On the Dirichlet problem associated to bounded perturbations of positively-(p, q)-homogeneous Hamiltonian systems

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Abstract

The existence of solutions for the Dirichlet problem associated to bounded perturbations of positively-(p, q)-homogeneous Hamiltonian systems is considered both in nonresonant and resonant situations. In order to deal with the resonant case, the existence of a couple of lower and upper solutions is assumed. Both the well-ordered and the non-well-ordered cases are analysed. The proof is based on phase-plane analysis and topological degree theory.

1 Introduction

Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable positively-(p, q)-homogeneous positive-definite function. By this we mean that, for some $p > 1$ and $q > 1$ with $(1/p) + (1/q) = 1$,

$$H(\lambda^q x, \lambda^p y) = \lambda^{p+q} H(x, y) > 0\,, \tag{1}$$

for every $\lambda > 0$ and $(x, y) \neq (0, 0)$. We are interested in Dirichlet problems of the type

$$\begin{cases} x' = \dfrac{\partial H}{\partial y}(x, y) + \phi(t, x, y)\,, \quad y' = -\dfrac{\partial H}{\partial x}(x, y) + \psi(t, x, y) \\ x(a) = 0 = x(b)\,, \end{cases} \tag{2}$$

where the functions $\phi, \psi : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and bounded.

The autonomous system

$$x' = \frac{\partial H}{\partial y}(x, y)\,, \quad y' = -\frac{\partial H}{\partial x}(x, y) \tag{3}$$

is isochronous (see [25]). More precisely, $(0, 0)$ is a global center, and all solutions are periodic of the same period, which we denote by τ. Moreover, if $y_0 > 0$, for all solutions (x, y) starting with $(x(0), y(0)) = (0, y_0)$ there is a first time $\tau_+ > 0$ for which $x(\tau_+) = 0$, while $x(t) > 0$ for every $t \in]0, \tau_+[$, and this time τ_+ is independent of $y_0 > 0$. Symmetrically, if $y_0 < 0$, there is a first time $\tau_- > 0$ for which $x(\tau_-) = 0$, while $x(t) < 0$ for every $t \in]0, \tau_-[$. Clearly enough, $\tau_+ + \tau_- = \tau$.

First of all, let us state the following *nonresonance* result.

Theorem 1. *Assume that there is an integer $n \geq 0$ such that one of the following alternatives hold*

$$n\tau < b - a < n\tau + \min\{\tau_-, \tau_+\}\,, \tag{4}$$

$$n\tau + \max\{\tau_-, \tau_+\} < b - a < (n+1)\tau\,. \tag{5}$$

Then, problem (2) *has a solution.*

Its proof relies on the so called *shooting method*, following the ideas presented in [10] (see also [3, 23]), and will be given in Section 3.

As a possible example we may consider the function

$$H(x,y) = \frac{1}{q}\Big(\delta(y^+)^q + \gamma(y^-)^q\Big) + \frac{1}{p}\Big(\mu(x^+)^p + \nu(x^-)^p\Big)\,,$$

for some positive constants δ, γ, μ, ν (we use the standard notation where $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$). If we choose $\delta = \gamma = 1$ and $\phi \equiv 0$, our problem is then equivalent to

$$\begin{cases} (|x'|^{p-2}x')' + \mu|x^+|^{p-2}x^+ - \nu|x^-|^{p-2}x^- = h(t,x,x') \\ x(a) = 0 = x(b)\,, \end{cases} \tag{6}$$

with $h(t,x,v) = -\psi(t,x,|v|^{p-2}v)$. In this case, $\tau_+ = \pi_p \mu^{-1/p}$ and $\tau_- = \pi_p \nu^{-1/p}$, cf. [18], where

$$\pi_p = \frac{2(p-1)^{1/p}}{p\sin(\pi/p)}\,\pi\,,$$

and the assumptions of Theorem 1 are illustrated in Figure 1. If $p = 2$, the differential equation in (6) models an asymmetric oscillator, with $\tau_+ = \pi/\sqrt{\mu}$, $\tau_- = \pi/\sqrt{\nu}$, and it is well known that some care on the choice of μ, ν must be taken in order to avoid resonance phenomena; when h is bounded, the existence of a solution depends on the position of (μ, ν) with respect to the Fučík spectrum, cf. [10]. In Figure 1, the assumption of Theorem 1 can be visualized, when $b - a = \pi_p$, taking the values $(\mu^{-1/p}, \nu^{-1/p})$ in the white regions.

A different way of avoiding resonance would be the assumption of the existence of a well-ordered couple of lower/upper solutions $\alpha \leq \beta$, together with some Nagumo-type conditions. This method goes back to the pioneering papers [20, 21, 22]. Some more care is needed in the non-well-ordered case $\alpha \not\leq \beta$, see [1, 8, 9, 16, 17]. We refer to the book [7] for an extensive exposition on the theory of lower and upper solutions for scalar second order differential equations.

The concept of lower and upper solution has been recently extended to planar systems in [15] for the periodic problem (see also [12]), and in [14] for Sturm–Liouville type problems, including the Dirichlet problem. We recall in Section 4 the main definitions in this case.

Here is our result for problem (2), in the well-ordered case.

Theorem 2. *Assume H to be a continuously differentiable positively-(p,q)-homogeneous positive-definite function, for some $p > 1$ and $q > 1$ with $(1/p) + (1/q) = 1$. Let ϕ, ψ be uniformly bounded continuous functions, and let (α, β) be a well-ordered pair of lower/upper solutions of problem* (2). *Then, there exists a solution (x,y) of* (2) *such that $\alpha \leq x \leq \beta$.*

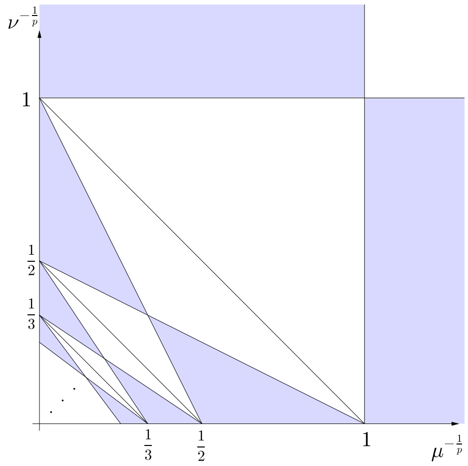

Figure 1: The Fučík spectrum.

The proof will be given in Section 5. We will first need some properties of positively-(p,q)-homogeneous Hamiltonian systems, which we provide in Section 2. Then, the main issue will be the construction of some guiding curves in the phase plane so to enter the framework of [14, Theorem 11].

In the sequel, we denote by $C^{j,\ell}_{loc}$ the space of C^j-smooth real functions whose j-th derivative is locally ℓ-Hölder continuous, with $\ell > 0$. Moreover, defining the function

$$\varphi_1(t) = \sin\left(\frac{t-a}{b-a}\,\pi\right),$$

we introduce the following order relation: for any continuous function $x : [a,b] \to \mathbb{R}$, we write

$$x \gg 0 \tag{7}$$

if and only if

$$\text{there exists } \epsilon > 0 \text{ such that } x(t) \geq \epsilon\, \varphi_1(t)\,, \text{ for every } t \in [a,b]\,.$$

We will write either $u \gg v$ or $v \ll u$ when $u - v \gg 0$.

Concerning the non-well-ordered case, we recall that the existence of a pair of lower/upper solutions does not guarantee the existence of a solution, since resonance phenomena can occur with respect to the higher part of the spectrum. However, at least for scalar second order equations, it is well known that resonance can be handled with respect to the first eigenvalue, cf. [7]. This observation leads us to assume, in the non-well-ordered case, that

$$b - a \leq \min\{\tau_+, \tau_-\}\,. \tag{8}$$

Here is our result, in the non-well-ordered case.

Theorem 3. *Let $\ell > 0$ and assume H to be a $C^{1,\ell}_{loc}$-smooth positively-(p,q)-homogeneous positive-definite function, for some $p > 1$ and $q > 1$ with $(1/p) + (1/q) = 1$. Let ϕ, ψ be uniformly bounded $C^{0,\ell}_{loc}$-smooth functions, and let (α, β)*

be a non-well-ordered pair of lower/upper solutions of (2)*. Assume moreover that*

$$\frac{\partial H}{\partial y}(x_0, \cdot) \text{ is a strictly increasing function, for every } x_0 \in \mathbb{R}. \tag{9}$$

Then, if (8) *holds, there exists a solution* (x, y) *of* (2) *such that* $\alpha \not\lll x$ *and* $x \not\lll \beta$.

The above theorem generalizes [14, Theorem 19]. Its proof is provided in Section 6 by the use of topological degree techniques, which require the above regularity assumptions. It would be interesting to know whether the result still holds when the functions ϕ, ψ are only assumed to be continuous. Notice that assumption (9) is surely verified for problems like (6), where $\frac{\partial H}{\partial y}(x, y) = |y|^{q-2}y$.

In the final section of the paper we extend the previous results to systems in $\mathbb{R}^{2N}$ which can be considered as weakly coupled planar systems of the above type. The different planar systems involved could have either well-ordered or non-well-ordered lower and upper solutions. We are able to deal with this mixed type of situations, still obtaining an existence result, thus carrying out the investigation opened in [14]. However, for the non-well-ordered case, we need to ask the lower/upper solutions to be strict, a concept we will introduce in Section 7.

2 Elementary properties of positively-(p, q)-homogeneous Hamiltonian systems

For the autonomous system (3) the origin $(0, 0)$ is an isochronous center, all solutions having minimal period $\tau > 0$. We denote by $S(t) = (S_1(t), S_2(t))$ the periodic solution such that $S_1(0) = 0$ and $S_2(0) > 0$, with $H(S_1(t), S_2(t)) = 1$ for every t. Then, the periodic solutions of system (3) having energy $H(x, y) = E$ can be written as

$$\big(x(t), y(t)\big) = \big(E^{\frac{1}{p}} S_1(t + \sigma), E^{\frac{1}{q}} S_2(t + \sigma)\big)\,,$$

for some $\sigma \in \mathbb{R}$. We will use the notations

$$\begin{aligned} \overline{S} &= \max\{|S_1(t)| + |S_2(t)| : t \in \mathbb{R}\}\,, \\ \overline{S'} &= \max\{|S_1'(t)| + |S_2'(t)| : t \in \mathbb{R}\}\,. \end{aligned} \tag{10}$$

For any function $u = (x, y) : [a, b] \to \mathbb{R}^2 \setminus \{(0, 0)\}$ we can introduce the *generalized polar coordinates*

$$\begin{cases} x(t) = r(t)^{\frac{1}{p}} S_1(\theta(t))\,, \\ y(t) = r(t)^{\frac{1}{q}} S_2(\theta(t))\,, \end{cases} \tag{11}$$

where $r(t) \ge 0$. If $u(t) = (0, 0)$, we set $r(t) = 0$, while $\theta(t)$ is not defined. Let

$$\mathcal{N}_p(u) = \|r\|_\infty^{1/p} = \sup\{r(t)^{\frac{1}{p}} : t \in [a, b]\}\,.$$

For every $D > 0$, one has that

$$\mathcal{N}_p(u) \le D \quad \Rightarrow \quad \|x\|_\infty \le D\,\overline{S} \quad \text{and} \quad \|y\|_\infty \le D^{\frac{p}{q}}\,\overline{S}\,.$$

Condition (1) can be rewritten as

$$H(\lambda^{\frac{1}{p}}x, \lambda^{\frac{1}{q}}y) = \lambda H(x,y) > 0\,,$$

for every $\lambda > 0$ and $(x,y) \neq (0,0)$. Then,

$$\frac{\partial H}{\partial x}(\lambda^q x, \lambda^p y) = \lambda^{q(p-1)}\frac{\partial H}{\partial x}(x,y) = \lambda^p \frac{\partial H}{\partial x}(x,y)\,, \tag{12}$$

$$\frac{\partial H}{\partial y}(\lambda^q x, \lambda^p y) = \lambda^{p(q-1)}\frac{\partial H}{\partial y}(x,y) = \lambda^q \frac{\partial H}{\partial y}(x,y)\,. \tag{13}$$

We can write the generalized Euler formula

$$\left(\frac{x}{p}\,, \frac{y}{q}\right)\cdot \nabla H(x,y) = H(x,y)\,. \tag{14}$$

We can rewrite (12) and (13) as

$$\frac{\partial H}{\partial x}(x, \mu y) = \mu \frac{\partial H}{\partial x}(\mu^{-\frac{q}{p}}x, y)\,, \tag{15}$$

$$\frac{\partial H}{\partial y}(x, \mu y) = \mu^{\frac{q}{p}} \frac{\partial H}{\partial y}(\mu^{-\frac{q}{p}}x, y)\,, \tag{16}$$

for every $\mu > 0$. We will need the following property.

Lemma 4. *For every $L > 0$, we have that*

$$\lim_{y\to\pm\infty} \frac{\partial H}{\partial y}(x,y) = \pm\infty \text{ uniformly with respect to } x \in [-L, L].$$

Proof. We first prove that there are $c_0 > 0$ and $y_0 \geq 1$ such that

$$|x| \leq L \quad \Rightarrow \quad \frac{\partial H}{\partial y}(x, y_0) > c_0 \quad \text{and} \quad \frac{\partial H}{\partial y}(x, -y_0) < -c_0\,.$$

From (14), we have both

$$\frac{\partial H}{\partial y}(0,1) > 0 \quad \text{and} \quad \frac{\partial H}{\partial y}(0,-1) < 0\,.$$

Hence, we can find $\delta_0 > 0$ and $c_0 > 0$ such that

$$|u| \leq \delta_0 \quad \Rightarrow \quad \frac{\partial H}{\partial y}(u, 1) > c_0 \quad \text{and} \quad \frac{\partial H}{\partial y}(u, -1) < -c_0\,. \tag{17}$$

If we set

$$y_0 = \max\left\{(L/\delta_0)^{\frac{p}{q}}\,, 1\right\},$$

then, for every $x \in [-L, L]$ we get, using (16) and (17),

$$\frac{\partial H}{\partial y}(x, y_0) = y_0^{\frac{q}{p}} \frac{\partial H}{\partial y}\left(y_0^{-\frac{q}{p}}x, 1\right) > y_0^{\frac{q}{p}} c_0 \geq c_0\,,$$

$$\frac{\partial H}{\partial y}(x, -y_0) = y_0^{\frac{q}{p}} \frac{\partial H}{\partial y}\left(y_0^{-\frac{q}{p}}x, -1\right) < -y_0^{\frac{q}{p}} c_0 \leq -c_0\,.$$

So, for any $x \in [-L, L]$ and $\mu > 1$, from (16) we obtain

$$\frac{\partial H}{\partial y}(x, \mu y_0) = \mu^{\frac{q}{p}} \frac{\partial H}{\partial y}\left(\mu^{-\frac{q}{p}}x, y_0\right) \geq \mu^{\frac{q}{p}} c_0\,,$$

$$\frac{\partial H}{\partial y}(x, -\mu y_0) = \mu^{\frac{q}{p}} \frac{\partial H}{\partial y}\left(\mu^{-\frac{q}{p}}x, -y_0\right) \leq -\mu^{\frac{q}{p}} c_0\,.$$

The conclusion follows. $\square$

3 Proof of Theorem 1

In order to prove Theorem 1, let us first assume the validity of the alternative (4).

Introducing the *generalized polar coordinates* (11), we can compute (cf. [25, Section 2])

$$\theta'(t) = 1 + \frac{1}{r(t)}\left[\frac{1}{q}y(t)\phi(t,x(t),y(t)) - \frac{1}{p}x(t)\psi(t,x(t),y(t))\right],$$
$$r'(t) = \psi(t,x(t),y(t))r^{\frac{1}{p}}(t)S_1'(\theta(t)) - \phi(t,x(t),y(t))r^{\frac{1}{q}}(t)S_2'(\theta(t)).$$

Since the functions $\phi, \psi : [a,b]\times\mathbb{R}^2 \to \mathbb{R}$ are bounded, there is a constant K for which

$$|\phi(t,x,y)| \le K\,, \quad |\psi(t,x,y)| \le K\,, \quad \text{for every } (t,x,y)\in[a,b]\times\mathbb{R}^2. \tag{18}$$

Setting $\omega = \min\{\frac{1}{p},\frac{1}{q}\}$ and using (10), we have

$$|\theta'(t) - 1| \le K\overline{S}\,r(t)^{-\omega}\,, \tag{19}$$
$$|r'(t)| \le 2K\overline{S'}\,r(t)\,, \tag{20}$$

when $r(t) > 1$. As a consequence of (20), all the solutions of

$$x' = \frac{\partial H}{\partial y}(x,y) + \phi(t,x,y)\,, \quad y' = -\frac{\partial H}{\partial x}(x,y) + \psi(t,x,y) \tag{21}$$

are globally defined. Since we are assuming (4), choosing $\delta > 0$ sufficiently small, we have

$$n\tau < (b-a)(1-\delta) < (b-a)(1+\delta) < n\tau + \min\{\tau_-,\tau_+\}\,. \tag{22}$$

Correspondingly, we can fix $\overline{R} > 1$ such that

$$K\overline{S}\,\overline{R}^{-\omega} < \delta\,.$$

From (19), a solution of (21) satisfying $r(t) \ge \overline{R}$ for every $t \in [a,b]$ is such that

$$1-\delta < \theta'(t) < 1+\delta\,,$$

for every $t\in[a,b]$. Hence, recalling assumption (4), from (22) we get

$$n\tau < \theta(b) - \theta(a) < n\tau + \min\{\tau_-,\tau_+\}\,. \tag{23}$$

Finally, from (20) and Gronwall lemma, we can find $R_1 > R_0 > \overline{R}$ with the following property: if a solution of (21) satisfies $r(t_0) = R_0$ for some $t_0 \in [a,b]$, then

$$\overline{R} < r(t) < R_1 \text{ for every } t\in[a,b]\,. \tag{24}$$

In particular, the solution is globally defined on $[a,b]$. Let us set

$$y_0^+ = R_0^{\frac{1}{q}}S_2(0) > 0 \quad\text{and}\quad y_0^- = R_0^{\frac{1}{q}}S_2(\tau_+) < 0\,.$$

For any $\sigma \in J := [y_0^-, y_0^+]$, we consider the Cauchy problem

$$\begin{cases} x' = \dfrac{\partial H}{\partial y}(x,y) + \phi(t,x,y)\,, \quad y' = -\dfrac{\partial H}{\partial x}(x,y) + \psi(t,x,y)\,, \\ \\ x(a) = 0\,, \qquad y(a) = \sigma\,. \end{cases} \tag{25}$$

We recall that all the solutions of (25) are defined in the interval $[a,b]$. Then, by [6, Corollary 2.3], the set

$$\mathcal{K} = \{(\sigma, z) \in J \times C([a,b], \mathbb{R}^2) \,:\, z = (x,y) \text{ is a solution of (25)}\}$$

is closed, connected and its projection on J coincides with J.

So, we can find some $(y_0^-, (x^-, y^-)) \in \mathcal{K}$ and $(y_0^+, (x^+, y^+)) \in \mathcal{K}$. The solutions $(x^\pm, y^\pm)$ satisfy (23) and

$$\overline{R} < r(t) < R_1 \text{ for every } t \in [a,b]\,.$$

Indeed, denoting by $(r^\pm, \theta^\pm)$ the modified polar coordinates of $(x^\pm, y^\pm)$, we have $r^\pm(a) = R_0$ and so (24) holds. Then, if we consider the initial data $y^+(a) = y_0^+$, [resp. $y^-(a) = y_0^-$], we can choose $\theta^+(a) = 0$ [resp. $\theta^-(a) = \tau_+$].

From (23), we get

$$n\tau < \theta^+(b) < n\tau + \tau_+\,, \qquad n\tau + \tau_+ < \theta^-(b) < (n+1)\tau\,,$$

so that

$$x^-(b) \cdot x^+(b) < 0\,. \tag{26}$$

Hence, defining the continuous function $\mathcal{Z} : \mathcal{K} \to \mathbb{R}$ as $\mathcal{Z}(\sigma, (x,y)) = x(b)$, from (26) we easily deduce that

$$\mathcal{Z}(y_0^-, (x^-, y^-)) \cdot \mathcal{Z}(y_0^+, (x^+, y^+)) < 0\,.$$

By continuity, $\mathcal{Z}(\mathcal{K})$ is an interval and we find the existence of $(\overline{\sigma}, (\overline{x}, \overline{y})) \in \mathcal{K}$ satisfying $\mathcal{Z}(\overline{\sigma}, (\overline{x}, \overline{y})) = 0$. Hence, $(\overline{x}, \overline{y})$ is the solution we were looking for.

The proof of Theorem 1 is thus completed if (4) holds. If (5) holds, the proof is similar: we need to replace (22) by

$$n\tau + \max\{\tau_-, \tau_+\} < (b-a)(1-\delta) < (b-a)(1+\delta) < (n+1)\tau\,,$$

and (23) by

$$n\tau + \max\{\tau_-, \tau_+\} < \theta(b) - \theta(a) < (n+1)\tau\,.$$

Finally, we get

$$n\tau + \tau_+ < \theta_n^+(b) < (n+1)\tau\,, \quad (n+1)\tau < \theta_n^-(b) < (n+1)\tau + \tau_+\,,$$

which gives (26), permitting us to conclude as above.

The proof of Theorem 1 is thus completed. □

Remarks. Other types of boundary conditions can be considered, leading to similar results. The case of a Neumann-type problem, with boundary conditions $y(a) = 0 = y(b)$, is nothing but the previous Dirichlet-type problem by a simple switch in the variables $x \leftrightharpoons y$. The mixed problem

$$\begin{cases} x' = \dfrac{\partial H}{\partial y}(x,y) + \phi(t,x,y)\,, \quad y' = -\dfrac{\partial H}{\partial x}(x,y) + \psi(t,x,y)\,, \\ x(a) = 0 = y(b) \end{cases} \tag{27}$$

can be considered, as well. Going back to the solution $S = (S_1, S_2)$ of the unperturbed system (3), we can find the positive values $\tau_1, \tau_2, \tau_3, \tau_4$ such that

$$\begin{array}{ll} S_1(0) = 0\,, & S_2(0) > 0 \\ S_1(\tau_1) > 0\,, & S_2(\tau_1) = 0\,, \\ S_1(\tau_1 + \tau_2) = 0\,, & S_2(\tau_1 + \tau_2) < 0\,, \\ S_1(\tau_1 + \tau_2 + \tau_3) < 0\,, & S_2(\tau_1 + \tau_2 + \tau_3) = 0\,, \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 = \tau\,, & \end{array}$$

leading to the following.

Theorem 5. *Assume that there is an integer $n \geq 0$ such that one of the following alternatives hold*

$$n\tau < b - a < n\tau + \min\{\tau_1, \tau_3\}\,,$$
$$n\tau + \tau_1 + \tau_3 + \max\{\tau_2, \tau_4\} < b - a < (n+1)\tau\,.$$

Then, problem (27) *has a solution.*

Similarly, we can consider the boundary conditions $y(a) = 0 = x(b)$, as well, thus obtaining the corresponding existence result. Finally, Carathéodory conditions on ϕ and ψ can be considered. We avoid entering into details, for briefness.

4 The definition of lower and upper solutions

Let us consider the boundary value problem

$$\begin{cases} x' = f(t,x,y)\,, \quad y' = g(t,x,y)\,, \\ x(a) = 0 = x(b)\,, \end{cases} \tag{28}$$

where $f, g : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, and recall the definitions introduced in [14].

Definition 6. *A continuously differentiable function $\alpha : [a,b] \to \mathbb{R}$ is said to be a* lower solution *for problem* (28) *if there exists a continuously differentiable function $y_\alpha : [a,b] \to \mathbb{R}$ such that, for every $t \in [a,b]$,*

$$\begin{cases} y < y_\alpha(t) \quad \Rightarrow \quad f(t,\alpha(t),y) < \alpha'(t)\,, \\ y > y_\alpha(t) \quad \Rightarrow \quad f(t,\alpha(t),y) > \alpha'(t)\,, \end{cases}$$

$$y_\alpha'(t) \geq g(t,\alpha(t),y_\alpha(t))\,,$$

and

$$\alpha(a) \leq 0\,, \quad \alpha(b) \leq 0\,.$$

Definition 7. *A continuously differentiable function $\beta : [a, b] \to \mathbb{R}$ is said to be an* upper solution *for problem* (28) *if there exists a continuously differentiable function $y_\beta : [a, b] \to \mathbb{R}$ such that, for every $t \in [a, b]$,*

$$\begin{cases} y < y_\beta(t) & \Rightarrow \quad f(t, \beta(t), y) < \beta'(t)\,, \\ y > y_\beta(t) & \Rightarrow \quad f(t, \beta(t), y) > \beta'(t)\,, \end{cases}$$

$$y'_\beta(t) \le g(t, \beta(t), y_\beta(t))\,,$$

and

$$\beta(a) \ge 0\,, \quad \beta(b) \ge 0\,.$$

We say that (α, β) is a well-ordered pair of lower/upper solutions of problem (28), if α and β are respectively a lower and an upper solution for problem (28) and they satisfy

$$\alpha(t) \le \beta(t)\,, \quad \text{for every } t \in [a, b]\,.$$

On the other hand, if the above inequality does not hold, we say that the pair (α, β) is non-well-ordered.

5 Proof of Theorem 2

Define

$$f(t, x, y) = \frac{\partial H}{\partial y}(x, y) + \phi(t, x, y)\,, \quad g(t, x, y) = -\frac{\partial H}{\partial x}(x, y) + \psi(t, x, y)\,,$$

and set $m_\alpha = \min \alpha$ and $M_\beta = \max \beta$. In order to apply [14, Theorem 11], we need to construct some guiding curves $\gamma^\pm_{1,2} : [m_\alpha, M_\beta] \to \mathbb{R}$ such that, for every $t \in [a, b]$ and $x \in [\alpha(t), \beta(t)]$,

$$\gamma_i^-(x) < \min\{y_\alpha(t), y_\beta(t)\} \le \max\{y_\alpha(t), y_\beta(t)\} < \gamma_i^+(x)\,, \quad i = 1, 2\,,$$

$$g\big(t, x, \gamma_1^+(x)\big) > f\big(t, x, \gamma_1^+(x)\big)(\gamma_1^+)'(x)\,, \tag{29}$$

$$g\big(t, x, \gamma_2^+(x)\big) < f\big(t, x, \gamma_2^+(x)\big)(\gamma_2^+)'(x)\,, \tag{30}$$

$$g\big(t, x, \gamma_1^-(x)\big) < f\big(t, x, \gamma_1^-(x)\big)(\gamma_1^-)'(x)\,, \tag{31}$$

$$g\big(t, x, \gamma_2^-(x)\big) > f\big(t, x, \gamma_2^-(x)\big)(\gamma_2^-)'(x)\,. \tag{32}$$

This can be obtained as an immeditate consequence of the next two lemmas.

Lemma 8. *For any $L > 0$ and $y_0 > 0$ we can define two continuously differentiable functions $\gamma_1^+, \gamma_2^+ : [-L, L] \to \mathbb{R}$ satisfying*

$$\gamma_i^+(x) \ge y_0\,, \quad i = 1, 2\,, \quad \textit{for all } x \in [-L, L]\,,$$

and such that (29) *and* (30) *hold for every $t \in [a, b]$ and $x \in [-L, L]$.*

Proof. Since both ϕ and ψ are bounded, let us consider $K > 0$ as in (18). From Lemma 4 we can find $y_1 \ge y_0$ such that

$$\frac{\partial H}{\partial y}(x, y) > K + 1, \text{ for all } (x, y) \in [-L, L] \times [y_1, +\infty[\,. \tag{33}$$

Since H is C^1, we can find a positive constant c_1 such that

$$\left|\frac{\partial H}{\partial x}(x,y)\right| \le c_1\,, \quad \text{for every } (x,y)\in[-L,L]\times[y_1,y_1+2L]\,. \tag{34}$$

Now, since $2-q<1$, we can find $M>1$ sufficiently large so to have

$$\big(c_1 M^{2-q}+1\big)(K+1) < M\,. \tag{35}$$

We define $\gamma_{1,2}^+ : [-L,L]\to\mathbb{R}$ as

$$\gamma_1^+(x) = M(y_1+L-x)\,, \quad \gamma_2^+(x) = M(y_1+L+x)\,. \tag{36}$$

Notice that $\gamma_{1,2}^+(x)\ge y_1\ge y_0$, for every $x\in[-L,L]$. If we show that, for $i=1,2$,

$$\frac{\left|\frac{\partial H}{\partial x}(x,\gamma_i^+(x))\right|+K}{\frac{\partial H}{\partial y}(x,\gamma_i^+(x))-K} < M\,, \quad \text{for every } x\in[-L,L]\,, \tag{37}$$

then, since from (36) and (33) the denominator in (37) is positive, we get both

$$\begin{aligned} M\left(\frac{\partial H}{\partial y}(x,\gamma_1^+(x))+\phi(t,x,\gamma_1^+(x))\right) &\ge M\left(\frac{\partial H}{\partial y}(x,\gamma_1^+(x))-K\right)\\ &> \left|\frac{\partial H}{\partial x}(x,\gamma_1^+(x))\right|+K \ge \frac{\partial H}{\partial x}(x,\gamma_1^+(x))-\psi(t,x,\gamma_1^+(x))\,, \end{aligned}$$

and

$$\begin{aligned} M\left(\frac{\partial H}{\partial y}(x,\gamma_2^+(x))+\phi(t,x,\gamma_2^+(x))\right) &\ge M\left(\frac{\partial H}{\partial y}(x,\gamma_2^+(x))-K\right)\\ &> \left|\frac{\partial H}{\partial x}(x,\gamma_2^+(x))\right|+K \ge -\frac{\partial H}{\partial x}(x,\gamma_2^+(x))+\psi(t,x,\gamma_2^+(x))\,, \end{aligned}$$

for every $t\in[a,b]$ and $x\in[-L,L]$ and the lemma will be proved. Hence, we need to prove the validity of (37).

Using equality (15) and the estimate (34), we have that

$$\left|\frac{\partial H}{\partial x}(x,Mv)\right| = M\left|\frac{\partial H}{\partial x}\left(M^{-\frac{q}{p}}x,v\right)\right| \le Mc_1\,,$$
$$\text{for every } x\in[-L,L] \text{ and } v\in[y_1,y_1+2L]\,. \tag{38}$$

Moreover, using equality (16) and recalling (33), we have that

$$M^{-\frac{q}{p}}\frac{\partial H}{\partial y}(x,Mv) = \frac{\partial H}{\partial y}\left(M^{-\frac{q}{p}}x,v\right) > 1\,,$$
$$\text{for every } x\in[-L,L] \text{ and } v\in[y_1,y_1+2L]\,. \tag{39}$$

From (38) and (39),

$$\left|\frac{\partial H}{\partial x}(x,Mv)\right| \le Mc_1 < c_1 M^{1-\frac{q}{p}}\frac{\partial H}{\partial y}(x,Mv) = c_1 M^{2-q}\frac{\partial H}{\partial y}(x,Mv)\,,$$
$$\text{for every } x\in[-L,L] \text{ and } v\in[y_1,y_1+2L]\,.$$

Since, using (36), for all $x \in [-L, L]$ we have $\frac{1}{M}\gamma_i^+(x) \in [y_1, y_1 + 2L]$, $i = 1, 2$, then we get

$$\left|\frac{\partial H}{\partial x}(x, \gamma_i^+(x))\right| < c_1 M^{2-q} \frac{\partial H}{\partial y}(x, \gamma_i^+(x)) .$$

Hence, recalling (33) and (36), we have

$$\begin{aligned}
\frac{\left|\frac{\partial H}{\partial x}(x, \gamma_i^+(x))\right| + K}{\frac{\partial H}{\partial y}(x, \gamma_i^+(x)) - K} &< \frac{c_1 M^{2-q} \frac{\partial H}{\partial y}(x, \gamma_i^+(x)) + K}{\frac{\partial H}{\partial y}(x, \gamma_i^+(x)) - K} \\
&= \frac{c_1 M^{2-q} + \frac{K}{\frac{\partial H}{\partial y}(x, \gamma_i^+(x))}}{1 - \frac{K}{\frac{\partial H}{\partial y}(x, \gamma_i^+(x))}} \\
&\leq \big(c_1 M^{2-q} + 1\big)(K + 1) < M ,
\end{aligned}$$

where the last estimate is given by (35). We have thus proved (37), and so the proof of the lemma is completed. $\square$

Lemma 9. *For any $L > 0$ and $y_0 > 0$ we can define two continuously differentiable functions $\gamma_1^-, \gamma_2^- : [-L, L] \to \mathbb{R}$ satisfying*

$$\gamma_i^-(x) \leq -y_0 , \quad i = 1, 2 , \quad \textit{for all } x \in [-L, L] ,$$

and such that (31) *and* (32) *hold for every $t \in [a, b]$ and $x \in [-L, L]$.*

The proof of this lemma is analogous to the previous one, so we omit it, for briefness.

Let us now conclude the proof of Theorem 2. Choosing

$$L = \max\{\|\alpha\|_\infty , \|\beta\|_\infty\} , \qquad y_0 > \max\{\|y_\alpha\|_\infty , \|y_\beta\|_\infty\} ,$$

we can apply Lemmas 8 and 9 in order to get the existence of the needed curves $\gamma_i^\pm$. Moreover, for every $t \in [a, b]$,

$$\begin{aligned}
y \geq \min\{\gamma_1^+(0), \gamma_2^+(0)\} \quad &\Rightarrow \quad f(t, 0, y) > 0 , \\
y \leq \max\{\gamma_1^-(0), \gamma_2^-(0)\} \quad &\Rightarrow \quad f(t, 0, y) < 0 .
\end{aligned}$$

This is an immediate consequence of Lemma 4, choosing a larger value of y_0, if necessary. Then, [14, Theorem 11] applies, thus completing the proof of Theorem 2.

6 Proof of Theorem 3

As a first step, we need the following two lemmas, where the ordering relation defined in the Introduction is used.

Lemma 10 ([14, Lemma 17])**.** *Given a continuous function $\varphi : [a, b] \to \mathbb{R}$, the sets*

$$\{x \in C_0^1([a, b]) : \varphi \ll x\} , \quad \{x \in C_0^1([a, b]) : x \ll \varphi\}$$

are open in $C_0^1([a, b]) = \{x \in C^1([a, b]) : x(a) = 0 = x(b)\}$.

Lemma 11 ([14, Lemma 18]). *Given a continuously differentiable function $\varphi : [a,b] \to \mathbb{R}$, we have that*

$$\begin{aligned} \max\{\varphi(a),\varphi(b)\} \le 0 \quad &\Rightarrow \quad \exists \hat{c} > 0 \,:\, \varphi \ll \hat{c}\varphi_1\,, \\ \min\{\varphi(a),\varphi(b)\} \ge 0 \quad &\Rightarrow \quad \exists \hat{c} > 0 \,:\, \varphi \gg -\hat{c}\varphi_1\,. \end{aligned}$$

We now use the notation introduced in Section 2, and set $\widetilde{S}_1(t) = S_1(t+\tau_+)$.

Lemma 12. *Given a continuously differentiable function $\varphi : [a,b] \to \mathbb{R}$, we have that*

$$\begin{aligned} \max\{\varphi(a),\varphi(b)\} \le 0 \quad &\Rightarrow \quad \exists c > 0 \,:\, \varphi \ll cS_1\,, \\ \min\{\varphi(a),\varphi(b)\} \ge 0 \quad &\Rightarrow \quad \exists c > 0 \,:\, \varphi \gg c\widetilde{S}_1\,. \end{aligned}$$

Proof. It is an immediate consequence of the previous lemma since $S_1 \gg 0$ and $\widetilde{S}_1 \ll 0$. □

Let us introduce a C^∞-smooth cut-off function $\chi : \mathbb{R} \to [0,1]$ such that

$$\chi(s) = \begin{cases} 1 & \text{if } |s| \le 1\,, \\ 0 & \text{if } |s| \ge 2\,, \end{cases} \tag{40}$$

and, for every $d \ge 1$, consider the modified problem

$$\begin{cases} x' = \widehat{f}_d(t,x,y)\,, \quad y' = \widehat{g}_d(t,x,y)\,, \\ x(a) = 0 = x(b)\,, \end{cases} \tag{41}$$

where

$$\begin{aligned} \widehat{f}_d(t,x,y) &= \frac{d-1}{d}\,\frac{\partial H}{\partial y}(x,y) + \widetilde{\phi}_d(t,x,y)\,, \\ \widehat{g}_d(t,x,y) &= -\frac{d-1}{d}\,\frac{\partial H}{\partial x}(x,y) + \widetilde{\psi}_d(t,x,y)\,, \end{aligned}$$

with

$$\begin{aligned} \widetilde{\phi}_d(t,x,y) &= \chi\Big(\frac{x}{d\overline{S}}\Big)\chi\Big(\frac{y}{d^{p/q}\overline{\overline{S}}}\Big)\left(\frac{1}{d}\frac{\partial H}{\partial y}(x,y) + \phi(t,x,y)\right), \\ \widetilde{\psi}_d(t,x,y) &= \chi\Big(\frac{x}{d\overline{S}}\Big)\chi\Big(\frac{y}{d^{p/q}\overline{\overline{S}}}\Big)\left(-\frac{1}{d}\frac{\partial H}{\partial x}(x,y) + \psi(t,x,y)\right). \end{aligned}$$

Notice that (2) and (41) coincide in the set $[-d\overline{S}, d\overline{S}] \times [-d^{\frac{p}{q}}\overline{\overline{S}}, d^{\frac{p}{q}}\overline{\overline{S}}]$. In particular, if a solution $u = (x,y)$ of (41) is such that $\mathcal{N}_p(u) < d$, then u is necessarily a solution of (2).

Lemma 13. *There exists $D > 1$ with the following property: if $u = (x,y)$ is any solution of* (41)*, with $d \in [D,+\infty]$, satisfying $\alpha \not\gg x$ and $x \not\gg \beta$, then $\mathcal{N}_p(u) < D$.*

Proof. Assume by contradiction that there exist a diverging sequence $(d_n)_n$ and some solutions $u_n = (x_n,y_n)$ of (41), with $d = d_n$, such that $\alpha \not\gg x_n$, $x_n \not\gg \beta$ and $\mathcal{N}_p(u_n) > n$. We introduce the functions

$$v_n = \frac{x_n}{\mathcal{N}_p(u_n)} \quad \text{and} \quad w_n = \frac{y_n}{\mathcal{N}_p(u_n)^{p-1}}\,.$$

Notice that, from (11) and (10), we have

$$\|v_n\|_\infty \le \overline{S} \quad \text{and} \quad \|w_n\|_\infty \le \overline{S}\,. \tag{42}$$

By (12) and (13), we see that (v_n, w_n) solves

$$\begin{cases} v_n' = \dfrac{d_n - 1}{d_n}\,\dfrac{\partial H}{\partial y}(v_n, w_n) + \Phi_n(t, v_n, w_n)\,, \\ w_n' = -\dfrac{d_n - 1}{d_n}\,\dfrac{\partial H}{\partial x}(v_n, w_n) + \Psi_n(t, v_n, w_n)\,, \\ v_n(a) = 0 = v_n(b)\,, \end{cases} \tag{43}$$

where

$$\Phi_n(t, v_n, w_n) = \frac{1}{\mathcal{N}_p(u_n)}\widetilde{\phi}_{d_n}(t, \mathcal{N}_p(u_n)v_n, \mathcal{N}_p(u_n)^{p-1}w_n)\,,$$
$$\Psi_n(t, v_n, w_n) = \frac{1}{\mathcal{N}_p(u_n)^{p-1}}\widetilde{\psi}_{d_n}(t, \mathcal{N}_p(u_n)v_n, \mathcal{N}_p(u_n)^{p-1}w_n)\,.$$

Then, by (42), since (v_n, w_n) solves (43), it is bounded in $C^1 \times C^1$. By a standard compactness argument, a subsequence converges to some $(\overline{v}, \overline{w})$ in $C^1 \times C^1$, and

$$\begin{cases} \overline{v}' = \dfrac{\partial H}{\partial y}(\overline{v}, \overline{w})\,, \quad \overline{w}' = -\dfrac{\partial H}{\partial x}(\overline{v}, \overline{w}) \\ \overline{v}(a) = 0 = \overline{v}(b)\,. \end{cases}$$

Hence, we have either $(\overline{v}(t), \overline{w}(t)) = S(t)$, or $(\overline{v}(t), \overline{w}(t)) = S(t + \tau_+)$. More precisely, using (8), the first case is possible if and only if $b - a = \tau_+$, the second one if and only if $b - a = \tau_-$. In the first case, since v_n C^1-converges to $S_1 \gg 0$, we have that $v_n \gg \frac{1}{2}S_1$ when n is sufficiently large, so that, recalling Lemma 12, we get the contradiction

$$x_n = \mathcal{N}_p(u_n)v_n \gg \frac{1}{2}\mathcal{N}_p(u_n)S_1 \gg cS_1 \gg \alpha\,.$$

In the second case, since v_n C^1-converges to $\widetilde{S}_1 = S_1(\,\cdot\, + \tau_+) \ll 0$, we have that $v_n \ll \frac{1}{2}\widetilde{S}_1$ when n is sufficiently large, thus providing the contradiction

$$x_n = \mathcal{N}_p(u_n)v_n \ll \frac{1}{2}\mathcal{N}_p(u_n)\widetilde{S}_1 \ll c\widetilde{S}_1 \ll \beta\,.$$

The proof is thus completed. $\square$

We now fix $D > 1$ as in Lemma 13, assuming also

$$\begin{aligned} &\max\{\|\alpha\|_\infty\,, \|\beta\|_\infty\} < D\overline{S}\,, \\ &\max\{\|y_\alpha\|_\infty\,, \|y_\beta\|_\infty\} < D\overline{S}^{\,p/q}\,. \end{aligned} \tag{44}$$

From Lemma 13, if $u = (x, y)$ is any solution of (41) with $d \ge D$, satisfying both $\alpha \not\ll x$ and $x \not\ll \beta$, then u is a solution of (2), too.

Lemma 14. *The functions α and β are a lower and an upper solution of* (41)*, respectively, provided that d is chosen large enough.*

Proof. From Lemma 4 and the boundedness of ϕ, we deduce the existence of a constant $\widetilde{K} > 0$ such that, for every $d \geq 1$, we have both $\widetilde{\phi}_d(t, \alpha(t), y) \geq -\widetilde{K}$ when $y > 0$, and $\widetilde{\phi}_d(t, \alpha(t), y) \leq \widetilde{K}$ when $y < 0$, for every $t \in [a, b]$. Moreover, using again Lemma 4, we can take d large enough so that, if $y > d^{p/q}\overline{S}$ then

$$\frac{d-1}{d}\frac{\partial H}{\partial y}(\alpha(t), y) + \widetilde{\phi}_d(t, \alpha(t), y) \geq \frac{d-1}{d}\frac{\partial H}{\partial y}(\alpha(t), y) - \widetilde{K} > \alpha'(t)\,,$$

and if $y < -d^{p/q}\overline{S}$ then

$$\frac{d-1}{d}\frac{\partial H}{\partial y}(\alpha(t), y) + \widetilde{\phi}_d(t, \alpha(t), y) \leq \frac{d-1}{d}\frac{\partial H}{\partial y}(\alpha(t), y) + \widetilde{K} < \alpha'(t)\,.$$

On the other hand, if $y \in [-d^{p/q}\overline{S}, d^{p/q}\overline{S}]$, then $(\alpha(t), y)$ belongs to the region where the problem has not been modified and the desired estimates hold.

The inequality for $y'_\alpha(t)$ still holds since, for every $t \in [a, b]$, also $(\alpha(t), y_\alpha(t))$ belongs to the region where the problem has not been modified. $\square$

Our aim now is to construct a lower solution $\widehat{\alpha}$ and an upper solution $\widehat{\beta}$ such that $\widehat{\alpha} \ll \beta$ and $\alpha \ll \widehat{\beta}$. In order to do so, let us consider, for every $d > 1$, the *slowed* autonomous system

$$x' = \frac{d-1}{d}\,\frac{\partial H}{\partial y}(x, y)\,, \quad y' = -\frac{d-1}{d}\,\frac{\partial H}{\partial x}(x, y)\,. \tag{45}$$

It is isochronous, all solutions being periodic with minimal period $\tau d/(d-1) > \tau$.

Lemma 15. *For every $\xi > 0$, the problem*

$$\begin{cases} x' = \dfrac{d-1}{d}\,\dfrac{\partial H}{\partial y}(x, y)\,, \quad y' = -\dfrac{d-1}{d}\,\dfrac{\partial H}{\partial x}(x, y)\,, \\ x(a) = \xi = x(b)\,, \end{cases}$$

has a unique solution (x_ξ, y_ξ). Moreover, $x_\xi(t) > \xi$ for every $t \in]a, b[$. Similarly, the problem

$$\begin{cases} x' = \dfrac{d-1}{d}\,\dfrac{\partial H}{\partial y}(x, y)\,, \quad y' = -\dfrac{d-1}{d}\,\dfrac{\partial H}{\partial x}(x, y)\,, \\ x(a) = -\xi = x(b)\,, \end{cases}$$

has a unique solution $(x_{-\xi}, y_{-\xi})$, satisfying $x_{-\xi}(t) < -\xi$ for every $t \in]a, b[$.

Proof. If we parametrize the solutions (x, y) with the energy $E = \frac{d-1}{d}H(x, y)$, we see that they cross the line $\mathcal{L} = \{(x, y) \in \mathbb{R}^2 : x = \xi\}$ if and only if E is greater of a well determined value $\overline{E}_\xi > 0$, and assumption (9) ensures that there are exactly two crossing points $(\xi, \bar{y}_\xi^+)$ and $(\xi, \bar{y}_\xi^-)$, with $\bar{y}_\xi^+ > \bar{y}_\xi^-$. Since $b - a \leq \tau_- < \tau_- d/(d-1)$, the solution (x, y) we are looking for cannot follow the path to the left of $\mathcal{L}$, hence, if it exists, it has to be $x_\xi(t) > \xi$ for every $t \in]a, b[$.

Let us denote by $T_\xi(E)$ the time needed to go from (ξ, y_ξ^+) to (ξ, y_ξ^-). We have thus defined a function $T_\xi :]\overline{E}_\xi, +\infty[\to \mathbb{R}$ which is continuous, positive and strictly increasing. Surely $T_\xi(E) < b - a$ if E is in a small right neighbourhood of $\overline{E}_\xi$. Hence, since $\lim_{E\to+\infty} T_\xi(E) = \tau_+ d/(d-1) > b - a$, there is a unique value $E_\xi > \overline{E}_\xi$ for which $T_\xi(E_\xi) = b - a$, and this value of the energy determines the solution we are looking for. □

We are now ready to define the lower and upper solutions $\widehat{\alpha}$ and $\widehat{\beta}$.

Lemma 16. *Taking $d > D$ and $\xi > 2d\overline{S}$, the functions $\widehat{\alpha} = x_{-\xi}$ and $\widehat{\beta} = x_\xi$ are a lower and an upper solution of* (41)*, respectively. Moreover,*

$$\widehat{\alpha} \ll \beta \quad \text{and} \quad \alpha \ll \widehat{\beta}\,.$$

Proof. Let us show that $\widehat{\beta} = x_\xi$ is an upper solution, with associated function $y_{\widehat{\beta}} = y_\xi$. Recalling that $\widetilde{\phi}_d(t,x,y)$ and $\widetilde{\psi}_d(t,x,y)$ vanish when $x \notin [-2d\overline{S}, 2d\overline{S}]$, since $\xi > 2d\overline{S}$, using assumption (9), we have

$$\begin{aligned} y < y_\xi(t) \quad &\Rightarrow \quad \frac{d-1}{d}\frac{\partial H}{\partial y}(x_\xi(t), y) < \frac{d-1}{d}\frac{\partial H}{\partial y}(x_\xi(t), y_\xi(t)) = x_\xi'(t)\,,\\ y > y_\xi(t) \quad &\Rightarrow \quad \frac{d-1}{d}\frac{\partial H}{\partial y}(x_\xi(t), y) > \frac{d-1}{d}\frac{\partial H}{\partial y}(x_\xi(t), y_\xi(t)) = x_\xi'(t)\,. \end{aligned}$$

Moreover,

$$y_\xi'(t) \le -\frac{d-1}{d}\frac{\partial H}{\partial x}(x_\xi(t), y_\xi(t))\,,$$

since equality holds. Finally, $x_\xi(a) \ge 0$ and $x_\xi(b) \ge 0$, thus proving that x_ξ is an upper solution. Analogously one proves that $x_{-\xi}$ is a lower solution.

Now, since from (44) we have $\xi > 2d\overline{S} > \max\{\|\alpha\|_\infty\,, \|\beta\|_\infty\}$, one has

$$x_{-\xi}(t) \le -\xi < \beta(t)\,, \quad x_\xi(t) \ge \xi > \alpha(t)\,, \qquad \text{for every } t \in [a,b]\,,$$

thus ending the proof of the lemma. □

Let us now fix $d > D$ sufficiently large in order to ensure the validity of Lemma 14. We thus have three well-ordered pairs of lower/upper solutions of (41):

$$(\widehat{\alpha}, \widehat{\beta})\,, \quad (\alpha, \widehat{\beta})\,, \quad (\widehat{\alpha}, \beta)\,.$$

Let $\bar{a} = \min \widehat{\alpha}$ and $\bar{b} = \max \widehat{\beta}$. By Lemma 4, we can find $\overline{M} > 0$ such that, if $t \in [a,b]$ and $x \in [\bar{a}, \bar{b}]$, then

$$\begin{aligned} y \ge \overline{M} \quad &\Rightarrow \quad \widehat{f}_d(t,x,y) > 0\,,\\ y \le -\overline{M} \quad &\Rightarrow \quad \widehat{f}_d(t,x,y) < 0\,. \end{aligned} \tag{46}$$

We now need to introduce the guiding curves.

Lemma 17. *There are four continuously differentiable functions $\gamma_i^{\pm} : [\bar{a}, \bar{b}] \to \mathbb{R}$, with $i = 1, 2$, satisfying*

$$
\begin{aligned}
\widehat{g}_d\big(t, x, \gamma_1^+(x)\big) &> \widehat{f}_d\big(t, x, \gamma_1^+(x)\big)(\gamma_1^+)'(x)\,, \\
\widehat{g}_d\big(t, x, \gamma_2^+(x)\big) &< \widehat{f}_d\big(t, x, \gamma_2^+(x)\big)(\gamma_2^+)'(x)\,, \\
\widehat{g}_d\big(t, x, \gamma_1^-(x)\big) &< \widehat{f}_d\big(t, x, \gamma_1^-(x)\big)(\gamma_1^-)'(x)\,, \\
\widehat{g}_d\big(t, x, \gamma_2^-(x)\big) &> \widehat{f}_d\big(t, x, \gamma_2^-(x)\big)(\gamma_2^-)'(x)\,,
\end{aligned}
\tag{47}
$$

and such that

$$
\begin{aligned}
\gamma_i^-(x) &< \min\{-\overline{M}, y_\alpha(t), y_{\widehat{\alpha}}(t), y_\beta(t), y_{\widehat{\beta}}(t)\} \\
&\quad \le \max\{\overline{M}, y_\alpha(t), y_{,\widehat{\alpha}}(t), y_\beta(t), y_{\widehat{\beta}}(t)\} < \gamma_i^+(x)\,,
\end{aligned}
\tag{48}
$$

for every $t \in [a, b]$ and $x \in [\bar{a}, \bar{b}]$.

Proof. It follows the lines of the proofs of Lemmas 8 and 9. □

Let us now introduce our functional setting for the problem (41).

Set $\mathcal{I} = [a, b]$, denote by $C^{0,\ell}(\mathcal{I})$ the space of ℓ-Hölder continuous functions and by $C^{1,\ell}(\mathcal{I})$ the space of functions having derivative belonging to $C^{0,\ell}(\mathcal{I})$. Moreover, define

$$
C_0^{1,\ell}(\mathcal{I}) = \{x \in C^{1,\ell}(\mathcal{I}) \,:\, x(a) = 0 = x(b)\}\,.
$$

We consider the linear operator

$$
\begin{aligned}
&L : C_0^{1,\ell}(\mathcal{I}) \times C^{1,\ell}(\mathcal{I}) \to C^{0,\ell}(\mathcal{I}) \times C^{0,\ell}(\mathcal{I}) \\
&L(x, y) = (x' - y\,,\, y')\,,
\end{aligned}
$$

and the Nemitskii operator

$$
\begin{aligned}
&N_d : C_0^1(\mathcal{I}) \times C^1(\mathcal{I}) \to C^{0,\ell}(\mathcal{I}) \times C^{0,\ell}(\mathcal{I}) \\
&N_d(x, y)(t) = \Big(\widehat{f}_d(t, x(t), y(t)) - y(t)\,,\, \widehat{g}_d(t, x(t), y(t))\Big)\,.
\end{aligned}
$$

Problem (41) is then the same as

$$
Lu = N_d u\,,
$$

with

$$
u \in X := C_0^1(\mathcal{I}) \times C^1(\mathcal{I})\,.
$$

Lemma 18. *The operator L is invertible with continuous inverse, and problem (41) is equivalent to $u = L^{-1} N_d u$. Moreover, the operator*

$$
L^{-1} N_d : X \to X
$$

is completely continuous.

Proof. It is rather standard, using the fact that $C^{1,\ell}(\mathcal{I})$ is compactly imbedded in $C^1(\mathcal{I})$. □

Let us define the sets

$$\mathcal{V}_1 = \mathcal{V}(\widehat{\alpha}, \widehat{\beta}) \,, \quad \mathcal{V}_2 = \mathcal{V}(\widehat{\alpha}, \beta) \,, \quad \mathcal{V}_3 = \mathcal{V}(\alpha, \widehat{\beta}) \,,$$

with the notation

$$\begin{aligned} \mathcal{V}(\varphi, \eta) = \{ (x,y) \in X \, : \, & \varphi \ll x \ll \eta \text{ and} \\ & \gamma^-(x(t)) < y(t) < \gamma^+(x(t)) \,, \text{ for every } t \in \mathcal{I} \} \,, \end{aligned}$$

where

$$\gamma^-(x) = \min\{ \gamma_i^-(x) \,, \, i = 1,2 \} \,, \qquad \gamma^+(x) = \max\{ \gamma_i^+(x) \,, \, i = 1,2 \} \,. \tag{49}$$

These sets are open in $X = C_0^1(\mathcal{I}) \times C^1(\mathcal{I})$, by Lemma 10.

We say that an open set $\Omega \subset X$ is *admissible* if $L^{-1}N_d$ has no fixed points on $\partial\Omega$ and the set of fixed points of $L^{-1}N_d$ in Ω is bounded, i.e., it is contained in some open ball B_ρ. In this case, we can define

$$\deg(I - L^{-1}N_d, \Omega) = d_{LS}(I - L^{-1}N_d, \Omega \cap B_\rho) \,,$$

where d_{LS} denotes the Leray–Schauder degree. By excision, this definition does not depend on the choice of ρ.

Our aim is to prove that, if there are no solutions of (41) on $\partial\mathcal{V}_j$, then $\mathcal{V}_j$ is admissible and

$$\deg(I - L^{-1}N_d, \mathcal{V}_j) = 1 \,, \quad j = 1,2,3.$$

This fact will be proved in Lemma 22.

In the following we denote by (φ, η) any of the three pairs

$$(\widehat{\alpha}, \widehat{\beta}) \,, \quad (\alpha, \widehat{\beta}) \,, \quad (\widehat{\alpha}, \beta) \,.$$

We need to truncate the functions $\widehat{f}_d$ and $\widehat{g}_d$ so to modify system (41). Define, for any $\mu \leq \nu$,

$$\zeta(s, \mu, \nu) = \begin{cases} \mu \,, & \text{if } s < \mu \,, \\ s \,, & \text{if } \mu \leq x \leq \nu \,, \\ \nu \,, & \text{if } s > \nu \,. \end{cases} \tag{50}$$

Fix $\overline{Y} > 0$ such that

$$-\overline{Y} \leq \gamma^-(x) \leq \gamma^+(x) \leq \overline{Y} \,,$$

for every $x \in [\bar{a}, \bar{b}]$. Let

$$\begin{aligned} f_{\varphi,\eta}(t,x,y) &= \widehat{f}_d\big(t, \zeta(x, \varphi(t), \eta(t)), \zeta(y, -\overline{Y}, \overline{Y})\big) - \zeta(y, -\overline{Y}, \overline{Y}) \,, \\ g_{\varphi,\eta}(t,x,y) &= \widehat{g}_d\big(t, \zeta(x, \varphi(t), \eta(t)), \zeta(y, -\overline{Y}, \overline{Y})\big) - \zeta(x, \varphi(t), \eta(t)) \,, \end{aligned}$$

and consider the problem

$$\begin{cases} x' = y + f_{\varphi,\eta}(t,x,y) \,, \quad y' = x + g_{\varphi,\eta}(t,x,y) \\ x(a) = 0 = x(b) \,, \end{cases} \tag{51}$$

which is equivalent to $Lu = N_{\varphi,\eta}u$, with the appropriate Nemitskii operator. Notice that the functions $f_{\varphi,\eta}$ and $g_{\varphi,\eta}$ are bounded. Moreover, if (x,y) is a solution of (51) satisfying $\varphi \leq x \leq \eta$ and $-\overline{Y} < y < \overline{Y}$, then it is a solution of (41), too.

Lemma 19. *Each (φ, η) is a well-ordered pair of lower/upper solutions for* (51)*.*

Proof. Assume for instance that $(\varphi, \eta) = (\alpha, \widehat{\beta})$. Then, if $y \in [-\overline{Y}, \overline{Y}]$,

$$\begin{aligned} y > y_\alpha(t) \quad &\Rightarrow \quad y + f_{\alpha,\eta}(t, \alpha(t), y) = \widehat{f}_d(t, \alpha(t), y) > \alpha'(t)\,, \\ y < y_\alpha(t) \quad &\Rightarrow \quad y + f_{\alpha,\eta}(t, \alpha(t), y) = \widehat{f}_d(t, \alpha(t), y) < \alpha'(t)\,, \end{aligned}$$

while

$$\begin{aligned} y > \overline{Y} \quad &\Rightarrow \quad y + f_{\alpha,\eta}(t, \alpha(t), y) > \overline{Y} + f_{\alpha,\eta}(t, \alpha(t), \overline{Y}) > \alpha'(t)\,, \\ y < -\overline{Y} \quad &\Rightarrow \quad y + f_{\alpha,\eta}(t, \alpha(t), y) < -\overline{Y} + f_{\alpha,\eta}(t, \alpha(t), -\overline{Y}) < \alpha'(t)\,. \end{aligned}$$

Since $-\overline{Y} < y_\alpha(t) < \overline{Y}$ for every $t \in [a, b]$ from the choice (48), the inequality for $y_\alpha'(t)$ still holds, since $(\alpha(t), y_\alpha(t))$ belongs to the region where the problem has not been modified. All the other cases can be treated similarly. ☐

Lemma 20. *Every solution $u = (x, y)$ of* (51) *satisfies $\varphi \le x \le \eta$.*

Proof. Let us define the following regions

$$\begin{aligned} A_{NE} &= \{(t, x, y) \,:\, t \in [a, b]\,,\, x > \eta(t)\,,\, y > y_\eta(t)\}\,, \\ A_{SE} &= \{(t, x, y) \,:\, t \in [a, b]\,,\, x > \eta(t)\,,\, y < y_\eta(t)\}\,, \\ A_{SW} &= \{(t, x, y) \,:\, t \in [a, b]\,,\, x < \varphi(t)\,,\, y < y_\varphi(t)\}\,, \\ A_{NW} &= \{(t, x, y) \,:\, t \in [a, b]\,,\, x < \varphi(t)\,,\, y > y_\varphi(t)\}\,. \end{aligned}$$

If $u = (x, y)$ is a solution of

$$x' = y + f_{\varphi,\eta}(t, x, y)\,, \quad y' = x + g_{\varphi,\eta}(t, x, y)\,,$$

then, for any $t_0 \in [a, b]$,

$$\begin{aligned} (t_0, u(t_0)) \in A_{SE} \quad &\Rightarrow \quad (t, u(t)) \in A_{SE} \text{ for every } t \in [a, t_0]\,, \\ (t_0, u(t_0)) \in A_{NW} \quad &\Rightarrow \quad (t, u(t)) \in A_{NW} \text{ for every } t \in [a, t_0]\,, \\ (t_0, u(t_0)) \in A_{NE} \quad &\Rightarrow \quad (t, u(t)) \in A_{NE} \text{ for every } t \in [t_0, b]\,, \\ (t_0, u(t_0)) \in A_{SW} \quad &\Rightarrow \quad (t, u(t)) \in A_{SW} \text{ for every } t \in [t_0, b]\,. \end{aligned}$$

Moreover, for any $t_0 \in [a, b]$, if

$$x(t_0) < \varphi(t_0) \quad \text{and} \quad y(t_0) = y_\varphi(t_0)\,,$$

then there exists $\delta > 0$ such that

$$\begin{aligned} t_0 \ne a\,,\, t \in]t_0 - \delta, t_0[\quad &\Rightarrow \quad (t, u(t)) \in A_{NW}\,, \\ t_0 \ne b\,,\, t \in]t_0, t_0 + \delta[\quad &\Rightarrow \quad (t, u(t)) \in A_{SW}\,. \end{aligned}$$

Similarly, if, for any $t_0 \in [a, b]$,

$$x(t_0) > \eta(t_0) \quad \text{and} \quad y(t_0) = y_\eta(t_0)\,,$$

then there exists $\delta > 0$ such that

$$\begin{aligned} t_0 \ne a\,,\, t \in]t_0 - \delta, t_0[\quad &\Rightarrow \quad (t, u(t)) \in A_{SE}\,, \\ t_0 \ne b\,,\, t \in]t_0, t_0 + \delta[\quad &\Rightarrow \quad (t, u(t)) \in A_{NE}\,. \end{aligned}$$

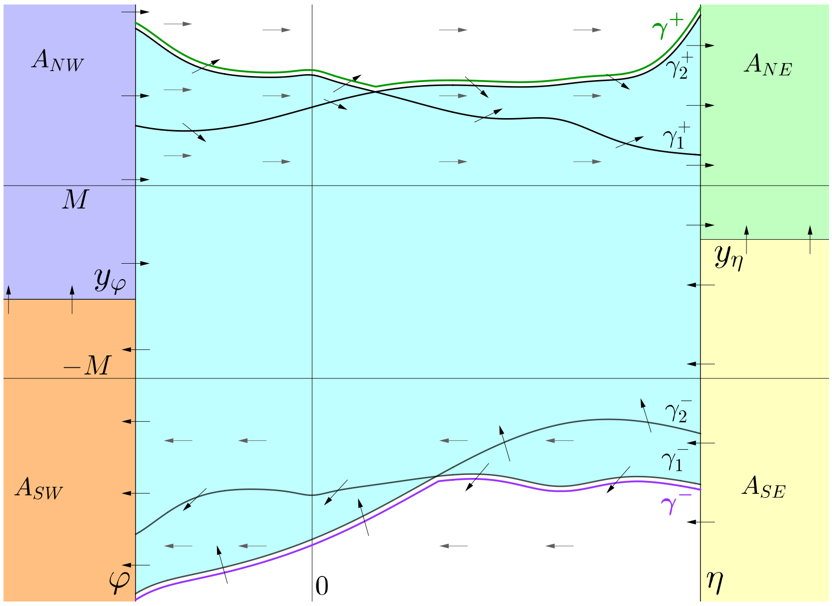

Figure 2: A section at a fixed time of the regions introduced in Lemmas 20 and 21, describing the dynamics of (51).

By contradiction, let $u = (x, y)$ be a solution of (51) such that $x(t_0) < \varphi(t_0)$, for some $t_0 \in [a, b]$. Since $x(a) = 0 \geq \varphi(a)$ and $x(b) = 0 \leq \varphi(b)$, then $t_0 \in]a, b[$ and by the above considerations it cannot be that $(t_0, u(t_0)) \in A_{NW} \cup A_{SW}$. Hence $y(t_0) = y_\varphi(t_0)$, and there exists $\delta > 0$ such that $(t, u(t)) \in A_{NW}$ for $t \in]t_0 - \delta, t_0[$, which leads to a contradiction. The case $x(t_0) > \eta(t_0)$ leads to a similar contradiction, as well. $\square$

Lemma 21. *Every solution $u = (x, y)$ of (51) satisfies*

$$\gamma^-(x(t)) < y(t) < \gamma^+(x(t)), \quad \textit{for every } t \in [a, b].$$

Proof. We introduce the functions $G_i^\pm(t) = y(t) - \gamma_i^\pm(x(t))$, with $i = 1, 2$. We first show that $y(a) < \gamma^+(x(a))$ and $y(b) < \gamma^+(x(b))$.

Let us prove that we cannot have $y(a) \geq \gamma_1^+(x(a))$. At first, assume that $y(t) \geq \gamma_1^+(x(t))$ for every $t \in [a, b]$. Then, from (48) we get $y(t) \geq \overline{M}$ for every $t \in [a, b]$, so that (46) gives $x(b) > 0$, a contradiction. So, there exists $t_0 \in [a, b[$ such that $G_1^+(t_0) = 0$ and $G_1^+(t) < 0$ in a right neighborhood of t_0. We can compute, recalling (47),

$$\begin{aligned}
(G_1^+)'(t_0) &= y'(t_0) - (\gamma_1^+)'(x(t_0))x'(t_0) \\
&= \widehat{g}_d\big(t_0, x(t_0), \gamma_1^+(x(t_0))\big) - (\gamma_1^+)'(x(t_0))\widehat{f}_d\big(t_0, x(t_0), \gamma_1^+(x(t_0))\big) \\
&= \widehat{g}_d\big(t_0, x(t_0), \gamma_1^+(x(t_0))\big) - (\gamma_1^+)'(x(t_0))\widehat{f}_d\big(t_0, x(t_0), \gamma_1^+(x(t_0))\big) > 0,
\end{aligned}$$

getting again a contradiction. Hence, $y(a) < \gamma_1^+(x(a)) \leq \gamma^+(x(a))$, recalling (49).

Similarly one shows that $y(b) < \gamma_2^+(x(b)) \leq \gamma^+(x(b))$, going backwards in time.

We now prove that $y(t) < \gamma^+(x(t))$ for every $t \in [a,b]$. Assume by contradiction that there is $t_0 \in]a,b[$ such that $y(t_0) \geq \gamma^+(x(t_0))$. We distinguish two possibilities. First, if $x(t_0) \geq 0$, then the solution remains above γ_1^+ for all $t \in]t_0,b]$, hence $y(t) > \overline{M}$ and $x'(t) > 0$ for all $t \in]t_0,b]$, leading to $x(b) > 0$, which is impossible. Second, if $x(t_0) < 0$, then there must exist a $t_1 \in [a,t_0[$ such that $y(t_1) \leq \overline{M}$. But then the solution remains below γ_2^+ for all $t \in [t_1,b]$, in contradiction with the assumption.

Similarly one proves that $y(t) > \gamma^-(x(t))$ for every $t \in [a,b]$. □

Lemma 22. *If there are no solutions of* (41) *on* $\partial\mathcal{V}_j$*, then* $\mathcal{V}_j$ *is admissible and*

$$\deg(I - L^{-1}N_d, \mathcal{V}_j) = 1\,, \quad j = 1,2,3.$$

Proof. For any sufficiently large $\rho > 0$, denoting by B_ρ the open ball in X with radius ρ, centered at the origin, we claim that

$$\deg(I - L^{-1}N_{\varphi,\eta}, B_\rho) = 1\,.$$

Indeed, let us show that there is a $\rho > 0$ such that, for every $\lambda \in [0,1]$, every solution of $Lu = \lambda N_{\varphi,\eta}u$ satisfies $\|u\|_{C^1} < \rho$. By contradiction, if this is not true, there exist a sequence $(\lambda_n)_n$ in $[0,1]$ and some solutions $u_n = (x_n,y_n)$ of $Lu_n = \lambda_n N_{\varphi,\eta}u_n$ such that $\|u_n\|_{C^1} \to \infty$. Let $v_n = x_n/\|u_n\|_{C^1}$ and $w_n = y_n/\|u_n\|_{C^1}$. By a standard argument it can be seen that, for a subsequence, $\lambda_n \to \bar\lambda \in [0,1]$, while $(v_n,w_n) \to (\bar v,\bar w)$ in X. Moreover, $\bar v' = \bar w$, $\bar w' = \bar\lambda\bar v$, so that $\bar v'' = \bar\lambda \bar v$, and since $\bar v(a) = 0 = \bar v(b)$, this implies $\bar v \equiv 0$, hence also $\bar w \equiv 0$, a contradiction. By homotopy invariance, the degree is then equal to 1.

Fix $\rho > 0$ as above. Since there are no solutions of (41) on $\partial\mathcal{V}(\varphi,\eta)$, we also have that there are no solutions of (51) on $\partial\mathcal{V}(\varphi,\eta)$. Hence, $\mathcal{V}(\varphi,\eta)$ is admissible and, by excision,

$$\deg(I - L^{-1}N_{\varphi,\eta}, \mathcal{V}(\varphi,\eta)) = \deg(I - L^{-1}N_{\varphi,\eta}, B_\rho) = 1\,.$$

Now, since $N_{\varphi,\eta} = N_d$ on $\mathcal{V}(\varphi,\eta)$, the result follows. □

Lemma 23. *There are no solutions of* (41) *on* $\partial\mathcal{V}_1$.

Proof. Let $u = (x,y)$ be a solution belonging to the closure of $\mathcal{V}_1$. Then $\widehat\alpha(t) \leq x(t) \leq \widehat\beta(t)$, for every $t \in \mathcal{I}$. Assume by contradiction that $x \not\ll \widehat\beta$. Since $x(a) = 0 < \xi = \widehat\beta(a)$ and $x(b) = 0 < \xi = \widehat\beta(b)$, there must be a $t_0 \in]a,b[$ such that

$$x(t_0) = \widehat\beta(t_0) > \xi\,, \quad x'(t_0) = \widehat\beta'(t_0)\,.$$

Hence both $(x(t),y(t))$ and $(x_\xi(t),y_\xi(t))$ solve the autonomous system (45) in a neighborhood of t_0. Recalling that $\widehat\beta = x_\xi$ by definition, since

$$x'(t_0) = \frac{d-1}{d}\frac{\partial H}{\partial y}(x(t_0),y(t_0))\,,$$

and

$$x'_\xi(t_0) = \frac{d-1}{d}\frac{\partial H}{\partial y}(x_\xi(t_0),y_\xi(t_0))\,,$$

being $x'(t_0) = x'_\xi(t_0)$ and $x(t_0) = x_\xi(t_0)$, by assumption (9) it has to be that $y(t_0) = y_\xi(t_0)$. Since autonomous planar Hamiltonian systems have the uniqueness property for Cauchy problems when the initial value is not an equilibrium (cf. [19, Theorem 1]), then the two solutions $(x(t), y(t))$ and $(x_\xi(t), y_\xi(t))$ coincide, as long as they remain in $[\xi, +\infty[\times\mathbb{R}$, leading to a contradiction. Hence, $x \ll \widehat{\beta}$. Similarly one proves that $\widehat{\alpha} \ll x$. So, there are no solutions of (41) on $\partial\mathcal{V}_1$. $\square$

Now, if there is a solution $u = (x, y)$ of (41) on $\partial\mathcal{V}_2$, then $x \leq \beta$ and $x \not\ll \beta$. Since $\alpha \not\leq \beta$, there is a t_0 such that $x(t_0) \leq \beta(t_0) < \alpha(t_0)$, implying that $\alpha \not\ll x$. So, u is the solution of (28) we are looking for.

A similar argument shows that if $u = (x, y)$ is a solution of (41) on $\partial\mathcal{V}_3$, then u is the solution of (28) we are looking for.

Finally, if there are no solutions of (41) on $\partial\mathcal{V}_2 \cup \partial\mathcal{V}_3$, then

$$\begin{aligned}&\deg(I - L^{-1}N_d, \mathcal{V}_1 \setminus \overline{\mathcal{V}_2 \cup \mathcal{V}_3}) = \\ &= \deg(I - L^{-1}N_d, \mathcal{V}_1) - \Big(\deg(I - L^{-1}N_d, \mathcal{V}_2) + \deg(I - L^{-1}N_d, \mathcal{V}_3)\Big) = -1\,.\end{aligned}$$

Then, there is a solution of (41) in $\mathcal{V}_1 \setminus \overline{\mathcal{V}_2 \cup \mathcal{V}_3}$, and this is the solution of (28) we are looking for.

The proof of Theorem 3 is thus completed.

7 Higher dimensional systems

Let us now introduce a higher dimensional version of problem (2). We will write $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, $y = (y_1, \dots, y_N) \in \mathbb{R}^N$, and assume that the continuously differentiable function $H : \mathbb{R}^{2N} \to \mathbb{R}$ is of the type

$$H(x, y) = \sum_{n=1}^{N} H_n(x_n, y_n)\,. \tag{52}$$

Moreover, for every $n \in \{1, \dots, N\}$, we assume that there exist $p_n > 1$ and $q_n > 1$, with $(1/p_n) + (1/q_n) = 1$, such that

$$H_n(\lambda^{q_n} u, \lambda^{p_n} v) = \lambda^{p_n + q_n} H_n(u, v) > 0\,,$$

for every $\lambda > 0$ and $(u, v) \neq (0, 0)$. We consider the problem

$$\begin{cases} x' = \dfrac{\partial H}{\partial y}(x, y) + \phi(t, x, y)\,, \quad y' = -\dfrac{\partial H}{\partial x}(x, y) + \psi(t, x, y) \\ x(a) = 0 = x(b)\,, \end{cases} \tag{53}$$

where the functions $\phi, \psi : [a, b] \times \mathbb{R}^{2N} \to \mathbb{R}^N$ are continuous and bounded. Equivalently, writing $\phi = (\phi_1, \dots, \phi_N)$ and $\psi = (\psi_1, \dots, \psi_N)$,

$$\begin{cases} x_n' = \dfrac{\partial H_n}{\partial y_n}(x_n, y_n) + \phi_n(t, x_1, \dots, x_N, y_1, \dots, y_N)\,, \\ y_n' = -\dfrac{\partial H_n}{\partial x_n}(x_n, y_n) + \psi_n(t, x_1, \dots, x_N, y_1, \dots, y_N)\,, \\ x_n(a) = 0 = x_n(b)\,, \quad n = 1, \dots, N\,. \end{cases}$$

For $\xi, \upsilon \in \mathbb{R}^N$ we write $\xi \preceq \upsilon$ (or $\upsilon \succeq \xi$) if

$$\xi_n \leq \upsilon_n \text{ for every } n \in \{1, \ldots, N\},$$

and in this case we define

$$\langle\!\langle \xi, \upsilon \rangle\!\rangle = \{u \in \mathbb{R}^N \, : \, \xi \preceq u \preceq \upsilon\}.$$

We now adapt the definition of lower/upper solutions given in [14, Definition 31] to the higher dimensional problem

$$\begin{cases} x' = f(t, x, y), \qquad y' = g(t, x, y), \\ x(a) = 0 = x(b), \end{cases} \tag{54}$$

where $f, g : [a, b] \times \mathbb{R}^{2N} \to \mathbb{R}^N$ are continuous functions. As usual, we write $f = (f_1, \ldots, f_N)$ and $g = (g_1, \ldots, g_N)$. Similarly for the vector-valued functions considered below.

Definition 24. *Given two C^1-functions $\alpha, \beta : [a, b] \to \mathbb{R}^N$, we say that (α, β) is a* well-ordered pair of lower/upper solutions *of problem* (54) *if*

$$\alpha(t) \preceq \beta(t), \quad \text{for every } t \in [a, b],$$

$$\alpha(a) \preceq 0 \preceq \beta(a), \qquad \alpha(b) \preceq 0 \preceq \beta(b),$$

and there exist two C^1-functions $y^\alpha, y^\beta : [a, b] \to \mathbb{R}^N$ such that, for every $x, y \in \mathbb{R}^N$, $n \in \{1, \ldots, N\}$ and $t \in [a, b]$,

$$\begin{cases} f_n(t, x, y) < \alpha_n'(t) & \text{when } x_n = \alpha_n(t) \text{ and } y_n < y_n^\alpha(t), \\ f_n(t, x, y) > \alpha_n'(t) & \text{when } x_n = \alpha_n(t) \text{ and } y_n > y_n^\alpha(t), \end{cases} \tag{55}$$

$$\begin{cases} f_n(t, x, y) < \beta_n'(t) & \text{when } x_n = \beta_n(t) \text{ and } y_n < y_n^\beta(t), \\ f_n(t, x, y) > \beta_n'(t) & \text{when } x_n = \beta_n(t) \text{ and } y_n > y_n^\beta(t), \end{cases} \tag{56}$$

and

$$(y_n^\alpha)'(t) \geq g_n(t, x, y), \\ \text{when } x \in \langle\!\langle \alpha(t), \beta(t) \rangle\!\rangle, \; x_n = \alpha_n(t) \text{ and } y_n = y_n^\alpha(t), \tag{57}$$

$$(y_n^\beta)'(t) \leq g_n(t, x, y), \\ \text{when } x \in \langle\!\langle \alpha(t), \beta(t) \rangle\!\rangle, \; x_n = \beta_n(t) \text{ and } y_n = y_n^\beta(t). \tag{58}$$

Here is our result in the well-ordered case.

Theorem 25. *Assume H to be as in* (52)*, with components H_n being positively-(p_n, q_n)-homogeneous positive-definite continuously differentiable functions, for some $p_n > 1$ and $q_n > 1$ with $(1/p_n) + (1/q_n) = 1$. Let ϕ, ψ be uniformly bounded continuous functions, and let (α, β) be a well-ordered pair of lower/upper solutions of problem* (53)*. Then, there exists a solution (x, y) of* (53) *such that $\alpha(t) \preceq x(t) \preceq \beta(t)$, for every $t \in [a, b]$.*

Proof. One proceeds like in the proof of [14, Theorem 32]. The main difference here is that the functions ϕ and ψ depend on *all* variables x and y. However, setting

$$
\begin{aligned}
m_\alpha &= \min\{\alpha_n(t) : t \in [a,b], n = 1, \ldots, N\}\,,\\
M_\beta &= \max\{\beta_n(t) : t \in [a,b], n = 1, \ldots, N\}\,,
\end{aligned}
$$

the fact that ϕ and ψ are bounded permits to recover, for every $n = 1, \ldots, N$, the required guiding curves $\gamma^{\pm}_{1,n}, \gamma^{\pm}_{2,n} : [m_\alpha, M_\beta] \to \mathbb{R}$, with $i = 1, 2$, and the result follows. $\square$

For the non-well-ordered case, we need to introduce the notion of *strict* lower and upper solutions. To this aim we will follow the ideas developed in [11], and distinguish the components which are well-ordered from the others.

The couple $(\mathcal{J}, \mathcal{K})$ is a partition of $\{1, \ldots, N\}$ if and only if $\mathcal{J} \cap \mathcal{K} = \varnothing$ and $\mathcal{J} \cup \mathcal{K} = \{1, \ldots, N\}$. We denote by $\#\mathcal{J}$ and $\#\mathcal{K}$ the cardinality of the sets $\mathcal{J}$ and $\mathcal{K}$. For a given partition $(\mathcal{J}, \mathcal{K})$, a vector

$$
x = (x_1, \ldots, x_N) = (x_n)_{n \in \{1,\ldots,N\}} \in \mathbb{R}^N
$$

can be decomposed as $x = (x_\mathcal{J}, x_\mathcal{K})$ where $x_\mathcal{J} = (x_j)_{j \in \mathcal{J}} \in \mathbb{R}^{\#\mathcal{J}}$ and $x_\mathcal{K} = (x_k)_{k \in \mathcal{K}} \in \mathbb{R}^{\#\mathcal{K}}$. Similarly, we can decompose every function $\mathcal{F} : \mathcal{D} \to \mathbb{R}^N$ as $\mathcal{F}(x) = \big(\mathcal{F}_\mathcal{J}(x), \mathcal{F}_\mathcal{K}(x)\big)$ with $\mathcal{F}_\mathcal{J} : \mathcal{D} \to \mathbb{R}^{\#\mathcal{J}}$ and $\mathcal{F}_\mathcal{K} : \mathcal{D} \to \mathbb{R}^{\#\mathcal{K}}$, for any domain $\mathcal{D}$. Moreover, for $\xi, \upsilon \in \mathbb{R}^N$ we write

$$
\langle\!\langle \xi, \upsilon \rangle\!\rangle_\mathcal{J} = \{u = (u_\mathcal{J}, u_\mathcal{K}) \in \mathbb{R}^N \;:\; \xi_\mathcal{J} \preceq u_\mathcal{J} \preceq \upsilon_\mathcal{J}\}\,.
$$

Definition 26. *Let $\alpha, \beta : [a,b] \to \mathbb{R}^N$ be two C^1-functions. We will say that (α, β) is a* pair of lower/upper solutions of (54) related to the partition $(\mathcal{J}, \mathcal{K})$ of $\{1, \ldots, N\}$ *if the following conditions hold:*

1. *$\alpha_j \le \beta_j$, for any $j \in \mathcal{J}$;*
2. *$\alpha_k \not\le \beta_k$, for any $k \in \mathcal{K}$;*
3. *$\alpha(a) \preceq 0 \preceq \beta(a)$ and $\alpha(b) \preceq 0 \preceq \beta(b)$;*
4. *there are two C^1-functions $y^\alpha, y^\beta : [a,b] \to \mathbb{R}^N$ such that (55) and (56) hold for every $n \in \{1, \ldots, N\}$ and $t \in [a,b]$;*
5. *for every $n \in \{1, \ldots, N\}$ and $t \in [a,b]$ one has*

$$
(y_n^\alpha)'(t) \ge g_n(t,x,y)\,, \qquad \text{when } x \in \langle\!\langle \alpha(t)\,,\, \beta(t) \rangle\!\rangle_\mathcal{J}\,,\; x_n = \alpha_n(t) \text{ and } y_n = y_n^\alpha(t)\,,
$$

and

$$
(y_n^\beta)'(t) \le g_n(t,x,y)\,, \qquad \text{when } x \in \langle\!\langle \alpha(t)\,,\, \beta(t) \rangle\!\rangle_\mathcal{J}\,,\; x_n = \beta_n(t) \text{ and } y_n = y_n^\beta(t)\,.
$$

In the following definition we will use the relation $\gg$ introduced in (7).

Definition 27. *Let* (α, β) *be a pair of lower/upper solutions of* (54) *related to the partition* $(\mathcal{J}, \mathcal{K})$. *We will say that this pair of lower/upper solutions is* strict with respect to the j-th component, *with* $j \in \mathcal{J}$, *if* $\alpha_j \ll \beta_j$ *and, for every solution* (x, y) *of* (54),

$$\alpha_j \le x_j \le \beta_j \quad \Rightarrow \quad \alpha_j \ll x_j \ll \beta_j \,.$$

We will say that this pair of lower/upper solutions is strict with respect to the k-th component, *with* $k \in \mathcal{K}$ *if, for every solution* (x, y) *of* (54),

$$\alpha_k \le x_k \ \Rightarrow \ \alpha_k \ll x_k \,, \quad \text{and} \quad x_k \le \beta_k \ \Rightarrow \ x_k \ll \beta_k \,.$$

In order to prove the existence of a solution of (53), once a pair of lower/upper solutions (α, β) is given, we need to ask the *strictness* property with respect to the non-well-ordered components α_k, β_k. Moreover, we will need to ask more regularity on the functions ϕ and ψ: we will ask them to be locally ℓ-Hölder continuous for a certain $\ell > 0$. Here is our result.

Theorem 28. *Let* $\ell > 0$ *and assume* $H : \mathbb{R}^{2N} \to \mathbb{R}$ *to be as in* (52), *with components* H_n *being* $C^{1,\ell}_{loc}$*-smooth positively-*(p_n, q_n)*-homogeneous positive-definite functions, for some* $p_n > 1$ *and* $q_n > 1$ *with* $(1/p_n) + (1/q_n) = 1$. *Let* (α, β) *be a pair of lower/upper solutions of* (53) *related to the partition* $(\mathcal{J}, \mathcal{K})$ *of* $\{1, \dots, N\}$ *which is strict with respect to the* k*-th component, for every* $k \in \mathcal{K}$. *Moreover, assume that for every* $k \in \mathcal{K}$,

$$\frac{\partial H_k}{\partial y}(x_0, \cdot) \text{ is a strictly increasing function, for every } x_0 \in \mathbb{R}. \tag{59}$$

Let ϕ, ψ *be uniformly bounded* $C^{0,\ell}_{loc}$*-smooth functions. If*

$$b - a \le \min\{\tau_k^+, \tau_k^- \ : \ k \in \mathcal{K}\} \,,$$

then (53) *has a solution* (x, y) *with the following properties:*

(J) *for any* $j \in \mathcal{J}$, $\alpha_j(t) \le x_j(t) \le \beta_j(t)$, *for every* $t \in [a, b]$;

(K) *for any* $k \in \mathcal{K}$, *there exist* $t_k^1, t_k^2 \in]a, b[$ *such that* $x_k(t_k^1) < \alpha_k(t_k^1)$ *and* $x_k(t_k^2) > \beta_k(t_k^2)$.

Proof. The case $\mathcal{K} = \varnothing$ has been already treated in Theorem 25. In order to simplify the exposition, we assume both $\mathcal{J} \neq \varnothing$ and $\mathcal{K} \neq \varnothing$, and that $\mathcal{J} = \{1, \dots, M\}$ and $\mathcal{K} = \{M + 1, \dots N\}$ for a certain $M \in \{1, \dots, N - 1\}$. The proof can be easily adapted to the case $\mathcal{J} = \varnothing$.

As in Section 2, for every $n \in \{1, \dots, N\}$, let $S_n(t) = (S_{1,n}(t), S_{2,n}(t))$ be the periodic solution of the planar autonomous system

$$u' = \frac{\partial H_n}{\partial v}(u, v) \,, \quad v' = -\frac{\partial H_n}{\partial u}(u, v)$$

such that $S_{1,n}(0) = 0$ and $S_{2,n}(0) > 0$, with $H_n(S_{1,n}(t), S_{2,n}(t)) = 1$ for every t. We set

$$\overline{S}_n = \max\{|S_{1,n}(t)| + |S_{2,n}(t)| : t \in \mathbb{R}\} \,,$$

and

$$\overline{S} = \max \left\{ \overline{S}_n : n \in \{1 \ldots, N\} \right\} . \tag{60}$$

For each $n \in \{1, \ldots, N\}$ we will mainly follow the procedure developed in Section 5 if $n \in \mathcal{J}$, the one in Section 6 if $n \in \mathcal{K}$, so that the couple of variables (x_n, y_n) will overshadow the remaining ones, which will essentially act as parameters. We first need to modify problem (53) both in the k-variables, following the lines of the proof of Theorem 3, and in the j-variables, in order to apply a topological degree argument, as in the proof of [11, Theorem 10].

Let us rewrite (53) as

$$\begin{cases} x_j' = f_j(t,x,y)\,, \qquad y_j' = g_j(t,x,y)\,, \qquad j \in \{1, \ldots, M\}\,, \\ x_k' = f_k(t,x,y)\,, \qquad y_k' = g_k(t,x,y)\,, \qquad k \in \{M+1, \ldots, N\}\,, \\ x(a) = 0 = x(b)\,. \end{cases}$$

We introduce the functions

$$\zeta_j(t, x_j) = \zeta(x_j, \alpha_j(t), \beta_j(t))\,,$$

where ζ was defined in (50), and

$$\Gamma(t,x) = \big(\zeta_1(t,x_1)\,, \ldots\,, \zeta_M(t,x_M)\,, x_{M+1}\,, \ldots\,, x_N\big)\,.$$

Then we set

$$\widehat{f}_j(t,x,y) = f_j(t, \Gamma(t,x), y)\,, \qquad \widehat{g}_j(t,x,y) = g_j(t,\Gamma(t,x),y) + x_j - \zeta_j(t,x_j)\,.$$

Concerning the non-well-ordered components, we need to consider a positive parameter d, which will be fixed later, following the construction of problem (41) in Section 6. We set

$$\begin{aligned} \widehat{f}_{k,d}(t,x,y) &= \frac{d-1}{d}\,\frac{\partial H_k}{\partial y_k}(x_k,y_k) + \widetilde{\phi}_{k,d}(t,x,y)\,, \\ \widehat{g}_{k,d}(t,x,y) &= -\frac{d-1}{d}\,\frac{\partial H_k}{\partial x_k}(x_k,y_k) + \widetilde{\psi}_{k,d}(t,x,y)\,, \end{aligned}$$

with

$$\begin{aligned} \widetilde{\phi}_{k,d}(t,x,y) &= \chi\Big(\frac{x_k}{d\overline{S}}\Big)\chi\Big(\frac{y_k}{d^{p_k/q_k}\overline{S}}\Big)\left(\frac{1}{d}\frac{\partial H_k}{\partial y_k}(x_k,y_k) + \phi_k(t,\Gamma(t,x),y)\right), \\ \widetilde{\psi}_{k,d}(t,x,y) &= \chi\Big(\frac{x_k}{d\overline{S}}\Big)\chi\Big(\frac{y_k}{d^{p_k/q_k}\overline{S}}\Big)\left(-\frac{1}{d}\frac{\partial H_k}{\partial x_k}(x_k,y_k) + \psi_k(t,\Gamma(t,x),y)\right), \end{aligned}$$

where $\overline{S}$ is defined in (60) and the cut-off function χ in (40). We thus are led to the modified problem

$$\begin{cases} x_j' = \widehat{f}_j(t,x,y)\,, \qquad y_j' = \widehat{g}_j(t,x,y)\,, \qquad j \in \{1, \ldots, M\}\,, \\ x_k' = \widehat{f}_{k,d}(t,x,y)\,, \qquad y_k' = \widehat{g}_{k,d}(t,x,y)\,, \qquad k \in \{M+1, \ldots, N\}\,, \\ x(a) = 0 = x(b)\,. \end{cases} \tag{61}$$

We will now provide, working separately on every component, an indexed family of well-ordered pair of lower/upper solutions of the modified problem (61), which will be strict in every component.

Let us first operate on a component $j \in \{1, \dots, M\}$. We can argue as in Lemmas 8 and 9 so to find some guiding curves $\gamma^{\pm}_{1,j}$ and $\gamma^{\pm}_{2,j}$. To be more precise, for an illustrative purpose, the curve $\gamma^{+}_{1,j}$ will satisfy the analogue of (29), i.e.,

$$\widehat{g}_j(t, x, y) > \widehat{f}_j(t, x, y)(\gamma^{+}_{1,j})'(x_j)\,,$$

for every $t \in [a, b]$, every $x \in \mathbb{R}^n$ with $\alpha_j(t) \le x_j \le \beta_j(t)$ and every $y \in \mathbb{R}^N$ with $y_j = \gamma^{+}_{1,j}(x_j)$.

Then, we can follow the reasoning in [14, Theorem 11, Claim 3] and prove that all the solutions (x, y) of (61) satisfy

$$\alpha_j(t) \le x_j(t) \le \beta_j(t)\,, \tag{62}$$

and

$$\min\{\gamma^{-}_{1,j}(x(t))\,,\, \gamma^{-}_{2,j}(x(t))\} < y_j(t) < \max\{\gamma^{+}_{1,j}(x(t))\,,\, \gamma^{+}_{2,j}(x(t))\}\,, \tag{63}$$

for every $t \in [a, b]$. We then define the functions

$$\check{\alpha}_j(t) = \alpha_j(t) - 1 \quad \text{and} \quad \check{\beta}_j(t) = \beta_j(t) + 1\,,$$

and conclude that

(S1) for every $j \in \mathcal{J}$, the functions $\check{\alpha}_j$ and $\check{\beta}_j$ satisfy

$$\check{\alpha}_j \ll \check{\beta}_j\,, \quad \check{\alpha}_j(a) < 0 < \check{\beta}_j(a)\,, \quad \check{\alpha}_j(b) < 0 < \check{\beta}_j(b)\,.$$

Moreover, the conditions (55), (56), (57) and (58) hold with $n = j$, replacing $\alpha_j, \beta_j, f_j, g_j$ by $\check{\alpha}_j, \check{\beta}_j, \widehat{f}_j, \widehat{g}_j$, setting $y^{\check{\alpha}}_j = y^{\alpha}_j$ and $y^{\check{\beta}}_j = y^{\beta}_j$. Finally, every solution (x, y) of (61) satisfies $\check{\alpha}_j \ll x_j \ll \check{\beta}_j$ and (63).

Let us now focus our attention on a component $k \in \mathcal{K}$. Arguing as in Lemma 13 we can prove that

(S2) there exists $D > 1$ with the following property: if $u = (x, y)$ is a solution of (61), with $d \ge D$, such that $\alpha_k \not\ll x_k$ and $x_k \not\ll \beta_k$, then $\mathcal{N}_{p_k}(u_k) < D$.

We can surely take the same constant D for every $k \in \mathcal{K}$. Now, as in Lemma 14, enlarging D if necessary we can prove that

(S3) for every $k \in \mathcal{K}$ and $d \ge D$, the functions α_k and β_k still satisfy conditions (55), (56), (57) and (58), replacing f_k, g_k by $\widehat{f}_{k,d}, \widehat{g}_{k,d}$, respectively.

Then, Lemma 16 suggests us how to continue the proof once we have fixed $d > D$ sufficiently large:

(S4) for every $k \in \mathcal{K}$, there are two couples of functions $(\widehat{\alpha}_k, y^{\widehat{\alpha}}_k)$ and $(\widehat{\beta}_k, y^{\widehat{\beta}}_k)$ such that $\widehat{\alpha}_k \ll \widehat{\beta}_k$, $\widehat{\alpha}_k \ll \beta_k$, and $\alpha_k \ll \widehat{\beta}_k$ satisfying the conditions (55), (56), (57) and (58), replacing in all formulas α_k, y^{α}_k, β_k, y^{β}_k, f_k, g_k by $\widehat{\alpha}_k$, $y^{\widehat{\alpha}}_k$, $\widehat{\beta}_k$, $y^{\widehat{\beta}}_k$, $\widehat{f}_{k,d}$, $\widehat{g}_{k,d}$, respectively.

Now a further step is needed. We are going to prove that

(S5) for every $k \in \mathcal{K}$, if (x, y) is a solution of (61) such that $\widehat{\alpha}_k \le x_k$, then $\widehat{\alpha}_k \ll x_k$. Analogously, if (x, y) is a solution of (61) such that $x_k \le \widehat{\beta}_k$, then $x_k \ll \widehat{\beta}_k$.

We prove the second assertion, the proof of the first one being similar. Following the proof of Lemma 23, assume by contradiction that there is a solution (x, y) of (61) such that $x_k \le \widehat{\beta}_k$ but $x_k \not\ll \widehat{\beta}_k$. Since $x_k(a) = 0 < \widehat{\beta}_k(a)$ and $x_k(b) = 0 < \widehat{\beta}_k(b)$, there exists $t_0 \in]a, b[$ such that $x_k(t_0) = \widehat{\beta}_k(t_0)$ and $x_k'(t_0) = \widehat{\beta}_k'(t_0)$. Using assumption (59), since

$$\frac{d-1}{d}\frac{\partial H_k}{\partial y_k}(x_k(t_0), y_k^{\widehat{\beta}}(t_0)) = \widehat{\beta}_k'(t_0) = x_k'(t_0) = \frac{d-1}{d}\frac{\partial H_k}{\partial y_k}(x_k(t_0), y_k(t_0)),$$

we get $y_k^{\widehat{\beta}}(t_0) = y_k(t_0)$. Recalling that $\widetilde{\phi}_{k,d} = 0$ and $\widetilde{\psi}_{k,d} = 0$ in a neighborhood of $\{(\widehat{\beta}_k, y_k^{\widehat{\beta}})(t) : t \in [a, b]\}$, arguing as in the proof of Lemma 23, we conclude that $(\widehat{\beta}_k, y_k^{\widehat{\beta}})$ and (x_k, y_k) coincide on $[a, b]$, leading to a contradiction with the fact that $x_k(a) = x_k(b) = 0$. Hence, **(S5)** is proved.

For every multi-index $\mu = (\mu_{M+1}, \dots, \mu_N) \in \{1, 2, 3\}^{N-M}$, we define the couple of functions (φ^μ, η^μ) by components: for every $j \in \mathcal{J}$ we set

$$(\varphi_j^\mu, \eta_j^\mu) = (\check{\alpha}_j, \check{\beta}_j),$$

and, for every $k \in \mathcal{K}$,

$$\begin{aligned}
&\text{if } \mu_k = 1 \text{ we set } (\varphi_k^\mu, \eta_k^\mu) = (\widehat{\alpha}_k\,, \widehat{\beta}_k),\\
&\text{if } \mu_k = 2 \text{ we set } (\varphi_k^\mu, \eta_k^\mu) = (\widehat{\alpha}_k\,, \beta_k),\\
&\text{if } \mu_k = 3 \text{ we set } (\varphi_k^\mu, \eta_k^\mu) = (\alpha_k\,, \widehat{\beta}_k).
\end{aligned}$$

From **(S1)**, **(S3)**, **(S4)** and **(S5)**, we can verify that, for every $\mu \in \{1, 2, 3\}^{N-M}$, the couple (φ^μ, η^μ) is a well-ordered pair of lower/upper solutions of problem (61) which is strict with respect to all its components. Let

$$\Xi = \left\{(\varphi^\mu, \eta^\mu) : \mu \in \{1, 2, 3\}^{N-M}\right\}.$$

As in Lemmas 8, 9 and 17, we can construct some guiding curves $\gamma_{1,n}^\pm$, $\gamma_{2,n}^\pm$, for every $n \in \{1, \dots, N\}$. Then, for every couple $(\varphi^\mu, \eta^\mu) \in \Xi$, we can modify system (61), only in the components $k \in \mathcal{K}$, exactly as we did to define problem (51), and obtain the new problem

$$\begin{cases} x_j' = \widehat{f}_j(t, x, y), \quad y_j' = \widehat{g}_j(t, x, y), \quad j \in \mathcal{J}, \\ x_k' = y_k + (f_k)_{\varphi^\mu, \eta^\mu}(t, x, y), \quad y_k' = x_k + (g_k)_{\varphi^\mu, \eta^\mu}(t, x, y), \quad k \in \mathcal{K}, \\ x(a) = 0 = x(b). \end{cases} \tag{64}$$

With the same procedure we can show that every couple (φ^μ, η^μ) is a well-ordered pair of lower/upper solutions for the new problem, too. Moreover, as in Lemmas 20 and 21, we can show that each solution of (64) satisfies, for every $t \in [a, b]$,

$$\varphi^\mu(t) \preceq x(t) \preceq \eta^\mu(t),$$

and, for every $n \in \{1, \dots, N\}$,

$$\min\{\gamma_{1,n}^-(x(t))\,,\, \gamma_{2,n}^-(x(t))\} < y_n(t) < \max\{\gamma_{1,n}^+(x(t))\,,\, \gamma_{2,n}^+(x(t))\}\,.$$

We need now to introduce the functional setting. We denote by $\widehat{f}_d$ and $\widehat{g}_d$ the functions

$$\widehat{f}_d = (\widehat{f}_1, \dots, \widehat{f}_M, \widehat{f}_{M+1,d}, \dots, \widehat{f}_{N,d})\,, \quad \widehat{g}_d = (\widehat{g}_1, \dots, \widehat{g}_M, \widehat{g}_{M+1,d}, \dots, \widehat{g}_{N,d})\,,$$

which describe system (61). Recalling the notations introduced in Section 6, we consider the linear operator

$$\begin{aligned} &L : [C_0^{1,\ell}(\mathcal{I})]^N \times [C^{1,\ell}(\mathcal{I})]^N \to [C^{0,\ell}(\mathcal{I})]^N \times [C^{0,\ell}(\mathcal{I})]^N \\ &L(x,y) = (x' - y\,,\, y')\,, \end{aligned}$$

and the Nemitskii operator

$$\begin{aligned} &N_d : [C_0^{1}(\mathcal{I})]^N \times [C^{1}(\mathcal{I})]^N \to [C^{0,\ell}(\mathcal{I})]^N \times [C^{0,\ell}(\mathcal{I})]^N \\ &N_d(x,y)(t) = \Big(\widehat{f}(t, x(t), y(t)) - y(t)\,,\, \widehat{g}(t, x(t), y(t))\Big)\,. \end{aligned}$$

Then, the analogue of Lemma 18 holds and $u = (x, y)$ is a solution of (61) if and only if it solves

$$Lu = N_d u\,,$$

with

$$u \in X := [C_0^1(\mathcal{I})]^N \times [C^1(\mathcal{I})]^N.$$

For every $(\varphi^\mu, \eta^\mu) \in \Xi$, we introduce the set

$$\begin{aligned} \mathcal{V}^\mu = \big\{(x,y) \in X \,:\, &\varphi_n^\mu \ll x_n \ll \eta_n^\mu \;\text{ and} \\ &\gamma_n^-(x_n(t)) < y_n(t) < \gamma_n^+(x_n(t))\,,\ \text{for every } t \in [a,b]\,, n \in \{1, \dots, N\}\big\}\,, \end{aligned}$$

where

$$\gamma_n^-(x) = \min\{\gamma_{i,n}^-(x)\,,\, i = 1,2\}\,, \qquad \gamma_n^+(x) = \max\{\gamma_{i,n}^+(x)\,,\, i = 1,2\}\,.$$

Notice that $\mathcal{V}^\mu$ is open in X.

Fix any $(\varphi^\mu, \eta^\mu) \in \Xi$. From **(S1)** and **(S5)**, we see that there are no solutions of (61) on $\partial\mathcal{V}^\mu$. Then, arguing as in the proof of Lemma 22, the set $\mathcal{V}^\mu$ is admissible and

$$\deg\big(I - L^{-1}N_d, \mathcal{V}^\mu\big) = 1\,.$$

Then, we can follow the procedure in [11, Section 3.1] and define, for every multi-index $\mu = (\mu_{M+1}, \dots, \mu_N) \in \{1,2,3,4\}^{N-M}$, the open set

$$\begin{aligned} \Omega^\mu := \big\{(x,y) \in X :\ &(\mathcal{A}_j^0),\ (\mathcal{A}_k^{\mu_k}),\ (\mathcal{A}_j^\gamma) \text{ and } (\mathcal{A}_k^\gamma) \\ &\text{hold for every } j \in \mathcal{J} \text{ and } k \in \mathcal{K}\big\}\,, \end{aligned}$$

where the conditions $(\mathcal{A}_j^0)$, $(\mathcal{A}_k^{\mu_k})$, and $(\mathcal{A}_n^\gamma)$ read as

$(\mathcal{A}_j^0)$ $\check{\alpha}_j \ll x_j \ll \check{\beta}_j\,;$

($\mathcal{A}_k^1$) $\widehat{\alpha}_k \ll x_k \ll \widehat{\beta}_k$;

($\mathcal{A}_k^2$) $\widehat{\alpha}_k \ll x_k \ll \beta_k$;

($\mathcal{A}_k^3$) $\alpha_k \ll x_k \ll \widehat{\beta}_k$;

($\mathcal{A}_k^4$) $\widehat{\alpha}_k \ll x_k \ll \widehat{\beta}_k$, and there are $t_k^1, t_k^2 \in]a,b[$ such that $x_k(t_k^1) < \alpha_k(t_k^1)$ and $x_k(t_k^2) > \beta_k(t_k^2)$;

($\mathcal{A}_n^\gamma$) $\gamma_n^-(x_n(t)) < y_n(t) < \gamma_n^+(x_n(t))$, for every $t \in [a,b]$.

Notice that

$$\Omega^\mu = \mathcal{V}^\mu\,, \quad \text{for every } \mu \in \{1,2,3\}^{N-M}\,.$$

Moreover, for any $\mu \in \{1,2,3,4\}^{N-M}$ and $\kappa \in \{M+1,\dots,N\}$, using the notation

$$\Omega^\mu_{\kappa,i} = \Omega^{(\mu_{M+1},\dots,\mu_{\kappa-1},i,\mu_{\kappa+1},\dots,\mu_N)}, \quad i = 1,2,3,4\,,$$

the arguments in [14, Proposition 23] show that

$$\overline{\Omega^\mu_{\kappa,4}} \cap \Omega^\mu_{\kappa,1} = [\Omega^\mu_{\kappa,1} \setminus \Omega^\mu_{\kappa,2}] \cap [\Omega^\mu_{\kappa,1} \setminus \Omega^\mu_{\kappa,3}]\,,$$

giving easily, passing to the complementary, $\Omega^\mu_{\kappa,1} \setminus \overline{\Omega^\mu_{\kappa,4}} = \Omega^\mu_{\kappa,2} \cup \Omega^\mu_{\kappa,3}$ and consequently

$$\Omega^\mu_{\kappa,4} = \Omega^\mu_{\kappa,1} \setminus \overline{\Omega^\mu_{\kappa,2} \cup \Omega^\mu_{\kappa,3}}\,.$$

Arguing as in [11, Propositions 15–18], we can prove that $\deg\big(I - L^{-1}N_d, \Omega^\mu\big)$ is well defined for every $\mu \in \{1,2,3,4\}^{N-M}$, and it is equal to $(-1)^m$, where m is the number of times the number 4 appears in the multi-index μ. In particular, we have that

$$\deg\big(I - L^{-1}N_d, \Omega^{(4,4,\dots,4,4)}\big) = (-1)^{N-M} \neq 0\,.$$

So, there exists a solution (x,y) of (61) belonging to $\Omega^{(4,4,\dots,4,4)}$. This solution satisfies ($\mathcal{A}_k^4$), for every $k \in \mathcal{K}$, hence $\alpha_k \not\ll x_k$ and $x_k \not\ll \beta_k$, for every $k \in \mathcal{K}$. Then, from **(S2)** we conclude that $\mathcal{N}_{p_k}(u_k) < D$, so that (x,y) is indeed a solution of the original problem (53), since the differential equation defining the two problems coincide in the set $\{u \in \mathbb{R}^{2N} : \mathcal{N}_{p_k}(u_k) < D\}$. Moreover, from (62), we have $\alpha_{\mathcal{J}}(t) \preceq x_{\mathcal{J}}(t) \preceq \beta_{\mathcal{J}}(t)$ for every $t \in [a,b]$. The proof is thus completed. □

As an example of application, consider the Dirichlet problem

$$\begin{cases} (|x_n'|^{p-2}x_n')' + \mu_n|x_n^+|^{p-2}x_n^+ - \nu_n|x_n^-|^{p-2}x_n^- = h_n(t,x_1,\dots,x_N,x_1',\dots,x_N') \\ x_n(0) = 0 = x_n(\pi_p)\,, \qquad n = 1,\dots,N\,, \end{cases} \tag{65}$$

where $p > 1$, $\mu_n > 0$ and $\nu_n > 0$. In this case, if $\alpha : [0,\pi_p] \to \mathbb{R}^N$ is a lower solution, then taking $y_n^\alpha = |\alpha_n'|^{p-2}\alpha_n'$, for every $n = 1,\dots,N$, conditions (55) are always satisfied, while (57) reads as

$$\begin{aligned} (|\alpha_n'|^{p-2}\alpha_n')'(t) + |\alpha_n^+(t)|^{p-2}\alpha_n^+(t) - \nu_n|\alpha_n^-(t)|^{p-2}\alpha_n^-(t) \geq \\ \geq h_n(t,x_1,\dots,\alpha_n(t),\dots,x_N,y_1,\dots,|\alpha_n'(t)|^{p-2}\alpha_n'(t),\dots,y_N)\,, \end{aligned} \tag{66}$$

for every $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$. Similarly for an upper solution.

As an immediate consequence of Theorem 28, we have the following.

Corollary 29. *Let $\ell > 0$ and h_n be a uniformly bounded $C^{0,\ell}$-smooth function, for every n. Assume that (α, β) is a pair of lower/upper solutions of* (65) *related to the partition $(\mathcal{J}, \mathcal{K})$ of $\{1, \ldots, N\}$, which is strict with respect to the k-th component, for every $k \in \mathcal{K}$. If $\max\{\mu_k\,, \nu_k \,:\, k \in \mathcal{K}\} \leq 1$, then problem* (65) *has a solution x with the following properties:*

(J) *for any $j \in \mathcal{J}$, $\alpha_j(t) \leq x_j(t) \leq \beta_j(t)$, for every $t \in [0, \pi_p]$;*

(K) *for any $k \in \mathcal{K}$, there exist $t_k^1, t_k^2 \in]0, \pi_p[$ such that $x_k(t_k^1) < \alpha_k(t_k^1)$ and $x_k(t_k^2) > \beta_k(t_k^2)$.*

The above result should be compared with [5, Theorem 2.4] (see also [2] and the references therein) where some monotonicity assumptions on the nonlinearities were assumed.

Notice that, since we are dealing with second order differential equations, the *strictness* property asked in the statement can be verified by a standard argument, e.g., simply verifying that a strict inequality holds in (66).

As an illustrative example, we suggest the system

$$\begin{cases} (|x_n'|^{p-2}x_n')' + \mu_n|x_n^+|^{p-2}x_n^+ - \nu_n|x_n^-|^{p-2}x_n^- \\ \qquad\qquad\qquad\qquad = \dfrac{s_n x_n}{1+|x_n|} + \widetilde{h}_n(t, x_1, \ldots, x_N, x_1', \ldots, x_N') \\ x_n(0) = 0 = x_n(\pi_p)\,, \qquad n = 1, \ldots, N\,, \end{cases}$$

where $s_n \in \{-1, +1\}$ and $\|\widetilde{h}_n\|_\infty < 1$ for every n. Moreover, for those components having $s_n = -1$, we ask that $\mu_n \leq 1$ and $\nu_n \leq 1$.

In this example, the pair of lower/upper solutions can be defined by components as follows. The set $\mathcal{J}$ is made of those n for which $s_n = +1$, while the set $\mathcal{K}$ consists of those n with $s_n = -1$. If $n \in \mathcal{J}$, we simply need to choose some sufficiently large constants $\beta_n = -\alpha_n > 0$. If $n \in \mathcal{K}$, we can argue as in [14, Proposition 26], where the case $p = 2$ is treated, in order to find α_n and β_n such that $\alpha_n \not\leq \beta_n$.

Remark 30. A generalization of Theorem 28 can be obtained removing the strictness assumption on one of the components $\kappa \in \mathcal{K}$. Indeed, the above proof can be easily adapted following the ideas in the proof of [11, Theorem 19]. In such a situation, the conclusion **(K)** of the statement must be slightly changed concerning the estimates on the κ-th component: the solution will be such that

$$\alpha_\kappa \not\lll x_\kappa \quad \textit{and} \quad x_\kappa \not\lll \beta_\kappa\,.$$

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