Vacuum energy in (2+1) NC scalar field theory

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Abstract

A scalar field in $(2+1)$ dimensional Minkowski space-time is considered. Postulating noncommutative spatial coordinates, one is able to determine the (UV finite) vacuum expectation value of the quantum field energy momentum tensor. Calculation for the $(3+1)$ case has been performed considering only two noncommutative coordinates. The results lead to a vacuum energy with a lowered degree of divergence, with respect to that of ordinary commutative theory.

Recent cosmological data [1] show that the expansion of the universe is presently accelerating. This acceleration effect is driven by a still undetermined sort of dark mass/energy. It is a common belief that the cosmological constant, or vacuum energy, provides a relevant contribution to the dark energy [2]. On the contrary of what was believed in the past, the cosmological constant is not zero but gives a contribution to the total mass energy of the universe, which is comparable to the one from visible matter. By means of the value of Λ, coming from observations of Type Ia supernovae, one can estimate the cosmological energy density to be [3]

$$|\mathcal{E}_\Lambda| = |\Lambda| / (8\pi G) \sim 5 \times 10^{-31} g/cm^3. \tag{1}$$

Since the Zel'dovich seminal paper [4], the cosmological constant is interpreted as the macroscopic effect produced by quantum matter vacuum fluctuations. Unfortunately, quantum field theory and even string theory are presently unable to reproduce the aforementioned value (1). Technically, the main difficulty encountered is related to the bad short distance behavior of point-like objects, which is only partially cured in string theory. On the other hand the great interest towards noncommutative geometry [5] could be explained with the hope that a correct formulation of noncommutative field theory could be free from UV divergences. In this framework one expects to find a finite value for the vacuum energy and thus for the cosmological constant. Many authors faced the problem of obtaining the vacuum expectation value for the energy momentum tensor (EMT) in a noncommutative space-time [6]: at present for (3+1) dimensional fields there are theories based on the Moyal $\star$-product, which are notoriously affected by some important problems, such as unitarity, Lorentz invariance breaking and UV/IR mixing. Exploiting the coherent states formalism, an alternative approach has been proposed [7] to formulate a (2+1) dimensional noncommutative

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field theory, which exhibits some positive aspects: the formalism is handy, the UV divergences can be cured, a minimal length is introduced in a clear and evident way. At the light of the above considerations, we retain useful to face the problem of the vacuum energy due to a scalar field in (2+1) dimensional space-time, when only spatial coordinates noncommute, while the time is still commutative. This is a preliminary step to a more realistic computation in (3+1) dimensions.

The expression for the EMT is generally given by

$$T_{\alpha\beta} = \phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\lambda\delta}\phi_{,\lambda}\phi_{,\delta} + \frac{1}{2}m^2\phi^2\eta_{\alpha\beta}. \tag{2}$$

After expansion on Fourier modes, one can compute its quantum expectation value on Fock space. Special interest attaches to the vacuum state $|0\rangle$, for which one can write [8]

$$\langle 0|T_{\alpha\beta}|0\rangle = \int d\mathbf{k} \;\; T_{\alpha\beta}[u(\mathbf{k}), u^*(\mathbf{k})] \tag{3}$$

where $T_{\alpha\beta}[u(\mathbf{k}), u^*(\mathbf{k})]$ denotes the bilinear expression (2) for $T_{\alpha\beta}$ and

$$u(\mathbf{k}) = \frac{1}{\sqrt{2\,(2\pi)^2\,k^0}} \exp\left(-ik_\mu x^\mu\right), \tag{4}$$

with the wave vector $k^\mu = (k^0, \mathbf{k})$. The temporal components, namely $\langle T_{00}\rangle$, gives the vacuum energy density

$$\mathcal{E} = \int d\mathbf{k} \;\; (\mathbf{k}^2 + m^2)\; u(\mathbf{k})u^*(\mathbf{k}) \tag{5}$$

a quantity that is non surprisingly different from zero. It is well known that to the zeroth order in the coupling, the field theory corresponds to an infinite collection of harmonic oscillators with the nth oscillator having a zero-point, or ground state energy, given by $\frac{1}{2}\omega_n$ and the sum $\sum_n \frac{1}{2}\omega_n$ diverges. This represents a well known and open problem when gravity is present, because the standard procedure of (infinite) rescaling of ground state energy is no more viable. Regarding this problem noncommutative geometry could provide a natural UV cut-off, namely a minimal length in space-time.

The simplest NC manifold is a 2D plane with spatial coordinate operators $\hat{x}_i$ subject to the following relations

$$[\hat{x}_i, \hat{x}_j] = i\theta\epsilon_{ij} \quad i, j = 1, 2 \tag{6}$$

θ has dimension of a length squared and measures the noncommutativity of spatial coordinates, while the time like coordinate x_0 remains commutative. As a consequence of (6) the noncommutative plane is divided into plaquette of area θ. One cannot speak of points anymore and the space becomes blurry. Then the best we can do is to define average coordinates x_i in an appropriate coherent state basis. The difference with respect the commutative theory is that the operators $\hat{x}_1$ and $\hat{x}_2$ have no common position eigenvector for (6). The convenient and physically meaningful choice is to deal with the following set of operators

$$\hat{z} \equiv \frac{1}{\sqrt{2}}\left(\hat{x}_1 + i\hat{x}_2\right) \tag{7}$$

$$\hat{z}^\dagger \equiv \frac{1}{\sqrt{2}}\left(\hat{x}_1 - i\hat{x}_2\right) \tag{8}$$

$$\tag{9}$$

in order to obtain

$$\left[\hat{z}, \hat{z}^{\dagger}\right] = \theta. \tag{10}$$

We can see that this is a familiar relation: θ plays the analogue role of $\hbar$ in ordinary quantum mechanics. As shown in [9] the operator $\hat{z}$ admits eigenstates

$$\hat{z}|z\rangle = z|z\rangle \tag{11}$$

$$\langle z|\hat{z}^{\dagger} = \langle z|\bar{z} \tag{12}$$

where z is a complex eigenvalue. $|z\rangle$ are coherent states and can be used as a basis of the Fock space. They are given by

$$|z\rangle \equiv \exp\left(-\frac{z\bar{z}}{2\theta}\right)\exp\left(-\frac{z}{\theta}\hat{z}\right)|0\rangle \tag{13}$$

where $|0\rangle$ is the vacuum state annihilated by $\hat{z}$. In order to formulate a field theory we assign ordinary function to any operator $F\left(\hat{x}_1, \hat{x}_2\right)$ via expectation values

$$F(z) \equiv \langle z|F\left(\hat{x}_1, \hat{x}_2\right)|z\rangle. \tag{14}$$

As particular cases we have

$$\langle z|\hat{x}_1|z\rangle = \sqrt{2}\,\mathcal{R}ez \tag{15}$$

$$\langle z|\hat{x}_2|z\rangle = \sqrt{2}\,\mathcal{I}mz. \tag{16}$$

The algebraic structure of the defining noncommutative relation (6) is properly taken care of, namely

$$\langle z|\left[\hat{x}_1, \hat{x}_2\right]|z\rangle = i\theta. \tag{17}$$

Assuming that momenta commute among themselves, the noncommutative version of the Fourier transform is given by

$$F\left(z\right) = \int \frac{d^2k}{2\pi}\tilde{F}\left(k\right)\langle z|\exp(ik^i\hat{x}_i)|z\rangle. \tag{18}$$

By means of Hausdorff decomposition of the above exponential we are left with

$$F\left(z\right) = \int \frac{d^2k}{2\pi}\tilde{F}\left(k\right)\exp\left[-\frac{\theta}{4}\left(k_1^2 + k_2^2\right)\right]\exp\left[i\frac{k_1}{\sqrt{2}}(z+\bar{z}) + \frac{k_2}{\sqrt{2}}\left(z-\bar{z}\right)\right]. \tag{19}$$

The above result shows that noncommutativity produces a Gaussian damping factor. It naturally descends the definition of a quantum field on a noncommutative plane. A scalar field of mass m is represented through the expansion

$$\phi(\mathbf{x}, t) = \frac{1}{2\pi}\int \frac{d\mathbf{k}}{\sqrt{2k^0}}\left[a^{\dagger}(\mathbf{k})\exp(-ik^0t)\langle z|\exp(ik^i\hat{x}_i)|z\rangle + h.c.\right] \tag{20}$$

where $a^{\dagger}(\mathbf{k})$, $a(\mathbf{k})$ are usual creation/annihilation operators on Fock states with definite energy and momentum. They are the same as in the commutative field theory since momenta commute among themselves. This field theory has been developed on Fourier modes, which exhibit an additional term responsible of a gaussian damping factor [7]

$$u(\mathbf{k}) = \frac{1}{2\pi}\frac{e^{-ik_\mu x^\mu - \theta\mathbf{k}^2/4}}{\sqrt{2k^0}}. \tag{21}$$

We stress that this exponential damping factor has not been put by hand: it is a direct consequence of the noncommutative behavior of coordinate operators. Substituting the explicit form for $u(\mathbf{k})$ given in (21) one is left with the following integral

$$\mathcal{E}_\theta = \frac{1}{8\pi^2}\int d\mathbf{k}\ \ e^{-\theta\mathbf{k}^2/2}\ \ \sqrt{\mathbf{k}^2+m^2}. \tag{22}$$

The calculation leads to the following result

$$\mathcal{E}_\theta = \frac{1}{8\pi\theta^{3/2}}\left\{\theta^{1/2}m + e^{\theta m^2}\frac{\sqrt{\pi}}{2}Erfc\left(\theta^{1/2}m\right)\right\} \tag{23}$$

where $Erfc(x)$ is the complementary error function [12]. For the massless field one obtains

$$\mathcal{E}_\theta = \frac{1}{8\pi\theta^{3/2}}\frac{\sqrt{\pi}}{2}. \tag{24}$$

We stress that $\mathcal{E}_\theta$ is finite and does not need any renormalization procedure.

The above calculation can be repeated in the (3+1) dimensional case. Of course one is naturally tempted to consider a full noncommutative space-time in which each pair of coordinates mutually anticommutes according to an homogeneous minimal length set by an unique θ parameter. A natural prescription could be given by a generalization of (6) to a multidimensional (anti)commutator in a similar fashion to what has been argued by Nambu in one of his famous papers [10]

$$[\hat{x}_\alpha, \hat{x}_\beta, \hat{x}_\gamma, \hat{x}_\delta] = \theta\epsilon_{\alpha\beta\gamma\delta}. \tag{25}$$

In absence of a viable field theory based on commutation relation of this kind one is left with the "binary" noncommutativity coming from rule (6), which forces the space-time to be foliated among noncommutative planes. Unwillingly we follow the usual wisdom to keep time as a commutative coordinate in order to avoid problems with unitarity [11]. So the best we can consider is still the noncommutativity in a plane, for instance the (x_1, x_2) plane. It is of course unpleasant having to do with a inhomogeneous space-time, even if such assumption can be accepted because the loss of homogeneity regards only the extremely high energies in which noncommutativity should play his fundamental role. Furthermore the Lorentz invariance breaking is an occurrence that cannot be excluded for so small scales of length. Mimicking calculations of the (2+1) dimensional case one adopts the following modes

$$u(\mathbf{k}) = \frac{1}{\sqrt{2\,(2\pi)^3\,k^0}}e^{-ik_\mu x^\mu - \theta\left(k_1^2+k_2^2\right)} \tag{26}$$

to find

$$\mathcal{E}_\theta = \frac{1}{16\pi^3}\int\ dk_3\ d\mathbf{q}\ \sqrt{k_3^2+\vec{q}^2+m^2}\exp(-\theta\vec{q}^2) \tag{27}$$

where $\mathbf{k} = (k_1, k_2, k_3)$ and $\mathbf{q} \equiv (k_1, k_2)$. Then one obtains

$$\mathcal{E}_\theta = \frac{1}{16\pi^2\theta^{3/2}}\int_{-\infty}^{\infty}\ dk_3\left\{\theta^{1/2}\left(k_3^2+m^2\right)^{1/2} + \frac{\sqrt{\pi}}{2}\ e^{\theta\left(k_3^2+m^2\right)}\ Erfc\left(\theta^{1/2}(k_3^2+m^2)^{1/2}\right)\right\}. \tag{28}$$

For the massless field the analogous expression is

$$\mathcal{E}_\theta = \frac{1}{16\pi^2\theta^{3/2}}\int_{-\infty}^{\infty}\ dk_3\left\{\theta^{1/2}k_3 + \frac{\sqrt{\pi}}{2}\ e^{\theta k_3^2}\ Erfc\left(\theta^{1/2}k_3\right)\right\}. \tag{29}$$

Even if the energy results to be clearly not finite, we can observe that the degree of divergence has been reduced to the one of a (1+1) dimensional theory. This is evident looking at the large argument behavior of the integrand, in which the Erfc term is suppressed [12]

$$\frac{\sqrt{\pi}}{2}\, e^{\theta x^2}\; Erfc\left(\theta^{1/2} x\right) \sim \frac{1}{2\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{(-1)^k \Gamma(k+1/2)}{(\theta x^2)^{k+1/2}} + O\left(\frac{1}{(\theta x^2)^{n+1/2}}\right) \qquad large \quad x \tag{30}$$

where x is equal to k_3 or $(k_3^2 + m^2)^{1/2}$ depending on the case. Then the asymptotic value of the energy density is

$$\mathcal{E}_\theta \sim \frac{1}{8\pi^2 \theta^{3/2}} \int^{\infty} dk_3 \left(\theta^{1/2} \sqrt{k_3^2 + m^2} + \frac{1}{2\, \theta^{1/2} \sqrt{k_3^2 + m^2}} \right) \tag{31}$$

and analogously for the massless case. We have got the divergence relative to the remaining (commutative) coordinates x_3. Such divergence can be cured through customary renormalization methods.

In this letter we adopted a model, that, even if simplified, is able to give a finite vacuum energy for a (2+1) scalar field. For (3+1) case we are not yet able to reproduce a finite vacuum energy, but we find that the degree of divergence of the (3+1) theory is lowered down to the degree of divergence of an effective (1+1) field theory. It would be interesting to deal with a nonrenormalizable theory since the beginning, rather than with the "good" scalar field theory. The hope would be to obtain (1+1) theory, whose divergences can be removed via usual renormalization procedures. Finally, we are left with the need of extending the present (2+1) method to higher dimensions in order to perform a more realistic calculation of the vacuum energy density.

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