Vortex nucleation in NC supercurrent

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Abstract

We face the problem of spontaneous nucleation of particle-antiparticle pairs in a (2+1) dimensional system, due to a static electromagnetic external field. We recall the relevance of this calculation for the vortex-antivortex nucleation in a thin superconducting film. Postulating noncommutativity for the spatial coordinates, we determine the vortex nucleation rate in a superconductor due to its planar supercurrent decay by a path integral approach.

Some authors had investigated the phenomenology of superfluids and superconductors by means of a relativistic field theory approach [1]. In particular the phenomenon of the decay of a supercurrent through homogeneous nucleation of vortex-antivortex pairs in a 2D like superconductor or superfluid has been analyzed via a quantum electrodynamic formulation for the decay of the 2D vacuum [2]. The formalism adopted corresponds to the two-dimensional version of the well-known Schwinger calculation [3] for the three-dimensional quantum electrodynamic problem of vacuum decay by production of electron-positron pairs that are taken to infinity upon creation [4]. The probability for the vacuum decay in time T is given by

$$e^{-iTW_0} \equiv \langle 0|e^{-i\hat{H}T}|0\rangle \underset{T\to\infty}{=} \langle 0|S|0\rangle \tag{1}$$

where $W_0 = \varepsilon_{vac} - i\frac{\Gamma}{2}$ is taken to give the energy of the vacuum ε_{vac} and is decay rate Γ. The probability amplitude is given by the functional integral over field configurations

$$e^{-T_E W_0} = \mathcal{N}\int \mathcal{D}\phi \exp\left\{-i\int dt d\mathbf{x}\, \phi^*\left(-D_0^2 + \mathbf{D}^2 - m^2\right)\phi\right\} \tag{2}$$

which is more conveniently evaluated in the Euclidean metric ($x_3 \equiv it$) by means of

$$W_0 = \lim_{T_E\to\infty} \frac{1}{T_E} Tr \ln(-D_3^2 - \mathbf{D}^2 + m^2). \tag{3}$$

The above expression properly corresponds to the standard contribution to the vacuum energy of a complex scalar field, whose quanta constitute the particles-antiparticles. Vacuum instability will manifest itself when, in presence of an external field, the virtual loops extend to infinity, giving rise

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to an imaginary part for the vacuum energy W_0. D is here the covariant derivative, containing the electromagnetic field A_μ, while m is the activation energy, that is the quanta rest mass. By means of known identity one gets

$$Tr \ln \frac{-D^2+m^2}{\Lambda} = -\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{d\tau}{\tau}\left(Tr e^{-(-D^2+m^2)\tau} - Tr e^{-\Lambda\tau}\right) \tag{4}$$

where we have defined $D^2 \equiv \mathbf{D}^2 + D_3^2$. The evaluation of the trace is straightforward with the method of the Feynman path integral; then one obtains

$$W_0[A_\mu] = \lim_{T_E\to\infty} -\frac{1}{T_E}\int_0^\infty \frac{d\tau}{\tau} e^{-m^2\tau}\mathcal{N}\int_{q_\mu(0)=q_\mu(\tau)} \mathcal{D}q_\mu e^{-\int_0^\tau ds\mathcal{L}_E} \tag{5}$$

where $\mathcal{L}_E$ is the Euclidean Lagrangian for the particle coupled to the external field A_μ

$$\mathcal{L}_E = \frac{1}{2}\mu\dot{q}_\mu^2 - i\dot{q}_\mu A_\mu(q) \tag{6}$$

with the mass parameter $\mu = 1/2$. In the uniform-field case calculation can be carried on exactly, being $A_\mu(q) = \frac{1}{2}F_{\mu\nu}q_\nu$, where the (2+1)-dimensional field tensor is given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & B & iE_x \\ -B & 0 & iE_y \\ -iE_x & -iE_y & 0 \end{pmatrix} \tag{7}$$

with $\mathbf{E} = (E_x, E_y, 0)$ and $\mathbf{B} = (0, 0, B_z)$ after analytic continuation to Euclidean time. The evaluation of the path integral give rise to the following expression

$$\frac{W_0}{L^2} = -\int_\epsilon^\infty \frac{d\tau}{\tau} e^{-m^2\tau}\left(\frac{1}{4\pi\tau}\right)^{3/2}\frac{M\tau}{\sin M\tau} \tag{8}$$

with $M^2 = E^2 - B^2$. Here we considered the renormalized expression obtained by subtraction of the vacuum energy in absence of electromagnetic fields. We observe that the integrand function is endowed of first order polar singularities for those values of τ s.t. $\sin M\tau = 0$. In order to estimate the production rate it is necessary to calculate the imaginary part of the vacuum energy, performing the sum on the residues of these poles. Such calculation demands the choice of the contour integral around singularities. The ambiguity can be solved by inserting a suitable $i\epsilon$ prescription on the mass. For this goal one defines $x \equiv m^2\tau$ and gets

$$\begin{aligned} Im\frac{1}{\sin(xM/m^2)} \underset{x\approx n\pi m^2/M}{\approx} & \; Im\frac{(-1)^n m^2/M}{x - n\pi m^2/M + i\epsilon} = \\ & = \frac{(-1)^{n+1}m^2\pi\delta(x - n\pi m^2/M)}{M} \end{aligned} \tag{9}$$

having used the property $Im\frac{1}{x-i\epsilon} = \pi\delta(x)$.

Finally one obtains

$$\frac{\Gamma}{2L^2} = \frac{\left(\sqrt{E^2-B^2}\right)^{3/2}}{8\pi^2}\sum_{n=1}^\infty (-1)^{n+1} n^{-3/2}\exp\left(-\frac{\pi m^2 n}{\sqrt{E^2-B^2}}\right) \tag{10}$$

in which we substitute the explicit form for M. We underline the main contribution of the first pole, while the others are exponentially suppressed. Let us notice that

$$E^2 - B^2 > 0 \tag{11}$$

is the threshold condition for the vacuum/supecurrent decay. Then for $B = 0$ one can write an approximate expression

$$\frac{\Gamma}{2L^2} = \frac{E^{3/2}}{8\pi^2} e^{-\frac{\pi m^2}{E}} \tag{12}$$

The kernel of the above calculation lies in Feynman path integral for virtual particle that give rise to vacuum fluctuation and its instability. Very recently some authors [5] had brilliantly formulated the Feynman path integral on a non commutative plane i.e. in a space-time where the following relation holds

$$[x_i, x_j] = i\Theta_{ij}. \tag{13}$$

The customary procedure for building a noncommutative physical theory consists in the introduction of the Moyal $*$-product [6] for the ordinary function of commuting variables. In spite of the correctness of this approach, its value and predictive power remain only on a formal ground, because the introduction of a natural UV cutoff in the theory does not clearly appear in final results. The reason is due to the fact that actual calculation are performed using truncated series expansions in the parameter characterizing the non commutativity of space: as a consequence the free particle propagator remains unaffected by the $*$-product, while interacting fields exhibits an even worse phenomenon i.e. UV/IR mixing. For these reasons the aforementioned formulation [5] is very promising, since bravely introduces the noncommutativity through coherent states in field theory instead of $*$-product. The final and most important result is the noncommutative version of the path integral for the propagation kernel

$$K_\theta\left(x-y,T\right) \equiv \mathcal{N}\int \mathcal{D}x\mathcal{D}p \exp\left\{ i\int_y^x \vec{p}\cdot d\vec{x} - \int_0^T d\tau \left(H(\vec{p},\vec{x}) + \frac{\theta}{2T}\vec{p}^2 \right) \right\} \tag{14}$$

where noncommutativity manifests itself via the extra term $\frac{\theta}{2T}\vec{p}^2$ in the action. Computing the propagator for a free particle one discovers that it short time limit is non more a delta function but a Gaussian for the fact that the best possible localization of the particle, in the non commutative space, is within a cell of area θ.

The main goal of this paper is to calculate the 2D pair nucleation rate for the vacuum/supercurrent dual decay, at the light of the this new formulation for the Feynman path integral: we expect to find a lower production rate, since noncommutativity has enlarged the volume of the Heisemberg cell i.e. has extended the quantum indetermination to the spatial coordinates too.

The starting point for the evaluation of the non commutative pair production rate is the expression for the vacuum energy

$$W_\theta = \lim_{T_E\to\infty} -\frac{1}{T_E}\int_0^\infty \frac{d\tau}{\tau} e^{-m^2\tau}\mathcal{N}\int_{q_\mu(0)=q_\mu(\tau)} \mathcal{D}q_\mu e^{-\int_0^\tau ds\mathcal{L}_\theta} \tag{15}$$

where we are considering a modified Lagrangian for a particle in a non commutative plane according to (14)

$$\mathcal{L}_\theta = \frac{1}{2}\mu\dot{q}_\mu^2 - i\dot{q}_\mu A_\mu(q) \tag{16}$$

with a redefined fictitious mass parameter

$$\mu \equiv \frac{1}{2} + \frac{\theta}{\tau}. \tag{17}$$

The explicit form of the path integral now reads

$$\mathcal{N} \int_{q_\mu(0)=q_\mu(\tau)} \mathcal{D}q_\mu e^{-\int_0^\tau ds \frac{1}{2}\mu \dot{q}_\mu^2 - i\dot{q}_\mu A_\mu}. \tag{18}$$

Without loss of generality we can choose the following conditions: $B = 0$ and $\mathbf{E} = (E, 0, 0)$ i.e.

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & iE \\ 0 & 0 & 0 \\ -iE & 0 & 0 \end{pmatrix}. \tag{19}$$

Bearing in mind the normalization for the free particle

$$\mathcal{N}^{(d)} \int_{q_\mu(0)=q_\mu(\tau)} \mathcal{D}q_\mu e^{-\int_0^\tau ds \frac{1}{2}\mu \dot{q}_\mu^2} \equiv \left(\frac{\mu}{2\pi\tau}\right)^{d/2} \tag{20}$$

we can integrate out the kinetic part in the direction without electric field. Then, expanding $q_1(s)$ in Fourier series

$$q_1(s) = q_{1(0)} + \sum_{n\neq 0} q_{1(n)} e^{2\pi i n s/\tau} \tag{21}$$

and using the antisymmetry of $F_{\mu\nu}$ one get

$$\mathcal{L}_\theta = \frac{1}{2}\mu \left(\dot{q}_1^2 + \dot{q}_3^2\right) + E\tilde{q}_1 \dot{q}_3 \tag{22}$$

where

$$\tilde{q}_1 \equiv \sum_{n\neq 0} q_{1(n)} e^{2\pi i n s/\tau}. \tag{23}$$

For the periodicity of the spatial coordinate we have

$$\int_0^\tau \tilde{q}_1 ds = 0 \tag{24}$$

from which descends that

$$\tilde{q}_1 = \frac{d}{ds} f(s) \tag{25}$$

where $f(s)$ is suitable function. Then one is left with the following expression for the path integral

$$\left(\frac{\mu}{2\pi\tau}\right)^{1/2} \mathcal{N}^{(2)} \int_{q_\mu(0)=q_\mu(\tau)} \mathcal{D}q_1 \mathcal{D}q_3 e^{-\int_0^\tau ds \frac{1}{2}\mu \dot{q}_1^2 - \int_0^\tau ds \left[\frac{d}{ds}\left(\frac{1}{2}\mu q_3 + 2Ef(s)\right)\right]^2 + \int_0^\tau ds E^2 \tilde{q}_1^2}. \tag{26}$$

Redefining in suitable manner the time coordinate we can integrate it getting another factor $\left(\frac{\mu}{4\pi\tau}\right)^{1/2}$. The last integration over $q_{1(n)}$ give rise to poles in variable τ which in turn, when inserted in the integration over τ with a suitable $i\epsilon$ prescription, give rise to an imaginary part for the vacuum energy.

$$\left(\frac{\mu}{2\pi\tau}\right)\mathcal{N}^{(1)}\int\prod d^2q_{1(n)}e^{2\tau\left[\frac{1}{2}\mu\left(\frac{2\pi n}{\tau}\right)^2-E^2\right]|q_{1(1)}|^2}=\left(\frac{\mu}{2\pi\tau}\right)^{3/2}\prod_{n=1}\frac{1}{1-\dfrac{\tau^2E^2}{2\pi^2n^2\mu}} \tag{27}$$

The dominant pole, corresponding to the minimal value of τ and thus giving the leading term in an expansion made up of further exponentially-suppressed contributions, comes from integrating over $q_{1(1)}$ or from an approximation of the above result for in the vicinity of the pole

$$\left(\frac{\mu}{2\pi\tau}\right)^{3/2}\prod_{n=1}\frac{1}{1-\dfrac{\tau^2E^2}{2\pi^2n^2\mu}}\approx\left(\frac{\mu}{2\pi\tau}\right)^{3/2}\frac{\pi}{\pi-\left(\dfrac{\tau^2E^2}{2\pi^2\mu}\right)^{1/2}}. \tag{28}$$

We stress that the noncommutative effect is believed to be a small perturbation i.e.

$$\theta m^2\ll 1 \tag{29}$$

so that expanding μ as a function of θ we obtain

$$\frac{\Gamma_\theta}{2L^2}=-\int_\epsilon^\infty\frac{d\tau}{\tau}e^{-m^2\tau}\left(\frac{\mu}{2\pi\tau}\right)^{3/2}\frac{\pi/E}{\tau-\dfrac{\pi}{E}-\theta+i\epsilon}. \tag{30}$$

Using $Im\frac{1}{x+i\epsilon}=-\pi\delta(x)$ one finally get the non commutative pair nucleation rate

$$\frac{\Gamma_\theta}{2L^2}=\frac{E^{3/2}}{8\pi^2}\left(1+\frac{\theta E}{\pi}\right)^{-5/2}\left(1+\frac{2}{\pi}\frac{\theta E}{\left(1+\dfrac{\theta E}{\pi}\right)}\right)^{3/2}e^{-(\frac{m^2\pi}{E}+m^2\theta)} \tag{31}$$

that can be rewritten as

$$\frac{\Gamma_\theta}{2L^2}=\frac{\Gamma}{2L^2}\frac{\left(1+\dfrac{2}{\pi}\dfrac{\theta E}{\left(1+\dfrac{\theta E}{\pi}\right)}\right)^{3/2}}{\left(1+\dfrac{\theta E}{\pi}\right)^{5/2}}e^{-m^2\theta}. \tag{32}$$

Since the Planck length is definitely small and the external electric field cannot reach the value of the mass parameter m, we can give the following expression for the regime $\theta E\ll 1$, $\theta m^2\ll 1$ and $m^2/\theta\gg 1$

$$\frac{\Gamma_\theta}{2L^2}\simeq\frac{\Gamma}{2L^2}\left(1-\theta m^2\right). \tag{33}$$

The smallness of the correcting term furnishes concern about the importance of a similar procedure in the context of vortex nucleation rate; on the other hand a similar treatment becomes fundamental in the general framework of a revisited quantum theory in presence of noncommutativity. The main result is the decreasing of the nucleation rate one expects from introducing further indetermination through (13). Even if the present status of path integral (14) regards the non commutative plane, one expects to extend the formalism to the space time volume in order to describe effects of cosmological interest.

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