# Two-point boundary value problems for planar systems: a lower and upper solutions approach

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**Abstract**

We extend the theory of lower and upper solutions to planar systems of ordinary differential equations with separated boundary conditions, both in the well-ordered and in the non-well-ordered cases. We are able to deal with general Sturm–Liouville boundary conditions in the well-ordered case, and we analyze the Dirichlet problem in the non-well-ordered case. Our results apply in particular to scalar second order differential equations, including those driven by the mean curvature operator. Higher dimensional systems are also treated, with the same approach.

## 1 Introduction

The method of lower and upper solutions has been developed for more than a century with the aim of studying boundary value problems associated with ordinary and partial differential equations of different types. It has been employed in thousands of papers and it still is one of the most useful tools for localizing solutions and providing information about their behaviour.

Since 1893, Picard [22] introduced lower and upper solutions in order to prove the existence of solutions for separated boundary value problems associated with scalar second order ordinary differential equations. The theory was then developed by Scorza-Dragoni [23] and Nagumo [17] in the thirties, thus reaching its modern form concerning classical solutions. It was then extended to different types of ordinary differential equations [5, 19], difference equations [4], and to some type of partial differential equations: elliptic [1], parabolic [9], and special kinds of hyperbolic equations like the transport equation [3] and the telegraph equation [21]. (The given references are obviously not exhaustive.)

Let us describe a typical situation by considering the Dirichlet problem

$$\begin{cases} x'' = g(t,x,x')\,, \\ x(a) = A\,, \quad x(b) = B\,. \end{cases} \tag{1}$$

A *classical lower solution* for this problem is a $C^2$-function $\alpha : [a,b] \to \mathbb{R}$ such that

$$\begin{cases} \alpha''(t) \geq g(t,\alpha(t),\alpha'(t))\,, \quad \text{for every } t \in [a,b]\,, \\ \alpha(a) \leq A\,, \quad \alpha(b) \leq B\,, \end{cases}$$

while a *classical upper solution* is a $C^2$-function $\beta : [a,b] \to \mathbb{R}$ such that

$$\begin{cases} \beta''(t) \leq g(t,\beta(t),\beta'(t))\,, \quad \text{for every } t \in [a,b]\,, \\ \beta(a) \geq A\,, \quad \beta(b) \geq B\,. \end{cases}$$

In [17], the following theorem was proved.

**Theorem 1.** *Assume the existence of a pair of classical lower/upper solutions $\alpha$, $\beta$ such that $\alpha \leq \beta$. Moreover, let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$|g(t,x,y)| \leq \varphi(|y|)\,, \qquad \text{for every } (t,x,y) \in [a,b] \times [\mu, \mathcal{M}] \times \mathbb{R}\,,$$

*with $\mu = \min \alpha$, $\mathcal{M} = \max \beta$, and*

$$\int_0^{+\infty} \frac{s}{\varphi(s)}\, ds = +\infty\,.$$

*Then, problem* (1) *has a solution such that $\alpha \leq x \leq \beta$.*

The above theorem has been generalized in several directions (see e.g. [7] and the references therein). In this paper we will extend it to planar systems in the spirit of [12, 13], where the periodic case was studied. To this aim, we will provide a definition of lower and upper solutions for a system of the type

$$x' = f(t,x,y)\,, \qquad y' = g(t,x,y)\,, \tag{2}$$

with general boundary conditions of Sturm–Liouville type; roughly speaking the starting point of the solution will lie on a straight line $\ell_S$ and the arrival point on another line $\ell_A$.

When $\alpha \leq \beta$, we say that the lower and upper solutions are *well-ordered.* Without this assumption the statement of Theorem 1 would not be true. For instance, there are no solutions of the problem

$$\begin{cases} x'' = -n^2 x + \sin(nt)\,, \\ x(0) = 0\,, \quad x(\pi) = 0\,, \end{cases}$$

when $n$ is a positive integer. However, when $n \geq 2$, the functions $\alpha(t) = c \sin t$ and $\beta(t) = -c\sin(t)$ are a lower and an upper solution, respectively, taking $c > 0$ sufficiently large; clearly, $\alpha \not\leq \beta$. This is why, in order to recover the existence of solutions when $\alpha$ and $\beta$ are not well-ordered, some nonresonance assumptions with respect to the higher part of the spectrum of the differential operator $-x''$ with Dirichlet boundary conditions are usually imposed.

For simplicity, in the non-well-ordered case $\alpha \not\leq \beta$ we will limit our analysis to nonlinearities that are bounded perturbations of linear ones, and to homogeneous Dirichlet boundary conditions $x(0) = 0 = x(\pi)$. It is well known that, if the associated linear system is non-resonant (i.e., it only has the trivial zero solution), the existence of a solution is an immediate consequence of the Schauder fixed point theorem. This is why we assume, on the contrary, that the associated linear system is at resonance. However, we must avoid the interaction with the higher order eigenvalues, as seen in the above example. So, we found it natural to choose in system (2) the functions

$$f(t,x,y) = y + p(t,x,y)\,, \qquad g(t,x,y) = -x + q(t,x,y)\,,$$

with $p, q$ bounded. In this setting, we will be able to prove the existence of a solution. We believe that there is still some work to be done in order to better understand this situation.

The paper is organized as follows.

In Section 2 we introduce the setting of our problem, together with the main definitions of lower and upper solutions for a system like (2) with starting point on a line $\ell_S$ and arrival point on a line $\ell_A$.

In Section 3 we provide our first existence results in the well-ordered case. The section is divided in four subsections: in the first one we deal with the case when both lines $\ell_S$ and $\ell_A$ are not vertical; in the second subsection, one of the two lines is allowed to be vertical, but not both; in the third one, the case when both lines are vertical is settled. Some applications are given, in particular for an equation involving the mean curvature operator.

In Section 4 we prove an existence result in the non-well-ordered case $\alpha \not\leq \beta$. We use the ideas introduced and developed in the papers [2, 8, 14, 15, 20]: after having constructed an extra lower solution $\hat{\alpha}$ and an extra upper solution $\hat{\beta}$ such that $\hat{\alpha} < \min\{\alpha, \beta\} \leq \max\{\alpha, \beta\} < \hat{\beta}$, the existence follows by topological degree arguments. We also provide an example of application when a condition of Landesman–Lazer type is assumed.

In Section 5 we suggest a possible extension of our results to higher dimensional systems of the type

$$x_j' = f_j(t, x_j, y_j)\,, \qquad y_j' = g_j(t, x_1, \ldots, x_N)\,,$$

with $j = 1, \ldots, N$. Some examples of applications are also suggested.

Finally, we postpone to the Appendix the proof of some technical claims stated in the text.

## 2 Setting of the problem

Let $\ell_S$ and $\ell_A$ be two lines in the plane, the "starting line" and the "arrival line", respectively. Given $a < b$, we are interested in the two-point problem

$$(P) \qquad \begin{cases} x' = f(t, x, y)\,, \quad y' = g(t, x, y)\,, \\ (x(a), y(a)) \in \ell_S\,, \quad (x(b), y(b)) \in \ell_A\,, \end{cases}$$

where $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

We denote the closed half-planes determined by $\ell_S$ and $\ell_A$ as follows:

$H_S^+$ is the one *above* $\ell_S$ or, when $\ell_S$ is vertical, the one to the *left* of it;

$H_S^-$ is the one *below* $\ell_S$ or, when $\ell_S$ is vertical, the one to the *right* of it;

$H_A^+$ is the one *above* $\ell_A$ or, when $\ell_A$ is vertical, the one to the *right* of it;

$H_A^-$ is the one *below* $\ell_A$ or, when $\ell_A$ is vertical, the one to the *left* of it.

Notice that the notations differ for the two lines when they are vertical.

**Definition 2.** *A continuously differentiable function $\alpha : [a, b] \to \mathbb{R}$ is said to be a* lower solution *for problem $(P)$ if there exists a continuously differentiable function $y_\alpha : [a, b] \to \mathbb{R}$ such that, for every $t \in [a, b]$,*

$$\begin{cases} y < y_\alpha(t) \quad \Rightarrow \quad f(t, \alpha(t), y) < \alpha'(t)\,, \\ y > y_\alpha(t) \quad \Rightarrow \quad f(t, \alpha(t), y) > \alpha'(t)\,, \end{cases} \tag{3}$$

$$y'_\alpha(t) \geq g(t, \alpha(t), y_\alpha(t)) \,, \tag{4}$$

*and*

$$(\alpha(a), y_\alpha(a)) \in H_S^+ \,, \quad (\alpha(b), y_\alpha(b)) \in H_A^- \,. \tag{5}$$

**Definition 3.** *A continuously differentiable function $\beta : [a, b] \to \mathbb{R}$ is said to be an* upper solution *for problem $(P)$ if there exists a continuously differentiable function $y_\beta : [a, b] \to \mathbb{R}$ such that, for every $t \in [a, b]$,*

$$\begin{cases} y < y_\beta(t) \quad \Rightarrow \quad f(t, \beta(t), y) < \beta'(t) \,, \\ y > y_\beta(t) \quad \Rightarrow \quad f(t, \beta(t), y) > \beta'(t) \,, \end{cases} \tag{6}$$

$$y'_\beta(t) \leq g(t, \beta(t), y_\beta(t)) \,, \tag{7}$$

*and*

$$(\beta(a), y_\beta(a)) \in H_S^- \,, \quad (\beta(b), y_\beta(b)) \in H_A^+ \,. \tag{8}$$

From (3) we have that

$$\alpha'(t) = f(t, \alpha(t), y_\alpha(t)) \,, \qquad \text{for every } t \in [a, b] \,, \tag{9}$$

and $y_\alpha(t)$ is the only value for which this identity holds. Similarly, from (6) we have

$$\beta'(t) = f(t, \beta(t), y_\beta(t)) \,, \qquad \text{for every } t \in [a, b] \,, \tag{10}$$

and $y_\beta(t)$ is uniquely defined on $[a, b]$ by this identity.

It is well known in the case of scalar second order equations that if a function is at the same time a lower and an upper solution, then it is a solution. Let us write the analogous statement in our situation.

**Proposition 4.** *Let $x : [a, b] \to \mathbb{R}$ be at the same time a lower and an upper solution for problem $(P)$. Then, there exists a function $y : [a, b] \to \mathbb{R}$ such that $(x, y)$ is a solution of problem $(P)$.*

*Proof.* If $x$ is at the same time a lower and an upper solution for problem $(P)$, from (9) and (10) we deduce that the functions $y_\alpha$ and $y_\beta$ given by Definitions 2 and 3 coincide. We set $y = y_\alpha = y_\beta$ and notice that $x'(t) = f(t, x(t), y(t))$, for every $t \in [a, b]$. Moreover, by (4) and (7), we have that $y'(t) = g(t, x(t), y(t))$, for every $t \in [a, b]$. Finally, from (5) and (8) we get $(x(a), y(a)) \in \ell_S$ and $(x(b), y(b)) \in \ell_A$, thus concluding the proof. □

We say that $(\alpha, \beta)$ is a *well-ordered* pair of lower/upper solutions of problem $(P)$ if $\alpha$ and $\beta$ are respectively a lower and an upper solution of problem $(P)$, and $\alpha(t) \leq \beta(t)$ for every $t \in [a, b]$.

## 3 Well-ordered lower/upper solutions

In this section we always assume that $(\alpha, \beta)$ is a well-ordered pair of lower/upper solutions of problem $(P)$. We will distinguish the cases when both lines $\ell_S$ and $\ell_A$ are not vertical, and those when one or both can be vertical.

### 3.1 The non-vertical case

We start assuming that both lines $\ell_S$ and $\ell_A$ are not vertical. Their equations are

$$y = m_S x + q_S \,, \qquad y = m_A x + q_A \,, \tag{11}$$

respectively. Here is our first existence result.

**Theorem 5.** *Assume the existence of a well-ordered pair $(\alpha, \beta)$ of lower/upper solutions of problem $(P)$, with the lines $\ell_S$ and $\ell_A$ having equations (11). Set $\mu = \min \alpha$ and $\mathcal{M} = \max \beta$, with $\mu < \mathcal{M}$. Let there exist two continuously differentiable functions $\gamma_\pm : [\mu, \mathcal{M}] \to \mathbb{R}$ such that, for every $t \in [a, b]$ and $x \in [\alpha(t), \beta(t)]$,*

$$\gamma_-(x) < \min\{y_\alpha(t), y_\beta(t)\} \le \max\{y_\alpha(t), y_\beta(t)\} < \gamma_+(x) \,, \tag{12}$$

*and*

$$g\big(t, x, \gamma_+(x)\big) > f\big(t, x, \gamma_+(x)\big)\gamma_+'(x) \,, \tag{13}$$

$$g\big(t, x, \gamma_-(x)\big) < f\big(t, x, \gamma_-(x)\big)\gamma_-'(x) \,. \tag{14}$$

*Assume moreover that*

$$\gamma_-(x) < m_A \xi + q_A < \gamma_+(x) \,, \quad \text{for every } x, \xi \in [\mu, \mathcal{M}] \,. \tag{15}$$

*Then, there exists a solution of problem $(P)$ such that*

$$\alpha(t) \le x(t) \le \beta(t) \text{ and } \gamma_-(x(t)) < y(t) < \gamma_+(x(t)) \,, \quad \text{for every } t \in [a, b] \,. \tag{16}$$

*Proof.* We are going to consider an auxiliary problem obtained by modifying both the vector field and the boundary conditions. In order to modify $f$ and $g$, we introduce the functions

$$\zeta(s; p, q) = \begin{cases} p \,, & \text{if } s < p \,, \\ s \,, & \text{if } p \le s \le q \,, \\ q \,, & \text{if } s > q \,, \end{cases} \tag{17}$$

$$e(s; p, q) = s - \zeta(s; p, q) = \begin{cases} s - p \,, & \text{if } s < p \,, \\ 0 \,, & \text{if } p \le s \le q \,, \\ s - q \,, & \text{if } s > q \,. \end{cases} \tag{18}$$

Let $D > 0$ be such that

$$|\gamma_\pm(x)| \le D \,, \quad \text{for every } x \in [\mu, \mathcal{M}] \,, \tag{19}$$

and define

$$\tilde{f}(t, x, y) = f\Big(t, \zeta\big(x; \alpha(t), \beta(t)\big) \,, \zeta\big(y; -D, D\big)\Big) + e\big(y; -D, D\big) \,,$$

$$\tilde{g}(t, x, y) = g\Big(t \,, \zeta\big(x; \alpha(t), \beta(t)\big) \,, \zeta\big(y; -D, D\big)\Big) + e\big(x; \alpha(t), \beta(t)\big) \,.$$

We now modify the starting line. We introduce the polygonal line $\widetilde{\ell}_S$ as follows: if $m_S \ge 0$, then $\widetilde{\ell}_S = \ell_S$ ; otherwise, if $m_S < 0$, then

$$\widetilde{\ell}_S = \{(x, y) \in \mathbb{R}^2 : y = m_S \zeta(x; \alpha(a), \beta(a)) + q_S\} \,.$$

Similarly, we introduce the polygonal line $\widetilde{\ell}_A$ as follows: if $m_A \leq 0$, then $\widetilde{\ell}_A = \ell_A$; otherwise, if $m_A > 0$, then

$$\widetilde{\ell}_A = \{(x,y) \in \mathbb{R}^2 : y = m_A \zeta(x; \alpha(b), \beta(b)) + q_A\}.$$

We consider the problem

$$(\widetilde{P}) \qquad \begin{cases} x' = \tilde{f}(t,x,y), \quad y' = \tilde{g}(t,x,y), \\ (x(a), y(a)) \in \widetilde{\ell}_S, \quad (x(b), y(b)) \in \widetilde{\ell}_A. \end{cases}$$

We will prove that problem $(\widetilde{P})$ has a solution, which satisfies (16). Hence, since the vector field and the starting and arrival lines have been modified only outside the region identified by (16), this solution of $(\widetilde{P})$ is indeed a solution of $(P)$.

Since we are going to prove the existence of a solution of $(\widetilde{P})$ by the use of degree theory, we need to construct a suitable homotopy.

We define, for every $\lambda \in [0,1]$, the polygonal lines $\ell_S^\lambda$ and $\ell_A^\lambda$ as follows. If $m_S \geq 0$, then $\ell_S^\lambda = \widetilde{\ell}_S = \ell_S$. Otherwise, if $m_S < 0$, let $\mathcal{Z}_S$ be the segment joining $(\alpha(a), y_\alpha(a))$ and $(\beta(a), y_\beta(a))$, possibly reduced to a single point, and let $P_S = (x_S, y_S)$ be an intersection point of $\mathcal{Z}_S$ with $\ell_S$ (there could be more than one); we set

$$\ell_S^\lambda = \Big\{(x,y) \in \mathbb{R}^2 : y = (1-\lambda)\Big(m_S \zeta(x; \alpha(a), \beta(a)) + q_S\Big) + \lambda y_S\Big\}.$$

Similarly, if $m_A \leq 0$, then $\ell_A^\lambda = \widetilde{\ell}_A = \ell_A$; otherwise, if $m_A > 0$, let $\mathcal{Z}_A$ be the segment joining $(\alpha(b), y_\alpha(b))$ and $(\beta(b), y_\beta(b))$, and let $P_A = (x_A, y_A)$ be an intersection point of $\mathcal{Z}_A$ with $\ell_A$; we choose

$$\ell_A^\lambda = \Big\{(x,y) \in \mathbb{R}^2 : y = (1-\lambda)\Big(m_A \zeta(x; \alpha(b), \beta(b)) + q_A\Big) + \lambda y_A\Big\}.$$

We now consider the problem

$$(\widetilde{P}_\lambda) \qquad \begin{cases} x' = \tilde{f}(t,x,y), \quad y' = \tilde{g}(t,x,y), \\ (x(a), y(a)) \in \ell_S^\lambda, \quad (x(b), y(b)) \in \ell_A^\lambda, \end{cases}$$

with $\lambda \in [0,1]$. Notice that $(\widetilde{P}_0)$ coincides with $(\widetilde{P})$.

**Claim 1.** All the solutions of $(\widetilde{P}_\lambda)$ satisfy (16).

We postpone the proof of Claim 1 to Section A.1.

Assuming that Claim 1 holds true, let us consider the problem $(\widetilde{P}_1)$. We are going to construct a second homotopy which transforms it into a linear problem whose only solution is the trivial one $(x,y) = (0,0)$.

Using this time $\sigma \in [0,1]$ as the homotopy parameter, we consider the problem

$$(Q_\sigma) \qquad \begin{cases} x' = (1-\sigma)\tilde{f}(t,x,y) + \sigma y, \quad y' = (1-\sigma)\tilde{g}(t,x,y) + \sigma x, \\ (x(a), y(a)) \in \ell_S^1(\sigma), \quad (x(b), y(b)) \in \ell_A^1(\sigma), \end{cases}$$

where the boundary conditions are constructed as follows. Recalling that the equation of the line $\ell_S^1$ is $y = m_S^1 x + q_S^1$, with

$$m_S^1 = \begin{cases} m_S\,, & \text{if } m_S \geq 0\,, \\ 0\,, & \text{if } m_S < 0\,, \end{cases} \qquad q_S^1 = \begin{cases} q_S\,, & \text{if } m_S \geq 0\,, \\ y_S\,, & \text{if } m_S < 0\,, \end{cases}$$

we define $\ell_S^1(\sigma)$ as the line of equation

$$y = Y_S^\sigma(x) := m_S^1 x + (1-\sigma) q_S^1\,.$$

Similarly, we define $\ell_A^1(\sigma)$ as the line of equation

$$y = Y_A^\sigma(x) := m_A^1 x + (1-\sigma) q_A^1\,.$$

Notice that $(Q_0)$ is the same as $(\widetilde{P}_1)$, while $(Q_1)$ is a linear problem whose only solution is the trivial one.

**Claim 2.** There is a $R > 0$ such that every solution $u = (x, y)$ of $(Q_\sigma)$ satisfies $\|u\|_\infty < R$.

We postpone the proof of Claim 2 to Section A.2.

Let us introduce our functional setting for the problem $(Q_\sigma)$. We define the linear operator

$$L : C^1([a,b],\mathbb{R}^2) \to C([a,b],\mathbb{R}^2) \times \mathbb{R} \times \mathbb{R}\,, \quad L\begin{pmatrix} x \\ y \end{pmatrix} = \left( \begin{pmatrix} x' \\ y' \end{pmatrix}, y(a), y(b) \right),$$

and the nonlinear operator

$$N_\sigma : C([a,b],\mathbb{R}^2) \to C([a,b],\mathbb{R}^2) \times \mathbb{R} \times \mathbb{R}\,,$$
$$N_\sigma \begin{pmatrix} x \\ y \end{pmatrix}(t) = \left( \begin{pmatrix} f_\sigma(t, x(t), y(t)) \\ g_\sigma(t, x(t), y(t)) \end{pmatrix}, Y_S^\sigma(x(a))\,, Y_A^\sigma(x(b)) \right),$$

where

$$f_\sigma(t,x,y) = (1-\sigma)\tilde{f}(t,x,y) + \sigma y\,, \qquad g_\sigma(t,x,y) = (1-\sigma)\tilde{g}(t,x,y) + \sigma x\,.$$

Setting $u = (x, y)$, problem $(Q_\sigma)$ is thus equivalent to

$$Lu = N_\sigma u\,.$$

By Mawhin's Coincidence Degree [16] theory, the operator $N_\sigma$ is $L$-completely continuous, and by Claim 2 the degree $D_L(L - N_\sigma, B_R)$ is well defined and its value is independent of $\sigma \in [0,1]$. Since $(Q_1)$ is linear and has only the trivial zero solution, we have that

$$D_L(L - N_0, B_R) = D_L(L - N_1, B_R) = \pm 1\,.$$

We now repeat the same procedure for the problem $(\widetilde{P}_\lambda)$. By Claim 1, we can enlarge the radius $R$, if necessary, so that any solution $u = (x, y)$ of $(\widetilde{P}_\lambda)$ satisfies $\|u\|_\infty < R$. Let us define

$$\widetilde{N}_\lambda : C([a,b],\mathbb{R}^2) \to C([a,b],\mathbb{R}^2) \times \mathbb{R} \times \mathbb{R},$$
$$\widetilde{N}_\lambda \begin{pmatrix} x \\ y \end{pmatrix}(t) = \left( \begin{pmatrix} \tilde{f}(t, x(t), y(t)) \\ \tilde{g}(t, x(t), y(t)) \end{pmatrix}, F_S^\lambda(x(a))\,, F_A^\lambda(x(b)) \right),$$

where

$$F_S^\lambda(x) = \begin{cases} m_S x + q_S\,, & \text{if } m_S \geq 0\,,\\ (1-\lambda)\Big(m_S\zeta(x;\alpha(a),\beta(a)) + q_S\Big) + \lambda y_S\,, & \text{if } m_S < 0\,, \end{cases}$$

and

$$F_A^\lambda(x) = \begin{cases} m_A x + q_A\,, & \text{if } m_A \leq 0\,,\\ (1-\lambda)\Big(m_A\zeta(x;\alpha(b),\beta(b)) + q_A\Big) + \lambda y_A\,, & \text{if } m_A > 0\,. \end{cases}$$

Problem $(\widetilde{P}_\lambda)$ is thus equivalent to

$$Lu = \widetilde{N}_\lambda u\,,$$

and we can conclude that the coincidence degree $D_L(L-\widetilde{N}_\lambda, B_R)$ is well defined and independent of $\lambda \in [0,1]$, hence

$$D_L(L-\widetilde{N}_0, B_R) = D_L(L-\widetilde{N}_1, B_R) = D_L(L-N_0, B_R) = \pm 1\,.$$

Therefore problem $(\widetilde{P}_0)$, which is the same as $(\widetilde{P})$, has a solution. By Claim 1, this solution solves $(P)$, since it satisfies (16). $\square$

## 3.2 Some extensions of Theorem 5

We first provide a variant of Theorem 5, concerning the curves $\gamma_\pm$.

**Theorem 6.** *Assume the existence of a well-ordered pair $(\alpha,\beta)$ of lower/upper solutions of problem $(P)$, with the lines $\ell_S$ and $\ell_A$ having equations* (11)*. Set $\mu = \min\alpha$ and $\mathcal{M} = \max\beta$, with $\mu < \mathcal{M}$. Let there exist two continuously differentiable functions $\widehat{\gamma}_\pm : [\mu,\mathcal{M}] \to \mathbb{R}$ such that, for every $t \in [a,b]$ and $x \in [\alpha(t),\beta(t)]$,*

$$\widehat{\gamma}_-(x) < \min\{y_\alpha(t), y_\beta(t)\} \leq \max\{y_\alpha(t), y_\beta(t)\} < \widehat{\gamma}_+(x)\,, \tag{20}$$

*and*

$$g\big(t,x,\widehat{\gamma}_+(x)\big) < f\big(t,x,\widehat{\gamma}_+(x)\big)\widehat{\gamma}_+'(x)\,, \tag{21}$$

$$g\big(t,x,\widehat{\gamma}_-(x)\big) > f\big(t,x,\widehat{\gamma}_-(x)\big)\widehat{\gamma}_-'(x)\,. \tag{22}$$

*Assume moreover that*

$$\widehat{\gamma}_-(x) < m_S\xi + q_S < \widehat{\gamma}_+(x)\,, \qquad \textit{for every } x,\xi \in [\mu,\mathcal{M}]\,. \tag{23}$$

*Then, there exists a solution of problem $(P)$ such that*

$$\alpha(t) \leq x(t) \leq \beta(t) \ \textit{and}\ \ \widehat{\gamma}_-(x(t)) < y(t) < \widehat{\gamma}_+(x(t))\,, \quad \textit{for every } t \in [a,b]. \tag{24}$$

*Proof.* The change of variables $u(t) = \mu + \mathcal{M} - x(a+b-t)$, $v(t) = y(a+b-t)$ transforms problem $(P)$ into

$$(\check{P}) \qquad \begin{cases} u' = \check{f}(t,u,v)\,, \quad v' = \check{g}(t,u,v)\,,\\ (u(a),v(a)) \in \check{\ell}_S\,, \quad (u(b),v(b)) \in \check{\ell}_A\,, \end{cases}$$

where

$$\check{f}(t,u,v) = f(a+b-t, \mu+\mathcal{M}-u, v)\,,$$
$$\check{g}(t,u,v) = -g(a+b-t, \mu+\mathcal{M}-u, v)\,,$$

and

$$\check{\ell}_S = \{(x,y) \in \mathbb{R}^2 : (\mu+\mathcal{M}-x, y) \in \ell_A\}\,,$$
$$\check{\ell}_A = \{(x,y) \in \mathbb{R}^2 : (\mu+\mathcal{M}-x, y) \in \ell_S\}\,.$$

Setting

$$\check{\alpha}(t) = \mu+\mathcal{M}-\beta(a+b-t)\,, \quad y_{\check{\alpha}}(t) = y_\beta(a+b-t)\,,$$

and

$$\check{\beta}(t) = \mu+\mathcal{M}-\alpha(a+b-t)\,, \quad y_{\check{\beta}}(t) = y_\alpha(a+b-t)\,,$$

we have a well-ordered pair of lower/upper solutions $(\check{\alpha}, \check{\beta})$ for problem $(\check{P})$. Setting

$$\gamma_\pm(x) = \widehat{\gamma}_\pm(\mu+\mathcal{M}-x)\,,$$

we recover the curves satisfying the assumptions of Theorem 5, which thus provides us the conclusion. □

**Remark 7.** In Theorems 5 and 6 the curves $\gamma_\pm$ and $\widehat{\gamma}_\pm$ can also be chosen with a different coupling. In Theorem 5 we had $(\gamma_-\,,\gamma_+)$, while in Theorem 6 we have taken $(\widehat{\gamma}_-\,,\widehat{\gamma}_+)$. However, we can also state another theorem with the coupling $(\gamma_-\,,\widehat{\gamma}_+)$, and a last theorem with $(\widehat{\gamma}_-\,,\gamma_+)$. We will not write the statements, for briefness.

We now extend Theorem 5 to the case when the equations of $\ell_S$ and $\ell_A$ are

$$x = x_S\,, \qquad y = m_A x + q_A\,, \tag{25}$$

respectively. Here is the precise statement.

**Theorem 8.** *Assume the existence of a well-ordered pair $(\alpha,\beta)$ of lower/upper solutions of problem $(P)$, with the lines $\ell_S$ and $\ell_A$ having equations* (25)*. Set $\mu = \min\alpha$ and $\mathcal{M} = \max\beta$, with $\mu < \mathcal{M}$. Let there exist two continuously differentiable functions $\gamma_\pm : [\mu,\mathcal{M}] \to \mathbb{R}$ such that* (12), (13) *and* (14) *hold, for every $t \in [a,b]$ and $x \in [\alpha(t),\beta(t)]$. Assume moreover that* (15) *holds. Then, there exists a solution of problem $(P)$ satisfying* (16).

*Proof.* It is almost exactly the same as the proof of Theorem 5, the only difference lying in the definition of the vertical line $\ell_S^1(\sigma)$, whose equation now is $x = (1-\sigma)x_S$. □

Now also Theorem 6 can be extended to the case when the equations of $\ell_S$ and $\ell_A$ are

$$y = m_S x + q_S\,, \qquad x = x_A\,, \tag{26}$$

respectively. Here is our existence result.

**Theorem 9.** *Assume the existence of a well-ordered pair $(\alpha, \beta)$ of lower/upper solutions of problem $(P)$, with the lines $\ell_S$ and $\ell_A$ having equations* (26)*. Set $\mu = \min \alpha$ and $\mathcal{M} = \max \beta$, with $\mu < \mathcal{M}$. Let there exist two continuously differentiable functions $\widehat{\gamma}_\pm : [\mu, \mathcal{M}] \to \mathbb{R}$ such that* (20), (21) *and* (22) *hold, for every $t \in [a, b]$ and $x \in [\alpha(t), \beta(t)]$. Assume moreover that* (23) *holds. Then, there exists a solution of problem $(P)$ satisfying* (24).

*Proof.* By the change of variables in the proof of Theorem 6, the assumptions of Theorem 8 are verified, and the result follows. □

As a consequence of the above results, we have the following.

**Corollary 10.** *Assume the existence of a well-ordered pair $(\alpha, \beta)$ of lower/upper solutions of problem $(P)$, where $\ell_S$ and $\ell_A$ are not both vertical lines. Set $\mu = \min \alpha$ and $\mathcal{M} = \max \beta$, with $\mu < \mathcal{M}$. Let the following assumptions hold:*

**(A1)** *there are a constant $d > 0$ and two continuous functions $f_+ : [d, +\infty[ \to \mathbb{R}$ and $f_- : ]-\infty, -d] \to \mathbb{R}$ such that*

$$\begin{cases} y \geq d & \Rightarrow \quad f(t,x,y) \geq f_+(y) > 0\,, \\ y \leq -d & \Rightarrow \quad f(t,x,y) \leq f_-(y) < 0\,, \end{cases}$$
$$\text{for every } (t,x) \in [a,b] \times [\mu, \mathcal{M}]\,;$$

**(A2)** *there is a positive continuous function $\varphi : [0, +\infty[ \to \mathbb{R}$ such that*

$$|g(t,x,y)| \leq \varphi(|y|)\,, \qquad \text{for every } (t,x,y) \in [a,b] \times [\mu, \mathcal{M}] \times \mathbb{R}\,;$$

**(A3)** *the above functions are such that*

$$\int_d^{+\infty} \frac{f_+(s)}{\varphi(s)}\,ds = +\infty\,, \qquad \int_{-\infty}^{-d} \frac{f_-(s)}{\varphi(|s|)}\,ds = -\infty\,.$$

*Then, there exists a solution of problem $(P)$ such that $\alpha \leq x \leq \beta$.*

*Proof.* The existence of the curves $\gamma_\pm$ and $\widehat{\gamma}_\pm$ follows from [12, Lemma 15] (see also [13, Theorem 3.1]), so that one of the previous theorems apply. □

## 3.3 Both vertical lines

We now consider the case when both lines $\ell_S$ and $\ell_A$ are vertical, having equations

$$x = x_S\,, \qquad x = x_A\,, \tag{27}$$

respectively. Here is our result.

**Theorem 11.** *Assume the existence of a well-ordered pair $(\alpha, \beta)$ of lower/upper solutions of problem $(P)$, with the lines $\ell_S$ and $\ell_A$ having equations* (27)*. Set $\mu = \min \alpha$ and $\mathcal{M} = \max \beta$, with $\mu < \mathcal{M}$. Let there exist four continuously differentiable functions $\gamma_\pm, \widehat{\gamma}_\pm : [\mu, \mathcal{M}] \to \mathbb{R}$ such that* (12), (13), (14) *and* (20), (21), (22) *hold, for every* $t \in [a, b]$ *and* $x \in [\alpha(t), \beta(t)]$. *Assume moreover that, for every* $t \in [a, b]$ *and* $x \in [x_A, x_S]$,

$$y \geq \min\{\gamma_+(x), \widehat{\gamma}_+(x)\} \quad \Rightarrow \quad f(t, x, y) > \frac{x_A - x_S}{b - a}, \tag{28}$$

$$y \leq \max\{\gamma_-(x), \widehat{\gamma}_-(x)\} \quad \Rightarrow \quad f(t, x, y) < \frac{x_A - x_S}{b - a}. \tag{29}$$

*Then, there exists a solution of problem* $(P)$ *such that*

$$\alpha(t) \leq x(t) \leq \beta(t), \tag{30}$$

*and*

$$\min\{\gamma_-(x(t)), \widehat{\gamma}_-(x(t))\} < y(t) < \max\{\gamma_+(x(t)), \widehat{\gamma}_+(x(t))\}, \tag{31}$$

*for every* $t \in [a, b]$.

*Proof.* Following the lines of the proof of Theorem 5, we introduce problem $(\widetilde{P})$ with $\widetilde{\ell}_S = \ell_S$ and $\widetilde{\ell}_A = \ell_A$, with (19) replaced by

$$\max\{|\gamma_\pm(x)|, |\widehat{\gamma}_\pm(x)|\} \leq D, \quad \text{for every } x \in [\mu, \mathcal{M}]. \tag{32}$$

It is not necessary in this situation to modify problem $(\widetilde{P})$ by introducing the family of problems $(\widetilde{P}_\lambda)$, and Claim 1 is replaced by the following.

**Claim 3.** All the solutions of $(\widetilde{P})$ satisfy (30) and (31), for every $t \in [a, b]$.

The proof of this claim is provided in Section A.3. We then introduce the family of problems $(Q_\sigma)$, where $\ell^1_S(\sigma)$ and $\ell^1_A(\sigma)$ have equations $x = (1-\sigma)x_S$ and $x = (1-\sigma)x_A$, respectively, and similarly prove the a priori bound given by Claim 2. The topological degree argument completes the proof. □

## 3.4 Some corollaries, in the well-ordered case

As a consequence of Theorems 5, 8, 9 and 11 we have the following result, which extends [7, Theorem II,1.3] where the case $f(t, x, y) = y$ was considered.

**Corollary 12.** *Assume the existence of a well-ordered pair* $(\alpha, \beta)$ *of lower/upper solutions of problem* $(P)$, *where* $\ell_S$ *and* $\ell_A$ *can be* any two lines *in the plane. Set* $\mu = \min \alpha$ *and* $\mathcal{M} = \max \beta$, *with* $\mu < \mathcal{M}$. *Let the assumptions A1, A2 and A3 of Corollary 10 hold, with the further requirement that*

$$\liminf_{y \to +\infty} f_+(y) > \frac{\mathcal{M} - \mu}{b - a}, \qquad \limsup_{y \to -\infty} f_-(y) < \frac{\mu - \mathcal{M}}{b - a}. \tag{33}$$

*Then, there exists a solution of problem* $(P)$ *such that* $\alpha \leq x \leq \beta$.

*Proof.* As observed in the proof of Corollary 10, the existence of the curves $\gamma_\pm$ and $\widehat{\gamma}_\pm$ follows from [12, Lemma 15] (see also [13, Theorem 3.1]). On the other hand, assumption (33) guarantees that (28) and (29) hold. The result then follows from Corollary 10 and Theorem 11. □

Let us provide a simpler version of the above result in the particular case when the starting and arrival lines are of the type

$$y = m_S x\,, \text{ with } m_S \geq 0\,, \quad \text{and} \quad y = m_A x\,, \text{ with } m_A \leq 0\,, \tag{34}$$

possibly including the cases when one or both are vertical, which will be identified assuming $m_S = +\infty$ or $m_A = -\infty$.

**Corollary 13.** *Assume the existence of two constants $\alpha, \beta$, with $\alpha < 0 < \beta$, such that*

$$f(t,\alpha,y)\,y > 0\,, \quad f(t,\beta,y)\,y > 0\,, \quad \textit{for every } t \in [a,b] \textit{ and } y \neq 0\,,$$

*and*

$$g(t,\alpha,0) \leq 0 \leq g(t,\beta,0)\,, \quad \textit{for every } t \in [a,b]\,.$$

*Moreover, let there exist two constants $r \geq 0$, $c > 0$ and a positive continuous function $\varphi : [0,+\infty[\to \mathbb{R}$ such that*

$$\liminf_{y\to\pm\infty} \frac{|f(t,x,y)|}{|y|^r} \geq c\,, \quad \textit{uniformly in } (t,x) \in [a,b] \times \mathbb{R}\,,$$

$$|g(t,x,y)| \leq \varphi(|y|)\,, \quad \textit{for every } (t,x,y) \in [a,b] \times [\alpha,\beta] \times \mathbb{R}\,,$$

*and*

$$\int_1^{+\infty} \frac{s^r}{\varphi(s)}\,ds = +\infty\,.$$

*Then, there exists a solution of problem $(P)$, when $\ell_S$ and $\ell_A$ are given by* (34)*, including the cases $m_S = +\infty$ or $m_A = -\infty$, such that $\alpha \leq x \leq \beta$.*

*Proof.* The constant functions $\alpha$ and $\beta$ are a well-ordered pair of lower/upper solutions, with corresponding functions $y_\alpha(t) = y_\beta(t) = 0$. Then, we can apply Corollary 12 to conclude. □

As an illustrative example of application, we propose the following:

$$\begin{cases} x' = F(t,x,y)\,|y|^{r-1}y\,, \\ y' = x^3 + G(t,x,y)|y|^{q-1}y + e(t)\,, \\ x(a) = 0 = x(b)\,, \end{cases}$$

where $q$ and $r$ are nonnegative constants with $q \leq r+1$, all functions $e, F, G$ being continuous, with

$$F(t,x,y) \geq c > 0\,, \qquad |G(t,x,y)| \leq C\,,$$

for every $(t,x,y) \in [a,b] \times \mathbb{R}^2$. One easily verifies that all the assumptions of Corollary 13 are satisfied, taking the constants $\alpha < 0 < \beta$, with $|\alpha|$ and $\beta$ sufficiently large. Hence, our problem has a solution.

### 3.5 The mean curvature equation

Consider now a problem of the type

$$\begin{cases} \big(\phi(x')\big)' = h(t,x,x')\,, \\ x(a) = x_S\,, \quad x(b) = x_A\,, \end{cases} \tag{35}$$

where $\phi : \mathbb{R} \to ]-1,1[$ is an increasing odd homeomorphism. Problem (35) is equivalent to problem $(P)$, with both vertical lines $\ell_S$ and $\ell_A$, taking

$$f(t,x,y) = \phi^{-1}(y)\,, \qquad g(t,x,y) = h\big(t,x,\phi^{-1}(y)\big)\,.$$

Notice however that these functions are now only defined on $[a,b]\times\mathbb{R}\times\,]-1,1[\,$.

**Corollary 14.** *Assume the existence of a well-ordered pair $(\alpha,\beta)$ of lower/upper solutions of problem (40), with $y_\alpha\,, y_\beta : [a,b] \to ]-1,1[\,$. Let there exist a constant $C > 0$ such that*

$$|h(t,x,z)| \le C\,, \qquad \text{for every } (t,x,z) \in [a,b]\times\mathbb{R}^2, \tag{36}$$

*and*

$$\phi\Big(\frac{|x_A - x_S|}{b-a}\Big) + C(b-a) < 1\,. \tag{37}$$

*Then, there exists at least one solution of problem (40) such that $\alpha \le x \le \beta$.*

*Proof.* For any $c \in ]0,1[$ we define the functions $f_c : \mathbb{R}\to\mathbb{R}$ and $g_c : [a,b]\times\mathbb{R}^2 \to \mathbb{R}$ as

$$f_c(y) = \begin{cases} \phi^{-1}(-c) + y + c\,, & \text{if } y < -c\,, \\ \phi^{-1}(y)\,, & \text{if } |y| \le c\,, \\ \phi^{-1}(c) + y - c\,, & \text{if } y > c\,, \end{cases}$$

$$g_c(t,x,y) = \begin{cases} g(t,x,-c)\,, & \text{if } y < -c\,, \\ g(t,x,y)\,, & \text{if } |y| \le c\,, \\ g(t,x,c)\,, & \text{if } y > c\,, \end{cases}$$

and we consider the problem

$$\begin{cases} x' = f_c(y)\,, \qquad y' = g_c(t,x,y)\,, \\ x(a) = x_S\,, \quad x(b) = x_A\,. \end{cases} \tag{38}$$

It is easy to see that, when $c$ is sufficiently near to 1, all the assumptions of Corollary 12 hold, so that problem (38) has a solution $(x,y)$ such that $\alpha \le x \le \beta$. We now show that, if $c$ satisfies

$$\phi\Big(\frac{|x_A - x_S|}{b-a}\Big) + C(b-a) < c < 1\,, \tag{39}$$

then $|y(t)| < c$ for every $t$, implying that $(x,y)$ is indeed a solution of problem (35). By Lagrange's Mean Value Theorem, there is a $\xi \in ]a,b[$ such that

$$x'(\xi) = \frac{x_A - x_S}{b-a}\,.$$

By (39), since $\phi\big(\frac{|x_A - x_S|}{b-a}\big) < c$, we have $x'(\xi) = \phi^{-1}(y(\xi))$ and hence, using also (36),

$$\begin{aligned} |y(t)| &= \Big| y(\xi) + \int_\xi^t y'(s)\, ds \Big| \\ &\le |\phi(x'(\xi))| + \Big| \int_\xi^t g_c(s, x(s), y(s))\, ds \Big| \\ &\le \phi\Big( \frac{|x_A - x_S|}{b-a} \Big) + C(b-a) < c\,, \end{aligned}$$

thus ending the proof. □

**Remark 15.** If $x_A = x_S$, condition (37) becomes

$$2C < \widehat{\lambda}_1 := \frac{2}{b-a}\,.$$

Note that $\widehat{\lambda}_1$ is the first eigenvalue of the minus 1-Laplace operator $-(\mathrm{sgn}(x'))'$ with homogeneous Dirichlet boundary conditions on $[a, b]$, see [6].

Corollary 14 applies in particular to the Dirichlet problem associated with the *mean curvature equation*

$$\begin{cases} \Big( \dfrac{x'}{\sqrt{1 + (x')^2}} \Big)' = h(t, x, x')\,, \\ x(a) = x_S\,, \quad x(b) = x_A\,. \end{cases} \tag{40}$$

It can be worth noticing that, if the function $h$ in (40) is constant, say $h(t, x, z) = C > 0$, then a solution of the differential equation $(\phi(x'))' = C$ is such that

$$x'(t) = \frac{Ct + K}{\sqrt{1 - (Ct + K)^2}}\,,$$

for some constant $K \in \mathbb{R}$, hence

$$\begin{aligned} x(b) - x(a) &= \int_a^b \frac{Ct + K}{\sqrt{1 - (Ct + K)^2}}\, dt \\ &= \frac{1}{C}\Big( \sqrt{1 - (Ca + K)^2} - \sqrt{1 - (Cb + K)^2} \Big). \end{aligned}$$

The function $K \mapsto \frac{1}{C}\Big( \sqrt{1 - (Ca + K)^2} - \sqrt{1 - (Cb + K)^2} \Big)$ is strictly increasing on its domain $[-1 - Ca, 1 - Cb]$, taking values in $[-\bar{c}, \bar{c}]$, where

$$\bar{c} = \sqrt{\frac{b-a}{C}\big(2 - C(b-a)\big)}\,.$$

Then, a necessary and sufficient condition for the existence of a solution of (40) with $h(t, x, x') = C > 0$ is

$$|x_A - x_S| \le \bar{c}\,,$$

which is equivalent to

$$\bar{c}\,\frac{|x_A - x_S|}{b-a} + b - a \le \frac{2}{C}\,. \tag{41}$$

The same is true if $h(t,x,x') = -C$, with $C > 0$. On the other hand, condition (37) is equivalent to

$$\bar{c}\,\frac{|x_A - x_S|}{b-a} + b - a < \frac{1}{C}\,.$$

The comparison with (41) naturally leads to the question whether our condition (37) could be improved. We propose this as an open problem.

**Remark 16.** The result in Corollary 14 should be compared with [18, Theorem 1.2], where the existence of a *bounded variation solution* was proved for the mean curvature equation. It was also shown in [18, Example 1.2] that, if $h(t,x,x') = -C$ with $C > \widehat{\lambda}_1$, then the problem does not have any bounded variation solution.

# 4 Non-well-ordered lower/upper solutions

In this section we study the case when the lower and upper solutions are such that $\alpha \not\leq \beta$. For simplicity, we only deal with the following homogeneous Dirichlet problem

$$(P_{Dir}) \qquad \begin{cases} x' = y + p(t,x)\,, \quad y' = -x + q(t,x,y)\,, \\ x(0) = 0 = x(\pi)\,, \end{cases}$$

where $p : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function and $q : [0,\pi] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, both functions being uniformly bounded. We will discuss in Section 4.2 the possibility of letting the function $p$ depend also on $y$.

We denote by $\varphi_1(t)$ the function $\sin t$, which is the first (positive) eigenfunction of the corresponding autonomous problem $x'' + x = 0$. For any continuous function $\varphi : [0,\pi] \to \mathbb{R}$, we will write $\varphi \gg 0$ if there exists an $\epsilon > 0$ such that

$$\varphi(t) \geq \epsilon\varphi_1(t)\,, \qquad \text{for every } t \in [0,\pi]\,,$$

and we write $\varphi \gg \psi$ (or $\psi \ll \varphi$) if $\varphi - \psi \gg 0$.

## 4.1 The existence result

Let us state our result in the non-well-ordered case.

**Theorem 17.** *Assume the existence of a non-well-ordered pair $(\alpha,\beta)$ of lower/upper solutions of problem $(P_{Dir})$, where $p(t,x)$ is a locally Lipschitz continuous function, $q(t,x,y)$ is continuous, and both functions are uniformly bounded. Then there exists a solution of problem $(P_{Dir})$ such that*

$$\alpha \not\ll x \quad \textit{and} \quad x \not\ll \beta\,.$$

*Proof.* For every $r \geq 1$, let $\phi_r : \mathbb{R} \to [0,1]$ be a continuously differentiable function such that

$$\phi_r(s) = 1 \quad \text{if } |s| \leq r\,, \qquad \phi_r(s) = 0 \quad \text{if } |s| \geq 2r\,.$$

for every $s \in \mathbb{R}$. We introduce the modified problems

$$(P_r) \qquad \begin{cases} x' = y + p_r(t,x)\,, \quad y' = -x + q_r(t,x,y)\,, \\ x(0) = 0 = x(\pi)\,, \end{cases}$$

where

$$\begin{aligned} p_r(t,x) &= \phi_r(x)p(t,x)\,, \\ q_r(t,x,y) &= \frac{1}{r}\,x + \phi_r(x)\phi_r(y)\Big(q(t,x,y) - \frac{1}{r}\,x\Big)\,. \end{aligned}$$

Moreover, we write $p_\infty(t,x) = p(t,x)$ and $q_\infty(t,x,y) = q(t,x,y)$.

**Claim 4.** There exists $R \geq 1$ such that, if $u = (x,y)$ is a solution of problem $(P_r)$ satisfying $\alpha \not\ll x$ and $x \not\ll \beta$, with $r \in [R, \infty]$, then $\|u\|_\infty < R$.

We postpone the proof of this claim to Section A.4. By Claim 4, in particular, if $u$ is a solution of $(P_r)$ satisfying $\alpha \not\ll x$ and $x \not\ll \beta$, with $r \geq R$, then it is a solution of $(P_{Dir})$.

We fix $r \geq R$ such that

$$r > \max\{\|\alpha\|_\infty, \|\beta\|_\infty, \|y_\alpha\|_\infty, \|y_\beta\|_\infty\}\,,$$

so that $\alpha$ and $\beta$ are lower/upper solutions of $(P_r)$, as well.

Let us introduce our functional setting for the problem $(P_r)$. We use the notation $C^{0,1}([0,\pi])$ for the space of Lipschitz continuous real functions, and define

$$\begin{aligned} C^1_0([0,\pi]) &= \{x \in C^1([0,\pi]) : x(0) = 0 = x(\pi)\}\,, \\ C^{1,1}_0([0,\pi]) &= \{x \in C^1_0([0,\pi]) : x' \in C^{0,1}([0,\pi])\}\,, \end{aligned}$$

the linear operator

$$L : C^{1,1}_0([0,\pi]) \times C^1([0,\pi]) \to C^{0,1}([0,\pi]) \times C([0,\pi])\,, \quad L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' - y \\ y' \end{pmatrix},$$

and the nonlinear operator

$$\begin{aligned} &N_r : C^1_0([0,\pi]) \times C([0,\pi]) \to C^{0,1}([0,\pi]) \times C([0,\pi])\,, \\ &N_r\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} p_r(t,x(t)) \\ -x(t) + q_r(t,x(t),y(t)) \end{pmatrix}. \end{aligned}$$

We will need the following three lemmas.

**Lemma 18.** *The operator $L$ is invertible, with a continuous inverse*

$$L^{-1} : C^{0,1}([0,\pi]) \times C([0,\pi]) \to C^{1,1}_0([0,\pi]) \times C^1([0,\pi])\,,$$

*and problem $(P_r)$ is equivalent to*

$$u = L^{-1}N_r u\,,$$

*where $u = (x,y) \in C^1_0([0,\pi]) \times C([0,\pi])$. Moreover, the operator*

$$L^{-1}N_r : C^1_0([0,\pi]) \times C([0,\pi]) \to C^1_0([0,\pi]) \times C([0,\pi])$$

*is completely continuous.*

*Proof.* If $(x, y) \in C_0^{1,1}([0, \pi]) \times C^1([0, \pi])$ satisfies $L(x, y) = (0, 0)$, then $(x, y)$ is constantly equal to $(0, 0)$. Hence, given $(h, k) \in C^{0,1}([0, \pi]) \times C([0, \pi])$, the problem

$$\begin{cases} x' = y + h(t)\,, \quad y' = k(t)\,, \\ x(0) = 0 = x(\pi)\,, \end{cases}$$

has a unique solution $(x, y) \in C_0^{1,1}([0, \pi]) \times C^1([0, \pi])$, and the function $(h, k) \mapsto (x, y)$ is continuous.

Problem $(P_r)$ is equivalent to $Lu = N_r u$, with $u = (x, y) \in C_0^{1,1}([0, \pi]) \times C^1([0, \pi])$. Then, it is also equivalent to $u = L^{-1} N_r u$, with $u = (x, y) \in C_0^1([0, \pi]) \times C([0, \pi])$.

The operator $N_r$ is continuous and transforms bounded sets of $C_0^1([0, \pi]) \times C([0, \pi])$ into bounded sets of $C^{0,1}([0, \pi]) \times C([0, \pi])$. Hence, $L^{-1} N_r$ is continuous and transforms bounded sets in $C_0^1([0, \pi]) \times C([0, \pi])$ into bounded sets in $C_0^{1,1}([0, \pi]) \times C^1([0, \pi])$. The conclusion follows, since the space $C_0^{1,1}([0, \pi]) \times C^1([0, \pi])$ is compactly imbedded into $C_0^1([0, \pi]) \times C([0, \pi])$. □

**Lemma 19.** *For any continuous function $\varphi : [0, \pi] \to \mathbb{R}$, the sets*

$$\{x \in C_0^1([0, \pi]) : \varphi \ll x\}\,, \qquad \{x \in C_0^1([0, \pi]) : x \ll \varphi\}$$

*are open in $C_0^1([0, \pi])$.*

*Proof.* Let us prove the first one, the second being similar. If $\varphi \ll x$, there is an $\epsilon > 0$ such that

$$\varphi(t) + \epsilon \varphi_1(t) \leq x(t)\,, \qquad \text{for every } t \in [0, \pi]\,.$$

It is easily seen that there is a $\delta > 0$ such that, for any continuously differentiable function $\psi : [0, \pi] \to \mathbb{R}$,

$$\|\psi\|_\infty + \|\psi'\|_\infty \leq \delta \quad \Rightarrow \quad |\psi| \ll \tfrac{1}{2} \epsilon \varphi_1\,. \tag{42}$$

Then, if $\tilde{x} \in C^1([0, \pi])$ is such that

$$\|\tilde{x} - x\|_\infty + \|\tilde{x}' - x'\|_\infty < \delta\,,$$

by (42) we have that

$$\varphi(t) + \tfrac{1}{2} \epsilon \varphi_1(t) \leq \tilde{x}(t)\,,$$

showing that $\varphi \ll \tilde{x}$. □

**Lemma 20.** *Let $\varphi : [0, \pi] \to \mathbb{R}$ be a continuously differentiable function. If*

$$\max\{\varphi(0), \varphi(\pi)\} \leq 0\,,$$

*then there exists a $C > 0$ such that $\varphi \ll C\varphi_1$. If, on the contrary,*

$$\min\{\varphi(0), \varphi(\pi)\} \geq 0\,,$$

*then there exists a $C > 0$ such that $\varphi \gg -C\varphi_1$.*

*Proof.* Take $C_1 > \max\{|\varphi'(0)|, |\varphi'(\pi)|\}$. Then, there is a $\delta > 0$ such that

$$\varphi(t) < C_1\varphi_1(t)\,, \qquad \text{for every } t \in ]0,\delta[\cup]\pi-\delta,\pi[\,.$$

On the other hand, there is a $C_2 > 0$ such that

$$\varphi(t) < C_2\varphi_1(t)\,, \qquad \text{for every } t \in [\delta,\pi-\delta]\,.$$

Taking $C = \max\{C_1, C_2\}$ we have the conclusion. □

By Lemma 20, we can fix a constant $C > 0$ such that $\alpha \ll C\varphi_1$ and $\beta \gg -C\varphi_1$. Let us now introduce the function $w_r : [0,\pi] \to \mathbb{R}$, defined as

$$w_r(t) = \frac{2r\cos\Big((t-\frac{\pi}{2})\sqrt{\frac{r-1}{r}}\Big)}{\cos\Big(\frac{\pi}{2}\sqrt{\frac{r-1}{r}}\Big)}\,.$$

**Lemma 21.** *The functions $\alpha_r, \beta_r : [0,\pi] \to \mathbb{R}$ defined by*

$$\alpha_r(t) = -(C\varphi_1(t) + w_r(t))\,, \qquad \beta_r(t) = C\varphi_1(t) + w_r(t)\,,$$

*are a lower and an upper solution of problem $(P_r)$, respectively. Moreover,*

$$\alpha_r \ll \beta \quad \text{and} \quad \alpha \ll \beta_r\,.$$

*Proof.* First notice that

$$w_r''(t) + \frac{r-1}{r}\,w_r(t) = 0\,, \qquad \text{for every } t \in [0,\pi]\,.$$

We set $y_{\alpha_r} = \alpha_r'$ and $y_{\beta_r} = \beta_r'$. Since $\beta_r(t) \geq 2r$ for every $t \in [0,\pi]$ and $p_r(t,x) = 0$ when $x \geq 2r$, conditions (6) and (8) are easily verified. Moreover,

$$\begin{aligned} y_{\beta_r}'(t) &= C\varphi_1''(t) + w_r''(t) = -C\varphi_1(t) - \frac{r-1}{r}\,w_r(t) = -\beta_r(t) + \frac{1}{r}w_r(t) \\ &\leq -\beta_r(t) + \frac{1}{r}\beta_r(t) = -\beta_r(t) + q_r(t,\beta_r(t),y_{\beta_r}(t))\,, \end{aligned}$$

so that (7) holds, too. A similar argument can be applied for $\alpha_r$. The last assertion in the statement of the lemma follows immediately from the choice of the constant $C$. □

Let us focus our attention on the three well-ordered pairs of lower/upper solutions $(\alpha_r, \beta_r)$, $(\alpha, \beta_r)$, and $(\alpha_r, \beta)$. Since problem $(P_r)$ satisfies hypotheses A1, A2, and A3 in Corollary 10, by [12, Lemma 15] we can find some curves $\gamma_\pm$ and $\widehat{\gamma}_\pm$ (the same for all the pairs) such that, for every $t \in [0,\pi]$ and $x \in [\alpha_r(t), \beta_r(t)]$, the conditions (13), (14), (21), (22) hold, together with

$$\begin{aligned} \gamma_-(x) &< \min\{y_\alpha(t), y_{\alpha_r}(t), y_\beta(t), y_{\beta_r}(t)\} \\ &\leq \max\{y_\alpha(t), y_{\alpha_r}(t), y_\beta(t), y_{\beta_r}(t)\} < \gamma_+(x)\,, \\ \widehat{\gamma}_-(x) &< \min\{y_\alpha(t), y_{\alpha_r}(t), y_\beta(t), y_{\beta_r}(t)\} \\ &\leq \max\{y_\alpha(t), y_{\alpha_r}(t), y_\beta(t), y_{\beta_r}(t)\} < \widehat{\gamma}_+(x)\,. \end{aligned}$$

By Lemma 19, the sets

$$\mathcal{V}_1 = \mathcal{V}(\alpha_r, \beta_r, \gamma_\pm, \widehat{\gamma}_\pm)\,, \quad \mathcal{V}_2 = \mathcal{V}(\alpha_r, \beta, \gamma_\pm, \widehat{\gamma}_\pm)\,, \quad \mathcal{V}_3 = \mathcal{V}(\alpha, \beta_r, \gamma_\pm, \widehat{\gamma}_\pm)\,,$$

with the notation

$$\begin{aligned}\mathcal{V}(\varphi, \psi, \gamma_\pm, \widehat{\gamma}_\pm) = \Big\{ & (x, y) \in C^1_0([0,\pi]) \times C([0,\pi]) : \varphi \ll x \ll \psi\,, \\ & \min\{\gamma_-(x(t)), \widehat{\gamma}_-(x(t))\} < y(t) < \max\{\gamma_+(x(t)), \widehat{\gamma}_+(x(t))\}\,, \\ & \text{for every } t \in [0,\pi] \Big\},\end{aligned}$$

are open in $C^1_0([0,\pi]) \times C([0,\pi])$.

Still denoting by $(\varphi, \psi)$ one of the three pairs $(\alpha_r, \beta_r)$, $(\alpha, \beta_r)$, and $(\alpha_r, \beta)$, we modify problem $(P_r)$. Set

$$g_r(t, x, y) = -x + q_r(t, x, y)\,,$$

and define

$$\begin{aligned}&\tilde{p}_{\varphi,\psi}(t, x) = p_r(t, \zeta(x; \varphi(t), \psi(t)))\,, \\ &\tilde{g}_{\varphi,\psi}(t, x, y) = g_r(t, \zeta(x; \varphi(t), \psi(t)), y) + e(x; \varphi(t), \psi(t))\,,\end{aligned}$$

with $\zeta(\cdot\,;\cdot,\cdot)$ and $e(\cdot\,;\cdot,\cdot)$ as in (17), (18), so to obtain

$$(\widetilde{P}_{\varphi,\psi}) \qquad \begin{cases} x' = y + \tilde{p}_{\varphi,\psi}(t, x)\,, \quad y' = \tilde{g}_{\varphi,\psi}(t, x, y)\,, \\ x(0) = 0 = x(\pi)\,. \end{cases}$$

It can be verified that $(\varphi, \psi)$ is a well-ordered pair of lower/upper solutions of $(\widetilde{P}_{\varphi,\psi})$, for any of the three choices of $(\varphi, \psi)$. Define the associated nonlinear operator

$$\begin{aligned}&\widetilde{N}_{\varphi,\psi} : C^1_0([0,\pi]) \times C([0,\pi]) \to C^{0,1}([0,\pi]) \times C([0,\pi])\,, \\ &\widetilde{N}_{\varphi,\psi}\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} \tilde{p}_{\varphi,\psi}(t, x(t)) \\ \tilde{g}_{\varphi,\psi}(t, x(t), y(t)) \end{pmatrix}.\end{aligned}$$

Indeed, if $x \in C^1_0([0,\pi])$, then $\tilde{p}_{\varphi,\psi}(\cdot, x(\cdot)) \in C^{0,1}([0,\pi])$. Problem $(\widetilde{P}_{\varphi,\psi})$ is then equivalent to

$$u = L^{-1}\widetilde{N}_{\varphi,\psi} u\,,$$

where $u = (x, y) \in C^1_0([0,\pi]) \times C([0,\pi])$. Moreover, the operator

$$L^{-1}\widetilde{N}_{\varphi,\psi} : C^1_0([0,\pi]) \times C([0,\pi]) \to C^1_0([0,\pi]) \times C([0,\pi])$$

is completely continuous (see Lemma 18).

An analogue of Claim 3 in the proof of Theorem 11 holds, i.e., every solution $u = (x, y)$ of problem $(\widetilde{P}_{\varphi,\psi})$ satisfies

$$\varphi(t) \le x(t) \le \psi(t)\,, \tag{43}$$

and

$$\min\{\gamma_-(x(t)), \widehat{\gamma}_-(x(t))\} < y(t) < \max\{\gamma_+(x(t)), \widehat{\gamma}_+(x(t))\}\,, \tag{44}$$

for every $t \in [0,\pi]$. Moreover, for any sufficiently large $\rho > 0$, denoting by $B_\rho$ the open ball in $C^1_0([0,\pi]) \times C([0,\pi])$, centered at the origin, with radius $\rho$, it can be proved that

$$d_{LS}(I - L^{-1}\widetilde{N}_{\varphi,\psi}\,,\, B_\rho) = 1\,,$$

where $d_{LS}$ denotes the Leray–Schauder degree. Let us fix a $\rho > 0$ with this property, such that $\mathcal{V}(\varphi, \psi, \gamma_\pm, \widehat{\gamma}_\pm) \subseteq B_\rho$.

**Lemma 22.** *If there are no solutions of* $(P_r)$ *on the boundary of* $\mathcal{V}(\varphi,\psi,\gamma_\pm,\widehat{\gamma}_\pm)$, *then*

$$d_{LS}(I - L^{-1}N_r\,,\,\mathcal{V}(\varphi,\psi,\gamma_\pm,\widehat{\gamma}_\pm)) = 1\,. \tag{45}$$

*Proof.* Notice that, on the set $\mathcal{V}(\varphi,\psi,\gamma_\pm,\widehat{\gamma}_\pm)$, the two problems $(P_r)$ and $(\widetilde{P}_{\varphi,\psi})$ coincide. Since all the solutions of problem $(\widetilde{P}_{\varphi,\psi})$ satisfy (43) and (44), they belong to the closure of $\mathcal{V}(\varphi,\psi,\gamma_\pm,\widehat{\gamma}_\pm)$. So, if there are no solutions on the boundary of $\mathcal{V}(\varphi,\psi,\gamma_\pm,\widehat{\gamma}_\pm)$, by the excision property of the degree,

$$d_{LS}(I - L^{-1}\widetilde{N}_{\varphi,\psi}\,,\,\mathcal{V}(\varphi,\psi,\gamma_\pm,\widehat{\gamma}_\pm)) = d_{LS}(I - L^{-1}\widetilde{N}_{\varphi,\psi}\,,\,B_\rho) = 1\,.$$

Since $\widetilde{N}_{\varphi,\psi} = N_r$ on $\mathcal{V}(\varphi,\psi,\gamma_\pm,\widehat{\gamma}_\pm)$, we have that

$$d_{LS}(I - L^{-1}N_r\,,\,\mathcal{V}(\varphi,\psi,\gamma_\pm,\widehat{\gamma}_\pm)) = d_{LS}(I - L^{-1}\widetilde{N}_{\varphi,\psi}\,,\,\mathcal{V}(\varphi,\psi,\gamma_\pm,\widehat{\gamma}_\pm)) = 1\,,$$

and the lemma is thus proved. □

Now we prove that there are no solutions of $(P_r)$ on $\partial\mathcal{V}_1$. Let $u = (x,y)$ be a solution of $(P_r)$ belonging to the closure of $\mathcal{V}_1$. We then have that both $\alpha_r(t) \le x(t) \le \beta_r(t)$ and (44) hold, for every $t \in [0,\pi]$. Assume by contradiction that $x \not\ll \beta_r$. Since $x(0) = 0 < 2r = \beta_r(0)$ and $x(\pi) = 0 < 2r = \beta_r(\pi)$, there must be a $t_0 \in ]0,\pi[$ such that $x(t_0) = \beta_r(t_0) > 2r$. Then,

$$x'(t_0) = \beta_r'(t_0) \quad \text{and} \quad x''(t_0) \le \beta_r''(t_0)\,.$$

Moreover, there is a neighborhood $U_0$ of $t_0$ in $]0,\pi[$ on which $x(t) > 2r$, so that, being $p_r(t,x) = 0$ and $q_r(t,x,y) = \frac{1}{r}x$ for $|x| \ge 2r$, we have

$$x'(t) = y'(t)\,, \quad y'(t) = \frac{1-r}{r}x(t)\,, \quad \text{and hence} \quad x''(t) = \frac{1-r}{r}x(t)\,,$$

for every $t \in U_0$. On the other hand,

$$\begin{aligned}\beta_r''(t_0) &= \frac{1-r}{r}w_r(t_0) - C\varphi_1(t_0)\\ &= \frac{1-r}{r}\beta_r(t_0) - \frac{C}{r}\varphi_1(t_0)\\ &< \frac{1-r}{r}\beta_r(t_0) = \frac{1-r}{r}x(t_0) = x''(t_0)\,,\end{aligned}$$

a contradiction. Hence, $x \ll \beta_r$. Similarly one proves that $\alpha_r \ll x$.

If there is a solution $(x,y)$ of $(P_r)$ in $\partial\mathcal{V}_2$, then we have $x \not\ll \beta$, while $x \le \beta$ still holds. Since $\alpha$ and $\beta$ are non-well-ordered, then $x(t_0) \le \beta(t_0) < \alpha(t_0)$ for a certain $t_0 \in [0,\pi]$, implying $\alpha \not\ll x$. The theorem is thus proved in this case. A similar argument leads to the conclusion assuming the existence of a solution in $\partial\mathcal{V}_3$.

Finally, if there are no solutions of $(P_r)$ in $\partial\mathcal{V}_2 \cup \partial\mathcal{V}_3$, then, by Lemma 22,

$$\begin{aligned}&d_L\Big(I - L^{-1}N_r\,,\,\mathcal{V}_1 \setminus \overline{\mathcal{V}_2 \cup \mathcal{V}_3}\Big) =\\ &\quad = d_L(I - L^{-1}N_r\,,\,\mathcal{V}_1) - \Big(d_L(I - L^{-1}N_r\,,\,\mathcal{V}_2) + d_L(I - L^{-1}N_r\,,\,\mathcal{V}_3)\Big) = -1\,.\end{aligned}$$

Hence, there is a solution in $\mathcal{V}_1 \setminus \overline{\mathcal{V}_2 \cup \mathcal{V}_3}$, and the proof is thus completed. □

## 4.2 Remarks and further developments

The following proposition better clarifies the conclusion of Theorem 17.

**Proposition 23.** *Let $\alpha, \beta : [0, \pi] \to \mathbb{R}$ be two continuously differentiable functions satisfying*

$$\alpha(0) \le 0 \le \beta(0)\,, \qquad \alpha(\pi) \le 0 \le \beta(\pi)\,.$$

*If moreover there is a $t_0 \in ]0, \pi[$ such that $\alpha(t_0) > \beta(t_0)$, then the set*

$$\{x \in C_0^1([0, \pi]) : \alpha \not\ll x \text{ and } x \not\ll \beta\}$$

*coincides with the closure in $C_0^1([0, \pi])$ of the set*

$$\{x \in C_0^1([0, \pi]) : \exists t_1, t_2 \in [0, \pi] : x(t_1) < \alpha(t_1),\, x(t_2) > \beta(t_2)\}\,.$$

*Proof.* Let us denote by $A$ the first set, and by $B$ the second one. We want to prove that $A = \overline{B}$. Let us first show that $\overline{B} \subseteq A$. Let $x \in \overline{B}$, and assume by contradiction that $\alpha \ll x$. Let $(x_n)_n$ be a sequence in $C_0^1([0, \pi])$ such that $\alpha \not\le x_n$, $x_n \not\le \beta$, and $x_n \to x$ in $C_0^1([0, \pi])$. By Lemma 19, $\alpha \ll x_n$, for $n$ sufficiently large, contradicting $\alpha \not\le x_n$. In the same way one can see that $x \not\ll \beta$, as well. Hence, $\overline{B} \subseteq A$.

In order to prove that $A \subseteq \overline{B}$, fix $x \in A$. We consider three cases.

Case 1: $\beta(t_0) < x(t_0) < \alpha(t_0)$. Then, $x \in B$, and we have finished.

Case 2: $x(t_0) \le \beta(t_0)$. Then, $x(t_0) < \alpha(t_0)$. Moreover, since $x \not\ll \beta$, there is a $\bar{t} \in [0, \pi]$ such that $x(\bar{t}\,) = \beta(\bar{t}\,)$. If $\bar{t} \in ]0, \pi[$, it is possible to $C^1$-perturb $x$ so to obtain a $\hat{x}$ which satisfies $\hat{x}(t_0) < \alpha(t_0)$ and $\hat{x}(\bar{t}\,) > \beta(\bar{t}\,)$. If $\bar{t} = 0$, then necessarily $x'(0) = \beta'(0)$ and it is possible to $C^1$-perturb $x$ so to obtain a $\hat{x}$ which satisfies $\hat{x}(t_0) < \alpha(t_0)$ and $\hat{x}(t) > \beta(t)$ for $t > 0$ near 0. Similarly, if $\bar{t} = \pi$, then necessarily $x'(\pi) = \beta'(\pi)$ and it is possible to $C^1$-perturb $x$ so to obtain a $\hat{x}$ which satisfies $\hat{x}(t_0) < \alpha(t_0)$ and $\hat{x}(t) > \beta(t)$ for $t < \pi$ near $\pi$. So, in any case, $x$ can be $C^1$-approximated by some function $\hat{x} \in B$, hence $x \in \overline{B}$.

Case 3: $x(t_0) \ge \alpha(t_0)$. It is analogous to Case 2.

We have thus proved that $A \subseteq \overline{B}$, hence the conclusion. □

Notice that if $p(t, x) = p(x)$ is continuously differentiable, setting

$$\gamma(x) = -p'(x)\,, \qquad h(t, x, w) = q(t, x, w - p(x))\,,$$

the differential system in problem $(P_{Dir})$ becomes a Liénard equation

$$x'' + \gamma(x)x' + x = h(t, x, x')\,.$$

A more general system can be considered if we ask some more regularity on the function $q$. We can deal with the Dirichlet problem

$$(P^*_{Dir}) \qquad \begin{cases} x' = y + p(t, x, y)\,, \quad y' = -x + q(t, x, y)\,, \\ x(0) = 0 = x(\pi)\,. \end{cases}$$

**Corollary 24.** *Assume the existence of a non-well-ordered pair $(\alpha, \beta)$ of lower/upper solutions of problem $(P^*_{Dir})$, where $p(t,x,y)$ and $q(t,x,y)$ are locally Lipschitz continuous uniformly bounded functions. Then there exists a solution of problem $(P^*_{Dir})$ such that*

$$\alpha \not\lll x \quad \text{and} \quad x \not\lll \beta .$$

*Proof.* The only difference with the proof of Theorem 17 is that, in this case, the functional setting would involve the linear operator

$$L : C_0^{1,1}([0,\pi]) \times C^{1,1}([0,\pi]) \to C^{0,1}([0,\pi]) \times C^{0,1}([0,\pi]) , \quad L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' - y \\ y' \end{pmatrix},$$

and the nonlinear operator

$$\begin{aligned} &N : C_0^1([0,\pi]) \times C^1([0,\pi]) \to C^{0,1}([0,\pi]) \times C^{0,1}([0,\pi]) , \\ &N\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} p(t,x(t),y(t)) \\ -x(t) + q(t,x(t),y(t)) \end{pmatrix}. \end{aligned}$$

We omit the details, for briefness. □

## 4.3 Two examples of applications

In this section, we will provide two examples in which non-well-ordered lower and upper solutions can be constructed. They are inspired by [14].

**Example 1.** We consider the problem

$$\begin{cases} x' = y + p(t,x,y) , \\ y' = -x + q(t,x,y) + h(t) , \\ x(0) = 0 = x(\pi) , \end{cases} \tag{46}$$

where the functions $p, q$ are locally Lipschitz continuous, and $h$ is continuous.

**Proposition 25.** *Assume that $p$ has a compact support, $q$ is uniformly bounded,*

$$q(t,x,y)\, x \le 0 , \quad \text{for every } (t,x,y) \in [0,\pi] \times \mathbb{R}^2, \tag{47}$$

*and*

$$\int_0^\pi h(t) \sin t \, dt = 0 .$$

*Then, there exists a solution of problem (46).*

*Proof.* Let $w(t)$ be the unique solution of

$$\begin{cases} w'' + w = h(t) , \\ w(0) = 0 = w(\pi) . \end{cases}$$

By the change of variables $u = x - w(t)$, $v = y - w'(t)$, problem (46) becomes

$$\begin{cases} u' = v + \tilde{p}(t,u,v) , \\ v' = -u + \tilde{q}(t,u,v) , \\ u(0) = 0 = u(\pi) , \end{cases} \tag{48}$$

where

$$\tilde{p}(t,u,v) = p(t, u + w(t), v + w'(t)), \quad \tilde{q}(t,u,v) = q(t, u + w(t), v + w'(t)).$$

Notice that the functions $\tilde{p}, \tilde{q}$ are locally Lipschitz continuous. Since $\tilde{p}$ has compact support, there exists $R > 0$ such that

$$\tilde{p}(t,u,v) = 0, \quad \text{when } \; u^2 + v^2 \geq R^2.$$

As a consequence, there is a constant $P \geq 0$ for which

$$|\tilde{p}(t,u,v)| \leq P, \quad \text{for every } (t,u,v) \in [0,\pi] \times \mathbb{R}^2.$$

Let us check that, for $k > 0$ sufficiently large, the functions $\alpha(t) = k \sin t$ and $\beta(t) = -k \sin t$ are lower/upper solutions for problem (48), respectively, with corresponding functions $v_\alpha(t) = \alpha'(t)$ and $v_\beta(t) = \beta'(t)$. (Here we use the notations $v_\alpha$ and $v_\beta$ instead of $y_\alpha$ and $y_\beta$ .)

We first check (3). Assume, for some $t \in [0,\pi]$, that $v < v_\alpha(t)$. If either $\alpha(t) \geq R$, or $0 \leq \alpha(t) < R$ and $|v| \geq R$, then $\tilde{p}(t, \alpha(t), v) = 0$, hence

$$v + \tilde{p}(t, \alpha(t), v) < v_\alpha(t) = \alpha'(t).$$

Assume now $0 \leq \alpha(t) < R$ and $|v| < R$. Taking $k \geq \sqrt{2}(R + P)$, we have

$$v + \tilde{p}(t, \alpha(t), v) < R + P \leq k\frac{\sqrt{2}}{2}.$$

Moreover, recalling that $0 \leq \alpha(t) < R$, it is $\sin t < \frac{R}{k} \leq \frac{\sqrt{2}}{2}$, hence $\cos t \geq \frac{\sqrt{2}}{2}$ (since $-R < v < v_\alpha(t) = k \cos t$, the case $\cos t \leq -\frac{\sqrt{2}}{2}$ is forbidden), and we have

$$k\frac{\sqrt{2}}{2} < k \cos t = \alpha'(t).$$

We have thus verified that, if $v < v_\alpha(t)$, then $v + \tilde{p}(t, \alpha(t), v) < \alpha'(t)$. In a similar way we can see that, if $v > v_\alpha(t)$, then $v + \tilde{p}(t, \alpha(t), v) > \alpha'(t)$. We thus proved (3). Moreover, taking $k$ so large to ensure that $k \sin t + w(t) \geq 0$ for every $t \in [0,\pi]$, using (47) we have

$$\begin{aligned} v_\alpha'(t) &= -k \sin t = -\alpha(t) \\ &\geq -\alpha(t) + q(t, \alpha(t) + w(t), v_\alpha(t) + w'(t)) = -\alpha(t) + \tilde{q}(t, \alpha(t), v_\alpha(t)), \end{aligned}$$

hence (4) holds. This proves that $\alpha$ is a lower solution for problem (48). Similarly one proves that $\beta$ is an upper solution. Corollary 24 can then be applied, to conclude the proof. □

**Example 2.** Let us now consider the problem

$$\begin{cases} x' = y + p(t,x,y), \\ y' = -x + q(t,x), \\ x(0) = 0 = x(\pi), \end{cases} \tag{49}$$

where the functions $p, q$ are locally Lipschitz continuous. We will assume here a *Landesman–Lazer* condition.

**Proposition 26.** *Assume that $p$ has a compact support, $q$ is uniformly bounded, and that*

$$\int_0^\pi \liminf_{x\to-\infty} q(t,x)\sin t\,dt > 0 > \int_0^\pi \limsup_{x\to+\infty} q(t,x)\sin t\,dt\,. \tag{50}$$

*Then, there exists a solution of problem* (49).

*Proof.* Let us construct a nonnegative lower solution $\alpha$. First of all, being $q$ bounded, there is a $Q > 0$ such that

$$|q(t,x)| \le Q\,, \qquad \text{for every } (t,x) \in [0,\pi]\times\mathbb{R}\,.$$

As in [14] (see also [10, Proposition 3.1]), there are a constant $s_1 > 0$ and a function $\eta \in L^1(0,\pi)$ such that

$$q(t,s) \le \eta(t)\,, \qquad \text{for every } s \ge s_1\,,$$

and

$$\int_0^\pi \eta(t)\sin t\,dt < 0\,.$$

Let $\delta > 0$ be such that

$$\int_0^\pi \eta(t)\sin t\,dt < \int_{[0,\delta]\cup[\pi-\delta,\pi]} (\eta(t)-Q)\sin t\,dt\,,$$

and define the function $\widetilde{\eta} \in L^1(0,\pi)$ as

$$\widetilde{\eta}(t) = \begin{cases} \eta(t)\,, & \text{if } t \in\,]\delta,\pi-\delta[\,,\\ Q\,, & \text{if } t \in [0,\delta]\cup[\pi-\delta,\pi]\,. \end{cases}$$

Notice that

$$\int_0^\pi \widetilde{\eta}(\tau)\sin\tau\,d\tau < 0\,.$$

Let $w(t)$ be the unique solution of

$$\begin{cases} w'' + w = \widetilde{\eta}(t) - \dfrac{2}{\pi}\Big(\displaystyle\int_0^\pi \widetilde{\eta}(\tau)\sin\tau\,d\tau\Big)\sin t\,,\\ w(0) = 0 = w(\pi)\,. \end{cases}$$

Let us check that, if $k > 0$ is sufficiently large, the function $\alpha(t) = w(t)+k\sin t$ is a lower solution of problem (49), such that $\alpha \gg 0$, with corresponding function $y_\alpha(t) = \alpha'(t)$. Using a similar reasoning as in the proof of Proposition 25, it can be seen that condition (3) holds true. Moreover,

$$\begin{aligned} y_\alpha'(t) &= -\alpha(t) + \widetilde{\eta}(t) - \frac{2}{\pi}\Big(\int_0^\pi \widetilde{\eta}(\tau)\sin\tau\,d\tau\Big)\sin t\\ &\ge -\alpha(t) + \widetilde{\eta}(t)\,. \end{aligned}$$

In order to obtain (4), we need to show that $q(t,\alpha(t)) \le \widetilde{\eta}(t)$, for every $t \in [0,\pi]$. If $t \in [0,\delta]\cup[\pi-\delta,\pi]$, this is an immediate consequence of the choice of the constant $Q$ and the definition of $\widetilde{\eta}(t)$. If $t \in\,]\delta,\pi-\delta[\,$, taking $k > 0$ sufficiently large we have that $\alpha(t) \ge s_1$, hence $q(t,\alpha(t)) \le \eta(t) = \widetilde{\eta}(t)$.

A similar construction can be made so to find an upper solution $\beta \ll 0$. Corollary 24 then applies, and the proof is completed. $\square$

**Remark 27.** We recall that, when $p$ is identically equal to zero and $q(t,\cdot)$ is strictly decreasing, the Landesman–Lazer condition (50) is necessary and sufficient for the existence of a solution to problem (49). Indeed, after noticing that the differential system is in this case equivalent to the scalar second order equation $x'' + x = q(t,x)$, assuming the existence of a solution $x(t)$ such that $x(0) = 0 = x(\pi)$, multiplying the equation by $\sin t$ and integrating we get

$$\int_0^\pi q(t,x(t)) \sin t \, dt = 0 \, .$$

Since, for every $t \in \,]0,\pi[\,$,

$$\lim_{x\to-\infty} q(t,x)\sin t > q(t,x(t))\sin t > \lim_{x\to+\infty} q(t,x)\sin t \, ,$$

by the Monotone Convergence Theorem these functions are integrable, and hence

$$\int_0^\pi \lim_{x\to-\infty} q(t,x)\sin t \, dt > 0 > \int_0^\pi \lim_{x\to+\infty} q(t,x)\sin t \, dt \, .$$

# 5 Higher order systems

Let us start by considering a system of $N$ second order scalar differential equations with Dirichlet boundary conditions,

$$(S_{Dir}) \qquad \begin{cases} x'' = g(t,x) \, , \\ x(a) = x_S \, , \; x(b) = x_A \, , \end{cases}$$

where $g : [a,b] \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function. We use the notation

$$x_S = (x_1^S, \dots, x_N^S) \in \mathbb{R}^N \, , \quad x_A = (x_1^A, \dots, x_N^A) \in \mathbb{R}^N \, .$$

Here is our definition of a well-ordered pair of lower and upper solutions, in this case.

**Definition 28.** *Given two $\mathcal{C}^2$-functions $\alpha, \beta : [a,b] \to \mathbb{R}^N$, we say that $(\alpha,\beta)$ is a* well-ordered pair of lower/upper solutions *of problem $(S_{Dir})$ if, for every $j \in \{1,\dots,N\}$ and $t \in [a,b]$,*

$$\alpha_j(t) \le \beta_j(t) \, ,$$

$$\alpha_j(a) \le x_j^S \le \beta_j(a) \, , \quad \alpha_j(b) \le x_j^A \le \beta_j(b) \, ,$$

*and, for every $x \in \prod_{m=1}^N [\alpha_m(t), \beta_m(t)]$,*

$$\begin{aligned} \alpha_j''(t) &\ge g_j(t, x_1, \dots, x_{j-1}, \alpha_j(t), x_{j+1}, \dots, x_N) \, , \\ \beta_j''(t) &\le g_j(t, x_1, \dots, x_{j-1}, \beta_j(t), x_{j+1}, \dots, x_N) \, . \end{aligned}$$

A similar situation has been studied in [11] for the periodic problem. Let us state, for example, the analogue of [11, Theorem 2].

**Theorem 29.** *If there exists a well-ordered pair of lower/upper solutions* $(\alpha, \beta)$*, then problem* $(S_{Dir})$ *has a solution* $x(t)$ *such that*

$$\alpha_j(t) \le x_j(t) \le \beta_j(t)\,, \quad \text{for every } j \in \{1, \dots, N\} \text{ and } t \in [a,b]\,. \tag{51}$$

Instead of providing the proof of this result, let us generalize it, considering the problem

$$(S^*_{Dir}) \qquad \begin{cases} x_j' = f_j(t, x_j, y_j)\,, \\ y_j' = g_j(t, x_1, \dots, x_N)\,, \qquad j = 1, \dots, N\,, \\ x_j(a) = x_j^S\,,\ x_j(b) = x_j^A\,, \end{cases}$$

where the functions $f_j : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ and $g_j : [a,b] \times \mathbb{R}^N \to \mathbb{R}$ are continuous. The definition of a well-ordered pair of lower and upper solutions now becomes the following.

**Definition 30.** *Given two* $\mathcal{C}^1$*-functions* $\alpha, \beta : [a,b] \to \mathbb{R}^N$*, we say that* $(\alpha, \beta)$ *is a* well-ordered pair of lower/upper solutions *of problem* $(S^*_{Dir})$ *if, for every* $j \in \{1, \dots, N\}$ *and* $t \in [a,b]$*,*

$$\alpha_j(t) \le \beta_j(t)\,,$$

$$\alpha_j(a) \le x_j^S \le \beta_j(a)\,, \quad \alpha_j(b) \le x_j^A \le \beta_j(b)\,,$$

*and there exist two* $\mathcal{C}^1$*-functions* $y^\alpha, y^\beta : [a,b] \to \mathbb{R}^N$ *such that, for every* $j \in \{1, \dots, N\}$ *and* $t \in [a,b]$*,*

$$\begin{cases} s < y_j^\alpha(t) \quad \Rightarrow \quad f_j(t, \alpha_j(t), s) < \alpha_j'(t)\,, \\ s > y_j^\alpha(t) \quad \Rightarrow \quad f_j(t, \alpha_j(t), s) > \alpha_j'(t)\,, \end{cases} \tag{52}$$

$$\begin{cases} s < y_j^\beta(t) \quad \Rightarrow \quad f_j(t, \beta_j(t), s) < \beta_j'(t)\,, \\ s > y_j^\beta(t) \quad \Rightarrow \quad f_j(t, \beta_j(t), s) > \beta_j'(t)\,, \end{cases} \tag{53}$$

*and, for every* $x \in \prod_{m=1}^N [\alpha_m(t), \beta_m(t)]$*,*

$$(y_j^\alpha)'(t) \ge g_j(t, x_1, \dots, x_{j-1}, \alpha_j(t), x_{j+1}, \dots, x_N)\,, \tag{54}$$

$$(y_j^\beta)'(t) \le g_j(t, x_1, \dots, x_{j-1}, \beta_j(t), x_{j+1}, \dots, x_N)\,. \tag{55}$$

Let us prove the following generalization of Theorem 29.

**Theorem 31.** *Assume the existence of a well-ordered pair* $(\alpha, \beta)$ *of lower/upper solutions of problem* $(S^*_{Dir})$*, and that, for every* $j = 1, \dots, N$*,*

$$\lim_{s \to -\infty} f_j(t, \xi, s) = -\infty\,, \qquad \lim_{s \to +\infty} f_j(t, \xi, s) = +\infty\,, \tag{56}$$

*uniformly for* $(t, \xi) \in [a,b] \times [\min \alpha_j, \max \beta_j]$*. Then, there exists a solution of problem* $(S^*_{Dir})$ *such that*

$$\alpha_j(t) \le x_j(t) \le \beta_j(t)\,, \tag{57}$$

*for every* $j \in \{1, \dots, N\}$ *and* $t \in [a,b]$*.*

*Proof.* We can easily adapt the proof of Corollary 10 to this context and, like in [12, Lemma 15], recover the curves $\gamma_{\pm,j}, \widehat{\gamma}_{\pm,j}$ such that, for every $t \in [a,b]$ and $x \in \prod_j [\alpha_j(t), \beta_j(t)]$,

$$\gamma_{-,j}(x_j) < \min\{y_j^\alpha(t), y_j^\beta(t)\} \le \max\{y_j^\alpha(t), y_j^\beta(t)\} < \gamma_{+,j}(x_j)\,,$$
$$\widehat{\gamma}_{-,j}(x_j) < \min\{y_j^\alpha(t), y_j^\beta(t)\} \le \max\{y_j^\alpha(t), y_j^\beta(t)\} < \widehat{\gamma}_{+,j}(x_j)\,,$$

and

$$f_j\big(t, x_j, \gamma_+(x_j)\big)\gamma_+'(x_j) < g_j(t,x) < f_j\big(t, x_j, \gamma_-(x_j)\big)\gamma_-'(x_j)\,,$$
$$f_j\big(t, x_j, \widehat{\gamma}_-(x_j)\big)\widehat{\gamma}_-'(x_j) < g_j(t,x) < f_j\big(t, x_j, \widehat{\gamma}_+(x_j)\big)\widehat{\gamma}_+'(x_j)\,.$$

Moreover, using (56), these curves can be chosen so that

$$y_j \ge \min\{\gamma_{+,j}(x_j), \widehat{\gamma}_{+,j}(x_j)\} \quad \Rightarrow \quad f_j(t, x_j, y_j) > \frac{x_j^A - x_j^S}{b-a}\,,$$
$$y_j \le \max\{\gamma_{-,j}(x_j), \widehat{\gamma}_{-,j}(x_j)\} \quad \Rightarrow \quad f_j(t, x_j, y_j) < \frac{x_j^A - x_j^S}{b-a}\,.$$

Following the main ideas of the proof of Theorem 11, after taking a constant $D$ such that

$$\|\gamma_{\pm,j}\|_\infty \le D\,, \quad \|\widehat{\gamma}_{\pm,j}\|_\infty \le D\,, \quad \text{for every } j \in \{1, \dots, N\}\,,$$

we can introduce the modified problem

$$(\widetilde{S}^*_{Dir}) \qquad \begin{cases} x_j' = \tilde{f}_j(t, x_j, y_j)\,, \\ y_j' = \tilde{g}_j(t, x_1, \dots, x_N)\,, \qquad j = 1, \dots, N\,, \\ x_j(a) = x_j^S\,,\ x_j(b) = x_j^A\,, \end{cases}$$

where

$$\tilde{f}_j(t, x_j, y_j) = f_j\Big(t, \zeta\big(x_j; \alpha_j(t), \beta_j(t)\big)\,,\, \zeta\big(y_j; -D, D\big)\Big) + e\big(y_j; -D, D\big)\,,$$
$$\tilde{g}_j(t, x_1, \dots, x_N) = g_j\Big(t\,,\, \zeta\big(x_1; \alpha_1(t), \beta_1(t)\big)\,, \dots\,,\, \zeta\big(x_N; \alpha_N(t), \beta_N(t)\big)\Big) + $$
$$+\, e\big(x_j; \alpha_j(t), \beta_j(t)\big)\,.$$

An analogue of Claim 3 holds, working separately on every pair of coordinates $(x_j, y_j)$ while considering the remaining components as parameters. Hence, we can prove that all the solutions of $(\widetilde{S}^*_{Dir})$ satisfy (57) and

$$\min\{\gamma_{-,j}(x_j(t)), \widehat{\gamma}_{-,j}(x_j(t))\} < y_j(t) < \max\{\gamma_{+,j}(x_j(t)), \widehat{\gamma}_{+,j}(x_j(t))\}\,, \quad (58)$$

for every $j \in \{1, \dots, N\}$ and $t \in [a,b]$. In such a way, following the lines of the proof of Theorem 11, one easily concludes. □

As an example of application, we consider the Dirichlet-Neumann problem

$$\begin{cases} (\phi_1(x_1'))' + \gamma_1(t, x_1, x_2, x_1', x_2')x_1' - |x_1|^{\sigma_1 - 1}x_1 = p_1(t, x_1, x_2, x_1', x_2')\,, \\ (\phi_2(x_2'))' + \gamma_2(t, x_1, x_2, x_1', x_2')x_2' - |x_2|^{\sigma_2 - 1}x_2 = p_2(t, x_1, x_2, x_1', x_2')\,, \\ x_1(a) = 0 = x_1(b)\,, \\ x_2'(a) = 0 = x_2'(b)\,, \end{cases} \quad (59)$$

ruled by some increasing homeomorphisms $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$, with $\phi_1(0) = \phi_2(0) = 0$. Here, $\sigma_1, \sigma_2$ are positive constants, the functions $\gamma_1, \gamma_2, p_1, p_2 : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ are continuous, and $p_1, p_2$ are also bounded. Choosing the constant functions $\alpha(t) = (-\|p_1\|_\infty, -\|p_2\|_\infty)$ and $\beta(t) = (\|p_1\|_\infty, \|p_2\|_\infty)$, with corresponding functions $y_\alpha(t) = y_\beta(t) = (0,0)$, we see that $(\alpha, \beta)$ is a well-ordered pair of lower and upper solutions of problem (59), which then has a solution $(x_1, x_2)$ such that

$$|x_1(t)| \leq \|p_1\|_\infty \quad \text{and} \quad |x_2(t)| \leq \|p_2\|_\infty , \qquad \text{for every } t \in [a,b] .$$

Concerning the non-well-ordered case, we can similarly deal with a system like

$$(S^0_{Dir}) \qquad \begin{cases} x_j' = y_j + p_j(t, x_j, y_j) , \\ y_j' = -x_j + q_j(t, x_1, \dots, x_N) , \qquad j = 1, \dots, N . \\ x_j(0) = 0 = x_j(\pi) , \end{cases}$$

In this case, the definition of lower and upper solutions must be given separately, assuming (54) and (55) to be true for every $x \in \mathbb{R}^N$. We can then state the following result.

**Theorem 32.** *Assume the existence of a non-well-ordered pair $(\alpha, \beta)$ of lower/upper solutions of problem $(S^0_{Dir})$, where $p_j : [0,\pi] \times \mathbb{R}^2 \to \mathbb{R}$ and $q_j : [0,\pi] \times \mathbb{R}^N \to \mathbb{R}$ are locally Lipschitz continuous and uniformly bounded, for every $j = 1, \dots, N$. Then, there exists a solution $(x, y)$ such that*

$$\alpha_j \not\ll x_j \quad \text{and} \quad x_j \not\ll \beta_j ,$$

*for every $j = 1, \dots, N$.*

*Proof.* It can be carried out following the lines of the proof of Theorem 17, working separately on every pair of coordinates $(x_j, y_j)$ while considering the remaining components as parameters. We omit the details, for briefness. □

It is now possible to provide a large number of examples, in the same spirit of systems (46) and (49).

# A Appendix. Proof of the claims

In this final section we provide the proofs of Claims 1–4 introduced in the previous sections in order to obtain Theorems 5, 11, and 17.

## A.1 Proof of Claim 1

Let us define the following regions

$$\begin{aligned} A_{NE} &= \{(t,x,y) : t \in [a,b] , x > \beta(t) , y > y_\beta(t)\} , \\ A_{SE} &= \{(t,x,y) : t \in [a,b] , x > \beta(t) , y < y_\beta(t)\} , \\ A_{SW} &= \{(t,x,y) : t \in [a,b] , x < \alpha(t) , y < y_\alpha(t)\} , \\ A_{NW} &= \{(t,x,y) : t \in [a,b] , x < \alpha(t) , y > y_\alpha(t)\} . \end{aligned}$$

**Lemma 33.** *Let $u = (x, y)$ be a solution of*

$$x' = \tilde{f}(t, x, y)\,, \qquad y' = \tilde{g}(t, x, y)\,. \tag{60}$$

*Then, for any $t_0 \in [a, b]$,*

$$\begin{array}{lcl} (t_0, u(t_0)) \in A_{SE} & \Rightarrow & (t, u(t)) \in A_{SE} \text{ for every } t \in [a, t_0]\,, \\ (t_0, u(t_0)) \in A_{NW} & \Rightarrow & (t, u(t)) \in A_{NW} \text{ for every } t \in [a, t_0]\,, \\ (t_0, u(t_0)) \in A_{NE} & \Rightarrow & (t, u(t)) \in A_{NE} \text{ for every } t \in [t_0, b]\,, \\ (t_0, u(t_0)) \in A_{SW} & \Rightarrow & (t, u(t)) \in A_{SW} \text{ for every } t \in [t_0, b]\,. \end{array}$$

*Proof.* From (3), (4), (6), and (7) we get, for every $t \in [a, b]$,

$$\begin{cases} \tilde{f}(t, x, y) < \alpha'(t)\,, & \text{if } x \le \alpha(t) \text{ and } y < y_\alpha(t)\,, \\ \tilde{f}(t, x, y) > \alpha'(t)\,, & \text{if } x \le \alpha(t) \text{ and } y > y_\alpha(t)\,; \end{cases} \tag{61}$$

$$\begin{cases} \tilde{f}(t, x, y) < \beta'(t)\,, & \text{if } x \ge \beta(t) \text{ and } y < y_\beta(t)\,, \\ \tilde{f}(t, x, y) > \beta'(t)\,, & \text{if } x \ge \beta(t) \text{ and } y > y_\beta(t)\,; \end{cases} \tag{62}$$

$$\begin{cases} \tilde{g}(t, x, y_\alpha(t)) < y'_\alpha(t)\,, & \text{if } x < \alpha(t)\,, \\ \tilde{g}(t, x, y_\beta(t)) > y'_\beta(t)\,, & \text{if } x > \beta(t)\,. \end{cases} \tag{63}$$

The proof can be obtained as an immediate consequence of the previous estimates. □

**Lemma 34.** *Let $u = (x, y)$ be a solution of* (60)*. If, for any $t_0 \in [a, b]$,*

$$x(t_0) < \alpha(t_0) \quad \text{and} \quad y(t_0) = y_\alpha(t_0)\,,$$

*then there exists $\delta > 0$ such that*

$$\begin{array}{lcl} t_0 \ne a\,,\ t \in \,]t_0 - \delta, t_0[ & \Rightarrow & (t, u(t)) \in A_{NW}\,, \\ t_0 \ne b\,,\ t \in \,]t_0, t_0 + \delta[ & \Rightarrow & (t, u(t)) \in A_{SW}\,. \end{array}$$

*Similarly, if, for any $t_0 \in [a, b]$,*

$$x(t_0) > \beta(t_0) \quad \text{and} \quad y(t_0) = y_\beta(t_0)\,,$$

*then there exists $\delta > 0$ such that*

$$\begin{array}{lcl} t_0 \ne a\,,\ t \in \,]t_0 - \delta, t_0[ & \Rightarrow & (t, u(t)) \in A_{SE}\,, \\ t_0 \ne b\,,\ t \in \,]t_0, t_0 + \delta[ & \Rightarrow & (t, u(t)) \in A_{NE}\,. \end{array}$$

*Proof.* The proof is an immediate consequence of (63). □

**Lemma 35.** *For each $\lambda \in [0, 1]$, all the solutions of $(\widetilde{P}_\lambda)$ satisfy*

$$\alpha(t) \le x(t) \le \beta(t)\,, \quad \text{for every } t \in [a, b]\,. \tag{64}$$

*Proof.* Recalling that $(\alpha(a), y_\alpha(a)) \in H_S^+$ and $(\beta(a), y_\beta(a)) \in H_S^-$, and applying Lemmas 33 and 34, we can prove that, for every $\lambda \in [0,1]$,

$$\begin{aligned}
(x(a), y(a)) \in \ell_S^\lambda\,,\; x(a) < \alpha(a) \quad &\Rightarrow \quad (a, x(a), y(a)) \in \overline{A_{SW}}\,, \quad x(a) < \alpha(a) \\
&\Rightarrow \quad (t, x(t), y(t)) \in A_{SW} \text{ for every } t \in ]a,b]\,, \\
(x(a), y(a)) \in \ell_S^\lambda\,,\; x(a) > \beta(a) \quad &\Rightarrow \quad (a, x(a), y(a)) \in \overline{A_{NE}}\,, \quad x(a) > \beta(a) \\
&\Rightarrow \quad (t, x(t), y(t)) \in A_{NE} \text{ for every } t \in ]a,b]\,.
\end{aligned}$$

Similarly, since $(\alpha(b), y_\alpha(b)) \in H_A^-$ and $(\beta(b), y_\beta(b)) \in H_A^+$ then, for every $\lambda \in [0,1]$,

$$\begin{aligned}
(x(b), y(b)) \in \ell_A^\lambda\,,\; x(b) < \alpha(b) \quad &\Rightarrow \quad (b, x(b), y(b)) \in \overline{A_{NW}}\,, \quad x(b) < \alpha(b) \\
&\Rightarrow \quad (t, x(t), y(t)) \in A_{NW} \text{ for every } t \in [a,b[\,, \\
(x(b), y(b)) \in \ell_A^\lambda\,,\; x(b) > \beta(b) \quad &\Rightarrow \quad (b, x(b), y(b)) \in \overline{A_{SE}}\,, \quad x(b) > \beta(b) \\
&\Rightarrow \quad (t, x(t), y(t)) \in A_{SE} \text{ for every } t \in [a,b[\,.
\end{aligned}$$

The only reasonable conclusion is that $\alpha(a) \leq x(a) \leq \beta(a)$ and $\alpha(b) \leq x(b) \leq \beta(b)$. Indeed, if $x(a) < \alpha(a)$ then $(t, x(t), y(t)) \in A_{SW}$ for every $t \in ]a,b]$. In particular $(b, x(b), y(b)) \in A_{SW}$ and so $x(b) < \alpha(b)$. Since $(x(b), y(b)) \in \ell_A^\lambda$ we get $(b, x(b), y(b)) \in \overline{A_{NW}}$, too. We get a contradiction since $A_{SW} \cap \overline{A_{NW}} = \varnothing$. A similar argument rules out the other three cases.

With a similar reasoning, the validity of Lemmas 33 and 34 forbids the existence of $t_0 \in ]a,b[$ such that $x(t_0) < \alpha(t_0)$ or $x(t_0) > \beta(t_0)$. □

**Lemma 36.** *For each $\lambda \in [0,1]$, all the solutions of $(\widetilde{P}_\lambda)$ satisfy*

$$\gamma_-(x(t)) < y(t) < \gamma_+(x(t))\,, \quad \textit{for every } t \in [a,b]\,. \tag{65}$$

*Proof.* By Lemma 35 we know that $\alpha(t) \leq x(t) \leq \beta(t)$ for every $t \in [a,b]$. Hence, (65) can be rephrased as

$$(t, x(t), y(t)) \in V\,, \qquad \text{for every } t \in [a,b]\,,$$

where

$$V = \{(t,x,y) \in [a,b] \times \mathbb{R}^2 : \alpha(t) \leq x \leq \beta(t)\,, \gamma_-(x) < y < \gamma_+(x)\}\,.$$

Assumption (15) ensures that

$$\gamma_-(x) < F_A^\lambda(x) < \gamma_+(x)\,, \qquad \text{for every } x \in [\mu, \mathcal{M}]\,.$$

In particular, $(b, x(b), y(b)) \in V$. By contradiction, assume the existence of $t_0 \in [a,b[$ such that $y(t_0) \geq \gamma_+(x(t_0))$. Then, defining $G_+(t) = y(t) - \gamma_+(x(t))$, since $G_+(t_0) \geq 0 > G_+(b)$ we can find $t_1 \in [t_0, b[$ such that $G_+(t_1) = 0$ and $G_+(t) < 0$ in a right neighborhood of $t_1$. Computing

$$\begin{aligned}
G'_+(t_1) &= y'(t_1) - \gamma'_+(x(t_1))x'(t_1) \\
&= \tilde{g}\big(t_1, x(t_1), \gamma_+(x(t_1))\big) - \gamma'_+(x(t_1))\tilde{f}\big(t_1, x(t_1), \gamma_+(x(t_1))\big) \\
&= g\big(t_1, x(t_1), \gamma_+(x(t_1))\big) - \gamma'_+(x(t_1))f\big(t_1, x(t_1), \gamma_+(x(t_1))\big) > 0\,,
\end{aligned} \tag{66}$$

we get a contradiction. The existence of a certain $t_0 \in [a,b[$ such that $y(t_0) \leq \gamma_-(x(t_0))$ analogously leads to a contradiction. □

Lemmas 35 and 36 complete the proof of Claim 1.

## A.2 Proof of Claim 2

System $(Q_\sigma)$ can be rewritten as

$$(Q_\sigma) \qquad \begin{cases} x' = y + (1-\sigma) f_b(t,x,y)\,, \quad y' = x + (1-\sigma) g_b(t,x,y)\,, \\ (x(a),y(a)) \in \ell^1_S(\sigma)\,, \quad (x(b),y(b)) \in \ell^1_A(\sigma)\,, \end{cases}$$

where $f_b(t,x,y) = \tilde f(t,x,y) - y$ and $g_b(t,x,y) = \tilde g(t,x,y) - x$ are bounded functions.

We argue by contradiction and assume the existence of two sequences $(\sigma_n)_n$ in $[0,1]$ and $(u_n)_n = (x_n,y_n)_n$ in $C^1([a,b],\mathbb{R}^2)$, with $\lim_n \|u_n\|_\infty = +\infty$, where $u_n$ is a solution of $(Q_{\sigma_n})$. We set $w_n = u_n/\|u_n\|_\infty$. Then, $w_n = (\xi_n, \upsilon_n)$ solves

$$\begin{cases} \xi' = \upsilon + (1-\sigma_n)\, f_{b,n}(t,\xi,\upsilon)\,, \\ \upsilon' = \xi + (1-\sigma_n)\, g_{b,n}(t,\xi,\upsilon)\,, \\ \|u_n\|_\infty (\xi(a),\upsilon(a)) \in \ell^1_S(\sigma)\,, \\ \|u_n\|_\infty (\xi(b),\upsilon(b)) \in \ell^1_A(\sigma)\,, \end{cases} \tag{67}$$

where

$$\begin{aligned} f_{b,n}(t,\xi,\upsilon) &= \frac{1}{\|u_n\|_\infty} f_b(t, \xi\|u_n\|_\infty, \upsilon\|u_n\|_\infty)\,, \\ g_{b,n}(t,\xi,\upsilon) &= \frac{1}{\|u_n\|_\infty} g_b(t, \xi\|u_n\|_\infty, \upsilon\|u_n\|_\infty)\,. \end{aligned}$$

Since $w_n \in C^1([a,b],\mathbb{R}^2)$ is such that $\|w_n\|_\infty = 1$ for every $n$, we can deduce that the sequence $(\|w_n'\|_\infty)_n$ is bounded. Hence, by a compactness argument there are $\bar\sigma \in [0,1]$ and $\bar w = (\bar\xi,\bar\upsilon) \in C([a,b],\mathbb{R}^2)$, with $\|\bar w\|_\infty = 1$, such that, up to a subsequence,

$$\lim_n \sigma_n = \bar\sigma\,, \quad \lim_n \|w_n - \bar w\|_\infty = 0\,.$$

Since $f_{b,n} \to 0$ and $g_{b,n} \to 0$ uniformly, then, passing to the limit as $n \to +\infty$, we see that $\bar w = (\bar\xi,\bar\upsilon)$ solves

$$\bar\xi' = \bar\upsilon\,, \quad \bar\upsilon' = \bar\xi\,. \tag{68}$$

Let us now focus our attention on the boundary conditions the function $\bar w = (\bar\xi,\bar\upsilon)$ must satisfy. We have

$$\begin{aligned} \upsilon_n(a) &= \tfrac{1}{\|u_n\|_\infty} Y^\sigma_S(\|u_n\|_\infty \xi_n(a)) = m^1_S\, \xi_n(a) + \tfrac{1}{\|u_n\|_\infty}(1-\sigma_n) q^1_S\,, \\ \upsilon_n(b) &= \tfrac{1}{\|u_n\|_\infty} Y^\sigma_A(\|u_n\|_\infty \xi_n(b)) = m^1_A\, \xi_n(b) + \tfrac{1}{\|u_n\|_\infty}(1-\sigma_n) q^1_A\,, \end{aligned}$$

so that, passing to the limit as $n \to +\infty$,

$$\bar\upsilon(a) = m^1_S\, \bar\xi(a)\,, \quad \bar\upsilon(b) = m^1_A\, \bar\xi(b)\,,$$

Recalling that $m^1_S \geq 0$, and so $\bar\xi(a)\bar\upsilon(a) \geq 0$, since $\bar w \neq 0$ solves (68) we conclude that $\bar\xi(t)\bar\upsilon(t) > 0$ for every $t \in ]a,b]$. Similarly, since $m^1_A \leq 0$ we get $\bar\xi(t)\bar\upsilon(t) < 0$ for every $t \in [a,b[$ . We get a contradiction.

## A.3 Proof of Claim 3

Since $\alpha(a) \le \min\{x_S, x_A\} \le \max\{x_S, x_A\} \le \beta(a)$ and $\alpha(b) \le \min\{x_S, x_A\} \le \max\{x_S, x_A\} \le \beta(b)$, recalling the validity of Lemmas 33 and 34 also in the present situation, we can show that all the solutions of $(\widetilde{P})$ satisfy (30) for every $t \in [a, b]$. We are going now to prove the validity of (31) for every $t \in [a, b]$, too.

We argue by contradiction and we assume the existence of a solution of $(\widetilde{P})$, such that $y(t_0) \ge \max\{\gamma_+(x(t_0)), \widehat{\gamma}_+(x(t_0))\}$, for a certain $t_0 \in [a, b]$. Recalling the procedure adopted in order to get the contradicting estimate in (66), we can prove that

$$\begin{aligned} y(t) &\ge \gamma_+(x(t))\,, \quad \text{for every } t \in [t_0, b]\,, \\ y(t) &\ge \widehat{\gamma}_+(x(t))\,, \quad \text{for every } t \in [a, t_0]\,, \end{aligned}$$

thus obtaining

$$y(t) \ge \min\{\gamma_+(x(t))\,,\, \widehat{\gamma}_+(x(t))\}\,, \quad \text{for every } t \in [a, b]\,. \tag{69}$$

We can find an interval $[t_1, t_2] \subseteq [a, b]$ with the following property: $x(t_1) = x_S$, $x(t_2) = x_A$, and $\min\{x_S, x_A\} \le x(t) \le \max\{x_S, x_A\}$ for every $t \in [t_1, t_2]$. Recalling the definition of $\tilde{f}$, with $D$ as in (32), and the hypothesis (28), since (69) holds, we get

$$x'(t) = \tilde{f}(t, x(t), y(t)) > \frac{x_A - x_S}{b - a}\,, \tag{70}$$

for every $t \in [t_1, t_2]$. If $x_S = x_A$ then $x'(t) > 0$ when $x(t) = x_A = x_S$ thus giving a contradiction. Otherwise, the interval $[t_1, t_2]$ is not trivial, and, integrating in this interval, we get

$$(x_A - x_S)\left(1 - \frac{t_2 - t_1}{b - a}\right) > 0\,. \tag{71}$$

If $x_A < x_S$ we get a contradiction, so we need to consider the remaining case $x_A > x_S$. The case $[t_1, t_2] = [a, b]$ is forbidden by (71). Moreover, from (70), we get $x'(t) > 0$ when $x(t) \in [x_S, x_A]$, so that $t_1 = a$ and $x(t) > x_A$ in a right neighborhood of $t_2$. So, since $x(b) = x_A$, we have necessarily the existence of $t_3 \in ]t_2, b]$ such that $x(t_3) = x_A$ and $x'(t_3) \le 0$ providing a contradiction also in this case.

We have thus proved that $y(t) < \max\{\gamma_+(x(t)), \widehat{\gamma}_+(x(t))\}$, for every $t \in [a, b]$. Analogously one proves that $y(t) > \min\{\gamma_-(x(t)), \widehat{\gamma}_-(x(t))\}$, for every $t \in [a, b]$.

## A.4 Proof of Claim 4

By contradiction, assume that there exist a sequence of numbers $r_n \ge n$ and some solutions $u_n = (x_n, y_n)$ of problems $(P_{r_n})$ such that $\alpha \not\ll x_n$, $x_n \not\ll \beta$, and $\|u_n\|_\infty > n$. Define $v_n = x_n/\|u_n\|_\infty$ and $w_n = y_n/\|u_n\|_\infty$. Then, $(v_n, w_n)$ solves

$$\begin{cases} v_n' = w_n + \frac{1}{\|u_n\|_\infty} p_{r_n}(t, x_n)\,, \quad w_n' = -v_n + \frac{1}{\|u_n\|_\infty} q_{r_n}(t, x_n, y_n)\,, \\ v_n(0) = 0 = v_n(\pi)\,. \end{cases}$$

By a standard compactness argument there is a subsequence, still denoted by $(v_n, w_n)_n$, such that, for some $(\overline{v}, \overline{w}) \in C^1([0,\pi]) \times C([0,\pi])$, we have that $v_n \to \overline{v}$ in $C^1([0,\pi])$ and $w_n \to \overline{w}$ uniformly; moreover, since $p_{r_n}$ and $q_{r_n}$ are bounded and both $r_n \to +\infty$ and $\|u_n\|_\infty \to +\infty$, we deduce that $\overline{w} \in C^1([0,\pi])$ and $(\overline{v}, \overline{w})$ solves

$$\begin{cases} \overline{v}\,' = \overline{w}\,, \quad \overline{w}\,' = -\overline{v}\,, \\ \overline{v}(0) = 0 = \overline{v}(\pi)\,. \end{cases}$$

Since $\|(\overline{v}, \overline{w})\|_\infty = 1$, it has to be either $(\overline{v}, \overline{w}) = (\varphi_1, \varphi_1')$, or $(\overline{v}, \overline{w}) = -(\varphi_1, \varphi_1')$. Assume that $(\overline{v}, \overline{w}) = (\varphi_1, \varphi_1')$. By Lemma 20, there exists $C > 0$ such that $\alpha \ll C\varphi_1$. Since $v_n \to \varphi_1$ in $C^1([0,\pi])$, for $n$ large enough it has to be $v_n \gg \frac{1}{2}\varphi_1$ and

$$x_n = \|u_n\|_\infty v_n \gg \tfrac{1}{2}\|u_n\|_\infty \varphi_1 \gg C\varphi_1 \gg \alpha\,,$$

a contradiction. A similar contradiction is reached if $(\overline{v}, \overline{w}) = -(\varphi_1, \varphi_1')$, thus completing the proof.

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