Periodic solutions of Hamiltonian systems coupling twist with generalized lower/upper solutions

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Abstract

The Hamiltonian systems considered in this paper are obtained by weakly coupling two systems having completely different behaviours. The first one satisfies the usual twist assumptions taylored for the application of the Poincaré–Birkhoff Theorem, while the second one presents the existence of some well-ordered lower and upper solutions. In the higher dimensional case, we also treat a coupling situation where the classical Hartman condition is assumed.

1 Main results in low dimension

In the first part of the paper we are interested in the periodic problem associated with a four-dimensional system of the type

$$\begin{cases} \dot q = \partial_p H(t,q,p) + \varepsilon\, \partial_p P(t,q,p,u,v)\,, \\ \dot p = -\partial_q H(t,q,p) - \varepsilon\, \partial_q P(t,q,p,u,v)\,, \\ \dot u = f(t,v) + \varepsilon\, \partial_v P(t,q,p,u,v)\,, \\ \dot v = g(t,u) - \varepsilon\, \partial_u P(t,q,p,u,v)\,. \end{cases} \tag{1.1}$$

The idea is to consider a Poincaré–Birkhoff situation for the system

$$\dot q = \partial_p H(t,q,p)\,, \qquad \dot p = -\partial_q H(t,q,p)\,, \tag{1.2}$$

and the existence of well-ordered lower/upper solutions for the system

$$\dot u = f(t,v)\,, \qquad \dot v = g(t,u)\,. \tag{1.3}$$

The coupling function $P = P(t,q,p,u,v)$ will be assumed to have a bounded gradient with respect to (q,p,u,v), and ε will be a small parameter. All functions involved are assumed to be continuous, and T-periodic in their first variable t.

In the second part of the paper we will extend our results to higher dimensional systems, both concerning the couple (q, p) and the couple (u, v). To this aim, for the couple (q, p) we will apply some recent generalizations of the Poincaré–Birkhoff Theorem (see [11, 12, 16, 17]), while for the couple (u, v) the treatment of lower and upper solutions will be based on two different situations. The first one comes from the recent papers [13, 15], while the second approach involves a classical condition by Hartman [19].

We could also have assumed some very general twist conditions, in the line of [10, 11, 12]. However, in this paper we preferred to present our ideas in some more concrete situations. The interested reader will have no difficulties in adapting our results to the more general setting.

Now, in order to better understand the spirit of our results, some historical hints may be useful.

One of the most brilliant results for the periodic problem associated with a Hamiltonian system was proved by Conley and Zehnder [6] in 1983, giving a partial answer to a conjecture by Arnold [1, 2]. They also mentioned a possible relation of their result with the Poincaré–Birkhoff Theorem [27]. The results in [6] have been developed by different researchers in several directions (see, e.g., [4, 8, 18, 20, 21, 23]).

Recently, a deeper relation between these results and the Poincaré–Birkhoff Theorem has been established by the first author and A.J. Urena [16]. The first author then extended the results of [16] jointly with P. Gidoni [11], introducing a very general twist condition in order to find periodic solutions. The same authors further extended the theory, in a second paper [12], to the case when the Hamiltonian function includes a nonresonant quadratic term. The possibility of resonance has also been studied in [5] by assuming some Ahmad–Lazer–Paul conditions.

On the other hand, the history of lower and upper solutions goes back to the pioneering work of Picard [26] in 1893. The first attempts towards a modern definition of lower and upper solutions were made by Scorza Dragoni [28] in 1939 for the following equation

$$\ddot{u} = g(t, u, \dot{u}). \tag{1.4}$$

A few years later Nagumo [25] provided the classical definition of lower solution α and upper solution β of (1.4) by assuming the inequalities

$$\ddot{\alpha}(t) \geq g(t, \alpha(t), \dot{\alpha}(t)), \qquad \ddot{\beta}(t) \leq g(t, \beta(t), \dot{\beta}(t)).$$

He also introduced an extra assumption, which we nowadays call *Nagumo condition*, so to find the existence of a solution.

The notion of lower and upper solutions has recently been extended in [13, 15] to planar systems. Moreover, the first author together with M. Garzon and A. Sfecci [10] further extended this fertile theory to coupled systems which contain both the periodicity-twist conditions and a pair of well-ordered lower and upper solutions. However, due to some technical problems, they only used *constant* lower and upper solutions, while proposing as an open problem the case of non-constant lower/upper solutions.

In this paper, we provide a partial answer to this open problem and extend the theory to systems which contain the periodicity-twist conditions together with generalized well-ordered lower/upper solutions, coupled by a perturbation term.

The paper is organized as follows.

In Section 2 we state our result in the low dimensional case by coupling "twist" and strict lower/upper solutions. The proof of this result is given in Section 3. In Section 4 we provide some consequences of the main result and an example of application.

In Section 5 we extend our previous theorem to higher dimensions, and provide some variants and an example of application. In Section 6 we prove a result by coupling "twist" with a Hartman-type condition [19] in higher dimensions. This condition extends the concept of *constant* lower and upper solutions to higher dimensions, and has been extensively studied by many authors (see [9] and the references therein).

Finally, in Section 7 we illustrate an application to the theory of perturbations of completely integrable systems.

2 A first multiplicity result

Let us first recall what we know about systems (1.2) and (1.3), separately.

- The Poincaré–Birkhoff Theorem

Here are our assumptions concerning system (1.2).

$A1$. The function $H(t,q,p)$ is 2π-periodic in q.

$A2$. There are $a<b$ such that all the solutions (q,p) of system (1.2) starting with $p(0)\in[a,b]$ are defined on $[0,T]$ and

$$\begin{cases} p(0)=a & \Rightarrow \quad q(T)-q(0)<0\,, \\ p(0)=b & \Rightarrow \quad q(T)-q(0)>0\,. \end{cases}$$

The following result was proved in [16].

Theorem 2.1. *Assume that* A1 *and* A2 *hold true. Then, system* (1.2) *has at least two geometrically distinct T-periodic solutions* (q,p) *such that* $p(0)\in\,]a,b[\,$.

Notice that, when a T-periodic solution (q,p) has been found, infinitely many others appear by just adding an integer multiple of 2π to the q-th component. We say that two solutions are geometrically distinct if they cannot be obtained from each other in this way.

We also want to remark here that the period 2π in assumption A1 is inessential; any period would be possible.

- Lower and upper solutions

Let us first recall the definitions of lower and upper solutions for the T-periodic problem associated with system (1.3).

Definition 2.2. *A T-periodic C^1-function $\alpha:\mathbb{R}\to\mathbb{R}$ is said to be a "lower solution" for the T-periodic problem associated with system* (1.3) *if there exists a T-periodic C^1-function $v_\alpha:\mathbb{R}\to\mathbb{R}$ such that*

$$\begin{cases} v<v_\alpha(t) & \Rightarrow \quad f(t,v)<\dot\alpha(t)\,, \\ v>v_\alpha(t) & \Rightarrow \quad f(t,v)>\dot\alpha(t)\,, \end{cases} \tag{2.1}$$

and

$$\dot v_\alpha(t)\geq g(t,\alpha(t))\,. \tag{2.2}$$

The lower solution is "strict" if the strict inequality in (2.2) *holds.*

Definition 2.3. *A T-periodic C^1-function $\beta : \mathbb{R} \to \mathbb{R}$ is said to be an "upper solution" for the T-periodic problem associated with system* (1.3) *if there exists a T-periodic C^1-function $v_\beta : \mathbb{R} \to \mathbb{R}$ such that*

$$\begin{cases} v < v_\beta(t) & \Rightarrow \quad f(t,v) < \dot{\beta}(t)\,, \\ v > v_\beta(t) & \Rightarrow \quad f(t,v) > \dot{\beta}(t)\,, \end{cases} \tag{2.3}$$

and

$$\dot{v}_\beta(t) \leq g(t, \beta(t))\,. \tag{2.4}$$

The upper solution is "strict" if the strict inequality in (2.4) *holds.*

The following result was proved in [13, 15]

Theorem 2.4. *Assume that there exist a lower solution α and an upper solution β for the T-periodic problem associated with system* (1.3)*, such that $\alpha \leq \beta$. Then, system* (1.3) *has a T-periodic solution (u, v) such that $\alpha \leq u \leq \beta$.*

- Back to the coupled system

Let us state our hypotheses. We will assume $A1$ and $A2$; moreover,

$A3$. There exist a strict lower solution α and a strict upper solution β for the T-periodic problem associated with system (1.3), such that $\alpha \leq \beta$.

$A4$. there exist positive constants c, d such that $|\dot{\alpha}(t)| < c$, $|\dot{\beta}(t)| < c$ for every $t \in [0, T]$, and

$$\begin{cases} f(t,v) \geq c\,, & \text{for } v \geq d\,, \\ f(t,v) \leq -c\,, & \text{for } v \leq -d\,. \end{cases}$$

$A5$. The second order partial derivatives $\partial^2_{tv} f(t,v)$ and $\partial^2_{vv} f(t,v)$ exist and are continuous; moreover, there exists $\lambda > 0$ such that

$$\partial_v f(t,v) \geq \lambda\,, \qquad \text{for every } (t,v) \in [0,T] \times \mathbb{R}\,.$$

$A6$. The function $P(t, q, p, u, v)$ is 2π-periodic in q and has a bounded gradient with respect to (q, p, u, v); moreover, the partial derivative $\partial_v P$ is independent of q and p, and the map $\partial_v P(t, u, v)$ is continuously differentiable.

Here is the main result of this section.

Theorem 2.5. *Assume that A1 – A6 hold true. Then there exists* $\bar{\varepsilon} > 0$ *such that, if* $|\varepsilon| \le \bar{\varepsilon}$*, there are at least two geometrically distinct* T*-periodic solutions of system* (1.1)*, with* $p(0) \in]a, b[$ *and* $\alpha \le u \le \beta$.

The proof of the Theorem 2.5 will be given in Section 3.

Remark 2.6. *Theorem 2.5 provides a partial answer to an open problem raised in [10], where only* constant *lower and upper solutions were considered. However, our result only applies to weakly coupled systems with strict lower and upper solutions. Hence, the problem raised in [10] remains open.*

Remark 2.7. *As already noticed in [16], instead of using a constant interval* $[a, b]$*, it is possible to deal with a varying interval* $[a(q), b(q)]$*, where* $a, b : \mathbb{R} \to \mathbb{R}$ *are continuous and* 2π*-periodic functions. Indeed, if* a *and* b *are continuously differentiable, then this case can be reduced to the previous one by the symplectic change of variables*

$$\psi(q, p) = \left(\int_0^q \frac{b(s) - a(s)}{2} ds \, , \frac{2p - b(q) - a(q)}{b(q) - a(q)} \right) .$$

On the other hand, if the functions a *and* b *are only continuous, then by the Fejer Theorem they can be replaced by smooth functions. Notice that the new Hamiltonian* $\widetilde{H}(t, \tilde{q}, \tilde{p}) = H(t, \psi^{-1}(\tilde{q}, \tilde{p}))$ *is periodic in* $\tilde{q}$ *with period* $\tau := \frac{1}{2} \int_0^{2\pi} (b(s) - a(s)) ds$.

Before going to the proof, we now present a variant of Theorem 2.5 which is more related to Poincaré-Birkhoff Theorem as originally stated by Poincaré [27].

We first recall the definition of "rotation number". Assume that $t_1 < t_2$ and let $\phi : [t_1, t_2] \to \mathbb{R}^2$ be a continuous curve such that $\phi(t) \neq (0, 0)$ for every $t \in [t_1, t_2]$. Writing $\phi(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$, where $\rho : \mathbb{R} \to]0, +\infty[$ and $\theta : \mathbb{R} \to \mathbb{R}$ are continuous, we define

$$\mathrm{Rot}(\phi; [t_1, t_2]) = -\frac{\theta(t_2) - \theta(t_1)}{2\pi} .$$

In the sequel, $\mathcal{D}(\Gamma)$ denotes the open bounded region delimited by a planar Jordan curve Γ. Here is the statement.

Theorem 2.8. *Let assumptions A3 – A6 hold true. Let k be any integer and assume that there exist $\rho > 0$ and two planar Jordan curves Γ_1, Γ_2, strictly star-shaped with respect to the origin, with*

$$0 \in \mathcal{D}(\Gamma_1) \subseteq \overline{\mathcal{D}(\Gamma_1)} \subseteq \mathcal{D}(\Gamma_2)\,,$$

such that the solutions of system (1.2) *starting with $(q(0), p(0)) \in \overline{\mathcal{D}(\Gamma_2)} \setminus \mathcal{D}(\Gamma_1)$ are defined on $[0, T]$ and satisfy*

$$(q(t), p(t)) \neq (0, 0)\,, \qquad \textit{for every } t \in [0, T]\,;$$

moreover,

$$\begin{cases} (q(0), p(0)) \in \Gamma_1 \quad \Rightarrow \quad \mathrm{Rot}((q, p); [0, T]) < k\,, \\ q(0), p(0)) \in \Gamma_2 \quad \Rightarrow \quad \mathrm{Rot}((q, p); [0, T]) > k\,. \end{cases} \tag{2.5}$$

Then system (1.1) *has at least two T-periodic solutions (q, p, u, v) such that*

$$\alpha \leq u \leq \beta\,,$$

$$(q(0), p(0)) \in \mathcal{D}(\Gamma_2) \setminus \overline{\mathcal{D}(\Gamma_1)}\,,$$

and

$$\mathrm{Rot}((q, p); [0, T]) = k\,.$$

The same is true if (2.5) *is replaced by the following:*

$$\begin{cases} (q(0), p(0)) \in \Gamma_1 \quad \Rightarrow \quad \mathrm{Rot}((q, p); [0, T]) > k\,, \\ (q(0), p(0)) \in \Gamma_2 \quad \Rightarrow \quad \mathrm{Rot}((q, p); [0, T]) < k\,. \end{cases}$$

In the above theorem, the Hamiltonian function H is not assumed to be periodic in the variable q. The 2π-periodicity can indeed be recovered when passing to some kind of polar coordinates. The proof is almost the same as in [10, Theorem 10], so we omit it, for briefness.

3 Proof of Theorem 2.5

Let $A = \min \alpha$ and $B = \max \beta$. Then there exists a constant $C > 0$ such that

$$|g(t, u)| \leq C\,, \qquad \text{for every} \qquad (t, u) \in [0, T] \times [A, B]\,.$$

We can find two straight lines $\gamma_\pm : \mathbb{R} \to \mathbb{R}$, whose equations are

$$\gamma_+(u) = \mu u + R\,, \qquad \gamma_-(u) = \mu u - R\,,$$

where $\mu < -C/c$ and $R > 0$ are chosen in such a way that

$$\gamma_-(u) < -d < d < \gamma_+(u)\,,$$

and

$$\gamma_-(u) \leq \dot\alpha(t), v_\alpha(t), \dot\beta(t), v_\beta(t) \leq \gamma_+(u)\,, \tag{3.1}$$

for every $(t,u) \in [0,T] \times [A,B]$.

Let us define the set

$$\mathcal{V} = \{(t,q,p,u,v) \in \mathbb{R}^5 : \ \alpha(t) \leq u \leq \beta(t)\,,\ \gamma_-(u) \leq v \leq \gamma_+(u)\}.$$

We can choose a constant $\hat d > \max\{c, d, C/|\mu|\}$ such that

$$-\hat d < \gamma_-(u) < \gamma_+(u) < \hat d\,, \qquad \text{for every } u \in [A,B]\,.$$

Consider the function $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined as

$$\eta(t,u) = \begin{cases} \alpha(t)\,, & \text{if } u \leq \alpha(t)\,, \\ u\,, & \text{if } \alpha(t) \leq u \leq \beta(t)\,, \\ \beta(t) & \text{if } u \geq \beta(t)\,. \end{cases}$$

Now define the functions

$$\tilde g(t,u) = g(t, \eta(t,u)) - \eta(t,u) + u\,, \tag{3.2}$$

and

$$\tilde f(t,v) = \begin{cases} v\,, \quad \text{if } v \leq -\hat d - 1\,, \\ f(t,v) - (v + \hat d)(v - f(t,v))\,, \quad \text{if } -\hat d - 1 \leq v \leq -\hat d\,, \\ f(t,v)\,, \quad \text{if } -\hat d \leq v \leq \hat d\,, \\ f(t,v) + (v - \hat d)(v - f(t,v))\,, \quad \text{if } \hat d \leq v \leq \hat d + 1\,, \\ v\,, \quad \text{if } v \geq \hat d + 1\,. \end{cases} \tag{3.3}$$

By $A3$, there exists a $\xi > 0$ such that

$$\dot v_\alpha(t) - g(t, \alpha(t)) > \xi\,, \qquad \text{for every } t \in [0,T]\,, \tag{3.4}$$

$$\dot{v}_\beta(t) - g(t,\beta(t)) < -\xi\,, \qquad \text{for every } t \in [0,T]\,. \tag{3.5}$$

By the global existence assumption in $A2$, we note that there exists a constant $C_1 > 0$ such that, for any solution (q,p) of (1.2) starting with $p(0) \in [a,b]$, one has that

$$|p(t)| \leq C_1\,, \qquad \text{for every } t \in [0,T]\,.$$

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that

$$\sigma(s) = \begin{cases} 1\,, & \text{if } \ s \leq C_1\,, \\ 0\,, & \text{if } \ s > C_1 + 1\,, \end{cases} \tag{3.6}$$

and set $\widehat{H}(t,q,p) = \sigma(|p|)H(t,q,p)$. Then $\widehat{H}$ has a bounded gradient with respect to (q,p). Now consider the modified system

$$\begin{cases} \dot{q} = \partial_p \widehat{H}(t,q,p) + \varepsilon\, \partial_p P(t,q,p,u,v)\,, \\ \dot{p} = -\partial_q \widehat{H}(t,q,p) - \varepsilon\, \partial_q P(t,q,p,u,v)\,, \\ \dot{u} = \tilde{f}(t,v) + \varepsilon\, \partial_v P(t,q,p,u,v)\,, \\ \dot{v} = \tilde{g}(t,u) - \varepsilon\, \partial_u P(t,q,p,u,v)\,, \end{cases} \tag{3.7}$$

where the new Hamiltonian function is defined as

$$\widetilde{H}(t,q,p,u,v) = \widehat{H}(t,q,p) + \int_0^v \tilde{f}(t,s)ds - \int_0^u \tilde{g}(t,s)ds + \varepsilon P(t,q,p,u,v)\,.$$

We can also write the modified system (3.7) as $\dot{z} = J\nabla\widetilde{H}(t,z)$, where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is the standard symplectic matrix, and $z = (q,p,u,v)$. Notice that

$$\widetilde{H}(t,z) = \frac{1}{2}(v^2 - u^2) + K(t,z)\,,$$

where K is a function having a bounded gradient with respect to z. Moreover, by $A2$, since $\partial_p P$ and $\partial_q P$ are bounded, if $|\varepsilon|$ is small enough, for any solution (q,p,u,v) of (3.7) one still has that

$$\begin{cases} p(0) = a \quad \Rightarrow \quad q(T) - q(0) < 0\,, \\ p(0) = b \quad \Rightarrow \quad q(T) - q(0) > 0\,. \end{cases}$$

Then, by [12, Corollary 2.4], we conclude that the modified system (3.7) has at least two geometrically distinct T-periodic solutions such that $p(0) \in \,]a,b[\,$, provided that $|\varepsilon|$ is small enough.

We now need to show that such solutions z are such that $(t, z(t)) \in \mathcal{V}$ for every $t \in [0, T]$. Let us first prove the following five lemmas.

Lemma 3.1. *There exist v_α^ε and v_β^ε such that*

$$f(t, v_\alpha^\varepsilon(t)) + \varepsilon\, \partial_v P\,(t, \alpha(t), v_\alpha^\varepsilon(t)) = \dot\alpha(t)\,, \tag{3.8}$$

$$f(t, v_\beta^\varepsilon(t)) + \varepsilon\, \partial_v P(t, \beta(t), v_\beta^\varepsilon(t)) = \dot\beta(t)\,, \tag{3.9}$$

for every $t \in [0, T]$. Moreover,

$$\lim_{\varepsilon\to 0} v_\alpha^\varepsilon = v_\alpha\,, \quad \lim_{\varepsilon\to 0} \dot v_\alpha^\varepsilon = \dot v_\alpha\,, \quad \lim_{\varepsilon\to 0} v_\beta^\varepsilon = v_\beta\,, \quad \lim_{\varepsilon\to 0} \dot v_\beta^\varepsilon = \dot v_\beta\,,$$

uniformly in $[0, T]$, i.e., $v_\alpha^\varepsilon \to v_\alpha$ and $v_\beta^\varepsilon \to v_\beta$ in $C^1([0, T], \mathbb{R})$, as $\varepsilon \to 0$.

Proof. We only prove the statement concerning v_α^ε, since the one for v_β^ε can be proved in a similar way. Consider the space $X = C^1([0, T], \mathbb{R})$. By *A*5 and *A*6, the functions f and $\partial_v P$ are continuously differentiable, and since

$$f(t, v_\alpha(t)) = \dot\alpha(t)\,, \qquad \text{for every } t \in \mathbb{R}\,,$$

we have that $\alpha \in C^2([0, T], \mathbb{R})$. We can then define the function $\widetilde F : X \times \mathbb{R} \to X$ by

$$\widetilde F(v, \varepsilon)(t) = f(t, v(t)) + \varepsilon\, \partial_v P(t, \alpha(t), v(t)) - \dot\alpha(t)\,. \tag{3.10}$$

Now, clearly $\widetilde F(v_\alpha, 0) = 0$, and for all $h \in X$, we have

$$\begin{aligned} \frac{\partial \widetilde F}{\partial v}(v_\alpha, 0)(h)(t) &= \lim_{\sigma\to 0} \frac{\widetilde F(v_\alpha + \sigma h, 0) - \widetilde F(v_\alpha, 0)}{\sigma}(t) \\ &= \lim_{\sigma\to 0} \frac{f\,(t, v_\alpha(t) + \sigma h(t)) - f(t, v_\alpha(t))}{\sigma} \\ &= \partial_v f(t, v_\alpha(t)) h(t)\,. \end{aligned}$$

Let us prove that $\widetilde F$ is differentiable with respect to its first variable at $(v_\alpha, 0)$, with

$$\big[d_v \widetilde F(v_\alpha, 0)(h)\big](t) = \partial_v f(t, v_\alpha(t)) h(t)\,.$$

Writing

$$\widetilde F(v, 0) = \widetilde F(v_\alpha, 0) + d_v \widetilde F(v_\alpha, 0)(v - v_\alpha) + r(v)\,, \tag{3.11}$$

we need to prove that

$$v \xrightarrow{C^1} v_\alpha \quad \Rightarrow \quad \frac{r(v)}{\|v - v_\alpha\|_{C^1}} \xrightarrow{C^1} 0\,. \tag{3.12}$$

Substituting (3.10) in (3.11), we obtain

$$f(t, v(t)) = f(t, v_\alpha(t)) + \partial_v f(t, v_\alpha(t))(v(t) - v_\alpha(t)) + r(v)(t) .$$

By the Lagrange Mean Value Theorem, for every $t \in [0, T]$ there exists $\zeta(t) \in [v_\alpha(t), v(t)]$ such that

$$f(t, v(t)) - f(t, v_\alpha(t)) = \partial_v f(t, \zeta(t))(v(t) - v_\alpha(t)) .$$

Then,

$$\begin{aligned} \frac{|r(v)(t)|}{\|v - v_\alpha\|_{C^1}} &= |\partial_v f(t, \zeta(t)) - \partial_v f(t, v_\alpha(t))| \frac{|v(t) - v_\alpha(t)|}{\|v - v_\alpha\|_{C^1}} \\ &\leq |\partial_v f(t, \zeta(t)) - \partial_v f(t, v_\alpha(t))| , \quad \text{for every } t \in [0, T] . \end{aligned}$$

If $v \to v_\alpha$ in C^1, then $v \to v_\alpha$ uniformly, hence also $\zeta \to v_\alpha$ uniformly. Since the partial derivative of f with respect to v is continuous, taking a constant $M > ||v_\alpha||_\infty$, the map $\partial_v f : [0, T] \times [-M, M] \to \mathbb{R}$ is uniformly continuous. It then follows that, if $v \to v_\alpha$ in C^1, then

$$\frac{r(v)(t)}{\|v - v_\alpha\|_{C^1}} \to 0 , \quad \text{uniformly for } t \in [0, T] .$$

It remains to be proved that, if $v \to v_\alpha$ in C^1, then

$$\frac{d}{dt} \left(\frac{r(v)(t)}{\|v - v_\alpha\|_{C^1}} \right) \to 0 , \quad \text{uniformly for } t \in [0, T].$$

We have

$$\begin{aligned} \frac{d}{dt} r(v)(t) &= \partial_t f(t, v(t)) + \partial_v f(t, v(t))\dot{v}(t) - \partial_t f(t, v_\alpha(t)) - \partial_v f(t, v_\alpha(t))\dot{v}_\alpha(t) \\ &\quad - \left(\partial^2_{tv} f(t, v_\alpha(t)) + \partial^2_{vv} f(t, v_\alpha(t))\dot{v}_\alpha(t)\right) (v(t) - v_\alpha(t)) \\ &\quad - \partial_v f(t, v_\alpha(t))(\dot{v}(t) - \dot{v}_\alpha(t)) \\ &= \left(\partial_t f(t, v(t)) - \partial_t f(t, v_\alpha(t)) - \partial^2_{tv} f(t, v_\alpha(t))(v(t) - v_\alpha(t))\right) \\ &\quad + \left(\partial_v f(t, v(t)) - \partial_v f(t, v_\alpha(t))\right) \dot{v}(t) - \partial^2_{vv} f(t, v_\alpha(t))\dot{v}_\alpha(t)(v(t) - v_\alpha(t)) . \end{aligned}$$

Again by using the Lagrange Mean Value Theorem twice, for every $t \in [0, T]$ there exist $\xi(t)$ and $\eta(t)$ in $[v_\alpha(t), v(t)]$ such that

$$\partial_t f(t, v(t)) - \partial_t f(t, v_\alpha(t)) = \partial^2_{tv} f(t, \xi(t))(v(t) - v_\alpha(t)) , \tag{3.13}$$

$$\partial_v f(t, v(t)) - \partial_v f(t, v_\alpha(t)) = \partial^2_{vv} f(t, \eta(t))(v(t) - v_\alpha(t)) . \tag{3.14}$$

Then,

$$\frac{d}{dt} r(v(t)) = \left(\partial^2_{tv} f(t, \xi(t)) - \partial^2_{tv} f(t, v_\alpha(t))\right) (v(t) - v_\alpha(t)) \\ + \left(\partial^2_{vv} f(t, \eta(t))\dot{v}(t) - \partial^2_{vv} f(t, v_\alpha(t))\dot{v}_\alpha(t)\right) (v(t) - v_\alpha(t)) \,.$$

If $v \to v_\alpha$ in C^1, the first term in the sum converges to 0 when divided by $||v - v_\alpha||_{C^1}$, uniformly on $[0, T]$, by the continuity of $\partial^2_{tv} f$. For the second term, we have

$$\left|\partial^2_{vv} f(t, \eta(t))\dot{v}(t) - \partial^2_{vv} f(t, v_\alpha(t))\dot{v}_\alpha(t)\right| \leq \\ \leq \left|\partial^2_{vv} f(t, \eta(t)) - \partial^2_{vv} f(t, v_\alpha(t))\right| |\dot{v}(t)| + \left|\partial^2_{vv} f(t, v_\alpha(t))\right| |\dot{v}(t) - \dot{v}_\alpha(t)| \,,$$

which converges uniformly to 0 when $v \to v_\alpha$ in C^1, since both $|\dot{v}(t)|$ and $\left|\partial^2_{vv} f(t, v_\alpha(t))\right|$ are bounded, $\dot{v} \to \dot{v}_\alpha$ uniformly on $[0, T]$, and the map $\partial^2_{vv} f$ is continuous.

We have thus proved (3.12). Therefore,

$$d_v \widetilde{F}(v_\alpha, 0) = \partial_v f(\cdot, v_\alpha(\cdot)) \, \mathrm{Id} \,,$$

where $\mathrm{Id} : X \to X$ is the identity map. By $A5$, we have that $\partial_v f(t, v_\alpha(t)) > 0$, for every $t \in [0, T]$, so the map $d_v \widetilde{F}(v_\alpha, 0) : X \to X$ is invertible.

By the Implicit Function Theorem, there exists an $\bar{\varepsilon} > 0$ and a map $\varphi :] - \bar{\varepsilon}, \bar{\varepsilon}[\to B_X(v_\alpha, \bar{\varepsilon})$, of class C^1, such that, for every $\varepsilon \in] - \bar{\varepsilon}, \bar{\varepsilon}[$ and $v \in B_X(v_\alpha, \bar{\varepsilon})$,

$$\widetilde{F}(v, \varepsilon) = 0 \iff v = \varphi(\varepsilon) \,.$$

Setting $v^\varepsilon_\alpha = \varphi(\varepsilon)$, the proof is completed. □

Lemma 3.2. *There exists $\tilde{\varepsilon} > 0$ such that, if $|\varepsilon| < \tilde{\varepsilon}$, then for every $t \in [0, T]$ and $u \in [A, B]$ the following inequalities hold:*

$$\begin{cases} \tilde{f}(t, v) + \varepsilon \, \partial_v P(t, u, v) < \dot{\alpha}(t) \,, & \textit{if } v < v^\varepsilon_\alpha(t) \,, \\ \tilde{f}(t, v) + \varepsilon \, \partial_v P(t, u, v) > \dot{\alpha}(t) \,, & \textit{if } v > v^\varepsilon_\alpha(t) \,, \end{cases} \tag{3.15}$$

$$\begin{cases} \tilde{f}(t, v) + \varepsilon \, \partial_v P(t, u, v) < \dot{\beta}(t) \,, & \textit{if } v < v^\varepsilon_\beta(t) \,, \\ \tilde{f}(t, v) + \varepsilon \, \partial_v P(t, u, v) > \dot{\beta}(t) \,, & \textit{if } v > v^\varepsilon_\beta(t) \,, \end{cases} \tag{3.16}$$

$$\begin{cases} \tilde{g}(t, u) - \varepsilon \, \partial_u P(t, q, p, u, v) < \dot{v}^\varepsilon_\alpha(t) \,, & \textit{if } u \leq \alpha(t) \,, \\ \tilde{g}(t, u) - \varepsilon \, \partial_u P(t, q, p, u, v) > \dot{v}^\varepsilon_\beta(t) \,, & \textit{if } u \geq \beta(t) \,. \end{cases} \tag{3.17}$$

Proof. We only prove the first inequality in (3.15), the proof of the second inequality in (3.15) and of the inequalities in (3.16) being similar.

We first want to prove that, for $|\varepsilon|$ small enough, we have

$$v < v_\alpha^\varepsilon(t) \quad \Rightarrow \quad f(t,v) + \varepsilon\, \partial_v P(t,u,v) < \dot\alpha(t)\,. \tag{3.18}$$

By $A5$, there exists $\lambda > 0$ such that $\partial_v f(t, v_\alpha^\varepsilon(t)) \geq \lambda$, and by $A6$ there exists a constant $\widehat{C} > 0$ such that

$$\left|\partial^2_{vv} P(t,u,v_\alpha(t))\right| \leq \widehat{C}\,, \qquad \text{for every } (t,u) \in [0,T] \times [A,B]\,.$$

So, if $2|\varepsilon|\widehat{C} < \lambda$, we have

$$\partial_v\big(f(t,v_\alpha^\varepsilon(t)) + \varepsilon\, \partial_v P(t,u,v_\alpha^\varepsilon(t))\big) \geq \frac{\lambda}{2}\,, \quad \text{for every } (t,u) \in [0,T] \times [A,B]\,.$$

By continuity, there exists a $\bar\delta > 0$ such that

$$|v - v_\alpha^\varepsilon(t)| < \bar\delta \quad \Rightarrow \quad \partial_v\big(f(t,v) + \varepsilon\, \partial_v P(t,u,v)\big) \geq \frac{\lambda}{4}\,,$$

for every $(t,u) \in [0,T] \times [A,B]$. So, by (3.8), there exists $\tau > 0$ such that

$$\begin{aligned} v \in [v_\alpha^\varepsilon(t) - \tau, v_\alpha^\varepsilon(t)] \quad &\Rightarrow \quad f(t,v) + \varepsilon\, \partial_v P(t,u,v) < \dot\alpha(t)\,, \\ v \in [v_\alpha^\varepsilon(t), v_\alpha^\varepsilon(t) + \tau] \quad &\Rightarrow \quad f(t,v) + \varepsilon\, \partial_v P(t,u,v) > \dot\alpha(t)\,. \end{aligned}$$

Without loss of generality, we assume for d in $A4$ that

$$-d < \dot\alpha(t), v_\alpha(t), \dot\beta(t), v_\beta(t) < d\,,$$

and take $|\varepsilon|, \tau$ small enough so that

$$-d < v_\alpha^\varepsilon(t) - \tau < v_\alpha^\varepsilon(t) + \tau < d\,.$$

By (2.1) and A4, there exists $\varrho > 0$ such that

$$\begin{aligned} f(t,v) - \dot\alpha(t) &\leq -\varrho\,, \quad \text{for } v \leq -d\,, \\ f(t,v) - \dot\alpha(t) &\geq \varrho\,, \quad \text{for } v \geq d\,. \end{aligned}$$

If $|\varepsilon|$ is small enough, since $\partial_v P$ is bounded, we have that

$$\begin{aligned} f(t,v) + \varepsilon\, \partial_v P(t,u,v) - \dot\alpha(t) &\leq -\frac{\varrho}{2}\,, \quad \text{for } v \leq -d\,, \\ f(t,v) + \varepsilon\, \partial_v P(t,u,v) - \dot\alpha(t) &\geq \frac{\varrho}{2}\,, \quad \text{for } v \geq d\,. \end{aligned}$$

Now it remains only to check what happens in the intervals $[v_\alpha^\varepsilon(t) + \tau, d]$ and $[-d, v_\alpha^\varepsilon(t) - \tau]$. Let us only consider the first interval, the argument being similar for the other one. If $v \in [v_\alpha^\varepsilon(t) + \tau, d]$, then, if $|\varepsilon|$ is small enough, by Lemma 3.1 and (2.1), using A5, we have

$$f(t,v) \geq f(t, v_\alpha^\varepsilon(t) + \tau) \geq f\Big(t, v_\alpha(t) + \frac{\tau}{2}\Big) > \dot\alpha(t)\,.$$

By Weierstrass Theorem, there exists a $m > 0$ such that

$$f(t,v) - \dot\alpha(t) \geq m\,,$$

for every $t \in [0,T]$ and $v \in [v_\alpha^\varepsilon(t) + \tau, d]$. Then, for $|\varepsilon|$ small enough,

$$f(t,v) + \varepsilon\,\partial_v P(t,u,v) - \dot\alpha(t) > 0\,,$$

for every $t \in [0,T]$ and $v \in [v_\alpha^\varepsilon(t) + \tau, d]$. We have thus proved (3.18).

Now, since $-\hat d < v_\alpha^\varepsilon(t) < \hat d$ for $|\varepsilon|$ small enough, we have the following three cases.

Case 1. If $-\hat d \leq v < v_\alpha^\varepsilon(t)$, then by (3.3) and (3.18) we have

$$\tilde f(t,v) + \varepsilon\,\partial_v P(t,u,v) = f(t,v) + \varepsilon\,\partial_v P(t,u,v) < \dot\alpha(t)\,.$$

Case 2. If $v \leq -\hat d - 1$, then by (3.3) we have

$$\begin{aligned}\tilde f(t,v) + \varepsilon\,\partial_v P(t,u,v) &= v + \varepsilon\,\partial_v P(t,u,v)\\ &\leq -\hat d + \varepsilon\,\partial_v P(t,u,v) < \dot\alpha(t)\,,\end{aligned}$$

for $|\varepsilon|$ small enough, since $\partial_v P$ is bounded.

Case 3. If $-\hat d - 1 \leq v < -\hat d$, then by (3.3) and (2.1) we have

$$\begin{aligned}\tilde f(t,v) + \varepsilon\partial_v P(t,u,v) &= f(t,v) - (v+\hat d)(v - f(t,v)) + \varepsilon\,\partial_v P(t,u,v)\\ &= (1 + (v+\hat d))f(t,v) - (v+\hat d)v + \varepsilon\,\partial_v P(t,u,v)\\ &< \dot\alpha(t)\,,\end{aligned}$$

for $|\varepsilon|$ small enough, since $-(v+\hat d) \in [0,1]$ and $f(t,v) < \dot\alpha(t)$, $v < \dot\alpha(t)$.

The proof of the first inequality in (3.15) is thus completed.

We now prove the first inequality in (3.17), the second one being analogous. Suppose $u \leq \alpha(t)$. By (3.2) and (3.4), we have

$$\begin{aligned}\tilde g(t,u) - \varepsilon\,\partial_u P(t,q,p,u,v) &= g(t,\alpha(t)) - \alpha(t) + u - \varepsilon\,\partial_u P(t,q,p,u,v)\\ &\leq g(t,\alpha(t)) - \varepsilon\,\partial_u P(t,q,p,u,v)\\ &< \dot v_\alpha(t) - \xi - \varepsilon\,\partial_u P(t,q,p,u,v)\\ &< \dot v_\alpha^\varepsilon(t)\,,\end{aligned}$$

for $|\varepsilon|$ small enough, since $\dot v_\alpha^\varepsilon \to \dot v_\alpha$ uniformly. The proof of the first inequality in (3.17) is thus completed. $\square$

Let us define the open sets

$$\begin{aligned}
A_{NW} &= \{(t,u,v) \in \mathbb{R}^3 \ : \ u < \alpha(t),\ v > v_\alpha^\varepsilon(t)\}\,,\\
A_{SW} &= \{(t,u,v) \in \mathbb{R}^3 \ : \ u < \alpha(t),\ v < v_\alpha^\varepsilon(t)\}\,,\\
A_{NE} &= \{(t,u,v) \in \mathbb{R}^3 \ : \ u > \beta(t),\ v > v_\beta^\varepsilon(t)\}\,,\\
A_{SE} &= \{(t,u,v) \in \mathbb{R}^3 \ : \ u > \beta(t),\ v < v_\beta^\varepsilon(t)\}\,.
\end{aligned}$$

Lemma 3.3. *For every solution $z = (q,p,u,v)$ of system* (3.7)*, the following assertions hold true:*

$$\begin{aligned}
(t_0,u(t_0),v(t_0)) \in A_{NW} \quad &\Rightarrow \quad (t,u(t),v(t)) \in A_{NW} \ \textit{ for every } t < t_0\,,\\
(t_0,u(t_0),v(t_0)) \in A_{SE} \quad &\Rightarrow \quad (t,u(t),v(t)) \in A_{SE} \ \textit{ for every } t < t_0\,,\\
(t_0,u(t_0),v(t_0)) \in A_{NE} \quad &\Rightarrow \quad (t,u(t),v(t)) \in A_{NE} \ \textit{ for every } t > t_0\,,\\
(t_0,u(t_0),v(t_0)) \in A_{SW} \quad &\Rightarrow \quad (t,u(t),v(t)) \in A_{SW} \ \textit{ for every } t > t_0\,.
\end{aligned}$$

Proof. We only prove the first assertion, since the remaining ones can be proved similarly. We suppose on contrary that there exists $t_1 < t_0$ such that

$$(t_0,u(t_0),v(t_0)) \in A_{NW}\,,$$

$$(t,u(t),v(t)) \in A_{NW}\,, \qquad \text{for } t \in]t_1,t_0[\,,$$

and

$$(t_1,u(t_1),v(t_1)) \in \partial A_{NW}\,.$$

Notice that

$$\begin{aligned}
\partial A_{NW} = \{t,u,v) \in \mathbb{R}^3 \ &: \ u = \alpha(t),\ v \geq v_\alpha^\varepsilon(t)\}\\
\cup\, \{t,u,v) \in \mathbb{R}^3 \ &: \ u \leq \alpha(t),\ v = v_\alpha^\varepsilon(t)\}\,.
\end{aligned} \tag{3.19}$$

Assume $v(t_1) > v_\alpha^\varepsilon(t_1)$. Without loss of generality, we may assume that there exists $\delta > 0$ such that $[t_1, t_1+\delta] \subseteq [t_1,t_0[$ and $v(t) > v_\alpha^\varepsilon(t)$ for every $t \in [t_1,t_1+\delta]$. Now define $w : [t_1,t_1+\delta] \to \mathbb{R}$ by $w(t) = u(t) - \alpha(t)$. Then, we have that $w(t_1+\delta) < 0$ and, by Lemma 3.2,

$$\dot w(t) = \dot u(t) - \dot\alpha(t) = \tilde f(t,v) + \varepsilon\, \partial_v P(t,u(t),v(t)) - \dot\alpha(t) > 0\,,$$

for every $t \in]t_1,t_1+\delta]$ and $|\varepsilon|$ small enough. Hence, $w(t_1) < 0$, implying that $u(t) < \alpha(t)$ for every $t \in [t_1,t_1+\delta]$. Then, by (3.19), we necessarily have that $v(t_1) = v_\alpha^\varepsilon(t_1)$. Now if we define the map $G(t) = v(t) - v_\alpha^\varepsilon(t)$, then G is continuous on $[t_1, t_0]$, $G(t_1) = 0$ and $G(t) > 0$ for every $t \in]t_1, t_0]$. But then, using (3.17) and the fact that $u(t) \le \alpha(t)$ for every $t \in [t_1, t_0]$, we have

$$\dot G(t_1) = \dot v(t_1) - \dot v_\alpha^\varepsilon(t_1) = \tilde g(t_1, u(t_1)) - \varepsilon\, \partial_u P(t, q, p, u, v) - \dot v_\alpha^\varepsilon(t_1) < 0\,,$$

for $|\varepsilon|$ small enough; a contradiction. □

We now define the sets

$$\begin{aligned} A_W &= \{(t,u,v) \in \mathbb{R}^3 \;:\; u < \alpha(t),\ v = v_\alpha^\varepsilon(t)\}\,,\\ A_E &= \{(t,u,v) \in \mathbb{R}^3 \;:\; u > \beta(t),\ v = v_\beta^\varepsilon(t)\}\,. \end{aligned}$$

Lemma 3.4. *If $z = (q,p,u,v)$ is a solution of system* (3.7) *such that $(t_0, u(t_0), v(t_0)) \in A_W$, then there exists a $\delta > 0$ such that*

$$\begin{aligned} t \in]t_0 - \delta, t_0[\quad &\Rightarrow \quad (t, u(t), v(t)) \in A_{NW}\,,\\ t \in]t_0, t_0 + \delta[\quad &\Rightarrow \quad (t, u(t), v(t)) \in A_{SW}\,. \end{aligned}$$

Similarly, if $(t_0, u(t_0), v(t_0)) \in A_E$, then there exists a $\delta > 0$ such that

$$\begin{aligned} t \in]t_0 - \delta, t_0[\quad &\Rightarrow \quad (t, u(t), v(t)) \in A_{SE}\,,\\ t \in]t_0, t_0 + \delta[\quad &\Rightarrow \quad (t, u(t), v(t)) \in A_{NE}\,. \end{aligned}$$

Proof. We give only the proof of the first part, the proof of the second part being similar. Let $z = (q,p,u,v)$ be a solution of system (3.7) such that $(t_0, u(t_0), v(t_0)) \in A_W$. Then $v(t_0) = v_\alpha^\varepsilon(t_0)$ and $u(t_0) < \alpha(t_0)$. Let us define a map $G(t) = v(t) - v_\alpha^\varepsilon(t)$. Then, G is continuous with $G(t_0) = 0$, and by (3.17) we have

$$\begin{aligned} \dot G(t_0) &= \dot v(t_0) - \dot v_\alpha^\varepsilon(t_0)\\ &= \tilde g(t_0, u(t_0)) - \varepsilon\, \partial_u P(t_0, q(t_0), p(t_0), u(t_0), v(t_0)) - \dot v_\alpha^\varepsilon(t_0) < 0\,, \end{aligned}$$

for $|\varepsilon|$ small enough. So, there exists $\delta > 0$ such that $G(t) > 0$ for every $t \in]t_0 - \delta, t_0[$, and $u(t) < \alpha(t)$ for every $t \in [t_0 - \delta, t_0 + \delta]$. The conclusion is thus proved. □

Lemma 3.5. *If $z = (q,p,u,v)$ is a T-periodic solution of system* (3.7)*, then $(t, z(t)) \in \mathcal{V}$, for every $t \in \mathbb{R}$.*

Proof. Let us first prove that, for every $t \in \mathbb{R}$, we have

$$\alpha(t) \le u(t) \le \beta(t) . \tag{3.20}$$

Suppose that there exists a solution $z = (q, p, u, v)$ of system (3.7) such that $u(t_0) < \alpha(t_0)$ for some $t_0 \in [0, T]$. If $(t_0, u(t_0), v(t_0)) \in A_{NW}$, then from Lemma 3.3 we have that $(t, u(t), v(t)) \in A_{NW}$ for every $t < t_0$. Then, by (3.15), we have

$$\frac{d}{dt}(u - \alpha)(t) = \tilde{f}(t, v(t)) + \varepsilon\, \partial_v P(t, u(t), v(t)) - \dot{\alpha}(t) > 0 ,$$

for $|\varepsilon|$ small enough and every $t < t_0$, which is clearly a contradiction, because $u - \alpha$ is a periodic solution. The same reasoning applies if $(t_0, u(t_0), v(t_0)) \in A_{SW}$. Finally, if $(t_0, u(t_0), v(t_0)) \in A_W$, then by Lemma 3.4 we know that the solution will be in A_{SW} or in A_{NW} at some time near t_0, hence we obtain a contradiction again. Then, $u(t) \ge \alpha(t)$ for every $t \in [0, T]$. In a similar way we can prove that $u(t) \le \beta(t)$ for every $t \in [0, T]$.

Finally we prove that

$$\gamma_-(u(t)) \le v(t) \le \gamma_+(u(t)) . \tag{3.21}$$

For such a solution $z = (q, p, u, v)$, by (3.2) and (3.20) we see that $\tilde{g}(t, u(t)) = g(t, u(t))$. Now, define the T-periodic function $H_-(t) = v(t) - \gamma_-(u(t))$. Let $t_m \in [0, T]$ be such that $H_-(t_m) = \min H_-$ and assume by contradiction that $H_-(t_m) < 0$. Then,

$$\begin{aligned}
\dot{H}_-(t_m) &= \dot{v}(t_m) - \gamma_-'(u(t_m))\dot{u}(t_m) \\
&= g(t_m, u(t_m)) - \varepsilon\, \partial_u P\,(t, q(t_m), p(t_m), u(t_m), v(t_m)) - \mu \dot{u}(t_m) \\
&= g(t_m, u(t_m)) - \mu \tilde{f}(t_m, v(t_m)) \\
&\quad - \varepsilon\,(\partial_u P(t_m, q(t_m), p(t_m), u(t_m), v(t_m)) + \mu\, \partial_v P(t_m, u(t_m), v(t_m))) .
\end{aligned}$$

We now consider the following cases:

Case 1. If $-\hat{d} \le v(t_m) \le \gamma_-(u(t_m))$, then $\tilde{f}(t_m, v(t_m)) = f(t_m, v(t_m))$ and so we have

$$\begin{aligned}
\dot{H}_-(t_m) &\le C - \mu \cdot (-c) \\
&\quad - \varepsilon\,(\partial_u P(t_m, q(t_m), p(t_m), u(t_m), v(t_m)) + \mu\, \partial_v P(t_m, u(t_m), v(t_m))) \\
&< 0 ,
\end{aligned}$$

for $|\varepsilon|$ small enough, since $g(t_m, u(t_m)) \le C$ and $\mu < -\frac{C}{c}$.

Case 2. If $v(t_m) < -\hat{d} - 1$, then $\tilde{f}(t_m, v(t_m)) = v(t_m)$ and so we have

$$
\begin{aligned}
\dot{H}_-(t_m) \le{} & C - \mu\, v(t_m) \\
& -\varepsilon\, (\partial_u P(t_m, q(t_m), p(t_m), u(t_m), v(t_m)) + \mu\, \partial_v P(t_m, u(t_m), v(t_m))) \\
<{} & C - \mu \cdot (-\hat{d} - 1) \\
& -\varepsilon\, (\partial_u P(t_m, q(t_m), p(t_m), u(t_m), v(t_m)) + \mu\, \partial_v P(t_m, u(t_m), v(t_m))) \\
<{} & 0\,,
\end{aligned}
$$

for $|\varepsilon|$ small enough, since $g(t_m, u(t_m)) \le C$ and $\hat{d} > C/|\mu|$.

Case 3. If $-\hat{d} - 1 \le v(t_m) \le -\hat{d}$, then $\tilde{f}(t_m, v(t_m))$ is a linear interpolation between $f(t_m, v(t_m))$ and $v(t_m)$, hence

$$
\min\{f(t_m, v(t_m)), v(t_m)\} \le f(t_m, v(t_m)) \le \max\{f(t_m, v(t_m)), v(t_m)\}\,,
$$

so we have

$$
\begin{aligned}
\dot{H}_-(t_m) \le{} & C - \mu \max\{f(t_m, v(t_m)), v(t_m)\} \\
& -\varepsilon\, (\partial_u P(t_m, q(t_m), p(t_m), u(t_m), v(t_m)) + \mu\, \partial_v P(t_m, u(t_m), v(t_m)))\,.
\end{aligned}
$$

If $f(t_m, v(t_m)) \le v(t_m)$, then

$$
\begin{aligned}
\dot{H}_-(t_m) \le{} & C + \mu \hat{d} \\
& -\varepsilon\, (\partial_u P(t_m, q(t_m), p(t_m), u(t_m), v(t_m)) + \mu\, \partial_v P(t_m, u(t_m), v(t_m))) \\
<{} & 0\,,
\end{aligned}
$$

for $|\varepsilon|$ small enough, since $\hat{d} > c$ and $\mu < -C/c$. On the other hand, if $v(t_m) < f(t_m, v(t_m))$, then again

$$
\begin{aligned}
\dot{H}_-(t_m) \le{} & C - \mu \cdot (-c) \\
& -\varepsilon\, (\partial_u P(t_m, q(t_m), p(t_m), u(t_m), v(t_m)) + \mu\, \partial_v P(t_m, u(t_m), v(t_m))) \\
<{} & 0\,,
\end{aligned}
$$

for $|\varepsilon|$ small enough.

In all the above three cases we obtain contradictions, hence we have proved that $v(t) \ge \gamma_-(u(t))$, for every $t \in [0, T]$. In a similar way we can prove that $v(t) \le \gamma_+(u(t))$, for every $t \in [0, T]$. □

We have thus proved that, if $z = (q, p, u, v)$ is a solution of system (3.7), then $(t, z(t)) \in \mathcal{V}$, for every $t \in \mathbb{R}$, and so z is a solution of system (1.1). This completes the proof of Theorem 2.5. □

4 Consequences and applications of Theorem 2.5

Let $\phi : \mathbb{R} \to \mathbb{R}$ be an increasing diffeomorphism with a bounded derivative, such that $\phi(0) = 0$. Consider the system

$$\begin{cases} \dot{q} = \partial_p H(t,q,p) + \varepsilon\, \partial_p P(t,q,p,u)\,, \\ \dot{p} = -\partial_q H(t,q,p) - \varepsilon\, \partial_q P(t,q,p,u)\,, \\ \frac{d}{dt}(\phi(\dot{u})) = g(t,u) - \varepsilon\, \partial_u P(t,q,p,u)\,, \end{cases} \tag{4.1}$$

where $P = P(t,q,p,u)$ is a perturbation term which is 2π-periodic in q and has a bounded gradient with respect to (q,p,u). As a direct consequence of the Theorem 2.5, we have the following result.

Corollary 4.1. *Let $A1$ and $A2$ hold. Moreover, let there exist two T-periodic C^2-functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ with $\alpha \le \beta$, such that*

$$\frac{d}{dt}(\phi(\dot{\alpha}))(t) > g(t,\alpha(t))\,, \qquad \frac{d}{dt}(\phi(\dot{\beta}))(t) < g(t,\beta(t))\,,$$

for every $t \in [0,T]$. Then there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \le \bar{\varepsilon}$, system (4.1) has at least two geometrically distinct T-periodic solutions, such that $p(0) \in\,]a,b[$ and $\alpha \le u \le \beta$.

Proof. Define $v_\alpha, v_\beta : \mathbb{R} \to \mathbb{R}$ as

$$v_\alpha(t) = \phi(\dot{\alpha}(t))\,, \qquad v_\beta(t) = \phi(\dot{\beta}(t))\,.$$

Setting $f(t,v) = \phi^{-1}(v)$, all the assumptions of Theorem 2.5 are satisfied, and so the conclusion follows. □

Notice that, taking $\phi(s) = s$ for all $s \in \mathbb{R}$, the last equation in (4.1) becomes

$$\ddot{u} = g(t,u) - \varepsilon\, \partial_u P(t,q,p,u)\,.$$

Example 4.2. Consider the following system

$$\begin{cases} -\ddot{q} = a \sin q + \varepsilon\, \partial_q P(t,q,u)\,, \\ -\ddot{u} = -g(t,u) + \varepsilon\, \partial_u P(t,q,u)\,, \end{cases} \tag{4.2}$$

where $a > 0$. Assume that P is 2π-periodic in q and has a bounded gradient with respect to (q,u), and the function g satisfies the Landesman–Lazer condition

$$\int_0^T \limsup_{u \to -\infty} g(t,u)dt < 0 < \int_0^T \liminf_{u \to +\infty} g(t,u)dt\,. \tag{4.3}$$

By using (4.3) and [14, Lemma 2], we get a strict lower solution α and a strict upper solution β of the equation $\ddot{u} = g(t, u)$. So, Corollary 4.1 applies, and thus system (4.2) has at least two geometrically distinct solutions.

5 The higher dimensional case

We now consider the following system

$$\begin{cases} \dot{q} = \nabla_p H(t, q, p) + \varepsilon \nabla_p P(t, q, p, u, v)\,, \\ \dot{p} = -\nabla_q H(t, q, p) - \varepsilon \nabla_q P(t, q, p, u, v)\,, \\ \dot{u_j} = f_j(t, v_j) + \varepsilon\, \partial_{v_j} P(t, q, p, u, v)\,, \quad j = 1, \dots, L\,, \\ \dot{v_j} = g_j(t, u_j) - \varepsilon\, \partial_{u_j} P(t, q, p, u, v)\,, \quad j = 1, \dots, L\,. \end{cases} \tag{5.1}$$

For $z = (q, p, u, v) \in \mathbb{R}^N$ we write

$$q = (q_1, \dots, q_M) \in \mathbb{R}^M, \quad p = (p_1, \dots, p_M) \in \mathbb{R}^M,$$
$$u = (u_1, \dots, u_L) \in \mathbb{R}^L, \quad v = (v_1, \dots, v_L) \in \mathbb{R}^L.$$

We assume all the involved functions to be continuous, and T-periodic in their first variable t.

We first recall the definition of lower and upper solution for the T-periodic problem associated with the system

$$\dot{u_j} = f_j(t, v_j)\,, \qquad \dot{v_j} = g_j(t, u_j)\,, \quad j = 1, \dots, L\,. \tag{5.2}$$

Definition 5.1. *A T-periodic C^1-function $\alpha : \mathbb{R} \to \mathbb{R}^L$ is said to be a "lower solution" for the T-periodic problem associated with system* (5.2) *if there exists a T-periodic C^1-function $v_\alpha : \mathbb{R} \to \mathbb{R}^L$ such that, for every $j = 1, \dots, L$ we have*

$$\begin{cases} s < v_{\alpha,j}(t) \quad \Rightarrow \quad f_j(t, s) < \dot{\alpha_j}(t)\,, \\ s > v_{\alpha,j}(t) \quad \Rightarrow \quad f_j(t, s) > \dot{\alpha_j}(t)\,, \end{cases} \tag{5.3}$$

and

$$\dot{v}_{\alpha,j}(t) \geq g_j(t, \alpha_j(t))\,. \tag{5.4}$$

The lower solution is "strict" if the strict inequalities in (5.4) hold.

Definition 5.2. *A T-periodic C^1-function $\beta : \mathbb{R} \to \mathbb{R}^L$ is said to be an "upper solution" for the T-periodic problem associated with system* (5.2) *if there exists a T-periodic C^1-function $v_\beta : \mathbb{R} \to \mathbb{R}^L$ such that, for every $j = 1, \dots, L$ we have*

$$\begin{cases} s < v_{\beta,j}(t) \quad \Rightarrow \quad f_j(t,s) < \dot{\beta}_j(t)\,, \\ s > v_{\beta,j}(t) \quad \Rightarrow \quad f_j(t,s) > \dot{\beta}_j(t)\,, \end{cases} \tag{5.5}$$

and

$$\dot{v}_{\beta,j}(t) \leq g_j(t, \beta_j(t))\,. \tag{5.6}$$

The upper solution is "strict" if the strict inequalities in (5.6) hold.

In the sequel, inequalities of n-tuples will be meant componentwise. It will be useful to introduce the vector $\mathbb{I} = (1, \dots, 1) \in \mathbb{R}^L$.

We say that $\mathcal{D}$ is a convex body of $\mathbb{R}^M$ if it is a closed convex bounded subset of $\mathbb{R}^M$ having nonempty interior. By assuming that $\mathcal{D}$ has a smooth boundary, we denote the unit outward normal at $\zeta \in \partial\mathcal{D}$ by $\nu_{\mathcal{D}}(\zeta)$. Moreover, we say that $\mathcal{D}$ is strongly convex if for any $p \in \partial\mathcal{D}$, the map $\mathcal{F} : \mathcal{D} \to \mathbb{R}$ defined by $\mathcal{F}(\xi) = \langle \xi - p, \nu_{\mathcal{D}}(p) \rangle$ has a unique maximum point at $\xi = p$.

Here are our hypotheses.

*A*1$'$. The function $H(t,q,p)$ is 2π-periodic in each variable $q_1, \dots, q_M$.

*A*2$'$. There are a strongly convex body $\mathcal{D}$ of $\mathbb{R}^M$ having a smooth boundary and a symmetric regular $M \times M$ matrix $\mathbb{A}$ such that all the solutions (q,p) of system

$$\dot{q} = \nabla_p H(t,q,p)\,, \qquad \dot{p} = -\nabla_q H(t,q,p)\,, \tag{5.7}$$

starting with $p(0) \in \mathcal{D}$ are defined on $[0,T]$, and

$$p(0) \in \partial\mathcal{D} \quad \Rightarrow \quad \langle q(T) - q(0)\,,\, \mathbb{A}\nu_{\mathcal{D}}(p(0)) \rangle > 0\,.$$

*A*3$'$. There exist a strict lower solution α and a strict upper solution β for the T-periodic problem associated with system (5.2), such that $\alpha \leq \beta$.

*A*4$'$. There exist positive constants c, d such that $|\dot{\alpha}_j(t)| < c$, $|\dot{\beta}_j(t)| < c$ for every $t \in [0,T]$, and

$$\begin{cases} f_j(t,s) \geq c\,, & \text{for } s \geq d\,, \\ f_j(t,s) \leq -c\,, & \text{for } s \leq -d\,, \end{cases}$$

for every $j = 1, \dots, L$.

*A*5′. The second order partial derivatives $\partial^2_{ts} f_j(t,s)$ and $\partial^2_{ss} f_j(t,s)$ exist and are continuous; moreover, there exists $\lambda > 0$ such that

$$\partial_s f_j(t,s) \geq \lambda\,, \qquad \text{for every } (t,s) \in [0,T] \times \mathbb{R}\,,$$

for every $j = 1, \dots, L$.

*A*6′. The function $P(t,q,p,u,v)$ is 2π-periodic in $q_1, \dots, q_M$ and has a bounded gradient with respect to (q,p,u,v); moreover, the partial derivative $\nabla_v P$ is independent of q and p, and the map $\nabla_v P(t,u,v)$ is continuously differentiable.

Here is our first generalization of Theorem 2.5.

Theorem 5.3. *Assume that A*1′ *– A*6′ *hold true. Then there exists* $\bar{\varepsilon} > 0$ *such that, if* $|\varepsilon| \leq \bar{\varepsilon}$*, there are at least* $M+1$ *geometrically distinct* T*-periodic solutions of system* (5.1)*, with* $p(0) \in \mathring{\mathcal{D}}$ *and* $\alpha \leq u \leq \beta$*.*

Proof. The arguments will be similar to the ones provided in Section 3, so we will be very brief. Let $A_j = \min \alpha_j$, $B_j = \max \beta_j$, for $j = 1, \dots, L$, and

$$A = \min\{A_1, \dots, A_L\}\,, \qquad B = \max\{B_1, \dots, B_L\}\,.$$

Then there exists a constant $C > 0$ such that

$$|g_j(t,s)| \leq C\,, \qquad \text{for every } (t,s) \in [0,T] \times [A,B]\,,$$

and every $j = 1, \dots, L$. We can find two straight lines $\gamma_\pm : \mathbb{R} \to \mathbb{R}$, whose equations are

$$\gamma_+(s) = \mu s + R\,, \qquad \gamma_-(s) = \mu s - R\,,$$

where $\mu < -C/c$ and $R > 0$ are chosen in such a way that

$$\gamma_-(u) < -d < d < \gamma_+(u)\,,$$

and

$$\gamma_-(u)\mathbb{I} \leq \dot{\alpha}(t)\,, v_\alpha(t)\,, \dot{\beta}(t)\,, v_\beta(t) \leq \gamma_+(u)\mathbb{I}\,, \tag{5.8}$$

for every $(t,u) \in [0,T] \times [A,B]$.

We define the set

$$\mathcal{V} = \{(t, q, p, u, v) \in \mathbb{R}^{2N+1} \ : \ \alpha(t) \le u \le \beta(t), \ \gamma_-(u)\mathbb{I} \le v \le \gamma_+(u)\mathbb{I}\}\,.$$

We can choose a constant $\hat{d} > d$ such that $\hat{d} > C/|\mu|$, $\hat{d} > c$ and for every $s \in [A, B]$ we have

$$-\hat{d} < \gamma_-(s) < \gamma_+(s) < \hat{d}\,.$$

Now for $j = 1, \dots, L$ we consider the function $\eta_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$\eta_j(t, s) = \begin{cases} \alpha_j(t)\,, & \text{if } s \le \alpha_j(t)\,, \\ s\,, & \text{if } \alpha_j(t) \le s \le \beta_j(t)\,, \\ \beta_j(t)\,, & \text{if } s \ge \beta_j(t)\,. \end{cases}$$

Now define the function $\tilde{g}_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as in (3.2) by using η_j instead of η and $\tilde{f}_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as defined in (3.3). Similarly, by a cut-off function σ as in (3.6) we set

$$\widehat{H}(t, q, p) = \sigma(|p|)H(t, q, p)\,.$$

Then, $\widehat{H}$ has a bounded gradient with respect to (q, p). Consider now the modified system

$$\begin{cases} \dot{q} = \nabla_p \widehat{H}(t, q, p) + \varepsilon\, \nabla_p P(t, q, p, u, v)\,, \\ \dot{p} = -\nabla_q \widehat{H}(t, q, p) - \varepsilon\, \nabla_q P(t, q, p, u, v)\,, \\ \dot{u_j} = \tilde{f}_j(t, v_j) + \varepsilon\, \partial_{v_j} P(t, q, p, u, v)\,, \quad j = 1, \dots, L\,, \\ \dot{v_j} = \tilde{g}_j(t, u_j) - \varepsilon\, \partial_{u_j} P(t, q, p, u, v)\,, \quad j = 1, \dots, L\,, \end{cases} \tag{5.9}$$

where the new Hamiltonian function $\widetilde{H} : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is defined as

$$\widetilde{H}(t, z) = \widehat{H}(t, q, p) + \sum_{j=1}^{L} \left(\int_0^{v_j} \tilde{f}_j(t, s)\, ds - \int_0^{u_j} \tilde{g}_j(t, s)\, ds \right) + \varepsilon P(t, z)\,,$$

where $z = (q, p, u, v)$. The modified system (5.9) can be written as $\dot{z} = J\nabla \widetilde{H}(t, z)$, where $J = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$ is the standard symplectic matrix. Notice that

$$\widetilde{H}(t, z) = \frac{1}{2}(|v|^2 - |u|^2) + K(t, z)\,,$$

where K is a function having a bounded gradient with respect to z. Moreover, by $A2'$, since $\nabla_p P$ and $\nabla_q P$ are bounded, if $|\varepsilon|$ is small enough, for any solution (q, p, u, v) of (5.9) one still has that

$$p(0) \in \partial\mathcal{D} \quad \Rightarrow \quad \langle q(T) - q(0)\,,\, \mathbb{A}\nu_{\mathcal{D}}(p(0)) \rangle > 0\,.$$

Then, by [12, Corollary 2.3], we get that the modified system (5.9) has at least $M+1$ geometrically distinct T-periodic solutions such that $p(0) \in \mathring{D}$, provided that $|\varepsilon|$ is small enough.

We can now prove the analogues of Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5.

In particular, all the $M+1$ geometrically distinct T-periodic solutions of system (5.9) are solutions of system (5.1). Thus system (5.1) has at least $M+1$ geometrically distinct solutions, with $p(0) \in \mathring{\mathcal{D}}$ and $\alpha \leq u \leq \beta$, and this completes the proof of Theorem 5.3. □

We now consider some variants of Theorem 5.3. Let us first state the following "avoiding rays" assumption.

*A*2$''$. There exists a convex body $\mathcal{D}$ of $\mathbb{R}^M$, having a smooth boundary, such that all the solutions (q,p) of system (5.7) starting with $p(0) \in \mathcal{D}$ are defined on $[0,T]$, and

$$p(0) \in \partial\mathcal{D} \quad \Rightarrow \quad q(T) - q(0) \notin \{\lambda \nu_{\mathcal{D}}(p(0)) \, : \, \lambda \geq 0\} \, .$$

Theorem 5.4. *If in the statement of Theorem 5.3 we replace assumption A*2$'$ *by A*2$''$*, the same conclusion holds.*

Proof. The argument of proof is the same as that of Theorem 5.3 with the only difference that instead of applying [12, Corollary 2.3], we apply [12, Corollary 2.1]. □

Now we consider the case when $\mathcal{D}$ is a rectangle in $\mathbb{R}^M$, i.e.

$$\mathcal{D} = [a_1, b_1] \times \cdots \times [a_M, b_M] \, .$$

We state the following assumption.

*A*2$'''$. There exists an M-tuple $\sigma = (\sigma_1, \ldots, \sigma_M) \in \{-1, 1\}^M$ such that all the solutions (q,p) of system (5.7) starting with $p(0) \in \mathcal{D}$ are defined on $[0,T]$, and, for every $i = 1, \ldots, M$, we have

$$\begin{cases} p_i(0) = a_i \quad \Rightarrow \quad \sigma_i(q_i(T) - q_i(0)) < 0 \, , \\ p_i(0) = b_i \quad \Rightarrow \quad \sigma_i(q_i(T) - q_i(0)) > 0 \, . \end{cases}$$

Theorem 5.5. *If in the statement of Theorem 5.3 we replace assumption A*2$'$ *by A*2$'''$*, the same conclusion holds.*

Proof. The argument of proof is the same as that of Theorem 5.3 with the only difference that instead of applying [12, Corollary 2.3], we apply [12, Corollary 2.4]. □

Remark 5.6. *Based on the Remark 2.7, we could have varying intervals* $[a_i(s), b_i(s)]$ *instead of the intervals* $[a_i, b_i]$ *in the rectangle* $\mathcal{D}$*, where* $a_i, b_i : \mathbb{R} \to \mathbb{R}$ *are* 2π*-periodic continuous functions.*

We now provide a higher dimensional version of the Theorem 2.8.

Theorem 5.7. *Assume that* $A3'$ – $A6'$ *hold true. Let* $k_1, k_2, \dots, k_M$ *be integers and assume that, for each* $i \in \{1, \dots, M\}$*, there exist two planar Jordan curves* Γ_1^i*,* Γ_2^i*, strictly star-shaped with respect to the origin, with*

$$0 \in \mathcal{D}(\Gamma_1^i) \subseteq \overline{\mathcal{D}(\Gamma_1^i)} \subseteq \mathcal{D}(\Gamma_2^i)\,,$$

such that the solutions of system (5.7) *with* $(q_i(0), p_i(0)) \in \overline{\mathcal{D}(\Gamma_2^i)} \setminus \mathcal{D}(\Gamma_1^i))$ *for every* $i \in \{1, \dots, M\}$ *are defined on* $[0, T]$ *and satisfy*

$$(q_i(t), p_i(t)) \neq (0, 0), \quad \forall t \in [0, T]\,,$$

and

$$\begin{cases} (q_i(0), p_i(0)) \in \Gamma_1^i & \Rightarrow \quad \mathrm{Rot}((q_i, p_i); [0, T]) < k_i\,, \\ (q_i(0), p_i(0)) \in \Gamma_1^i & \Rightarrow \quad \mathrm{Rot}((q_i, p_i); [0, T]) < k_i\,. \end{cases} \tag{5.10}$$

Then system (5.1) *has at least* $M + 1$ *geometrically distinct* T*-periodic solutions* $z^{(n)}(t)$ *for* $n = 1, \dots, M + 1$ *such that*

$$\alpha \leq u^{(n)} \leq \beta\,,$$

$$(q_i^{(n)}(0), p_i^{(n)}(0)) \in \mathcal{D}(\Gamma_2^i) \setminus \overline{\mathcal{D}(\Gamma_1^i)}\,,$$

and

$$\mathrm{Rot}((q_i^{(n)}, p_i^{(n)}); [0, T]) = k_i\,,$$

for $i = 1, \dots, M$*. The same is true if for some* $i \in \{1, \dots, M\}$ *the assumption* (5.10) *is replaced by the following*

$$\begin{cases} (q_i(0), p_i(0)) \in \Gamma_1^i & \Rightarrow \quad \mathrm{Rot}((q_i, p_i); [0, T]) < k_i\,, \\ (q_i(0), p_i(0)) \in \Gamma_1^i & \Rightarrow \quad \mathrm{Rot}((q_i, p_i); [0, T]) > k_i\,. \end{cases}$$

Example 5.8. Consider the following system

$$\begin{cases} -\ddot{q}_i = a_i \sin q_i + \varepsilon\, \partial_{q_i} P(t,q,u)\,, \quad i = 1,\dots,M\,, \\ -\ddot{u}_j = -g_j(t,u_j) + \varepsilon\, \partial_{u_j} P(t,q,u)\,, \quad j = 1,\dots,L\,, \end{cases} \tag{5.11}$$

where $a_i > 0$. Assume that P is 2π-periodic in $q_1,\dots,q_M$ and has a bounded gradient with respect to (q,u), and for each $j \in \{1,\dots,L\}$, the function g_j satisfies the Landesman–Lazer condition

$$\int_0^T \limsup_{s\to-\infty} g_j(t,s)\,dt < 0 < \int_0^T \liminf_{s\to+\infty} g_j(t,s)\,dt\,. \tag{5.12}$$

By using (5.12) and [14, Lemma 2], we get a strict lower solution α_j and a strict upper solution β_j of the equation $\ddot{u}_j = g_j(t,u_j)$, with $\alpha_j(t) < \beta_j(t)$. So all the assumptions of Theorem 5.3 hold and thus system (5.11) has at least $M+1$ geometrically distinct T-periodic solutions.

6 Twist with Hartman type condition

In this section we consider a system in $\mathbb{R}^{2M+2L}$ of the type

$$\begin{cases} \dot{q} = \nabla_p H(t,q,p) + \varepsilon\, \nabla_p P(t,q,p,u)\,, \\ \dot{p} = -\nabla_q H(t,q,p) - \varepsilon\, \nabla_q P(t,q,p,u)\,, \\ \dot{u} = v\,, \qquad \dot{v} = \nabla_u G(t,u) + \varepsilon\, \nabla_u P(t,q,p,u)\,. \end{cases} \tag{6.1}$$

Here again all functions involved are assumed to be continuous, and T-periodic in t. We will assume the periodicity condition $A1'$ and one of the twist conditions $A2'$, $A2''$ or $A2'''$, even if in the following statement we concentrate on $A2'$. We also assume condition $A6'$ which, in this setting, can be stated in the following simpler form.

$A6''$. The function $P(t,q,p,u)$ is 2π-periodic in $q_1,\dots,q_M$ and has a bounded gradient with respect to (q,p,u).

Here is our statement, involving a Hartman-type condition (see [9, 19] and the references therein).

Theorem 6.1. *Assume that $A1'$, $A2'$ and $A6''$ hold true and that there exists $R > 0$ such that*

$$|u| = R \quad \Rightarrow \quad \langle \nabla_u G(t,u), u\rangle > 0\,. \tag{6.2}$$

Then there exists $\bar{\varepsilon} > 0$ such that for $|\varepsilon| \leq \bar{\varepsilon}$, there are at least $M+1$ geometrically distinct T-periodic solutions of system (6.1) *such that* $p(0) \in \mathring{D}$ *and* $|u(t)| \leq R$ *for every* $t \in \mathbb{R}$.

Proof. First of all, we modify the function G outside the ball $B_R = \{u : |u| \leq R\}$. By (6.2) and the continuity of the inner product, there exists $\tilde{\rho} > 0$ and $\delta > 0$ such that

$$R \leq |u| \leq R + \tilde{\rho} \quad \Rightarrow \quad \langle \nabla_u G(t,u), u \rangle \geq \delta \,. \tag{6.3}$$

We can assume without loss of generality that

$$G(t,u) \leq 0 \,, \qquad \text{when } R \leq |u| \leq R + \tilde{\rho} \,. \tag{6.4}$$

Indeed, if it is not already the case, it is sufficient to replace $G(t,u)$ by $G(t,u) - M$, where

$$M = \max\{|G(t,u)| \;\;:\;\; 0 \leq t \leq T,\; R < |u| \leq R + \tilde{\rho}\} \,.$$

Its gradient will not be changed.

Moreover, as in the previous proofs, after a truncation we can from now on assume that H has a bounded gradient with respect to (q,p).

Now choose a C^∞-function $\eta : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{cases} \eta(s) = 1 \,, & \text{if } s \leq R \,, \\ \dot{\eta}(s) \leq 0 \,, & \text{if } R \leq s \leq R + \tilde{\rho} \,, \\ \eta(s) = 0 \,, & \text{if } s > R + \tilde{\rho} \,, \end{cases}$$

and define the function

$$\widetilde{G}(t,u) = \begin{cases} G(t,u) \,, & \text{if } \; |u| \leq R \,, \\ \eta(|u|)\, G(t,u) + (1 - \eta(|u|)) \frac{1}{2}|u|^2, & \text{if } \; R \leq |u| \leq R + \tilde{\rho} \,, \\ \frac{1}{2}|u|^2, & \text{if } \; |u| > R + \tilde{\rho} \,. \end{cases}$$

Notice that, outside the ball $B_{R+\tilde{\rho}}$, the system becomes almost linear.

We now consider the new system

$$\begin{cases} \dot{q} = \nabla_p H(t,q,p) + \varepsilon \, \nabla_p P(t,q,p,u) \,, \\ \dot{p} = -\nabla_q H(t,q,p) - \varepsilon \, \nabla_q P(t,q,p,u) \,, \\ \dot{u} = v, \qquad \dot{v} = \nabla_u \widetilde{G}(t,u) + \varepsilon \, \nabla_u P(t,q,p,u) \,, \end{cases} \tag{6.5}$$

where the new Hamiltonian function is

$$\widetilde{H}(t,q,p,u,v) = H(t,q,p) + \tfrac{1}{2}|v|^2 + \widetilde{G}(t,u) + \varepsilon\, P(t,q,p,u)\,.$$

Writing the modified system (6.5) as $\dot z = J\nabla \widetilde{H}(t,z)$, we see that

$$\widetilde{H}(t,z) = \frac{1}{2}(|v|^2 - |u|^2) + K(t,z)\,,$$

with $K(t,z)$ has a bounded gradient with respect to $z=(q,p,u,v)$. Then, by [12, Corollary 2.3], the modified system (6.5) has at least $M+1$ geometrically distinct T-periodic solutions, such that $p(0)\in \mathring{D}$, provided that $|\varepsilon|$ is small enough.

We need to prove that the T-periodic solutions of system (6.5) we have found are such that $|u(t)|\le R$ for every $t\in[0,T]$, so that they are indeed solutions of system (6.1).

Assume by contradiction that there exists $t_0\in\mathbb{R}$ such that

$$|u(t_0)| = \max\{|u(t)| \,:\, t\in[0,T]\} > R\,.$$

Consider the function $f(t)=|u(t)|^2$. We have that $\dot f(t_0)=0$ and $\ddot f(t_0)\le 0$. Being $\dot f(t) = \langle 2u(t),\dot u(t)\rangle$, we compute

$$\begin{aligned}\ddot f(t) &= 2\langle \dot u(t),\dot u(t)\rangle + 2\,\langle u(t),\ddot u(t)\rangle\\ &= 2|\dot u(t)|^2 + 2\left\langle u(t), \nabla_u\widetilde{G}(t,u(t)) + \varepsilon\,\nabla_u P(t,q(t),p(t),u(t))\right\rangle\\ &\ge 2\left\langle u(t), \nabla_u\widetilde{G}(t,u(t)) + \varepsilon\,\nabla_u P(t,q(t),p(t),u(t))\right\rangle.\end{aligned} \tag{6.6}$$

We have two cases.

Case 1. If $|u(t_0)|>R+\tilde\rho$, then by the Cauchy–Schwartz inequality and the fact that $|\nabla_u P(t,q,p,u)|<C$, the inequality (6.6) implies that

$$\begin{aligned}\ddot f(t_0) &\ge 2\,\langle u(t_0), u(t_0) + \varepsilon\,\nabla_u P(t_0,q(t_0),p(t_0),u(t_0))\rangle\\ &\ge 2|u(t_0)|^2 - 2\varepsilon|u(t_0)|\,|\nabla_u P(t_0,q(t_0),p(t_0),u(t_0))|\\ &\ge 2|u(t_0)|\big(|u(t_0)| - \varepsilon\,|\nabla_u P(t_0,q(t_0),p(t_0),u(t_0))|\big)\\ &> 2R^2 > 0\,,\end{aligned}$$

for $|\varepsilon|$ small enough, a contradiction.

Case 2. If $R < |u(t_0)| < R + \tilde{\rho}$, then again (6.6) implies that

$$
\begin{aligned}
\ddot{f}(t_0) \geq\ & 2 \left\langle u(t_0), \dot{\eta}\left(|u(t_0)|\right) \frac{u(t_0)}{|u(t_0)|} G(t_0, u(t_0)) + \eta\left(|u(t_0)|\right) \nabla_u G(t_0, u(t_0)) \right\rangle \\
& +2 \left\langle u(t_0), -\dot{\eta}\left(|u(t_0)|\right) \frac{u(t_0)}{|u(t_0)|} \frac{1}{2} |u(t_0)|^2 + \left(1 - \eta\left(|u(t_0)|\right)\right) u(t_0) \right\rangle \\
& -2|u(t_0)| \varepsilon \left| \nabla_u P(t_0, q(t_0), p(t_0), u(t_0)) \right| \\
=\ & 2\dot{\eta}\left(|u(t_0)|\right) |u(t_0)| G(t_0, u(t_0)) + 2\eta\left(|u(t_0)|\right) \langle u(t_0), \nabla_u G(t_0, u(t_0)) \rangle \\
& -\dot{\eta}\left(|u(t_0)|\right) |u(t_0)|^3 + 2\left(1 - \eta\left(|u(t_0)|\right)\right) |u(t_0)|^2 \\
& -2|u(t_0)| \varepsilon \left| \nabla_u P(t_0, q(t_0), p(t_0), u(t_0)) \right| .
\end{aligned}
$$

By (6.3) and (6.4), since $\dot{\eta}\left(|u(t_0)|\right) \leq 0$ and

$$
\eta\left(|u(t_0)|\right) \langle u(t_0), \nabla_u G(t_0, u(t_0)) \rangle + \left(1 - \eta\left(|u(t_0)|\right)\right) |u(t_0)|^2 \geq \min\{\delta, R^2\} ,
$$

we have that

$$
\ddot{f}(t_0) \geq 2 \min\{\delta, R^2\} - 2(R + \tilde{\rho}) \varepsilon \left| \nabla_u P(t_0, q(t_0), p(t_0), u(t_0)) \right| > 0 ,
$$

when $|\varepsilon|$ is small enough, a contradiction. The proof is thus completed. □

Remark 6.2. *In the case $L = 1$, writing $g(t, u) = \nabla_u G(t, u)$, the Hartman condition becomes*

$$
g(t, -R) < 0 < g(t, R) .
$$

It is thus seen that $\alpha = -R$ and $\beta = R$ are constant *strict lower/upper solutions, with $\alpha < \beta$.*

7 Perturbations of completely integrable systems

There is a very large literature on the periodic problem for perturbations of completely integrable systems (see, e.g., [3, 12] and references therein), starting from Poincaré, who referred to Hamiltonian perturbation theory as the "Problème général de la Dynamique".

We will add now an extra term to the Hamiltonian function, involving a Hartman-type situation. Consider the system

$$
\begin{cases}
\dot{\varphi} = \nabla \mathcal{K}(I) + \varepsilon \nabla_I P(t, \varphi, I, u) , \quad \dot{I} = -\varepsilon \nabla_\phi P(t, \varphi, I, u) , \\
-\ddot{u} = \nabla_u G(t, u) + \varepsilon \nabla_u P(t, \varphi, I, u) ,
\end{cases}
\tag{7.1}
$$

where $(\varphi, I) \in \mathbb{R}^{2M}$ and $u \in \mathbb{R}^L$. As usual we assume that all the involved functions are continuous and T-periodic in t. The perturbation function $P : \mathbb{R} \times \mathbb{R}^{2M+L} \to \mathbb{R}$ is assumed to have a bounded gradient with respect to (φ, I, u). Moreover, it is τ_i-periodic in each variable φ_i, i.e.

$$P(t, \dots, \varphi_i + \tau_i, \dots) = P(t, \dots, \varphi_i, \dots)\,,$$

and we assume that there exist some integers $m_1, \dots, m_M$ such that

$$T\nabla\mathcal{K}(I^0) = (m_1\tau_1, \dots, m_M\tau_M)\,.$$

We are thus dealing with a completely resonant torus. Here is our result.

Theorem 7.1. *In the above setting, assume that there exist* $I^0 \in \mathbb{R}^M$*, a symmetric invertible* $M \times M$ *matrix* $\mathbb{A}$ *and* $\rho > 0$ *such that*

$$0 < |I - I^0| \le \rho \quad \Rightarrow \quad \big\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0)\,,\, \mathbb{A}(I - I^0) \big\rangle > 0\,. \tag{7.2}$$

Moreover, let there exist $R > 0$ *such that*

$$|u| = R \quad \Rightarrow \quad \langle \nabla_u G(t, u), u \rangle > 0\,.$$

Then, for every $\sigma > 0$*, there exists* $\tilde{\varepsilon} > 0$ *such that, for* $|\varepsilon| < \tilde{\varepsilon}$*, there are at least* $M + 1$ *geometrically distinct solutions of system* (7.1)*, with*

$$\varphi(t + T) = \varphi(t) + T\nabla\mathcal{K}(I^0), \quad u(t + T) = u(t), \quad I(t + T) = I(t)\,,$$

$$|\varphi(t) - \varphi(0) - t\nabla\mathcal{K}(I^0)| + |I(t) - I^0| < \sigma\,,$$

and

$$|u(t)| \le R\,,$$

for every $t \in \mathbb{R}$*.*

The proof is based on Theorem 6.1, following the same reasoning as in [10, Theorem 23], so we omit it, for briefness.

Remark 7.2. *It can easily be seen that assumption* (7.2) *is satisfied if the function* $\mathcal{K}$ *is twice continuously differentiable at* I^0*, with*

$$\det \mathcal{K}''(I^0) \neq 0\,.$$

It is indeed sufficient to choose $\mathbb{A} = \mathcal{K}''(I^0)$*.*

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