiplicity equal to *two*. As in a generic ωN manifold the eigenspaces of N are even-dimensional, from the above assumption it follows that N (and the adjoint tensor N^*) can be put in diagonal form and therefore their minimal and characteristic polynomials are respectively

$$m_N(z) = \prod_{i=1}^n (z - \lambda_i) = z^n - c_1 z^{n-1} - c_2 z^{n-2} - \dots - c_n \quad (2.11)$$

$$\Delta_N(z) = (m_N(z))^2 \quad (2.12)$$

Hence, N^* (N) satisfy the following equation to be used in the sequel

$$N^{*n} = c_1 N^{*n-1} + \dots + c_{n-1} N^* + c_n \mathbb{I}_{2n}, \quad (2.13)$$

where \mathbb{I}_{2n} denotes the identity operator acting on T^*M .

Remarkable properties of the eigenvalues of N are that they are in involution w.r.t. the Poisson bracket (1.3) and that their differentials are eigenvectors of N^* (see, e.g., Falqui this volume)

$$\{\lambda_i, \lambda_j\} = 0 \quad , \quad N^* d\lambda_i = \lambda_i d\lambda_i \quad . \quad (2.14)$$

Definition 2.4. *Let p be a point of an ωN manifold. It will be called a regular point if the eigenvalues of N are functionally independent in p . An open set $U \subseteq M$ will be called regular if each point of U is a regular point.*

Proposition 2.5. (Magri & Marsico 1996) *Let (M, ω, N) be an ωN manifold. In each open neighborhood of a regular point, the n functions λ_i can be completed by quadratures with n functions μ_i such that the chart $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a Darboux chart for ω and moreover*

$$N^* d\mu_i = \lambda_i d\mu_i \quad . \quad (2.15)$$

As in the Darboux chart $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ the Nijenhuis tensor N takes the diagonal form

$$N = \begin{bmatrix} \boldsymbol{\Lambda}_n & \mathbf{0}_n \\ \mathbf{0}_n & \boldsymbol{\Lambda}_n \end{bmatrix}, \quad (2.16)$$

where $\boldsymbol{\Lambda}_n = \text{diag}(\lambda_1, \dots, \lambda_n)$ and $\mathbf{0}_n$ is the $n \times n$ matrix with zero entries, the coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are said to be Darboux–Nijenhuis (DN) coordinates. Needless to say that the Poisson tensor $P_1 := N(\omega^b)^{-1}$ in DN coordinates takes the form

$$P_1 = \begin{bmatrix} \mathbf{0}_n & \boldsymbol{\Lambda}_n \\ -\boldsymbol{\Lambda}_n & \mathbf{0}_n \end{bmatrix}. \quad (2.17)$$

Remark 2.6. *The definition of DN coordinates can be generalized to each set of Darboux coordinates in which N takes the diagonal form (2.16) (see, e.g., Falqui, this volume). With such more general definition, the cases in which the eigenvalues of N are not independent (for instance constant) can be managed (see Falqui & Musso 2003).*

3. BH systems and Lenard chains

The concept of BH formulation for a vector field was introduced firstly by Magri (1978) studying the Korteweg–de–Vries equation, in order to explain integrability of soliton equations from the standpoint of classical analytical mechanics. The BH formulation has been very effective and has given rise to hundreds of articles about integrable systems both continuous and discrete. Let us briefly recall some definitions.

Definition 3.1. *Let (M, P_0, P_1) be a bi-Hamiltonian manifold. A vector field X is said to admit a BH formulation w.r.t. the Poisson tensors (P_0, P_1) , if there exist two smooth functions H and K , such that*

$$X = P_0 dH = P_1 dK . \quad (3.1)$$

Due to (2.3) the vector field X is a symmetry both of P_0 and of P_1 .

$$\mathcal{L}_X(P_i) = 0 \quad i = 0, 1 . \quad (3.2)$$

Moreover from (3.1) and from the skew symmetry of the Poisson tensors it follows that the Hamiltonian functions of a BH vector field are automatically in involution w. r. t. both Poisson brackets

$$\begin{aligned} \{H, K\}_{P_0} &= - \langle dK, P_0 dH \rangle = - \langle dK, P_1 dK \rangle = 0 \\ \{H, K\}_{P_1} &= \langle dH, P_1 dK \rangle = \langle dH, P_0 dH \rangle = 0 \end{aligned} \quad (3.3)$$

Hence, if M is in particular a four-dimensional ωN manifold and X admits a BH formulation w.r.t. $P_0 := (\omega^\flat)^{-1}$ and $P_1 := NP_0$, then the dynamical system X is integrable after Liouville, provided that the two functions H and K are functionally independent. In the general case of a $2n$ -dimensional ωN manifold, n integrals of motion independent and in involution are needed in order to assure integrability. Fortunately, a BH vector field does not come alone but is always a member of an entire hierarchy of BH vector fields known as Lenard chain.

Definition 3.2. *Let (M, ω, N) be an ωN manifold and $\{H_i\}_{1 \leq i \leq n}$ smooth functions on M which satisfy the following recursion relations*

$$d_N H_i = N^* dH_i = dH_{i+1} . \quad (3.4)$$

They will be said forming the Lenard chain generated by H_1 .

Proposition 3.3. (Magri & Morosi 1984) *The vector fields $X_i := P_0 dH_i$ satisfy the recursion relations*

$$NX_i = X_{i+1} \quad (i = 1, \dots, n-1) \quad (3.5)$$

and are symmetries of N , whence they are said to belong to a Lenard chain of symmetries. Moreover, X_i ($i = 2, \dots$) are Hamiltonian vector fields also w.r.t. each Poisson tensor $P_k := N^k P_0$ for $k < i$

$$X_i = P_0 dH_i = P_k dH_{i-k} . \quad (3.6)$$

Finally, the functions H_i ($i = 1, \dots$) are in involution w.r.t. each Poisson bracket associated with the Poisson tensors P_k ($k = 1, \dots$) and therefore the Hamiltonian vector fields X_i ($i = 1, \dots$) form an abelian Lie algebra.

Proof. The sketch of the proof is as follows. Relations (3.5) are equivalent to (3.4). The invariance of N w.r.t. to each vector field of a Lenard chain follows from the fact that, due to the compatibility condition (2.10) between P_0 and N , it holds $\forall Y \in T^*M$ that

$$\mathcal{L}_{P_0 dH_i} Y = P_0 i_Y (d(d_N H_i)) \stackrel{(3.4)}{=} 0 \quad i = 1, \dots \quad (3.7)$$

Finally, the involutivity of $\{H_i\}_{1 < i}$ is due to the skew-symmetry of the Poisson tensors P_k and to the fact that

$$\{H_i, H_j\}_{P_k} = \langle dH_i, P_k dH_j \rangle \stackrel{(3.6)}{=} \langle dH_i, P_{k+j-i} dH_i \rangle = 0 . \quad (3.8)$$

□

Remark 3.4. Let us note that the Hamiltonian vector field $X_1 = P_0 dH_1$, whose Hamiltonian function is the generator H_1 of the Lenard chain (3.6), does not necessarily admit a BH formulation, unless N is invertible. Nevertheless, also X_1 is a symmetry of N thanks to (3.7).

The following proposition contains a necessary and sufficient condition in order that a function K be a generator of a Lenard chain.

Proposition 3.5. Let (M, ω, N) be an ωN manifold and K a smooth function on M . It is the generator of a Lenard chain iff

$$d(d_N K) = 0 \quad (3.9)$$

Proof. Denoting for convenience $H_1 := K$, condition (3.9) means that the one form $d_N H_1$ is locally exact, therefore a function H_2 exists defined by $dH_2 = d_N H_1$. But also H_2 satisfies condition (3.9) as

$$d(d_N H_2) \stackrel{(2.6)}{=} -d_N dH_2 = -d_N d_N H_1 \stackrel{(2.7)}{=} 0 , \quad (3.10)$$

hence the procedure can be iterated. □

Obviously enough, the Lenard chain generated by $K = H_1$ contains, at most, n independent functions $\{H_i\}_{1 \leq i \leq n}$ because from (2.13) and (3.4) follows that

$$dH_{n+1} = (N^*)^n dH_1 = c_n dH_1 + \dots + c_1 dH_n \quad (3.11)$$

In matrix form, a Lenard chain can be represented as follows

$$N^* \mathbf{dH} = \mathbf{F}_L \mathbf{dH} , \quad (3.12)$$

where N^* acts componentwise on the vector $\mathbf{dH} = [dH_1, \dots, dH_n]^T$ and \mathbf{F}_L (the index L means Lenard) is the matrix-valued function

$$\mathbf{F}_L = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ c_n & c_{n-1} & \cdots & \cdots & c_1 \end{bmatrix} . \quad (3.13)$$

It is of interest knowing the more general expression of the Hamiltonian functions forming a Lenard chain, in DN coordinates. Remarkably enough, it turns out that they take a very peculiar form. In fact, they are additively separated functions of the eigenvalues λ_i only.

Proposition 3.6. *In each set of DN coordinates, the Hamiltonian functions H_k of a Lenard chain take the form*

$$H_k = \sum_{i=1}^n \int^{\lambda_i} \xi^{k-1} g'_i(\xi) d\xi \quad (k = 1, \dots) \quad (3.14)$$

where $g_i(\lambda_i)$ are generic smooth functions depending only on λ_i . Viceversa, if H_k are n functions which take the form (3.14) in some set of DN coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu})$, then they belong to the Lenard chain generated by $H_1 = \sum_{i=1}^n g_i(\lambda_i)$.

Proof. As in DN coordinates N^* takes the diagonal form (2.16), condition (3.9) explicitly reads

$$\begin{aligned} d(d_N H_1) &= \sum_{i,j=1}^n \frac{\partial^2 H_1}{\partial \lambda_j \partial \lambda_i} \lambda_i d\lambda_j \wedge d\lambda_i + \sum_{i,j=1}^n \frac{\partial^2 H_1}{\partial \mu_j \partial \mu_i} \lambda_i d\mu_j \wedge d\mu_i \\ &+ \sum_{i=1}^n \frac{\partial H_1}{\partial \mu_i} d\lambda_i \wedge d\mu_i + \sum_{i,j=1}^n (\lambda_i - \lambda_j) \frac{\partial^2 H_1}{\partial \mu_j \partial \lambda_i} d\mu_j \wedge d\lambda_i \end{aligned} \quad (3.15)$$

Therefore, H_1 must solve the system ($i, j = 1, \dots, n$)

$$\frac{\partial^2 H_1}{\partial \lambda_j \partial \lambda_i} (\lambda_i - \lambda_j) = 0 \quad (3.16)$$

$$\frac{\partial^2 H_1}{\partial \mu_j \partial \mu_i} (\lambda_i - \lambda_j) = 0 \quad (3.17)$$

$$\delta_{ij} \frac{\partial H_1}{\partial \mu_j} + (\lambda_j - \lambda_i) \frac{\partial^2 H_1}{\partial \mu_j \partial \lambda_i} = 0 \quad (3.18)$$

The general solution of such a system is $H_1 = \sum_{i=1}^n g_i(\lambda_i)$ and acting with $(N^*)^{k-1}$ on dH_1 equations (3.14) follow. \square

In this case, DN coordinates are variables of action-angle type for each Hamiltonian vector field X_k and therefore are (trivial) separation coordinates.

A relevant example of Lenard chain is given by the normalized traces of the powers of N

$$I_k := \frac{1}{2k} \text{tr}(N^k) . \quad (3.19)$$

In fact, in terms of the eigenvalues of N they take the form $kI_k = \sum_{i=1}^n \lambda_i^k$, therefore they have the form (3.14) with $g_i(\lambda_i) = \lambda_i$.

Now we can show that each BH vector field can be inserted into a Lenard chain.

Proposition 3.7. *Let (M, ω, N) be an ωN manifold and $X = (\omega^b)^{-1} dH$ a Hamiltonian vector field. It admits a BH formulation w.r.t. $(P_0 := (\omega^b)^{-1}, P_1 := NP_0)$ iff there exists a smooth function K such that*

$$N^* dK = dH . \quad (3.20)$$

In this case, X is the second vector field of the Lenard chain generated by K .

Proof. From (3.20) it follows immediately that condition (3.9) is satisfied, therefore K is the generator of the Lenard chain with $H_1 = K$, $H_2 = H$ and $X_2 = X$. \square

Remark 3.8. *If one is interested in constructing a set of integrals of motion in involution and not a BH formulation for a given Hamiltonian vector field $X = P_0 dH$, one can limit himself to check if condition (3.9) is satisfied for H . If so, the integrals are furnished by the members of the Lenard chain generated by H .*

Below we present an example of BH system with three degrees of freedom.

(a) *The 3-particles Toda system*

Let us consider three particles on the line, with unitary masses and interacting with near neighbors by a repulsive force exponentially dependent on the mutual distance. Such a system is Hamiltonian w.r.t. the symplectic form $\omega = \sum_{i=1}^3 dp_i \wedge dq_i$, with Hamiltonian function

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + e^{q_1 - q_2} + e^{q_2 - q_3} , \quad (3.21)$$

where (q_1, q_2, q_3) denote the positions of the three particles and (p_1, p_2, p_3) their momenta. Consequently, the equations of motion read

$$\begin{aligned} \dot{q}_1 &= p_1 & \dot{p}_1 &= -e^{q_1 - q_2} \\ \dot{q}_2 &= p_2 & \dot{p}_2 &= e^{q_1 - q_2} - e^{q_2 - q_3} \\ \dot{q}_3 &= p_3 & \dot{p}_3 &= e^{q_2 - q_3} . \end{aligned} \quad (3.22)$$

The BH formulation of this system can be found in the work of Das and Okubo (1989). The Nijenhuis tensor is given by

$$N = \begin{bmatrix} p_1 & 0 & 0 & 0 & 1 & 1 \\ 0 & p_2 & 0 & -1 & 0 & 1 \\ 0 & 0 & p_3 & -1 & -1 & 0 \\ 0 & -e^{q_1 - q_2} & 0 & p_1 & 0 & 0 \\ e^{q_1 - q_2} & 0 & -e^{q_2 - q_3} & 0 & p_2 & 0 \\ 0 & e^{q_2 - q_3} & 0 & 0 & 0 & p_3 \end{bmatrix} . \quad (3.23)$$

Computing the traces of the powers of N , one finds the Lenard chain given by

$$I_1 = p_1 + p_2 + p_3 \quad \text{total momentum} \quad (3.24)$$

$$I_2 = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + e^{q_1 - q_2} + e^{q_2 - q_3} \quad \text{total energy} \quad (3.25)$$

$$I_3 = \frac{1}{3}(p_1^3 + p_2^3 + p_3^3) + p_1 e^{q_1 - q_2} + p_2 (e^{q_1 - q_2} + e^{q_1 - q_3}) + p_3 e^{q_2 - q_3} . \quad (3.26)$$

Such functions are independent and because of proposition 3.5, they are automatically in involution. Hence, the Toda system $X_2 = P_0 dI_2$, being $P_0 = (\omega^b)^{-1}$, is integrable after Liouville together with $X_1 = P_0 dI_1$ and $X_3 = P_0 dI_3$. Moreover, DN coordinates are separation coordinates for them.

4. QBH systems and Fröbenius chains

The notion of BH formulation and of Lenard chain for a vector field X have been very effective for integrable systems with infinite degrees of freedom (soliton equations) but less effective for classical (finite degrees of freedom) integrable systems. Indeed, seeking a BH formulation for classical integrable Hamiltonian systems in a symplectic manifold w.r.t. the (inverse of the) symplectic structure and to a Poisson structure compatible with the first one (called in the sequel *standard* BH structure), often has been unsuccessful. The theoretical reason for that has been discovered in the work of Brouzet (1993), where it was proved that a strong geometrical condition must be satisfied in order that an integrable Hamiltonian system admits such a BH formulation in a neighborhood of a Liouville torus. Then, some attempts have been made in order to encompass classical Hamiltonian systems in the BH theory; we have to quote at least the following formulations.

- A BH formulation of X w. r. t. *alternative* (i.e. not including P_0) Poisson structures, still compatible each other (see Vilasi 2001 and reference therein);
- a BH formulation of X w. r. t. a *degenerate* bi-Hamiltonian structure in an *extended* phase space;
- a QBH formulation of X w. r. t. a standard bi-Hamiltonian structure in its original phase space.

While the first two attempts weaken the request about the BH structure, the last one weakens the request on the formulation of X . As a matter of fact, the last two, which are connected to each other, have been shown to be very effective in the theory of SoV.

Definition 4.1. *Let (M, P_0, P_1) be a bi-Hamiltonian manifold. A vector field X is said to admit a QBH formulation w.r.t. a pair of Poisson tensors (P_0, P_1) if there exist three smooth functions H, K, ρ such that*

$$X = P_0 dH = \rho P_1 dK \quad (4.1)$$

Otherwise stated, X must be Hamiltonian w.r.t. P_0 with Hamiltonian function H and *quasi-Hamiltonian* w.r.t. P_1 with *quasi-Hamiltonian* function K and conformal factor ρ .

Remark 4.2. *Allowing X to be quasi-Hamiltonian w.r.t. both P_0 and P_1 , the bi-quasi-Hamiltonian formulation, introduced by Crampin & Sarlet (2002) studying cofactor pair systems (see Lundmark 2003 and reference therein), is obtained.*

Let us note that the quasi-Hamiltonian formulation of X w. r. t. a Poisson tensor (say P_1) is a quite natural generalization of the Hamiltonian one because it amounts to say that X is Hamiltonian w.r.t. P_1 , provided that a not trivial change in time: $dt \mapsto d\tau = \rho dt$ is introduced. Such change in time has been widely used in the literature (see, e.g., Perelomov 1990, page 99). Nevertheless, the invariance of P_1 w.r.t. X , occurring in the BH formulation, is not preserved. In fact, computing the Lie derivative of P_1 w.r.t. X we find

$$\mathcal{L}_X(P_1) = X \wedge P_1 d \log \rho . \quad (4.2)$$

Therefore we can infer that X it is no longer a symmetry of P_1 , unless X is parallel to the Hamiltonian vector field $P_1 d \log(\rho)$. However, if X admits the QBH formulation (4.1) the functions H and K are still in involution w. r. t. $\{, \}_{P_0}$ and $\{, \}_{P_1}$ as

$$\begin{aligned} \{H, K\}_{P_0} &= - \langle dK, P_0 dH \rangle = -\rho \langle dK, P_1 dK \rangle = 0 \\ \{H, K\}_{P_1} &= \langle dH, P_1 dK \rangle = \frac{1}{\rho} \langle dH, P_0 dH \rangle = 0 . \end{aligned} \quad (4.3)$$

Hence, if M is a four-dimensional ωN manifold the dynamical system of X is completely integrable, provided that H and K are globally defined and functionally independent. Indeed, the QBH formulation was just introduced for Hamiltonian systems with two degrees of freedom and in such a case it was shown by Brouzet et al. (1996) that each integrable system admits a QBH formulation in a neighborhood of a Liouville torus. What about the existence of QBH systems with more than two degrees of freedom on ωN manifolds? In the work of Magri & Marsico (1996) a relevant example of QBH system (unnoticed by the authors) is contained. In fact, they proved that the functions $\{f_i\}_{1 \leq i \leq n}$ implicitly defined in terms of Darboux–Nijenhuis coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ by

$$\mu_k = f_1 \lambda_k^{n-1} + \dots + f_{n-1} \lambda_k + f_n , \quad k = 1, \dots, n \quad (4.4)$$

are coordinates canonically conjugated with the functions $G_i := I_{n+1-i}$ (3.19) and moreover they satisfy the following recursion relations ($i = 1, \dots, n$)

$$N^* df_i = df_{i+1} + c_i df_1 , \quad \text{with } f_{n+1} = 0 , \quad (4.5)$$

where the c_i are the same of (2.13). As in matrix form such relations can be written

$$N^* \mathbf{d}f = \mathbf{F}_F \mathbf{d}f , \quad (4.6)$$

with $\mathbf{d}f = [df_1, \dots, df_n]^T$ and the matrix \mathbf{F}_F given by the Fröbenius matrix

$$\mathbf{F}_F = \begin{bmatrix} c_1 & 1 & 0 & \cdots & 0 \\ c_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ c_n & 0 & 0 & \cdots & 0 \end{bmatrix} , \quad (4.7)$$

they are said to be Fröbenius chains. Note that, due to the last equation (4.5), the vector field $P_0 df_1 = 1/c_n P_1 df_n$ is just a QBH vector field with conformal factor $\rho = 1/c_n$. Hence, we can infer the existence of a QBH vector field in each regular open set of an ωN manifold, where the eigenvalues of N take values different from zero. We want to stress that so far every example of QBH vector field has precisely the same conformal factor $\rho = 1/c_n$ (or a slight generalization of that, as in the work of Tondo & Morosi 1999) and belongs to a Fröbenius chain.

Definition 4.3. *Let (M, ω, N) be an ωN manifold and $\{H_i\}_{1 \leq i \leq n}$ smooth functions on M which satisfy the recursion relations*

$$d_N H_i = N^* dH_i = dH_{i+1} + c_i dH_1 \quad \text{with} \quad H_{n+1} = 0. \quad (4.8)$$

They will be said forming the Fröbenius chain generated by H_1 .

Proposition 4.4. *The elements of a Fröbenius chain are in involution w.r.t. the two Poisson brackets associated with the Poisson tensors $P_0 := (\omega^\flat)^{-1}$ and $P_1 := NP_0$.*

We postpone the proof of this proposition to remark (5.7), in the more general context of generalized Lenard chains.

Another relevant example of Fröbenius chains is given by the coefficient c_i of the minimal polynomial of N . In fact, exploiting the recursion relations (3.4) for the Lenard chain formed by I_k (3.19) and the Newton formula connecting c_i with the I_k

$$c_1 = I_1, \quad 2c_2 = 2I_2 - c_1 I_1, \quad kc_k = [kI_k - (k-1)c_1 I_{k-1} - \dots - c_{k-1} I_1] \quad (4.9)$$

it can be proved that ($i = 1, \dots, n$)

$$N^* dc_i = dc_{i+1} + c_i dc_1 \quad \text{with} \quad c_{n+1} = 0. \quad (4.10)$$

We note that, in contrast with the case of a generic Fröbenius chain (4.5), the vector field $P_0 dc_1 = 1/c_n P_1 dc_n$ is a BH vector field whose Hamiltonian function w. r. t. P_1 is the function $K = \log c_n$.

Remark 4.5. *The functions $\{c_i\}_{1 \leq i \leq n}$ together with the functions $\{f_i\}_{1 \leq i \leq n}$ defined by (4.4) form a set of coordinates (\mathbf{c}, \mathbf{f}) said Hankel–Fröbenius coordinates in the work of Falqui et al. (2000). In such coordinates P_0 and N take the matrix form*

$$P_0 = \begin{bmatrix} \mathbf{0}_n & -\mathbf{H} \\ \mathbf{H}^T & \mathbf{0}_n \end{bmatrix}, \quad N = \begin{bmatrix} \mathbf{F}_F & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{F}_F \end{bmatrix}, \quad (4.11)$$

where

$$\mathbf{H} = \begin{bmatrix} 0 & \dots & 0 & 1 \\ \vdots & \ddots & \ddots & -c_1 \\ 0 & \ddots & \ddots & \ddots \\ 1 & -c_1 & \ddots & -c_{n-1} \end{bmatrix}, \quad (4.12)$$

is an Hankel matrix and \mathbf{F}_F denotes the Fröbenius matrix (4.7), whence the denomination of Hankel–Fröbenius (HF) coordinates.

Proposition 4.6. (Magri, 2003) *Let (M, ω, N) be an ωN manifold and H a smooth function. It is the generator of a Fröbenius chain iff*

$$\mathrm{d}d_N H = \mathrm{d}c_1 \wedge \mathrm{d}H . \quad (4.13)$$

Proof. Denoting $H_1 := H$ for convenience, condition (4.13) means that the 1-form $\mathrm{d}_N H_1 - c_1 \mathrm{d}H_1$ is locally exact, therefore a function H_2 exists defined by $\mathrm{d}H_2 = \mathrm{d}_N H_1 - c_1 \mathrm{d}H_1$. But also H_2 satisfies a condition similar to (4.13) as

$$\begin{aligned} \mathrm{d}d_N H_2 &= -\mathrm{d}_N \mathrm{d}H_2 = -\mathrm{d}_N \mathrm{d}_N H_1 + \mathrm{d}_N c_1 \wedge \mathrm{d}H_1 - c_1 \mathrm{d}d_N H_1 \\ &\stackrel{(4.10)}{=} (\mathrm{d}c_2 + c_1 \mathrm{d}c_1) \wedge \mathrm{d}H_1 - c_1 \mathrm{d}c_1 \wedge \mathrm{d}H_1 = \mathrm{d}c_2 \wedge \mathrm{d}H_1 . \end{aligned} \quad (4.14)$$

Hence, iterating such a procedure we are able to construct a chain of functions satisfying the integrability conditions

$$\mathrm{d}d_N H_i = \mathrm{d}c_i \wedge \mathrm{d}H_1 \quad (i \geq 1) \quad (4.15)$$

and therefore the recursion relations (4.8). A Fröbenius chain stops with H_n . In fact, exploiting (4.8),

$$\begin{aligned} N^* \mathrm{d}H_n &= N^*(N^* \mathrm{d}H_{n-1} - c_{n-1} \mathrm{d}H_{n-1}) = \dots = \\ &((N^*)^n - c_1 (N^*)^{n-1} - \dots - c_{n-1} N^*) \mathrm{d}H_1 \stackrel{(2.13)}{=} c_n \mathrm{d}H_1 \end{aligned} \quad (4.16)$$

□

Remark 4.7. *Being $c_1 \equiv I_1$, it is the generator of the Lenard chain (3.19) and therefore it satisfies $\mathrm{d}_N c_1 = \mathrm{d}I_2$. However, due to (4.9), $I_2 = c_1^2/2 + c_2$, then $\mathrm{d}_N c_1 = \mathrm{d}I_2 = \mathrm{d}c_2 + c_1 \mathrm{d}c_1$. Hence $c_1 \equiv I_1$ is also the generator of the Fröbenius chain (4.10). This is an example of a more general situation. Indeed, it can be easily proved that each generator H_1 of a Lenard chain $\{H_i\}_{1 \leq i \leq n}$ generates also a (generalized) Fröbenius chain $\{h_i\}_{1 \leq i \leq n}$ recursively defined by*

$$h_1 := H_1 , \quad \mathrm{d}h_{i+1} = \mathrm{d}_N h_i - h_i \mathrm{d}h_1 . \quad (4.17)$$

The functions h_i are connected to H_i through Newton formula analogous to (4.9).

Let us consider the form taken by the QBH formulation of a vector field on an ωN manifold.

Proposition 4.8. *Let (M, ω, N) be an ωN manifold and $X = (\omega^\flat)^{-1} \mathrm{d}H$ a Hamiltonian vector field. It admits a QBH formulation w.r.t. $(P_0 := (\omega^\flat)^{-1}, P_1 := NP_0)$, with conformal factor $\rho = 1/c_n$, iff there exists a smooth function K such that*

$$N^* \mathrm{d}K = c_n \mathrm{d}H . \quad (4.18)$$

In this case, X is the last vector field in the Fröbenius chain generated by H (Falqui 2004, personal communication) .

Proof. Equation (4.18) is obvious. Let us check condition (4.13) for H .

$$\begin{aligned} \mathrm{d}d_N H &= -\mathrm{d}_N \mathrm{d}H = -\mathrm{d}_N \left(\frac{1}{c_n} N^* \mathrm{d}K \right) = -\mathrm{d}_N \left(\frac{1}{c_n} \right) \wedge N^* \mathrm{d}K - \frac{1}{c_n} \mathrm{d}_N \mathrm{d}_N \mathrm{d}K \\ &= \frac{1}{c_n^2} \mathrm{d}_N c_n \wedge N^* \mathrm{d}K = \frac{1}{c_n} \mathrm{d}c_1 \wedge N^* \mathrm{d}K = \mathrm{d}c_1 \wedge \mathrm{d}H \quad (4.19) \end{aligned}$$

Therefore H is the generator of a Fröbenius chain with $K = H_n$. \square

Now we are in the position to present the more general form which the members of a Fröbenius chain take in DN coordinates.

Proposition 4.9. *In each set of DN coordinates, the Hamiltonian functions H_k of a Fröbenius chain take the form:*

$$H_k = \sum_{i=1}^n \frac{\partial c_k}{\partial \lambda_i} \frac{g_i(\lambda_i, \mu_i)}{m'(\lambda_i)} \quad (j = 1, \dots, n), \quad (4.20)$$

where $g_i(\lambda_i, \mu_i)$ are generic smooth functions depending, at most, on the single pair (λ_i, μ_i) and $m'(\lambda_i) := m'(z)|_{z=\lambda_i} = \prod_{\substack{j=1 \\ j \neq i}}^n (\lambda_i - \lambda_j)$, $m(z)$ being the minimal polynomial of N . Furthermore, it is remarkable that the Hamilton–Jacobi equations corresponding to functions H_k are separable just in DN coordinates.

Proof. As in DN coordinates N^* takes the diagonal form (2.16), the left-hand side of (4.13) takes the form (3.15) whilst, taking into account that $c_1 = \sum_{i=1}^n \lambda_i$, the right-hand side reads

$$\mathrm{d}c_1 \wedge \mathrm{d}H_1 = \sum_{i,j=1}^n \frac{\partial H_1}{\partial \lambda_j} \mathrm{d}\lambda_i \wedge \mathrm{d}\lambda_j + \sum_{i,j=1}^n \frac{\partial H_1}{\partial \mu_j} \mathrm{d}\lambda_i \wedge \mathrm{d}\mu_j \quad (4.21)$$

Therefore, H_1 must solve the following system ($i, j = 1, \dots, n$)

$$\frac{\partial^2 H_1}{\partial \lambda_j \partial \lambda_i} (\lambda_i - \lambda_j) = \frac{\partial H_1}{\partial \lambda_i} - \frac{\partial H_1}{\partial \lambda_j} \quad (4.22)$$

$$\frac{\partial^2 H_1}{\partial \mu_j \partial \mu_i} (\lambda_i - \lambda_j) = 0 \quad (4.23)$$

$$\delta_{ij} \frac{\partial H_1}{\partial \mu_j} + \frac{\partial^2 H_1}{\partial \mu_j \partial \lambda_i} (\lambda_j - \lambda_i) = \frac{\partial H_1}{\partial \mu_j} \quad (4.24)$$

It can be proved that the general solution of such a system is given by (4.20) for $k = 1$ and acting with (N^*) on $\mathrm{d}H_{k-1}$ ($k = 2, \dots, n$) the expression for H_k follows. \square

The form of H_k for $k = 1$ and $k = n$ was found firstly by Morosi & Tondo (1997), who proved their separability applying the Levi-Civita test (1.9). For the other values of k the form of H_k was found by Błaszak (1998, pp.194–196).

Now we are able to exhibit the explicit form of functions $\{f_i\}_{1 \leq i \leq n}$ defined by (4.4). In fact, from proposition 4.9 it follows that

Corollary 4.10. *In DN coordinates, the functions $\{f_i\}_{1 \leq i \leq n}$ take the form (4.20) with $g_i(\lambda_i, \mu_i) = \mu_i$.*

From the standpoint of applications to the Kowalevski and to the Lagrange tops, let us consider explicitly a Fröbenius chain in a four-dimensional manifold.

Corollary 4.11. *In a four-dimensional ωN manifold, the only two elements of a Fröbenius chain in a DN chart $(\lambda_1, \lambda_2, \mu_1, \mu_2)$ take the normal form*

$$H(\lambda_1, \lambda_2, \mu_1, \mu_2) = \frac{1}{\lambda_1 - \lambda_2} (g_1(\lambda_1, \mu_1) - g_2(\lambda_2, \mu_2)) , \quad (4.25)$$

$$K(\lambda_1, \lambda_2, \mu_1, \mu_2) = \frac{1}{\lambda_1 - \lambda_2} (-\lambda_2 g_1(\lambda_1, \mu_1) + \lambda_1 g_2(\lambda_2, \mu_2)) . \quad (4.26)$$

Moreover, the Jacobi–Sklyanin separation equations (1.11) for H and K are

$$g_i(\lambda_i, \mu_i) - \lambda_i H - K = 0, \quad (i = 1, 2) . \quad (4.27)$$

Viceversa, if two functions H and K satisfy equations (4.27) for generic smooth functions $g_i(\lambda_i, \mu_i)$, they respectively are the first and the last member of a Fröbenius chain.

At the present, we know many significant examples with n degrees of freedom of QBH systems, hence separable in DN coordinates. There is mainly one source of QBH systems: Gel’fand–Zakharevich systems of maximal rank (see Gel’fand & Zakharevich 1993). They are BH systems w.r.t. degenerate Poisson brackets of maximal rank. After a suitable reduction to a generic symplectic leaf S of one of the Poisson brackets, this class of systems has been proved by Falqui & Pedroni (2003) to become QBH w.r.t. the ωN structure on S induced by the initial Poisson brackets. Among the examples of these systems we recall stationary flows of the Korteweg–de–Vries hierarchy (Morosi & Tondo 1997, Falqui et al. 2000) and Benenti systems with a special conformal Killing tensor (Ibort et al. 2000) such as Neumann and Neumann–Rosochatius systems (Błaszak 1998, Pedroni 2001, Bartocci et al. 2003). Below we present two classical systems which are proved for the first time to admit a QBH formulation.

(a) *The Jacobi–Rosochatius system*

Let us consider the motion of a particle in \mathbb{R}^{n+1} , with unitary mass and subjected to an isotropic elastic potential plus the Rosochatius potential. If x_i ($i = 0, 1, \dots, n$) are Cartesian coordinates, such a potential is given by

$$V(\mathbf{x}) = \frac{1}{2} \sum_{i=0}^n \left(k x_i^2 + \frac{d_i}{x_i^2} \right) \quad (4.28)$$

where k and d_i ($i = 0, 1, \dots, n$) are constant parameters. Being V of the form $V(\mathbf{x}) = \sum_{i=1}^n V(x_i)$, Cartesian coordinates in \mathbb{R}^{n+1} are separation variables for it. We will see below that also elliptic coordinates in \mathbb{R}^{n+1} are separated coordinates (see Benenti 1993 for a different proof).

In order to study the previous motion constrained to the n -dimensional ellipsoid $\mathcal{E}^n = \{\mathbf{x} \mid \sum_{i=0}^n x_i^2/a_i = 1\}$, where $a_0 < a_1 < a_2 \dots < a_n$ are positive real constants,

it is convenient to introduce in \mathbb{R}^{n+1} a system of coordinates adapted to \mathcal{E}^n , i.e. the generalized elliptic coordinates of Jacobi defined as the $(n+1)$ roots λ_i of the equation

$$\sum_{i=0}^n \frac{x_i^2}{z - a_i} + 1 = 0 \quad (4.29)$$

(see, e.g., Arnold et al. 1988, page 126). In these coordinates \mathcal{E}^n is simply the submanifold $\lambda_0 = 0$. The left-hand side of (4.29) can be written as the ratio of the two polynomials $p(z) := (z - \lambda_0)m(z)$, being $m(z) = \prod_{j=1}^n (z - \lambda_j)$, and $q(z) := \prod_{j=0}^n (z - a_j)$.

$$\sum_{i=0}^n \frac{x_i^2}{z - a_i} + 1 = \frac{p(z)}{q(z)}. \quad (4.30)$$

By computing the residue of both sides of (4.30) in the simple poles $z = a_i$, it follows that

$$x_i^2 = \frac{p(a_i)}{q'(a_i)}, \quad (4.31)$$

therefore the potential (4.28) can be written

$$V = \frac{1}{2} \sum_{i=0}^n k \frac{p(a_i)}{q'(a_i)} + \sum_{i=0}^n d_i \frac{q'(a_i)}{p(a_i)}. \quad (4.32)$$

Moreover, comparing the numerators of both sides in (4.30), it follows that $\sum_{i=0}^n x_i^2 - \sum_{i=0}^n a_i = -\sum_{i=0}^n \lambda_i$. Hence, the elastic term of the potential in elliptical coordinates takes the form (up to an additive constant term)

$$V_e = -\frac{1}{2} \sum_{i=0}^n k \lambda_i = -\frac{1}{2} \sum_{i=0}^n k \frac{\lambda_i^{n+1}}{p'(\lambda_i)}, \quad (4.33)$$

taking into account the Jacobi identity $\sum_{i=0}^n \lambda_i = \sum_{i=0}^n \frac{\lambda_i^{n+1}}{p'(\lambda_i)}$. By exploiting the decomposition in partial fraction $\frac{1}{p(z)} = \sum_j \frac{1/p'(\lambda_j)}{z - \lambda_j}$, the Rosochatius term in (4.32), takes the form

$$V_R = \sum_{i=0}^n \frac{1}{2} \frac{h(\lambda_i)}{p'(\lambda_i)}, \quad (4.34)$$

where $h(\lambda_i) = \sum_{j=0}^n d_j \frac{q'(a_j)}{(a_j - \lambda_i)}$. The kinetic energy T has been shown to take the form (see, e.g., Moser 1980, page 273)

$$T = \frac{1}{2} \sum_{i=0}^n \dot{x}_i^2 = -\frac{1}{8} \sum_{i=0}^n \left(\frac{p'(\lambda_i)}{q(\lambda_i)} \right) \dot{\lambda}_i^2. \quad (4.35)$$

Passing from the configuration space \mathbb{R}^{n+1} to the phase space $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, let us consider the canonical transformation $(x_i, \dot{x}_i) \mapsto (\lambda_i, \mu_i)$ induced by the point transformation $x_i \mapsto \lambda_i$. As from (4.35) $\mu_i = -\left(\frac{p'(\lambda_i)}{4q(\lambda_i)} \right) \dot{\lambda}_i$, in the chart (λ_i, μ_i) the Hamiltonian function $H = T + V$ reads

$$H = \frac{1}{2} \sum_{i=0}^n \frac{-4q(\lambda_i)\mu_i^2 - k\lambda_i^{n+1} + h(\lambda_i)}{p'(\lambda_i)} \quad (4.36)$$

As $q(\lambda_i)$ and $h(\lambda_i)$ are functions depending only on the single coordinate λ_i , the Hamiltonian function (4.36) possesses the form (4.20); therefore the corresponding flow admits a QBH formulation in \mathbb{R}^{n+1} . Hence, it is separable also in elliptical coordinates besides that in cartesian coordinates, as mentioned above. As the cotangent bundle of \mathcal{E}^n is simply $T^*\mathcal{E}^n = \{(\lambda, \mu) | \lambda_0 = 0, \mu_0 = 0\}$ and $p'(\lambda_i)|_{T^*\mathcal{E}^n} = \lambda_i m'(\lambda_i)$ ($i = 1, \dots, n$), the Hamiltonian function H restricted to $T^*\mathcal{E}^n$ keeps the normal form (4.20)

$$H|_{T^*\mathcal{E}^n} = \frac{1}{2} \sum_{i=1}^n \frac{4 \prod_{j=0}^n (a_j - \lambda_i) \lambda_i^{-1} \mu_i^2 - k \lambda_i^n + \sum_{j=0}^n d_j \frac{\prod_{k=0, k \neq j}^n (a_j - a_k)}{\lambda_i (a_j - \lambda_i)}}{m'(\lambda_i)} \quad (4.37)$$

where $\prod_{\substack{j=1 \\ j \neq i}}^n (\lambda_i - \lambda_j) := 1$ for $n = 1$ and $m'(\lambda_i) = \prod_{\substack{j=1 \\ j \neq i}}^n (\lambda_i - \lambda_j)$. Since $H|_{T^*\mathcal{E}^n}$ has just the form (4.20), we can infer immediately that the Jacobi-Rosochatius system restricted to the n -dimensional ellipsoid is separable and the Jacobi elliptic coordinates with their conjugated momenta are just a set of DN coordinates for an ωN structure in $T^*\mathcal{E}^n$, whose Nijenhuis tensor has the minimal polynomial $m(z)$.

(b) *The Kowalevski top*

It is one of the most beautiful examples of integrable systems and nowadays is considered as the best example to test the power of several modern methods. Its action-angle variables were first found by Novikov & Veselov (1985) within the general theory of algebro-geometric Poisson brackets on the universal bundle of hyperelliptic Jacobians; later they were rederived by Komarov and Kuznetsov (1987), using SoV. Basing on the later work, we limit ourselves to show that the Kowalevski top admits a QBH formulation.

Physically, the Kowalevski top is a heavy rigid body with a fixed point along the symmetry axis, the center of mass lying in the plane orthogonal to such an axis, and with the three principal inertia moments in the ratios $1 : 1 : 1/2$. In the phase space M_6 , parametrized by the coordinates $(l_1, l_2, l_3, \gamma_1, \gamma_2, \gamma_3)$, where l_i and γ_i ($i = 1, 2, 3$) are respectively the components of the angular momenta and of the vertical unit vector along the inertia axis in the comoving body-frame, the Euler-Poisson equations of motion are

$$\begin{aligned} \dot{l}_1 &= l_2 l_3 & \dot{\gamma}_1 &= (2l_3 \gamma_2 - l_2 \gamma_3) \\ \dot{l}_2 &= -(l_1 l_3 + b \gamma_3) & \dot{\gamma}_2 &= (-2l_3 \gamma_1 + l_1 \gamma_3) \\ \dot{l}_3 &= b \gamma_2 & \dot{\gamma}_3 &= (l_2 \gamma_1 - l_1 \gamma_2) \end{aligned} \quad (4.38)$$

Such a system is Hamiltonian w.r.t. the degenerate Poisson bracket of the Lie algebra $\mathfrak{e}(3)$

$$\{l_i, l_j\} = \epsilon_{ijk} l_k, \{l_i, \gamma_j\} = \epsilon_{ijk} \gamma_k, \{\gamma_i, \gamma_j\} = 0, \quad (4.39)$$

because the system (4.38) can be written $\frac{d}{dt} = \{H, \}$, with Hamiltonian function

$$H = \frac{1}{2} (l_1^2 + l_2^2 + 2l_3^2 - 2b\gamma_1). \quad (4.40)$$

Fixing the value of the two Casimir functions

$$l\gamma_1 + l_2\gamma_2 + l_3\gamma_3 = l \quad (4.41)$$

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = a^2 \quad (4.42)$$

the system (4.38) can be restricted to a four-dimensional generic symplectic leaf, S . It possesses other two independent integrals of motion in involution. One is the energy (4.40), the second is the famous quartic integral discovered by Kowalevski, which can be written

$$Kow = |(l_1 + il_2)^2 + 2b(\gamma_1 + i\gamma_2)|^2 \quad (4.43)$$

Hence, the Kowalevski top is integrable in the Liouville sense. Complexifying the variables l_i and γ_i by $l_{\pm} = l_1 \pm il_2$ and $\gamma_{\pm} = \gamma_1 \pm i\gamma_2$, Kowalevski introduced the separated coordinates $\lambda_i := w_i + 2H$ ($i = 1, 2$) where

$$w_{1,2} := \frac{R(l_+, l_-) \pm \sqrt{R(l_+, l_+)R(l_-, l_-)}}{(l_+ - l_-)^2} \quad (4.44)$$

and

$$R(l_+, l_-) = -l_+^2 l_-^2 + 4Hl_+ l_- + 4bl(l_+ - l_-) + 4b^2 - Kow. \quad (4.45)$$

The momenta conjugated to λ_i are given by

$$\mu_i = \frac{1}{2\sqrt{-2\lambda_i}} \log\left(\frac{z_i + \sqrt{z_i^2 + a_i^2}}{a_i}\right), \quad (4.46)$$

with $z_i := \sqrt{4R_5(\lambda_i)/\lambda_i}$, $a_i := 4b^2(1 - 2l^2/\lambda_i)$ and

$$R_5(\lambda) = (\lambda^2 - 4H\lambda - K - 2b^2)[\lambda(\lambda^2 - 4H\lambda - K + 2b^2) - 8b^2l^2], \quad (4.47)$$

where $K := Kow - 4H^2 - 2a^2b^2$. The Jacobi-Sklyanin separation equations (1.11) derived in Komarov and Kuznetsov (1987) are

$$2b^2(2l^2 - \lambda_i a^2) \cos(2\mu_i \sqrt{2\lambda_i}) - 4b^2l^2 + \lambda_i^3 - 4\lambda_i^2 H - \lambda_i K = 0 \quad i = 1, 2 \quad (4.48)$$

Comparing such equations with (4.27) we notice that the Hamiltonian function $4H$ and the second integral of motion K have just the form (4.25) with

$$g(\lambda_i, \mu_i) = \lambda_i^{-1} [2b^2(2l^2 - \lambda_i a^2) \cos(2\mu_i \sqrt{2\lambda_i}) - 4b^2l^2 + \lambda_i^3]. \quad (4.49)$$

Therefore, we can conclude that the Kowalevski top admits a QBH formulation in the symplectic manifold S , whose DN coordinates are just the separation variables discovered by Kowalevski (Tondo 1998, unpublished result).

Very recently the same QBH formulation has been recovered as an example of a more general construction by Magri (2004).

5. Nijenhuis systems and generalized Lenard chains.

Dealing with Boussinesq stationary flows which do not admit a BH or a QBH formulation, we have introduced a generalization of Lenard and Fröbenius chains, said generalized Lenard chains, that have been effective in the theory of SoV.

Definition 5.1. *Let (M, ω, N) be an ωN -manifold and $\{H_j\}_{1 \leq j \leq n}$ n independent functions which satisfy the following relation w.r.t. N^**

$$dH_j = p_j(N^*)dH_1 \quad j = 1, \dots, n, \quad (5.1)$$

where $p_j(N^*) := \sum_{k=1}^n a_{jk} (N^*)^{k-1}$ and the coefficients a_{jk} of the polynomials $p_j(N^*)$ are supposed being elements of an invertible matrix-valued function A . In matrix form condition (5.1) reads

$$\begin{bmatrix} dH_1 \\ dH_2 \\ \vdots \\ dH_n \end{bmatrix} = \mathbf{A} \begin{bmatrix} dH_1 \\ N^* dH_1 \\ \vdots \\ (N^*)^{n-1} dH_1 \end{bmatrix} \quad (5.2)$$

The functions H_j are said to belong to a generalized Lenard chain and the Hamiltonian vector fields $X_j = P_0 dH_j$ are said to be Nijenhuis systems as in Falqui et al. (2000).

Let us explain the meaning of this definition. Let \mathcal{F}_H be the foliation given by the level sets of the functions $\{H_i\}_{1 \leq i \leq n}$, \mathcal{D} the distribution generated by the vector fields $X_i = P_0 dH_i$, which are tangent to \mathcal{F}_H , and $\mathcal{D}^\circ = \text{Span}(dH_1, dH_2, \dots, dH_n)$ its dual distribution. Let $\mathcal{D}_{LC}^* = \text{Span}(dH_1, N^* dH_1, \dots, (N^*)^{n-1} dH_1)$ be the distribution generated by the cyclic exact 1-form dH_1 (LC means Levi-Civita). Equation (5.2) amounts to say that $\mathcal{D}^\circ \equiv \mathcal{D}_{LC}^*$ and \mathbf{A} is nothing but the transition matrix between the natural base $\{dH_1, dH_2, \dots, dH_n\}$ and the cyclic base $\{dH_1, N^* dH_1, \dots, (N^*)^{n-1} dH_1\}$. The relevance of generalized Lenard chains is based on the following theorem proved by Falqui et al (2000).

Proposition 5.2. *Let (M, ω, N) be an ωN manifold and (λ, μ) a set of DN coordinates. They are separation variables for each function H_j belonging to a generalized Lenard chain.*

Proof. It amounts to show that functions H_j are in separable involution (see (1.10)) in DN coordinates. To this end we note that, due to (5.1) and to the diagonal form of N^* in DN coordinates, it holds that

$$\frac{\partial H_j}{\partial \lambda_k} = p_j(\lambda_k) \frac{\partial H_1}{\partial \lambda_k} \quad (5.3)$$

$$\frac{\partial H_j}{\partial \mu_k} = p_j(\lambda_k) \frac{\partial H_1}{\partial \mu_k} \quad (5.4)$$

Therefore, $\{H_i, H_j\}_{P_0|k} = p_i(\lambda_k) \frac{\partial H_1}{\partial \lambda_k} p_j(\lambda_k) \frac{\partial H_1}{\partial \mu_k} - p_j(\lambda_k) \frac{\partial H_1}{\partial \lambda_k} p_i(\lambda_k) \frac{\partial H_1}{\partial \mu_k} = 0$. □

Corollary 5.3. *Under the assumption of the above proposition the functions $\{H_j\}_{1 \leq j \leq n}$ of a generalized Lenard chain are in bi-involution, i.e. are in involution also w.r.t. the Poisson bracket defined by the Poisson tensor $P_1 := NP_0$.*

Proof. From (2.17) and (5.3) it follows that

$$\{H_i, H_j\}_{P_1|k} = p_i(\lambda_k) \frac{\partial H_1}{\partial \lambda_k} \lambda_k p_j(\lambda_k) \frac{\partial H_1}{\partial \mu_k} - \lambda_k p_j(\lambda_k) \frac{\partial H_1}{\partial \lambda_k} p_i(\lambda_k) \frac{\partial H_1}{\partial \mu_k} = 0, \quad (5.5)$$

whence the thesis. □

Let us compare this proposition with the necessary and sufficient condition proved by Falqui & Pedroni (2003) (see also the work of Falqui in this volume).

Proposition 5.4. (Falqui & Pedroni 2003) *Let (M, ω, N) be an ωN manifold, N possessing n distinct eigenvalues. Let $\{H_j\}_{1 \leq j \leq n}$ be n independent functions in involution. The Hamiltonian functions $\{H_j\}_{1 \leq j \leq n}$ are separable if and only if*

i) \mathcal{D}° is invariant w.r.t. to the adjoint tensor N^* , i.e.

$$N^* \mathbf{dH} = \mathbf{F} \mathbf{dH} \quad (5.6)$$

ii) $N^*_{|\mathcal{D}^\circ_m}$ possesses distinct eigenvalues.

As the (transpose of the) matrix \mathbf{F} represents $N^*_{|\mathcal{D}^\circ_m}$ in the natural base $\{dH_j\}_{1 \leq j \leq n}$, it is called the control matrix of N . In order to compare the two above propositions it suffices to interpret the latter in terms of cyclic vector subspaces (see, e.g., Gantmacher 1998, page 184.) A possible characterization of such subspaces is the following

Definition 5.5. *Let V be a finite dimensional vector space, and $N^* : V \rightarrow V$ a linear operator. A subspace $W \subseteq V$ is said to be cyclic w.r.t. N^* , or N^* -cyclic, iff*

- it is invariant w.r.t. N^*

$$N^*(V) \subseteq V \quad (5.7)$$

- denoting $N^*_{|V}$ the restriction of N^* to V , the minimal polynomial of $N^*_{|V}$ coincides with its characteristic polynomial

$$m_{N^*_{|V}}(z) \equiv \Delta_{N^*_{|V}}(z) \quad (5.8)$$

Comparing the two items of the above definition with the two ones of proposition 5.4, it is easy to understand that the latter can be recast in the following equivalent form.

Proposition 5.6. *Let (M, ω, N) be an ωN manifold, N possessing n distinct eigenvalues. Let $\{H_j\}_{1 \leq j \leq n}$ be n independent functions in involution. The Hamiltonian functions $\{H_j\}_{1 \leq j \leq n}$ are separable in DN coordinates if and only if \mathcal{D}° is a distribution of N^* -cyclic subspaces of T^*M .*

Now we are in the position of explaining the connection between propositions 5.2 and 5.4. Let us denote \mathbf{C}_m the companion matrix of the minimal polynomial $m(z)$ of N

$$\mathbf{C}_m = \begin{bmatrix} 0 & \cdots & \cdots & 0 & c_n \\ 1 & \ddots & \cdots & \vdots & c_{n-1} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & c_2 \\ 0 & \cdots & 0 & 1 & c_1 \end{bmatrix}. \quad (5.9)$$

It represents $N_{|\mathcal{D}_m^\circ}$ on every cyclic base. As the matrix \mathbf{A} is nothing but the transition matrix between the natural base $\{dH_1, dH_2, \dots, dH_n\}$ and the cyclic base generated by dH_1 , i.e. $\{dH_1, N^*dH_1, \dots, (N^*)^{n-1}dH_1\}$, it follows that

$$\mathbf{F} = \mathbf{A} \mathbf{C}_m^T \mathbf{A}^{-1} . \quad (5.10)$$

For the special case of Lenard chains (3.4), the natural base being just a cyclic base, it is $\mathbf{A}_L = \mathbb{I}_n$ therefore $\mathbf{F}_L = \mathbf{C}_m^T$.

For Fröbenius chains, the natural base can be verified to coincide with the so-called *control base* generated by dH_1 (see Fuhrmann 1996, page 98), whose elements are defined by $e_i := m_i(N^*) dH_1$ ($i = 1, \dots, n$), where $m_i(z)$ are polynomials given by

$$m_i(z) := z^{i-1} - c_1 z^{i-2} - c_2 z^{i-3} - \dots - c_{i-2} z - c_{i-1} . \quad (5.11)$$

Consequently, for Fröbenius chains the transition matrix \mathbf{A}_F is the following Hankel matrix

$$\mathbf{A}_F = - \begin{bmatrix} -1 & 0 & \cdots & \cdots & 0 \\ c_1 & \ddots & \ddots & \ddots & \vdots \\ c_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ c_{n-1} & \cdots & c_2 & c_1 & -1 \end{bmatrix} \quad (5.12)$$

and the control matrix \mathbf{F}_F is given by (4.7).

Remark 5.7. *The above arguments and corollary (5.3) imply that the members of a Lenard chain and the elements of a Fröbenius chain (4.8) are mutually in bi-involution.*

Below we present two examples of separable Hamiltonian systems which are members of truly generalized Lenard chains (neither Lenard nor Fröbenius chains). For other examples see the work of Falqui & Musso (2003).

(a) *The Lagrange top*

Physically, the Lagrange top is a heavy symmetric rigid body with a fixed point and the center of mass lying along the symmetry axis. The ratio between the principal inertia momenta is $1 : 1 : b$, b being a constant parameter. In the phase space M_6 , parametrized by the coordinates $(\boldsymbol{\omega}, \boldsymbol{\gamma}) := (\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3)$, where ω_i and γ_i ($i = 1, 2, 3$) are respectively the components of the angular momenta and of the vertical unit vector along the inertia axes in the comoving body-frame, the Euler-Poisson equations of motion read

$$\begin{aligned} \dot{\omega}_1 &= -(b-1)\omega_2\omega_3 - \gamma_2 & \dot{\gamma}_1 &= \gamma_2\omega_3 - \gamma_3\omega_2 \\ \dot{\omega}_2 &= (b-1)\omega_3\omega_1 + \gamma_1 & \dot{\gamma}_2 &= \gamma_3\omega_1 - \gamma_1\omega_3 \\ \dot{\omega}_3 &= 0 & \dot{\gamma}_3 &= \gamma_1\omega_2 - \gamma_2\omega_1 \end{aligned} \quad (5.13)$$

The Lagrange top vector field X_L (the right hand side of system (5.13)) admits a degenerate BH formulation (see Morosi & Tondo 2002 and reference therein)

$$X_L = P_0 dh_0 = P_1 dh_1; \quad (5.14)$$

the compatible Poisson tensors P_k ($k = 0, 1$), written in matrix block form, are

$$P_0 = \begin{bmatrix} \mathbf{0}_3 & \mathbf{B} \\ \mathbf{B} & \mathbf{C} \end{bmatrix}, \quad P_1 = \begin{bmatrix} -\mathbf{B} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{\Gamma} \end{bmatrix}, \quad (5.15)$$

where \mathbf{B} , \mathbf{C} , $\mathbf{\Gamma}$, are 3×3 matrices

$$\mathbf{B} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & b\omega_3 & -\omega_2 \\ -b\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}, \quad \mathbf{\Gamma} = \begin{bmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{bmatrix}, \quad (5.16)$$

and the Hamiltonian functions are

$$h_0 = \frac{1}{2}F_4 + (b-1)F_1F_3 \quad (5.17)$$

$$h_1 = \frac{1}{2}b(b-1)F_1^3 - F_3 - (b-1)F_1F_2 \quad (5.18)$$

where

$$\begin{aligned} F_1 &= \omega_3 & F_2 &= \frac{1}{2}(\omega_1^2 + \omega_2^2 + \omega_3^2) - 2\gamma_3 \\ F_3 &= \omega_1\gamma_1 + \omega_2\gamma_2 + \omega_3\gamma_3 & F_4 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 \end{aligned} \quad (5.19)$$

The two Poisson brackets defined by P_0 and P_1 are degenerate. It has been proved by Morosi & Tondo (2003) that the Lagrange top is a Gel'fand–Zakharevich system of rank two and that the Poisson pencil $P_1 - \lambda P_0$ results a pure Kronecker pencil of type (1, 5) in the classification of Gel'fand & Zakharevich (2000). In fact it has two polynomial Casimir functions, one of degree 0 and one of degree 2.

Fixing the value of the two Casimir functions F_1 , F_2 of P_0 , the system (4.38) can be restricted to a generic four-dimensional symplectic leaf S_0

$$S_0 = \{m \in M \mid \omega_3 = C_1, \quad \omega_1^2 + \omega_2^2 + c\omega_3^2 - 2\gamma_3 = 2C_2\}, \quad (5.20)$$

together with P_0 . Unfortunately the other tensor P_1 does not restrict to S_0 . Nevertheless, a quite general reduction technique (Falqui et al. 2000; Falqui et al. 2001) given by the Marsden-Ratiu theorem (1994) can be applied; it has enabled us to construct on S_0 an ωN structure induced by the bi-Hamiltonian structure on M_6 . Moreover, the polynomial Casimir function of degree 2 restricted to S_0 gives rise to a generalized Lenard chain whose generator is the Hamiltonian function of the restricted vector field of the Lagrange top X_L^r . For further generalizations of such a type of reduction see the work of Falqui & Pedroni (2002).

In conclusion we have proved the following

Proposition 5.8. (Morosi & Tondo 2002) *The bi-Hamiltonian structure (P_0, P_1) of the Lagrange top is reducible to an ωN structure on each symplectic leaf S_0 of P_0 .*

To express the reduced tensors in a particularly simple and useful form, it is convenient to choose the chart $(\mathbf{c}, \mathbf{f}, \mathbf{w})$ in M_6 , related to $(\boldsymbol{\omega}, \boldsymbol{\gamma})$ by the map $\Phi : M \rightarrow M : (\boldsymbol{\omega}, \boldsymbol{\gamma}) \mapsto (\mathbf{c}, \mathbf{f}, \mathbf{w})$

$$c_1 = -b\omega_3 + i\omega_2, \quad c_2 = -i\gamma_2 + \gamma_3, \quad (5.21)$$

$$f_1 = \omega_1, \quad f_2 = -\gamma_1, \quad w_1 = i\omega_2 + b\omega_3, \quad w_2 = -i\gamma_2 - \gamma_3.$$

Taking into account the bi-Hamiltonian structure P_k given by (5.15) and the definition 5.20 of S_0 , a straightforward (though lengthy) calculation allows one to verify that the chart (\mathbf{c}, \mathbf{f}) gives a parametrization on each one of the symplectic leaves S_0 ; the reduced Poisson tensors P'_0 and the tensor $N := P'_1(P'_0)^{-1}$ take the form

$$P'_0 = i \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -c_1 \\ 0 & -1 & 0 & 0 \\ -1 & c_1 & 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} c_1 & 1 & 0 & 0 \\ c_2 & 0 & 0 & 0 \\ 0 & 0 & c_1 & 1 \\ 0 & 0 & c_2 & 0 \end{bmatrix}. \quad (5.22)$$

Remark 5.9. *The matrix representation of P'_0 and of N' are formed by Hankel and Fröbenius blocks, respectively, so that (\mathbf{c}, \mathbf{f}) are the HF coordinates defined in remark (4.5).*

Proposition 5.10. *Let us consider the map $\Psi : S_0 \rightarrow S_0 : (\mathbf{c}, \mathbf{f}) \mapsto (\boldsymbol{\lambda}, \boldsymbol{\mu})$*

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(c_1 - \sqrt{c_1^2 + 4c_2}), & \lambda_2 &= \frac{1}{2}(c_1 + \sqrt{c_1^2 + 4c_2}), & (5.23) \\ \mu_1 &= \frac{1}{2}(2f_2 + f_1c_1 - f_1\sqrt{c_1^2 + 4c_2}), & \mu_2 &= \frac{1}{2}(2f_2 + f_1c_1 + f_1\sqrt{c_1^2 + 4c_2}). \end{aligned}$$

The chart $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a Darboux-Nijenhuis chart for the ωN structure (5.22) on S_0 ; the reduced Poisson tensors P'_0 and N have the matrix block form

$$P'_0 = -i \begin{bmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} \boldsymbol{\Lambda}_2 & 0 \\ 0 & \boldsymbol{\Lambda}_2 \end{bmatrix}, \quad (5.24)$$

where $\boldsymbol{\Lambda}_2 = \text{diag}(\lambda_1, \lambda_2)$.

Proof. A straightforward computation, taking into account equations (5.22) and (5.23). \square

On the manifold S_0 , the restriction of the Hamiltonian function h_0 takes the form

$$\tilde{H}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = h_0|_{S_0} = (b-1)C_1 H(\boldsymbol{\lambda}, \boldsymbol{\mu}) - K(\boldsymbol{\lambda}, \boldsymbol{\mu}) \quad (5.25)$$

where $H(\boldsymbol{\lambda}, \boldsymbol{\mu}) = F_3|_{S_0}$ and $K(\boldsymbol{\lambda}, \boldsymbol{\mu}) = -\frac{F_4}{2}|_{S_0}$ have the form (4.25) with

$$g(\lambda_i, \mu_i) = -\frac{1}{2}\mu_i^2 + \frac{1}{2}\lambda_i^4 + bC_1\lambda_i^3 + \left(C_2 + \frac{1}{2}b(b-1)C_1^2\right)\lambda_i^2 \quad i = 1, 2. \quad (5.26)$$

Hence, H and K are members of the Fröbenius chain generated by H , whilst \tilde{H} and K are members of a generalized Lenard chain. In fact, denoting $H_1 := \tilde{H}$ and $H_2 := K$, we find that the transition matrix \mathbf{A} and the control matrix \mathbf{F} in terms of the HF coordinates (\mathbf{c}, \mathbf{f}) are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -\frac{c_1 - c_2/a}{(c_1 - c_2/a) + a} & \frac{1}{(c_1 - c_2/a) + a} \end{bmatrix} \quad (5.27)$$

$$\mathbf{F} = \begin{bmatrix} c_1 - c_2/a & (c_1 - c_2/a) + a \\ c_2/a & c_2/a \end{bmatrix} \quad (5.28)$$

where $a = (b-1)C_1$.

Remark 5.11. Taking into account the separation equations (4.27) for H , K and equations (5.25) it can be easily proved that the separation equations for H_1 and H_2 read

$$\mu_i^2 = \lambda_i^4 + 2bC_1\lambda_i^3 + [2C_2 + b(b-1)C_1^2]\lambda_i^2 - \frac{2}{a}(H_1 + H_2)\lambda_i - 2H_2, \quad (5.29)$$

($i = 1, 2$). Now, we are able to connect with the Sklyanin method of SoV. Indeed, comparing (5.29) with the spectral curve coming from the Lax pair (see, e.g., Gavrillov & Zhivkov 1998), it can be noticed that the DN separation coordinates (λ, μ) satisfy the equation of the elliptic spectral curve (5.29).

(b) *The t_5 -Boussinesq stationary flow*

It is well known that the Boussinesq equation $u_{t_2 t_2} = \frac{1}{3}(-u_{xxxx} + 4u_x^2 + 4uu_{xx})$ represented as a first order system of equations in the variables (u, v)

$$\begin{aligned} u_{t_2} &= -u_{xx} + 2v_x \\ v_{t_2} &= v_{xx} - 2/3 u_{xxx} + 2/3 uu_x \end{aligned} \quad (5.30)$$

is a member of an infinite hierarchy of evolution equations. These equations are defined as follows. In the space \mathcal{M}_3 of the third-order differential operators $L = \partial_x^3 + u(x)\partial_x + v(x)$, one considers the third root $L^{\frac{1}{3}}$ of L , i.e. a pseudodifferential operator of the form

$$L^{\frac{1}{3}} = \partial + \sum_{i=1}^{\infty} q_i(u, v)\partial^{-i} \quad (5.31)$$

satisfying $(L^{\frac{1}{3}})^3 = L$. By means of the powers of such a third root, one defines an infinite family of commutative flows on \mathcal{M}_3 as

$$\frac{\partial}{\partial t_p} L = [L, (L^{\frac{p}{3}})_+] \quad (5.32)$$

where $(\cdot)_+$ is the projection on the purely differential part of a pseudodifferential operator.

Stationary flows for such a system correspond to that subspace of operators L satisfying the (non-linear) relation $[L, L^{\frac{p}{3}}] = 0$ for some p .

Below we will present only the results of the case $p = 5$. The corresponding equation of the hierarchy reads:

$$\begin{aligned} u_{t_5} &= -1/9 u_{xxxxx} + 5/9 u_x u_{xx} + 5/9 uu_{xxx} + 5/3 u_{xx} v + \\ & 5/3 u_x v_x - 5/9 u^2 u_x - 10/3 v v_x \\ v_{t_5} &= -1/9 v_{xxxxx} + 10/9 u_{xxx} v + 20/9 u_{xx} v_x + 5/3 u_x v_{xx} + \\ & 5/9 uv_{xxx} + -5/3 v_x^2 - 5/3 v v_{xx} - 10/9 uvu_x - 5/9 u^2 v_x, \end{aligned} \quad (5.33)$$

and one can see that the stationary manifold $\mathcal{M}^{(5)}$, i.e. the set of the zeroes of $\frac{\partial}{\partial t_5}$ can be parametrized by the space of the Cauchy data of two fifth order ODE in two

variables. On such a ten-dimensional manifold we consider the (restriction of the) first *four* non-trivial flows of the Boussinesq hierarchy

$$\frac{\partial}{\partial t_p} L = [L, (L^{\frac{p}{3}})_+], \quad p = 1, 2, 4, 7. \quad (5.34)$$

It has been shown in Falqui et al. (2000) that on $\mathcal{M}^{(5)}$ one can construct a Poisson pencil $P_1 - \lambda P_0$ of pure Kronecker type (3, 7) after Gel'fand & Zakharevich (2000). In fact, such a pencil has two polynomial Casimir functions $H^{(1)}(\lambda) = H_1 \lambda + H_2$ and $H^{(2)}(\lambda) = H_3 \lambda^3 + H_4 \lambda^2 + H_5 \lambda + H_6$, the functions H_1 and H_3 being Casimir functions of P_0 , the functions H_2 and H_6 being Casimir functions of P_1 . Moreover, the functions H_2, H_4, H_5 and H_6 are the Hamiltonian functions, w.r.t. P_0 , of the four vector fields under scrutiny, while the functions H_1, H_3, H_4, H_5 are their Hamiltonian functions w.r.t. P_1 .

Also in this case, we have performed a Marsden-Ratiu reduction procedure similar to the one used in the case of the Lagrange top. Precisely, on each eight-dimensional symplectic leaf S_0 of P_0 , we have constructed an ωN structure inherited by the Poisson pencil on $\mathcal{M}^{(5)}$. Moreover, restricting to S_0 the two polynomial Casimir functions $H^{(1)}(\lambda)$ and $H^{(2)}(\lambda)$ we have obtained a generalized Lenard chain formed by the functions H_2, H_4, H_5, H_6 . The transition matrix \mathbf{A} and the control matrix \mathbf{F} in terms of the HF coordinates (\mathbf{c}, \mathbf{f}) are given by

$$\mathbf{A} = - \begin{bmatrix} -1 & 0 & 0 & 0 \\ f_1 c_1 + f_2 & -f_1 & 0 & 0 \\ f_1 c_2 + f_3 & f_1 c_1 & -f_1 & 0 \\ f_1 c_3 + f_4 & f_1 c_2 & f_1 c_1 & -f_1 \end{bmatrix} \quad (5.35)$$

$$\mathbf{F} = \frac{1}{f_1} \begin{bmatrix} f_1 c_1 + f_2 & 1 & 0 & 0 \\ -c_1 f_1 f_2 - f_2^2 + f_1^2 c_2 + f_3 f_1 & -f_2 & f_1 & 0 \\ -c_1 f_3 f_1 - f_3 f_2 + f_1^2 c_3 + f_4 f_1 & -f_3 & 0 & f_1 \\ f_1^2 c_4 - f_4 f_1 c_1 - f_4 f_2 & -f_4 & 0 & 0 \end{bmatrix}. \quad (5.36)$$

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