A dynamical approach to lower and upper solutions for planar systems

Alessandro Fonda and Rodica Toader

To the memory of Massimo Tarallo

Abstract. We prove the existence of bounded and periodic solutions for planar systems by introducing a notion of lower and upper solutions which generalizes the classical one for scalar second order equations. The proof relies on phase plane analysis; after suitably modifying the nonlinearities, the Ważewski theory provides a solution which remains bounded in the future. For the periodic problem, the Massera Theorem applies, yielding the existence result. We then show how our result generalizes some well known theorems on the existence of bounded and of periodic solutions. Finally, we provide some corollaries on the existence of almost periodic solutions for scalar second order equations.

1 Introduction

The notion of lower and upper solutions for some second order scalar ordinary differential equations, with separated boundary conditions, was already introduced by Picard [26] in 1893. Concerning the general equation

$$x'' = g(t, x, x'), \tag{1}$$

the first attempts towards a modern definition were made by Scorza Dragoni [30] in 1931, and few years later Nagumo [20] provided the by now classical one, requiring that the lower solution α and the upper solution β satisfy the familiar inequalities

$$\alpha''(t) \geq g(t, \alpha(t), \alpha'(t)), \qquad \beta''(t) \leq g(t, \beta(t), \beta'(t)).$$

In order to get the existence of a solution in the presence of lower and upper solutions $\alpha \leq \beta$, he also needed to assume what we call today a *Nagumo condition*: there is a positive continuous function $\varphi : [0, +\infty[\to \mathbb{R}$ such that

$$|g(t, x, y)| \leq \varphi(|y|), \qquad \text{for every } (t, x, y) \in \mathbb{R} \times [a, b] \times \mathbb{R},$$

where $a = \inf \alpha$, $b = \sup \beta$, and

$$\int_0^{+\infty} \frac{s}{\varphi(s)}\, ds = +\infty.$$

Many different variants of the Nagumo condition have been proposed since then, and it has been shown that, in general, such a condition cannot be completely avoided (see, e.g., [7, 12]). We refer to [5, 7] for a more complete historical account.

Surprisingly enough, the first results for the periodic problem associated to (1), due to Knobloch [17], appeared only in 1963. This is probably due to the method of proof, which relied on the search of fixed points of some nonlinear operators in a suitable Banach space of periodic functions, having to face the difficulty that the differential operator is not invertible in this case.

The problem of boundedness of the solutions of equation (1) has a long history, as well (see, e.g., [1, 11, 22, 25, 32] and the references therein). Although apparently different from usual boundary value problems, the existence of a bounded solution of (1) has also been proved by the use of lower and upper solutions [2, 19, 28].

The lower/upper solutions theory was developed in several directions, by different methods. The first proofs made use of iterative methods; fixed point theory played a central role, mainly by the use of topological degree; and, when the nonlinearity does not depend on the derivative, also variational methods have been proposed. The regularity of α and β has been considerably weakened. The ordering $\alpha \leq \beta$ has been exploited to provide minimal and maximal solutions. The case $\alpha \not\leq \beta$ has also been analyzed, assuming some nonresonance conditions with respect to the higher part of the spectrum of the differential operator. Other types of differential operators have been considered, even involving partial differential equations of elliptic or parabolic type (see, e.g., [9, 10] and the references therein).

In particular, much attention has been given to the equation

$$(\phi(x'))' = h(t, x, x'), \tag{2}$$

assuming for α and β the inequalities

$$(\phi(\alpha'))'(t) \geq h(t, \alpha(t), \alpha'(t)), \qquad (\phi(\beta'))'(t) \leq h(t, \beta(t), \beta'(t)).$$

Here, $\phi : I \to J$ is an increasing homeomorphism between two intervals I and J containing 0, and $\phi(0) = 0$. Typical examples in the applications involve the choice $\phi(v) = |v|^{p-2}v$, leading to the so-called "scalar p-Laplacian" operator, or $\phi(v) = v/\sqrt{1+v^2}$, providing a "mean curvature" operator, or $\phi(v) = v/\sqrt{1-v^2}$, providing a "relativistic" operator.

Notice that equation (2) can be written as an equivalent first order planar system

$$x' = \phi^{-1}(y), \qquad y' = g(t, x, y),$$

with $g(t, x, y) = h(t, x, \phi^{-1}(y))$. Concerning α and β, whose derivatives take their values in the domain of ϕ, we can define the functions $y_\alpha(t) = \phi(\alpha'(t))$

and $y_\beta(t) = \phi(\beta'(t))$, so that the inequalities characterizing the lower and upper solutions become

$$y'_\alpha(t) \geq g(t, \alpha(t), y_\alpha(t)), \qquad y'_\beta(t) \leq g(t, \beta(t), y_\beta(t)).$$

This simple observation will be our guide to provide a notion of lower and upper solutions for a general planar system

$$x' = f(t, x, y), \qquad y' = g(t, x, y). \tag{3}$$

Besides the fact that the function f may depend also on t and x, a major improvement is achieved in that we do not need any monotonicity with respect to y (cf. [3]).

The aim of this paper is twofold. On one hand, we are interested in finding a *bounded solution* of system (3), i.e., a solution for which

$$\sup\big\{|x(t)| + |y(t)| : t \in \mathbb{R}\big\} < +\infty.$$

On the other hand, assuming f and g to be T-periodic in their first variable t, we want to prove the existence of a T-*periodic solution* of (3). As a consequence of our results, we will also be able to deal with the problem of *almost periodic solutions*. This is a much more delicate argument, in view of a counter-example by Ortega and Tarallo [24] (see however [8, 33]).

Our results generalize some known theorems on the existence of bounded, periodic or almost periodic solutions for scalar second order equations. However, concerning our definition of lower and upper solutions, we will not aim at the greatest generality, in order to keep the exposition not too complicated.

Our approach is based on a dynamical study of the solutions of system (3). We define an open set V in $\mathbb{R}^3$ whose projection on the (x, y)-plane is bounded, and show that, after modifying $f(t, x, y)$ and $g(t, x, y)$ outside $\overline{V}$ (the closure of V), the Ważewski Theorem [35] can be applied. Having thus found a solution which is bounded in the future, a simple argument provides a solution which remains in V for all times. Concerning the periodic problem, once a bounded solution is found, the Massera Theorem [18] provides a T-periodic solution of the modified system. We then show that this solution remains in $\overline{V}$, so that it is indeed a solution of the original problem. In the case when the nonlinearity is strictly increasing in the x variable and almost periodic in t, a theorem by Corduneanu [6] will provide us an existence result for almost periodic solutions for equations (1) and (2) with mean curvature and relativistic operators.

The Ważewski method was already used by many authors to study the existence of solutions for boundary value problems [14, 15, 27]. Combined with the Massera Theorem, it has been proposed in [2, 16] for the search of periodic solutions of (1) in the framework of constant lower and upper solutions.

The paper is organized as follows. In Section 2 we specify our setting, introducing the notion of lower and upper solution for system (3), and we state our main theorem, whose proof is given in Section 4 (see also Section 5, where a more general version of the theorem will be proposed). In Section 3 we provide some corollaries and applications. In particular, we show how to deal with the case when the Nagumo condition is satisfied; then, we consider an equation with a mean curvature operator and provide existence results assuming some bound on the forcing term; finally, we consider an equation with a relativistic operator and prove that our results apply also to this case. In Section 6 we treat the problem of existence of almost periodic solutions for equations of the type (1) or (2) with mean curvature and relativistic operators. The paper ends with an Appendix in which, for the reader's convenience, we state and prove Corduneanu's Theorem on almost periodic solutions.

2 The main result

Our aim is to prove an existence result for bounded and for periodic solutions of system (3), where $f, g : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions. We first give our definition of lower and upper solutions.

Definition 1. *A continuously differentiable function $\alpha : \mathbb{R} \to \mathbb{R}$ is said to be a* lower solution *for* (3) *if the following properties hold:*

(i) *there exists a unique function $y_\alpha : \mathbb{R} \to \mathbb{R}$ such that*

$$\begin{cases} y < y_\alpha(t) & \Rightarrow \quad f(t, \alpha(t), y) < \alpha'(t)\,, \\ y > y_\alpha(t) & \Rightarrow \quad f(t, \alpha(t), y) > \alpha'(t)\,; \end{cases} \tag{4}$$

(ii) *y_α is continuously differentiable, and*

$$y'_\alpha(t) \geq g(t, \alpha(t), y_\alpha(t))\,, \qquad \text{for every } t \in \mathbb{R}\,; \tag{5}$$

(iii) *there are two positive constants δ, m such that, when $|y - y_\alpha(t)| \leq \delta$,*

$$\begin{cases} y < y_\alpha(t) - m|x - \alpha(t)| & \Rightarrow \quad f(t, x, y) < \alpha'(t)\,, \\ y > y_\alpha(t) + m|x - \alpha(t)| & \Rightarrow \quad f(t, x, y) > \alpha'(t)\,. \end{cases} \tag{6}$$

We say that α is a strict lower solution *if* (5) *holds with strict inequality.*

Notice that the inequalities in (4) imply the identity

$$f(t, \alpha(t), y_\alpha(t)) = \alpha'(t)\,, \qquad \text{for every } t \in \mathbb{R}\,. \tag{7}$$

Here and in the sequel, the curve $\Gamma_\alpha : \mathbb{R} \to \mathbb{R}^3$, defined as

$$\Gamma_\alpha(t) = (t, \alpha(t), y_\alpha(t))\,,$$

will play an important role. We illustrate in Figure 1 the inequalities appearing in (i) and (iii) above.

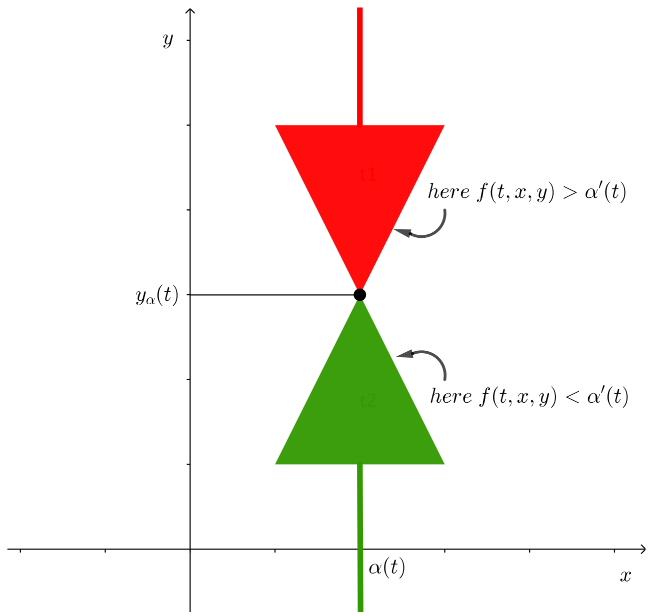

Figure 1: *In the red region* $f(t,x,y) > \alpha'(t)$, *in the green one* $f(t,x,y) < \alpha'(t)$

Remark 2. Whenever the function $f(t,x,y)$ does not depend on x, condition (i) implies (iii), hence (iii) does not need to be explicitly stated.

Remark 3. In the particular case when $f : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable, condition (iii) holds assuming

$$\frac{\partial f}{\partial y}(\Gamma_\alpha(t)) \geq c > 0\,, \qquad \text{for every } t \in \mathbb{R}\,.$$

In this case, by (7) and (ii), we have that a lower solution α is twice continuously differentiable and, differentiating in (7), we see that (5) is equivalent to

$$\alpha''(t) \geq \frac{\partial f}{\partial t}(\Gamma_\alpha(t)) + \frac{\partial f}{\partial x}(\Gamma_\alpha(t))\alpha'(t) + \frac{\partial f}{\partial y}(\Gamma_\alpha(t))g(\Gamma_\alpha(t))\,,$$

for every $t \in \mathbb{R}$. Clearly, we have the strict inequality if α is a strict lower solution.

We now give the analogous definition for an upper solution.

Definition 4. *A continuously differentiable function* $\beta : \mathbb{R} \to \mathbb{R}$ *is said to be an* upper solution *for* (3) *if the following properties hold:*

(i) *there exists a unique function* $y_\beta : \mathbb{R} \to \mathbb{R}$ *such that*

$$\begin{cases} y < y_\beta(t) & \Rightarrow \quad f(t,\beta(t),y) < \beta'(t)\,, \\ y > y_\beta(t) & \Rightarrow \quad f(t,\beta(t),y) > \beta'(t)\,; \end{cases} \tag{8}$$

(ii) y_β *is continuously differentiable, and*

$$y_\beta'(t) \leq g(t, \beta(t), y_\beta(t)) \,, \qquad \textit{for every } t \in \mathbb{R} \,; \tag{9}$$

(iii) there are two positive constants δ, m *such that, when* $|y - y_\beta(t)| \leq \delta$*,*

$$\begin{cases} y < y_\beta(t) - m|x - \beta(t)| & \Rightarrow \quad f(t,x,y) < \beta'(t) \,, \\ y > y_\beta(t) + m|x - \beta(t)| & \Rightarrow \quad f(t,x,y) > \beta'(t) \,. \end{cases} \tag{10}$$

We say that β *is a* strict upper solution *if* (9) *holds with strict inequality.*

Notice that the inequalities in (8) imply the identity

$$f(t, \beta(t), y_\beta(t)) = \beta'(t) \,, \qquad \text{for every } t \in \mathbb{R} \,. \tag{11}$$

Observe also that (4) and (8) are of the same type, as well as (6) and (10), while in (5) and (9) the inequalities are reversed. Again, if $f(t,x,y)$ does not depend on x, condition *(iii)* needs not to be explicitly stated.

As above, we can define $\Gamma_\beta : \mathbb{R} \to \mathbb{R}^3$ by

$$\Gamma_\beta(t) = (t, \beta(t), y_\beta(t)) \,.$$

Whenever $f : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable with

$$\frac{\partial f}{\partial y}(\Gamma_\beta(t)) \geq c > 0 \,, \qquad \text{for every } t \in \mathbb{R} \,,$$

we have that *(iii)* holds and, differentiating in (11), we see that (9) is equivalent to

$$\beta''(t) \leq \frac{\partial f}{\partial t}(\Gamma_\beta(t)) + \frac{\partial f}{\partial x}(\Gamma_\beta(t))\beta'(t) + \frac{\partial f}{\partial y}(\Gamma_\beta(t)) g(\Gamma_\beta(t)) \,,$$

with strict inequality if β is a strict upper solution.

We now state our main theorem for the existence of bounded and periodic solutions for system (3).

Theorem 5. *Let* $f, g : \mathbb{R}^3 \to \mathbb{R}$ *be continuous functions. Assume that there exist a bounded lower solution* α *and a bounded upper solution* β *for* (3)*, with* $\alpha(t) < \beta(t)$ *for every* $t \in \mathbb{R}$*, and let* $a = \inf \alpha$*,* $b = \sup \beta$*. Assume moreover that there exist two continuously differentiable functions* $\gamma_\pm : [a,b] \to \mathbb{R}$*, such that*

$$\gamma_-(x) < \min\{y_\alpha(t), y_\beta(t)\} \,, \qquad f(t, x, \gamma_-(x))\gamma_-'(x) > g(t, x, \gamma_-(x)) \,, \tag{12}$$

and

$$\gamma_+(x) > \max\{y_\alpha(t), y_\beta(t)\} \,, \qquad f(t, x, \gamma_+(x))\gamma_+'(x) < g(t, x, \gamma_+(x)) \,, \tag{13}$$

for every $t \in \mathbb{R}$ and $x \in [\alpha(t), \beta(t)]$. Then, there exists a bounded solution $(x(t), y(t))$ of (3) *satisfying*

$$\alpha(t) \le x(t) \le \beta(t) \quad \text{and} \quad \gamma_-(x(t)) \le y(t) \le \gamma_+(x(t)), \quad \text{for every } t \in \mathbb{R}. \tag{14}$$

If, moreover, f and g are T-periodic in their first variable t, and α, β are also T-periodic, then there exists a T-periodic solution $(x(t), y(t))$ of (3) *satisfying* (14).

The proof of this theorem is postponed to Section 4. In Section 5 we will extend Theorem 13 to a more general setting, where the functions $\gamma_\pm$ may also depend on t.

3 Some applications

In this section, divided in three subsections, we want to provide some more specific conditions in order to guarantee the existence of bounded and of periodic solutions to system (3). First, we analyze the Nagumo condition. Then, equations involving mean curvature-like operators are studied. Finally, equations of relativistic type are also treated.

3.1 The Nagumo condition

In the theorem below we introduce some Nagumo-type conditions for the functions f and g.

Theorem 6. *Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be continuous functions. Assume the existence of a bounded lower solution α and a bounded upper solution β for* (3), *with $\alpha(t) < \beta(t)$ for every $t \in \mathbb{R}$, and let $a = \inf \alpha$, $b = \sup \beta$. Moreover, let the following assumptions hold:*

A1. there are a constant $d > 0$ and two continuous functions $f_+ : [0, +\infty[\to \mathbb{R}$ and $f_- :]-\infty, 0] \to \mathbb{R}$ such that

$$\begin{cases} y \ge d \quad \Rightarrow \quad f(t,x,y) \ge f_+(y) > 0\,, \\ y \le -d \quad \Rightarrow \quad f(t,x,y) \le f_-(y) < 0\,, \end{cases} \qquad \text{for every } (t,x) \in \mathbb{R} \times [a,b]\,;$$

A2. there is a positive continuous function $\varphi : [0, +\infty[\to \mathbb{R}$ such that

$$|g(t,x,y)| \le \varphi(|y|)\,, \qquad \text{for every } (t,x,y) \in \mathbb{R} \times [a,b] \times \mathbb{R}\,; \tag{15}$$

A3. the above functions are such that

$$\int_0^{+\infty} \frac{f_+(s)}{\varphi(s)}\, ds = +\infty\,, \qquad \int_{-\infty}^0 \frac{f_-(s)}{\varphi(|s|)}\, ds = -\infty\,. \tag{16}$$

Then, there exists a bounded solution $(x(t), y(t))$ of (3), *with*

$$\alpha(t) \le x(t) \le \beta(t)\,, \qquad \text{for every } t \in \mathbb{R}\,. \tag{17}$$

If, moreover, f and g are T-periodic in their first variable t, and α, β are also T-periodic, then there exists a T-periodic solution $(x(t), y(t))$ of (3) *whose first component satisfies* (17).

Proof. We show that the assumptions A1, A2 and A3 permit us to construct the two functions $\gamma_\pm : [a,b] \to \mathbb{R}$ satisfying the conditions in the statement of Theorem 5.

Let us construct $\gamma_+(x)$. For any $y_0 > 0$, let $\mathcal{F}_{y_0} : [0, +\infty[\to \mathbb{R}$ be the function defined as

$$\mathcal{F}_{y_0}(\xi) = \int_{y_0}^{\xi} \frac{f_+(s)}{\varphi(s)}\, ds\,.$$

It is easy to check that $\mathcal{F}_{y_0}$ is strictly increasing on $[d, +\infty[$ and, by the first equality in (16), we see that

$$\lim_{\xi \to +\infty} \mathcal{F}_{y_0}(\xi) = +\infty\,.$$

Moreover, for any $M \geq 0$, using the first equality in (16) again,

$$\lim_{y_0 \to +\infty} \mathcal{F}_{y_0}(\xi) = -\infty\,, \qquad \text{uniformly for } \xi \in [0, M]\,. \tag{18}$$

Take $y_0 > 0$ large enough so that $\mathcal{F}_{y_0}(0) < -2(b-a)$. Then, for every $x \in [a,b]$ there is a unique $\xi \in]0, +\infty[$ such that $\mathcal{F}_{y_0}(\xi) = -2(x-a)$, and we define $\gamma_+(x) = \xi$; we thus have

$$\mathcal{F}_{y_0}(\gamma_+(x)) = -2(x-a)\,, \qquad \text{for every } x \in [a,b]\,. \tag{19}$$

By (18),

$$\lim_{y_0 \to +\infty} \gamma_+(x) = +\infty\,, \qquad \text{uniformly for } x \in [a,b]\,,$$

so that the first part of (13) holds, for $y_0 > d$ large enough. Differentiating in (19), we see that $\gamma_+'(x) < 0$ for every $x \in [a,b]$, and

$$\begin{aligned} f(t, x, \gamma_+(x))\gamma_+'(x) &\leq f_+(\gamma_+(x))\gamma_+'(x) \\ &= -2\varphi(\gamma_+(x)) < -\varphi(\gamma_+(x)) \\ &\leq g(t, x, \gamma_+(x))\,, \end{aligned}$$

thus proving also the second part of (13).

Analogously we can construct $\gamma_-(x)$ satisfying (12). Hence, Theorem 5 applies, yielding the conclusion. $\square$

Let us now provide some examples where Theorem 6 applies. Assume for instance that $f(t,x,y) = f(y)$, so that we are dealing with the system

$$x' = f(y)\,, \qquad y' = g(t,x,y)\,. \tag{20}$$

Corollary 7. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be continuous functions. Assume the existence of a bounded lower solution α and a bounded upper solution β for* (20)*, with $\alpha(t) < \beta(t)$ for every $t \in \mathbb{R}$, and let $a = \inf \alpha$, $b = \sup \beta$. Moreover, let there exist a constant $d > 0$ such that*

$$y \geq d \quad \Rightarrow \quad yf(y) > 0\,,$$

and a positive continuous function $\varphi : [0, +\infty[\to \mathbb{R}$ such that (15) *holds. If*

$$\int_0^{+\infty} \frac{f(s)}{\varphi(s)}\, ds = +\infty\,, \qquad \int_{-\infty}^0 \frac{f(s)}{\varphi(|s|)}\, ds = -\infty\,, \tag{21}$$

then the same conclusions of Theorem 6 hold.

Proof. We can take $f_- = f = f_+$, and apply Theorem 6. □

As a special case, assume $f : \mathbb{R} \to \mathbb{R}$ to be a strictly increasing continuous function, with $f(0) = 0$, and denote by $]\omega^-, \omega^+[$ its image $f(\mathbb{R})$. Setting $\phi = f^{-1} :]\omega^-, \omega^+[\to \mathbb{R}$, system (3) is equivalent to the scalar equation

$$(\phi(x'))' = h(t, x, x')\,, \tag{22}$$

where $h(t, x, z) = g(t, x, \phi(z))$. In this case, if α and β are lower/upper solutions, we have $y_\alpha(t) = \phi(\alpha'(t))$, $y_\beta(t) = \phi(\beta'(t))$, and

$$(\phi(\alpha'))'(t) \geq h(t, \alpha(t), \alpha'(t))\,, \qquad (\phi(\beta'))'(t) \leq h(t, \beta(t), \beta'(t))\,.$$

Assumptions (15) and (21) are satisfied if there is a positive continuous function $\psi : [0, +\infty[\to \mathbb{R}$ such that

$$|h(t, x, z)| \leq \psi(|z|)\,, \qquad \text{for every } (t, x, z) \in \mathbb{R} \times [a, b] \times]\omega^-, \omega^+[\,,$$

and

$$\int_0^{\omega^+} \frac{v\phi'(v)}{\psi(v)}\, dv = +\infty\,, \qquad \int_{\omega^-}^0 \frac{v\phi'(v)}{\psi(|v|)}\, dv = -\infty\,.$$

We thus ensure the existence of bounded or T-periodic solutions of equation (22).

A typical example is provided by the choice $f(y) = |y|^{q-2}y$, for some $q \in]1, 2]$. In this case, we have $\phi(v) = |v|^{p-2}v$, with $p \geq 2$ satisfying $(1/p)+(1/q) = 1$, so that we are considering a second order equation (22) with the so-called "scalar p-Laplacian" differential operator. We thus obtain, with a different approach, some well-known existence results (cf. [5, 34]).

Another well-studied example is provided by taking $f(y) = y/\sqrt{1 + y^2}$, so that $\phi(v) = v/\sqrt{1 - v^2}$. Here we are considering a second order scalar equation (22) with the so-called "relativistic" differential operator. We will come back to this equation at the end of this section, showing that, indeed, no Nagumo condition is necessary in this case (as already observed in [4]).

As another example, let $f(y) = \varrho(y)(1 + \eta \sin y)$, with $|\eta| < 1$ and $\varrho : \mathbb{R} \to \mathbb{R}$ an increasing homeomorphism such that $\varrho(0) = 0$. Notice that f is not invertible any more. However, we can still apply our result, taking, e.g., as α and β two constant functions. Indeed, assuming $\alpha < \beta$ and

$$g(t, \alpha, 0) \leq 0 \leq g(t, \beta, 0), \qquad \text{for every } t \in \mathbb{R},$$

we have that $\alpha' = \beta' = 0$, hence $y_\alpha = y_\beta = 0$, and α, β are a lower and an upper solution, respectively. In this case, we still have to assume (15), with a positive continuous function φ satisfying

$$\int_0^{+\infty} \frac{\varrho(s)}{\varphi(s)}\, ds = +\infty, \qquad \int_{-\infty}^{0} \frac{\varrho(s)}{\varphi(|s|)}\, ds = -\infty,$$

so to ensure the existence of a bounded or a T-periodic solution of system (20).

3.2 The mean curvature operator

We consider the scalar equation (22), where $\phi : \mathbb{R} \to]-1, 1[$ is an increasing homeomorphism such that $\phi(0) = 0$, and $h : \mathbb{R}^3 \to \mathbb{R}$ is continuous. Setting $f(y) = \phi^{-1}(y)$ and $g(t, x, y) = h(t, x, \phi^{-1}(y))$, we have that equation (22) is equivalent to system (20). Since $f :]-1, 1[\to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \times]-1, 1[\to \mathbb{R}$ are not defined on $\mathbb{R}$ and $\mathbb{R}^3$, respectively, we will need to modify and extend these functions, so to be able to apply Theorem 5.

For simplicity, we assume ϕ (hence also f) to be an odd function. We define the function $F :]-1, 1[\to \mathbb{R}$ as $F(y) = \int_0^y f(s)\, ds$, and set

$$F(1) = \int_0^1 f(s)\, ds = \lim_{y \to 1^-} F(y).$$

Notice that $F(1)$ could be $+\infty$.

Proposition 8. *In the above setting, assume the existence of a bounded lower solution α and a bounded upper solution β for* (20)*, with bounded derivatives, such that $\alpha(t) < \beta(t)$ for every $t \in \mathbb{R}$, and set $a = \inf \alpha$, $b = \sup \beta$. Assume moreover that there is a $c \geq 0$ such that*

$$|h(t, x, z)| \leq c, \qquad \text{for every } (t, x, z) \in \mathbb{R} \times [a, b] \times \mathbb{R}, \tag{23}$$

with

$$c < \frac{F(1) - F(\max\{\phi(\|\alpha'\|_\infty)), \phi(\|\beta'\|_\infty)\})}{b - a}. \tag{24}$$

Then, equation (22) *has a bounded solution $x(t)$ satisfying*

$$\alpha(t) \leq x(t) \leq \beta(t), \qquad \text{for every } t \in \mathbb{R}, \tag{25}$$

and

$$\sup\{|x'(t)| : t \in \mathbb{R}\} < +\infty. \tag{26}$$

If, moreover, g is T-periodic in its first variable t, and α, β are also T-periodic, then there exists a T-periodic solution $x(t)$ of (22) *satisfying* (25).

Proof. Notice that $y_\alpha(t) = \phi(\alpha'(t))$ and $y_\beta(t) = \phi(\beta'(t))$. By (24), we can fix $\delta \in]0,1[$ such that

$$F(1-\delta) > F(\max\{\phi(\|\alpha'\|_\infty)), \phi(\|\beta'\|_\infty)\}) + c(b-a)\,. \tag{27}$$

Since f is strictly increasing and $f(0)=0$, the function F is strictly increasing on $[0,1[$, and from the above inequality we deduce, in particular, that

$$\max\{\phi(\|\alpha'\|_\infty)), \phi(\|\beta'\|_\infty)\} < 1-\delta\,. \tag{28}$$

We define the functions $f_\delta : \mathbb{R} \to \mathbb{R}$ and $g_\delta : \mathbb{R}^3 \to \mathbb{R}$ as

$$f_\delta(y) = \begin{cases} f(-1+\delta) & \text{if } y < -1+\delta\,, \\ f(y) & \text{if } |y| \le 1-\delta\,, \\ f(1-\delta) & \text{if } y > 1-\delta\,, \end{cases}$$

and

$$g_\delta(t,x,y) = \begin{cases} g(t,x,-1+\delta) & \text{if } y < -1+\delta\,, \\ g(t,x,y) & \text{if } |y| \le 1-\delta\,, \\ g(t,x,1-\delta) & \text{if } y > 1-\delta\,, \end{cases}$$

and we consider the system

$$x' = f_\delta(y)\,, \qquad y' = g_\delta(t,x,y)\,. \tag{29}$$

By (28), α and β are a lower and an upper solution for system (29), as well.

Recalling (27), we can choose $\hat{c} > 0$ such that

$$c < \hat{c} < \frac{F(1-\delta) - F(\max\{\phi(\|\alpha'\|_\infty)), \phi(\|\beta'\|_\infty)\})}{b-a}\,. \tag{30}$$

We consider the Cauchy problem

$$\begin{cases} w' = -\dfrac{\hat{c}}{f_\delta(w)} \\ w(a) = 1-\delta\,. \end{cases} \tag{31}$$

As long as $w(x) > 0$, with $x \in [a,b]$, the function $w(x)$ is strictly decreasing, and the differential equation in (31) is equivalent to

$$\frac{d}{dx} F(w(x)) = -\hat{c}\,,$$

so that, integrating over $[a,x]$, we get

$$F(w(x)) = F(1-\delta) - \hat{c}(x-a) > F(1-\delta) - \hat{c}(b-a)\,.$$

In view of (30) and the fact that F is strictly increasing on $[0,1[$, we conclude that

$$\max\{\phi(\|\alpha'\|_\infty), \phi(\|\beta'\|_\infty)\} < w(x) \le 1-\delta\,, \qquad \text{for every } x \in [a,b]\,.$$

Hence, the function $\gamma_+(x) = w(x)$ satisfies the first part of (13) for system (29). On the other hand, using (23), from the differential equation in (31) we have

$$f_\delta(\gamma_+(x))\gamma_+'(x) = -\hat{c} < g_\delta(t,x,\gamma_+(x))\,,$$

so that also the second part of (13) is satisfied.

Similarly we can construct $\gamma_-(x)$ satisfying (12) for system (29). Theorem 5 thus applies, providing a solution $(x(t), y(t))$ of system (29) for which (14) holds. Since $-1+\delta \leq \gamma_-(x) < \gamma_+(x) \leq 1-\delta$ for every $x \in [a,b]$, we see that $x(t)$ is a solution of (22) satisfying (25) and

$$\sup\{|\phi(x'(t))| : t \in \mathbb{R}\} \leq 1-\delta\,.$$

Since the last inequality implies (26), the proof is completed. $\square$

Remark 9. The assumption (23) holds whenever $h(t,x,z) = h(t,x)$ is continuous on $\mathbb{R} \times [a,b]$ and T-periodic in its first variable. On the other hand, assumption (24) is surely satisfied if $F(1) = +\infty$. Notice also that, if α and β are constant functions, i.e.,

$$h(t,\alpha,0) < 0 < h(t,\beta,0)\,, \qquad \text{for every } t \in \mathbb{R}\,,$$

then condition (24) becomes

$$c < \frac{1}{b-a}\int_0^1 f(s)\,ds\,.$$

A typical example where Proposition 8 applies is provided by the choice $\phi(v) = v/\sqrt{1+v^2}$, leading to the equation

$$\left(\frac{x'}{\sqrt{1+(x')^2}}\right)' = h(t,x,x')\,, \tag{32}$$

in which the so-called "mean curvature" differential operator is involved. Notice that, in this case, $f(y) = y/\sqrt{1-y^2}$ and $F(y) = 1-\sqrt{1-y^2}$, so that $F(1) = 1$ and condition (24) becomes

$$c < \frac{1}{b-a}\min\left\{\frac{1}{\sqrt{1+\|\alpha'\|_\infty^2}}\,,\ \frac{1}{\sqrt{1+\|\beta'\|_\infty^2}}\right\}\,.$$

We recall that, in the case $h(t,x,z) = h(t,x)$, the existence of a *bounded variation* T-periodic solution of equation (32) was obtained in [21] without requiring condition (24). Moreover, in [21, Corollary 5.2], where $h(t,x)$ was supposed to be continuously differentiable and such that

$$\frac{\partial h}{\partial x}(t,x) \geq m > 0\,, \qquad \text{for every } (t,x) \in \mathbb{R} \times [a,b]\,,$$

that solution was proved to be *regular*, i.e., a *classical solution*.

3.3 The relativistic operator

Let us first consider the equation

$$x'' = \big(1-(x')^2\big)^{\sigma} p(t,x,x')\,, \tag{33}$$

where $\sigma > 1$ and $p : \mathbb{R} \times \mathbb{R} \times]-1,1[\to \mathbb{R}$ is continuous.

Proposition 10. *Assume the existence of a bounded lower solution α and a bounded upper solution β for the system*

$$x' = y\,, \qquad y' = \big(1-y^2\big)^\sigma p(t,x,y)\,,$$

with $\|\alpha'\|_\infty < 1$, $\|\beta'\|_\infty < 1$, such that $\alpha(t) < \beta(t)$ for every $t \in \mathbb{R}$, and set $a = \inf\alpha$, $b = \sup\beta$. Assume moreover that there is a $c \geq 0$ such that

$$|p(t,x,y)| \leq c\,, \qquad \text{for every } (t,x,y) \in \mathbb{R}\times[a,b]\times\,]-1,1[\,. \tag{34}$$

Then, equation (33) *has a bounded solution satisfying*

$$\alpha(t) \leq x(t) \leq \beta(t)\,, \qquad \text{for every } t \in \mathbb{R}\,, \tag{35}$$

and $\|x'\|_\infty < 1$. If, moreover, p is T-periodic in its first variable t, and α, β are also T-periodic, then there exists a T-periodic solution $x(t)$ of (33) *satisfying* (35).

Proof. We take $f(y) = y$, define $g_\delta : \mathbb{R}^3 \to \mathbb{R}$ as

$$g_\delta(t,x,y) = \begin{cases} (1-(1-\delta)^2)^\sigma p(t,x,1-\delta) & \text{if } y > 1-\delta\,, \\ (1-y^2)^\sigma p(t,x,y) & \text{if } |y| \leq 1-\delta\,, \\ (1-(-1+\delta)^2)^\sigma p(t,x,-1+\delta) & \text{if } y < -1+\delta\,, \end{cases}$$

for some $\delta \in\,]0,1[$ to be fixed, and consider the system

$$x' = y\,, \qquad y' = g_\delta(t,x,y)\,. \tag{36}$$

If $\delta > 0$ satisfies $1-\delta > \max\{\|\alpha'\|_\infty, \|\beta'\|_\infty\}$, the functions α and β are a lower and an upper solution for system (36), too.

We consider the Cauchy problem

$$\begin{cases} w' = -\hat{c}\,\dfrac{\big(1-w^2\big)^\sigma}{w} \\ w(a) = 1-\delta\,, \end{cases} \tag{37}$$

for a given $\hat{c} > c$. In this case, $w = 1$ is an equilibrium of the differential equation in (37). Hence, the solution $w(x) = w(x; a, 1-\delta)$ of (37) satisfies

$$\lim_{\delta\to 0^+} w(x) = 1\,, \qquad \text{uniformly in } x \in [a,b]\,.$$

Then, there exists a sufficiently small $\delta > 0$ such that

$$\max\{\|\alpha'\|_\infty, \|\beta'\|_\infty\} < w(x) \leq 1-\delta\,, \qquad \text{for every } x \in [a,b]\,.$$

Hence, the function $\gamma_+(x) = w(x)$ satisfies the first part of (13) for system (36). On the other hand, using (34), from the differential equation in (37) we have

$$f(\gamma_+(x))\gamma_+'(x) = -\hat{c}(1-\gamma_+^2(x))^\sigma < g_\delta(t,x,\gamma_+(x))\,,$$

so that also the second part of (13) is satisfied.

Similarly we can construct $\gamma_-(x)$ satisfying (12) for system (36). Theorem 5 thus applies, providing a solution $(x(t), y(t))$ of system (36) for which (14) holds. Since $-1 + \delta \le \gamma_-(x) < \gamma_+(x) \le 1 - \delta$ for every $x \in [a, b]$, the proof is completed. $\square$

In the particular case $\sigma = 3/2$, we have the equation

$$\left(\frac{x'}{\sqrt{1 - (x')^2}} \right)' = p(t, x, x'),$$

and we recover the result in [4, Theorem 3] for the T-periodic problem, since (34) holds when $p(t, x, y) = p(t, x)$ is continuous on $\mathbb{R} \times [a, b]$ and T-periodic in its first variable.

4 Proof of Theorem 5

The proof will be divided into four steps. In the first one, we assume that α, β are strict lower/upper solutions, and that $f(t, x, y)$, $g(t, x, y)$ are locally Lipschitz continuous in (x, y), i.e., for every $(t_0, x_0, y_0) \in \mathbb{R}^3$, there are $\delta_0 > 0$ and $L_0 \ge 0$ such that, if

$$\|(t, x, y) - (t_0, x_0, y_0)\| < \delta, \qquad \|(t, \xi, \eta) - (t_0, x_0, y_0)\| < \delta,$$

then

$$|f(t, x, y) - f(t, \xi, \eta)| \le L_0\big(|x - \xi| + |y - \eta|\big),$$

and

$$|g(t, x, y) - g(t, \xi, \eta)| \le L_0\big(|x - \xi| + |y - \eta|\big).$$

Under these additional assumptions, we prove the existence of a solution satisfying (14). In the second step, we only require f and g to be continuous, while maintaining the assumption that α and β are strict. In the third step, we conclude the proof of the existence of a solution satisfying (14), in the general case. Finally we prove the second statement of the theorem, concerning the existence of a periodic solution, still satisfying (14).

Without loss of generality, we assume that α and β satisfy the inequalities (6) and (10) with the same positive constants m and δ. Let us introduce the open set

$$V = \{(t, x, y) \in \mathbb{R}^3 : t \in \mathbb{R}, \, \alpha(t) < x < \beta(t), \, \gamma_-(x) < y < \gamma_+(x)\}.$$

We will indeed prove a more general result, assuming that the functions α, β and $\gamma_\pm$ are such that the inequalities in (4), (6), (8), (10) in the definitions of lower/upper solution hold only for $(t, x, y) \in \overline{V}$. For simplicity, we will continue to speak about lower/upper solutions even in this more general case. This observation will also lead us to a generalization of Theorem 5 stated in Section 5.

Step 1. Locally Lipschitz continuous and strict

We assume that α, β are strict lower/upper solutions, and that the restrictions of $f(t,x,y)$ and $g(t,x,y)$ to $\overline{V}$ are locally Lipschitz continuous in (x,y).

We modify f and g outside the set $\overline{V}$. For any $\mu < \nu$, let

$$\tau(x;\mu,\nu) = \begin{cases} \mu & \text{if } x \le \mu\,, \\ x & \text{if } \mu \le x \le \nu\,, \\ \nu & \text{if } x \ge \nu\,. \end{cases}$$

Define

$$\tilde f(t,x,y) = f\Big(t, \tau(x;\alpha(t),\beta(t)), \tau\big(y; \gamma_-(\tau(x;\alpha(t),\beta(t))), \gamma_+(\tau(x;\alpha(t),\beta(t)))\big)\Big),$$

and

$$\tilde g(t,x,y) = g\Big(t, \tau(x;\alpha(t),\beta(t)), \tau\big(y; \gamma_-(\tau(x;\alpha(t),\beta(t))), \gamma_+(\tau(x;\alpha(t),\beta(t)))\big)\Big).$$

Clearly, on $\overline{V}$ the functions $\tilde f, \tilde g$ coincide with f, g, respectively. From now on we concentrate our study on the system

$$x' = \tilde f(t,x,y)\,, \qquad y' = \tilde g(t,x,y)\,. \tag{38}$$

Notice that α and β are a lower and an upper solution of this system, as well.

The solutions to initial value problems are unique and globally defined, since for every compact interval $J = [t_1, t_2]$ there is a constant $c_J > 0$ such that

$$|\tilde f(t,x,y)| + |\tilde g(t,x,y)| \le c_J\,, \qquad \text{for every } t \in J\,.$$

Claim. *The set E of egress points of V can be written as*

$$E = E_1 \cup E_2 \cup E_3 \cup E_4\,,$$

with

$$\begin{aligned} E_1 &= \{(t,\alpha(t),y) : \gamma_-(\alpha(t)) < y < y_\alpha(t)\}\,, \\ E_2 &= \{(t,\beta(t),y) : y_\beta(t) < y < \gamma_+(\beta(t))\}\,, \\ E_3 &= \{(t,x,\gamma_-(x)) : \alpha(t) \le x < \beta(t)\}\,, \\ E_4 &= \{(t,x,\gamma_+(x)) : \alpha(t) < x \le \beta(t)\}\,, \end{aligned}$$

and all points of E are strict egress points.

We recall that a boundary point (t_0,x_0,y_0) of the open set V is said to be an *egress point* if there is an $\varepsilon > 0$ such that the solution $(x(t),y(t))$ with initial value $(x(t_0),y(t_0)) = (x_0,y_0)$ is such that $(t,x(t),y(t)) \in V$ for every $t \in \,]t_0-\varepsilon, t_0[\,$. It is said to be a *strict egress point* if, moreover, $(t,x(t),y(t)) \notin \overline{V}$ for every $t \in \,]t_0, t_0+\varepsilon[\,$. In Figure 2 we provide a pictorial view of a section of the set V at a fixed time t; the coloured region represents the corresponding section of E, the set of egress points of V.

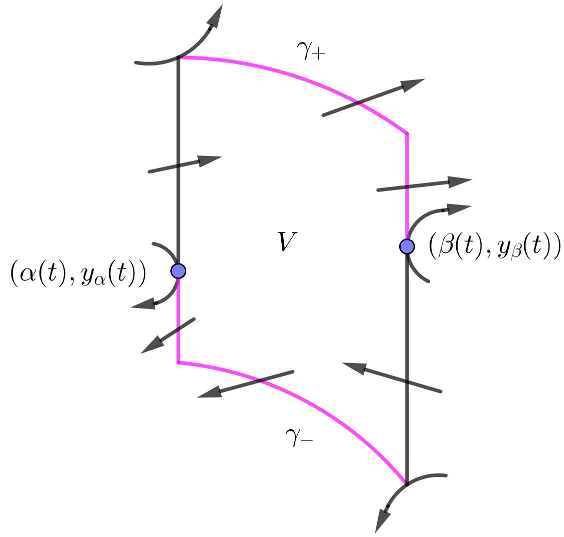

Figure 2: *A section of the set V at time t with its egress points*

We now show how to prove the existence of a solution satisfying (14) assuming that the Claim holds true. Once this is done, the proof of the Claim will be provided.

For every integer $n \geq 0$, let us define the set

$$Z_n = \{(-n, x, \gamma_n(x)) : \alpha(-n) \leq x \leq \beta(-n)\},$$

where $\gamma_n : [\alpha(-n), \beta(-n)] \to \mathbb{R}$ is given by

$$\gamma_n(x) = \frac{\beta(-n) - x}{\beta(-n) - \alpha(-n)}\, \gamma_-(x) + \frac{x - \alpha(-n)}{\beta(-n) - \alpha(-n)}\, \gamma_+(x).$$

Notice that $Z_n \subseteq V \cup E$, and

$$Z_n \cap E = \big\{\big(-n, \alpha(-n), \gamma_-(\alpha(-n))\big), \big(-n, \beta(-n), \gamma_+(\beta(-n))\big)\big\},$$

so that $Z_n \cap E$ is a retract of E but it is not a retract of Z_n. Therefore, by Ważewski Theorem (see [13, Chapter 10] or [31, page 596]), there is a $\bar{x}_n \in]\alpha(-n), \beta(-n)[$ such that the solution $(x_n(t), y_n(t))$ satisfying $x_n(-n) = \bar{x}_n$, $y_n(-n) = \gamma_n(\bar{x}_n)$ is such that

$$(t, x_n(t), y_n(t)) \in \overline{V}, \qquad \text{for every } t \geq -n.$$

Set $\hat{x}_n = x_n(0)$, $\hat{y}_n = y_n(0)$, and consider the sequence $(\hat{x}_n, \hat{y}_n)_n$. By compactness, there is a subsequence $(\hat{x}_{n_k}, \hat{y}_{n_k})_k$ which converges to some $(\hat{x}, \hat{y})$, with $(0, \hat{x}, \hat{y}) \in \overline{V}$. Let $(x(t), y(t))$ be the solution satisfying $x(0) = \hat{x}$, $y(0) = \hat{y}$. We will show that this solution satisfies (14), i.e.,

$$(t, x(t), y(t)) \in \overline{V}, \qquad \text{for every } t \in \mathbb{R}.$$

By contradiction, assume there exists a $\bar{t} \in \mathbb{R}$ such that $(\bar{t}, x(\bar{t}), y(\bar{t})) \notin \overline{V}$. Then, by continuous dependence on initial data, there is a positive integer k such that $\bar{t} > -n_k$ and $(\bar{t}, x_{n_k}(\bar{t}), y_{n_k}(\bar{t})) \notin \overline{V}$, in contradiction with the above.

We now prove the Claim.

If $(x(t), y(t))$ is a solution such that $(\bar{t}, x(\bar{t}), y(\bar{t})) \in E_1$, for some $\bar{t} \in \mathbb{R}$, then $x(\bar{t}) = \alpha(\bar{t})$ and $\gamma_-(x(\bar{t})) < y(\bar{t}) < y_\alpha(\bar{t})$, hence by (4),

$$x'(\bar{t}) = \tilde{f}(\bar{t}, \alpha(\bar{t}), y(\bar{t})) = f(\bar{t}, \alpha(\bar{t}), y(\bar{t})) < \alpha'(\bar{t})\,.$$

Then, there is $\varepsilon > 0$ such that

$$x(t) \begin{cases} > \alpha(t) & \text{if } t \in]\bar{t} - \varepsilon, \bar{t}[\,, \\ < \alpha(t) & \text{if } t \in]\bar{t}, \bar{t} + \varepsilon[\,, \end{cases}$$

and, by continuity and the first part of (12),

$$\gamma_-(x(t)) < y(t) < y_\alpha(t) < \gamma_+(x(t))\,, \qquad \text{for every } t \in]\bar{t} - \varepsilon, \bar{t}[\,.$$

This proves that the points of E_1 are strict egress points of V. Analogously one proves that the points of E_2 are strict egress points.

If $(x(t), y(t))$ is a solution such that $(\bar{t}, x(\bar{t}), y(\bar{t})) \in E_3$, for some $\bar{t} \in \mathbb{R}$, then $\alpha(\bar{t}) \le x(\bar{t}) < \beta(\bar{t})$ and $y(\bar{t}) = \gamma_-(x(\bar{t}))$, hence, setting $F(t) = y(t) - \gamma_-(x(t))$, by (12),

$$F'(\bar{t}) = g(\bar{t}, x(\bar{t}), \gamma_-(x(\bar{t}))) - f(\bar{t}, x(\bar{t}), \gamma_-(x(\bar{t})))\gamma_-'(x(\bar{t})) < 0\,.$$

Then, if $x(\bar{t}) > \alpha(\bar{t})$, there is $\varepsilon > 0$ such that

$$\alpha(t) < x(t) < \beta(t)\,, \qquad \text{for every } t \in]\bar{t} - \varepsilon, \bar{t} + \varepsilon[\,,$$

and $y(\cdot) - \gamma_-(x(\cdot))$ is strictly decreasing on $]\bar{t} - \varepsilon, \bar{t} + \varepsilon[$, hence

$$y(t) \begin{cases} > \gamma_-(x(t)) & \text{if } t \in]\bar{t} - \varepsilon, \bar{t}[\,, \\ < \gamma_-(x(t)) & \text{if } t \in]\bar{t}, \bar{t} + \varepsilon[\,, \end{cases}$$

proving that $(\bar{t}, x(\bar{t}), y(\bar{t}))$ is a strict egress point. On the other hand, if $x(\bar{t}) = \alpha(\bar{t})$, we need to consider the normal cone at $\bar{v} = (\bar{t}, \alpha(\bar{t}), \gamma_-(\alpha(\bar{t})))$, i.e.,

$$\mathcal{N}(\bar{v}) = \Big\{ c\big(\lambda(\alpha'(\bar{t}), -1, 0) + (1 - \lambda)(0, \gamma_-'(\alpha(\bar{t})), -1)\big) : c \ge 0, \lambda \in [0, 1] \Big\}\,.$$

We then easily verify, using (4) and (12), that

$$\big\langle \lambda(\alpha'(\bar{t}), -1, 0) + (1 - \lambda)(0, \gamma_-'(\alpha(\bar{t})), -1)\,,\, (1, f(\bar{v}), g(\bar{v})) \big\rangle > 0\,,$$

for every $\lambda \in [0, 1]$, proving that $\bar{v}$ is a strict egress point.

This shows that the points of E_3 are strict egress points of V. Analogously one proves that the points of E_4 are strict egress points.

We now need to show that there are no other egress points of V. To this aim, we write $\partial V \setminus (E_1 \cup E_2 \cup E_3 \cup E_4) = F_1 \cup F_2$, with

$$F_1 = \{(t, \alpha(t), y) : y_\alpha(t) \le y \le \gamma_+(\alpha(t))\}\,,$$
$$F_2 = \{(t, \beta(t), y) : \gamma_-(\beta(t)) \le y \le y_\beta(t)\}\,.$$

Let us prove that $F_1 \cap E = \emptyset$. Let $(x(t), y(t))$ be a solution such that $x(\bar t) = \alpha(\bar t)$ and $y_\alpha(\bar t) \le y(\bar t) \le \gamma_+(\alpha(\bar t))$, for some $\bar t \in \mathbb{R}$. If $y(\bar t) > y_\alpha(\bar t)$, then, by (4),

$$x'(\bar t) = \tilde f(\bar t, \alpha(\bar t), y(\bar t)) > \alpha'(\bar t)\,.$$

Then, there is $\varepsilon > 0$ such that

$$x(t) < \alpha(t)\,, \qquad \text{for every } t \in \,]\bar t - \varepsilon, \bar t\,[\,.$$

This proves that $(\bar t, x(\bar t), y(\bar t))$ is not an egress point of V.

Assume now $x(\bar t) = \alpha(\bar t)$ and $y(\bar t) = y_\alpha(\bar t)$. Then, defining

$$\xi(t) = x(t) - \alpha(t)\,, \qquad \eta(t) = y(t) - y_\alpha(t)\,,$$

by (7) we have that

$$\xi(\bar t) = 0 \quad \text{ and } \quad \xi'(\bar t) = f(\bar t, \alpha(\bar t), y_\alpha(\bar t)) - \alpha'(\bar t) = 0\,,$$

while, since α is a strict lower solution,

$$\eta(\bar t) = 0 \quad \text{ and } \quad \eta'(\bar t) = g(\bar t, \alpha(\bar t), y_\alpha(\bar t)) - y_\alpha'(\bar t) < 0\,.$$

Let m, δ be the positive constants introduced in (iii). Then, there is a $\bar\tau < \bar t$ such that

$$m|\xi(t)| < \eta(t) \le \delta\,, \qquad \text{for every } t \in \,]\bar\tau, \bar t\,[\,,$$

i.e., $y(t) - y_\alpha(t) \le \delta$ and $y(t) > y_\alpha(t) + m|x(t) - \alpha(t)|$, for every $t \in \,]\bar\tau, \bar t\,[$. By (6), we obtain

$$x'(t) = \tilde f(t, x(t), y(t)) > \alpha'(t)\,, \qquad \text{for every } t \in \,]\bar\tau, \bar t\,[\,,$$

and since $x'(\bar t) = \alpha'(\bar t)$, it follows that $x(t) < \alpha(t)$ for every $t \in \,]\bar\tau, \bar t\,[$. Hence, $(\bar t, x(\bar t), y(\bar t))$ can not be an egress point of V.

We have thus proved that $F_1 \cap E = \emptyset$. In an analogous way one shows that $F_2 \cap E = \emptyset$. The Claim is thus proved. □

Step 2. Only continuous and strict

Having proved the first statement of the theorem in the case when α, β are strict lower/upper solutions and the restrictions of $f(t, x, y)$ and $g(t, x, y)$ to $\overline{V}$ are locally Lipschitz continuous in (x, y), we now consider the case when f and g are only continuous, maintaining for the moment the assumption of α, β being strict. In this case, we will construct two sequences of continuous functions $f_n : \mathbb{R}^3 \to \mathbb{R}$, $g_n : \mathbb{R}^3 \to \mathbb{R}$, whose restrictions to $\overline{V}$ are locally Lipschitz continuous in (x, y), converging uniformly on compact sets towards f and g, respectively, and for which the assumptions of Theorem 5 still hold.

Concerning the function g, this approximation can be made by a standard regularization procedure, since we only have to guarantee that the strict inequalities in (5) and (9) still hold.

The approximation of the function f is more delicate, since we have to preserve the conditions (i) and (iii) in Definitions 1 and 4, and these conditions involve more subtle inequalities, because of (7) and (11). Let us now describe how such an approximation can be constructed.

Since the section of $\overline{V}$ at a fixed time $t \in \mathbb{R}$ is compact, by assumptions (i) and (iii) it is possible to find a continuously differentiable function $\epsilon : \mathbb{R} \to]0, 1]$ such that, for every $(t, x, y) \in \overline{V}$, we have that $f(t, x, y) < \alpha'(t)$ if either

$$y_\alpha(t) - \delta \le y < y_\alpha(t) - m|x - \alpha(t)|\,,$$

or

$$|x - \alpha(t)| \le \epsilon(t) \;\text{ and }\; y < y_\alpha(t) - \delta\,,$$

while $f(t, x, y) > \alpha'(t)$ if either

$$y_\alpha(t) + m|x - \alpha(t)| < y \le y_\alpha(t) + \delta\,,$$

or

$$|x - \alpha(t)| \le \epsilon(t) \;\text{ and }\; y > y_\alpha(t) + \delta\,,$$

and similarly for β, with the same $\epsilon(t)$.

For any fixed $t \in \mathbb{R}$, we now construct a triangularization of the plane xy, depending on t. Consider, for every positive integer n, the lines

$$y = \pm mx + \frac{k}{n}\epsilon(t)\,, \qquad \text{with } k \in \mathbb{Z}\,,$$

together with the four lines, varying with t,

$$y = \pm m(x - \alpha(t)) + y_\alpha(t)\,, \qquad y = \pm m(x - \beta(t)) + y_\beta(t)\,.$$

For any fixed $t \in \mathbb{R}$, all these lines form a grid in $\mathbb{R}^2$, with an infinite number of quadrilaterals. We can visualize this grid in Figure 3.

We now join the opposite vertices of each quadrilateral, so to obtain infinitely many triangles. On each of these triangles we make a convex interpolation of the function f. We thus obtain, for every positive integer n, a function $f_n : \mathbb{R}^3 \to \mathbb{R}$, whose restriction to the set V is locally Lipschitz continuous in (x, y). It can be seen that this function is continuous. Moreover, there is a $\delta' \in]0, \delta[$ such that, if n is sufficiently large, f_n satisfies the inequalities in (4), (6), (8), (10), with δ replaced by δ', for every $(t, x, y) \in \overline{V}$. Since f is uniformly continuous on any compact set, we conclude that the sequence $(f_n)_n$ converges to f uniformly on compact sets.

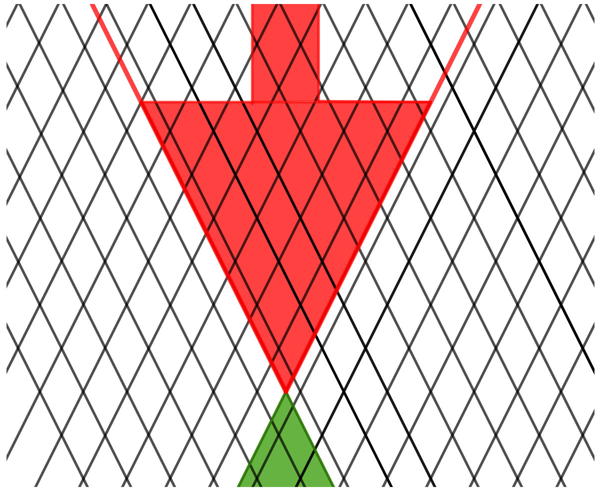

Figure 3: *The grid made of quadrilaterals near the point* $(\alpha(t), y_\alpha(t))$

Hence, if n is sufficiently large, α and β are still strict lower/upper solutions for the modified system

$$x' = f_n(t, x, y)\,, \qquad y' = g_n(t, x, y)\,, \tag{39}$$

meaning that the inequalities in the definitions of lower/upper solution hold for $(t, x, y) \in \overline{V}$. By Step 1, we know that there is a solution of (39), which we denote by $(x_n(t), y_n(t))$, and that

$$(t, x_n(t), y_n(t)) \in \overline{V}\,, \qquad \text{for every } t \in \mathbb{R}\,. \tag{40}$$

Set $z_n(t) = (x_n(t), y_n(t))$ and, for every positive integer m, denote by $z_{n,m}$ the restriction of z_n to the interval $[-m, m]$. Then, $(z_{n,m})_n$ is a sequence taking values in a compact set $\mathcal{K}_m$ of $\mathbb{R}^2$. Moreover, since the functions $z_{n,m}$ are solutions of (39) and the functions $\tilde{f}, \tilde{g}$ are bounded on $\overline{V} \cap ([-m, m] \times \mathbb{R}^2)$, their derivatives $z'_{n,m}$ are equi-uniformly bounded, so that $(z_{n,m})_n$ is equi-uniformly continuous.

We now proceed recursively. If $m = 1$, the Ascoli–Arzelà Theorem provides us a strictly increasing sequence of indices $(n_{k,1})_k$ such that the subsequence $(z_{n_{k,1},1})_k$ uniformly converges on $[-1, 1]$ to some continuous function $z_{*,1} : [-1, 1] \to \mathbb{R}^2$. To simplify the notation, we denote by $(z^1_n)_n$ the subsequence $(z_{n_{k,1}})_k$ of $(z_n)_n$ having the same indices $n_{k,1}$ as $(z_{n_{k,1},1})_k$.

By the same argument, if $m = 2$, we find a strictly increasing sequence of indices $(n_{k,2})_k$ such that the subsequence $(z^1_{n_{k,2},2})_k$ uniformly converges on $[-2, 2]$ to some continuous function $z_{*,2} : [-2, 2] \to \mathbb{R}^2$. Clearly, $z_{*,2}(t) = z_{*,1}(t)$ for every $t \in [-1, 1]$. We denote by $(z^2_n)_n$ the subsequence $(z^1_{n_{k,2}})_k$ of $(z_n)_n$ having the same indices $n_{k,2}$ as $(z^1_{n_{k,2},2})_k$.

We can thus define, for every m, the subsequence $(z^m_n)_n$ of $(z_n)_n$ which uniformly converges on $[-m, m]$ to some continuous function $z_{*,m} : [-m, m] \to$

$\mathbb{R}^2$. Then, the diagonal sequence $(z_n^n)_n$ converges uniformly on compact sets to some continuous function $z_*(t) = (x_*(t), y_*(t))$, which coincides with each $z_{*,m}(t)$ when $t \in [-m, m]$.

Writing the integral representation of the solutions of (39) and passing to the limit, we easily conclude that $(x_*(t), y_*(t))$ is the solution we are looking for. $\square$

Step 3. Conclusion of the proof for the bounded solution

Finally, having proved the first statement of the theorem in the case when α and β are strict lower/upper solutions, we now consider the general case. Define the functions

$$g_n(t,x,y) = g(t,x,y) + \frac{1}{n}\left(x - \frac{\alpha(t)+\beta(t)}{2}\right).$$

Since $\alpha(t) < \beta(t)$ for every $t \in \mathbb{R}$, it is easy to verify that, for n large enough, α and β are strict lower/upper solutions for system (39). Hence, by Step 2, we know that there is a solution of

$$x' = f(t,x,y)\,, \qquad y' = g_n(t,x,y)\,,$$

which we denote by $(x_n(t), y_n(t))$, satisfying (40) for every $t \in \mathbb{R}$. We now conclude by the Ascoli–Arzelà Theorem, as above. $\square$

Step 4. The periodic case

We now assume that f, g are T-periodic in their first variable t, and α, β are also T-periodic. As in Step 1, we first assume that α, β are strict lower/upper solutions, and that the restrictions of $f(t,x,y)$ and $g(t,x,y)$ to $\overline{V}$ are locally Lipschitz continuous in (x,y). We modify f and g as above, and consider system (38). Notice that $\tilde{f}$ and $\tilde{g}$ are T-periodic in their first variable. Having proved the existence of a bounded solution, by Massera Theorem, cf. [23, page 26], system (38) has a T-periodic solution $(x(t), y(t))$.

Let us prove that

$$\alpha(t) < x(t) < \beta(t)\,, \qquad \text{for every } t \in \mathbb{R}\,. \tag{41}$$

Assume by contradiction that there is a $\bar{t} \in \mathbb{R}$ such that $x(\bar{t}) \le \alpha(\bar{t})$. We have two cases.

<u>Case 1</u>. $y(\bar{t}) \le y_\alpha(\bar{t})$. Let us show that the set

$$A_1 = \{(t,x,y) : t \in \mathbb{R},\, x \le \alpha(t),\, y \le y_\alpha(t)\}$$

is positively invariant. Indeed, if $x(t) = \alpha(t)$ and $y(t) < y_\alpha(t)$, for some $t \in \mathbb{R}$, then, by (4),

$$x'(t) = \tilde{f}(t, \alpha(t), y(t)) = f\big(t, \alpha(t), \max\{y(t), \gamma_-(\alpha(t))\}\big) < \alpha'(t)\,.$$

On the other hand, if $x(t) \le \alpha(t)$ and $y(t) = y_\alpha(t)$, for some $t \in \mathbb{R}$, then, by the assumption that α is a *strict* lower solution,

$$y'(t) = \tilde{g}(t, x(t), y_\alpha(t)) = g(t, \alpha(t), y_\alpha(t)) < y_\alpha'(t)\,.$$

We thus have that A_1 is strongly positively invariant, so that

$$x(t) < \alpha(t) \quad \text{ and } \quad y(t) < y_\alpha(t)\,, \quad \text{ for every } t > \bar{t}\,.$$

Then, since $\max\{y(t), \gamma_-(\alpha(t))\} < y_\alpha(t)$ for every $t > \bar{t}$, it follows again from (4) that

$$x'(t) = \tilde{f}(t, x(t), y(t)) = f\big(t, \alpha(t), \max\{y(t), \gamma_-(\alpha(t))\}\big) < \alpha'(t)\,,$$

for every $t > \bar{t}$, and we get a contradiction, since both x and α are T-periodic.

<u>Case 2</u>. $y(\bar{t}) > y_\alpha(\bar{t})$. We can exclude the possibility that $x(t) \le \alpha(t)$ and $y(t) > y_\alpha(t)$ for every $t > \bar{t}$, since this would imply that $\min\{y(t), \gamma_+(\alpha(t))\} > y_\alpha(t)$, and hence, by (4),

$$x'(t) = \tilde{f}(t, x(t), y(t)) = f\big(t, \alpha(t), \min\{y(t), \gamma_+(\alpha(t))\}\big) > \alpha'(t)\,,$$

for every $t > \bar{t}$, yielding a contradiction. Then, since A_1 is strongly positively invariant, there is a $\hat{t} \in]\bar{t}, \bar{t}+T[$ such that $x(\hat{t}) > \alpha(\hat{t})$ and $y(\hat{t}) > y_\alpha(\hat{t})$. By the T-periodicity of $x(t)$, there must be a $\check{t} \in]\hat{t}, \bar{t}+T]$ such that $x(\check{t}) = \alpha(\check{t})$ and $x(t) > \alpha(t)$ for every $t \in]\hat{t}, \check{t}[$. Since A_1 is strongly positively invariant, $y(\check{t}) \le y_\alpha(\check{t})$ is excluded, so that $y(\check{t}) > y_\alpha(\check{t})$ and, by (4),

$$x'(\check{t}) = \tilde{f}(\check{t}, \alpha(\check{t}), y(\check{t})) = f\big(\check{t}, \alpha(\check{t}), \min\{y(\check{t}), \gamma_+(\alpha(\check{t}))\}\big) > \alpha'(\check{t})\,,$$

a contradiction.

We have thus proved that $x(t) > \alpha(t)$, for every $t \in \mathbb{R}$. The proof that $x(t) < \beta(t)$ is analogous, by the use of (8). So, (41) holds.

Let us now prove that

$$\gamma_-(x(t)) < y(t) < \gamma_+(x(t))\,, \quad \text{ for every } t \in \mathbb{R}\,. \tag{42}$$

Assume by contradiction that there is a $\bar{t} \in \mathbb{R}$ such that $y(\bar{t}) \le \gamma_-(x(\bar{t}))$. Since (41) holds and E_3 is made of strict egress points, it has to be that $y(t) < \gamma_-(x(t))$ for every $t > \bar{t}$. Therefore,

$$x'(t) = f(t, x(t), \gamma_-(x(t)))\,, \qquad y'(t) = g(t, x(t), \gamma_-(x(t)))\,,$$

hence, by (12),

$$\frac{d}{dt}\gamma_-(x(t)) = f(t, x(t), \gamma_-(x(t)))\gamma_-'(x(t)) > g(t, x(t), \gamma_-(x(t))) = y'(t)\,,$$

for every $t > \bar{t}$. Since both functions $\gamma_-(x(t))$ and $y(t)$ are T-periodic, this leads to a contradiction.

We have thus proved that $y(t) > \gamma_-(x(t))$, for every $t \in \mathbb{R}$. Similarly one proves that $y(t) < \gamma_+(x(t))$, for every $t \in \mathbb{R}$. So, (42) holds, and the proof of the existence of a T-periodic solution of (3) is completed in this case.

Concerning the general case when f, g are only continuous, and α, β are not necessarily strict, one proceeds by approximation, exactly as in Steps 2 and 3. The proof is thus completed. □

5 Towards a general definition of lower/upper solutions for planar systems

In this section we briefly explain how, in view of the ideas in the above proof, Theorem 5 can be generalized. We consider system (3) where, as usual, $f, g : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions. Let us introduce the concept of vector lower/upper solution.

Definition 11. *Let $\vec{\alpha} : \mathbb{R} \to \mathbb{R}^2$ be a continuously differentiable function, with $\vec{\alpha}(t) = (\alpha_1(t), \alpha_2(t))$. We say that $\vec{\alpha}$ is a* vector lower solution *of* (3) *if*

$$\alpha_1'(t) = f(t, \alpha_1(t), \alpha_2(t)), \qquad \alpha_2'(t) \geq g(t, \alpha_1(t), \alpha_2(t)),$$

for every $t \in \mathbb{R}$, and there are two continuously differentiable functions $\mathcal{A}_\pm : \mathbb{R} \to \mathbb{R}$, with

$$\mathcal{A}_-(t) < \alpha_2(t) < \mathcal{A}_+(t), \quad \text{for every } t \in \mathbb{R},$$

and two continuous functions $\delta : \mathbb{R} \to]0, 1[$, $m : \mathbb{R} \to]0, +\infty[$, with the following properties. For every $t \in \mathbb{R}$,

$$\begin{cases} \mathcal{A}_-(t) < y < \alpha_2(t) & \Rightarrow \quad f(t, \alpha_1(t), y) < \alpha_1'(t), \\ \alpha_2(t) < y < \mathcal{A}_+(t) & \Rightarrow \quad f(t, \alpha_1(t), y) > \alpha_1'(t); \end{cases}$$

$$\begin{cases} \alpha_2(t) - \delta(t) < y < \alpha_2(t) - m(t)|x - \alpha_1(t)| & \Rightarrow \quad f(t, x, y) < \alpha_1'(t), \\ \alpha_2(t) + m(t)|x - \alpha_1(t)| < y < \alpha_2(t) + \delta(t) & \Rightarrow \quad f(t, x, y) > \alpha_1'(t). \end{cases}$$

Definition 12. *Let $\vec{\beta} : \mathbb{R} \to \mathbb{R}^2$ be a continuously differentiable function, with $\vec{\beta}(t) = (\beta_1(t), \beta_2(t))$. We say that $\vec{\beta}$ is a* vector lower solution *of* (3) *if*

$$\beta_1'(t) = f(t, \beta_1(t), \beta_2(t)), \qquad \beta_2'(t) \leq g(t, \beta_1(t), \beta_2(t)),$$

for every $t \in \mathbb{R}$, and there are two continuously differentiable functions $\mathcal{B}_\pm : \mathbb{R} \to \mathbb{R}$, with

$$\mathcal{B}_-(t) < \beta_2(t) < \mathcal{B}_+(t), \quad \text{for every } t \in \mathbb{R},$$

and two continuous functions $\delta : \mathbb{R} \to]0, 1[$, $m : \mathbb{R} \to]0, +\infty[$, with the following properties. For every $t \in \mathbb{R}$,

$$\begin{cases} \mathcal{B}_-(t) < y < \beta_2(t) & \Rightarrow \quad f(t, \beta_1(t), y) < \beta_1'(t), \\ \beta_2(t) < y < \mathcal{B}_+(t) & \Rightarrow \quad f(t, \beta_1(t), y) > \beta_1'(t); \end{cases}$$

$$\begin{cases} \beta_2(t) - \delta(t) < y < \beta_2(t) - m(t)|x - \beta_1(t)| & \Rightarrow \quad f(t,x,y) < \beta_1'(t)\,, \\ \beta_2(t) + m(t)|x - \beta_1(t)| < y < \beta_2(t) + \delta(t) & \Rightarrow \quad f(t,x,y) > \beta_1'(t)\,. \end{cases}$$

We are now in the position to state our general result.

Theorem 13. *Let $\vec{\alpha}, \vec{\beta}$ be two bounded vector lower and upper solutions, with $\alpha_1(t) < \beta_1(t)$ for every $t \in \mathbb{R}$, and consider the set*

$$D = \{(t,x) : t \in \mathbb{R}\,, \alpha_1(t) \le x \le \beta_1(t)\}\,.$$

Let $\gamma_\pm : D \to \mathbb{R}$ be two continuously differentiable bounded functions such that for every $t \in \mathbb{R}$,

$$\mathcal{A}_-(t) < \gamma_-(t, \alpha_1(t)) < \alpha_2(t) < \gamma_+(t, \alpha_1(t)) < \mathcal{A}_+(t)\,,$$

$$\mathcal{B}_-(t) < \gamma_-(t, \beta_1(t)) < \beta_2(t) < \gamma_+(t, \beta_1(t)) < \mathcal{B}_+(t)\,,$$

and for every $(t,x) \in D$,

$$\frac{\partial \gamma_-}{\partial t}(t,x) + \frac{\partial \gamma_-}{\partial x}(t,x) f(t,x,\gamma_-(t,x)) > g(t,x,\gamma_-(t,x))\,,$$

$$\frac{\partial \gamma_+}{\partial t}(t,x) + \frac{\partial \gamma_+}{\partial x}(t,x) f(t,x,\gamma_+(t,x)) < g(t,x,\gamma_+(t,x))\,.$$

Then, there exists a solution $(x(t), y(t))$ of (3) *satisfying*

$$\alpha_1(t) \le x(t) \le \beta_1(t) \quad \text{and} \quad \gamma_-(t, x(t)) \le y(t) \le \gamma_+(t, x(t))\,, \tag{43}$$

for every $t \in \mathbb{R}$. If, moreover, f, g, and $\gamma_\pm$ are T-periodic in their first variable t, and α, β are also T-periodic, then there exists a T-periodic solution $(x(t), y(t))$ of (3) *satisfying* (43).

The assumptions in the above theorem remind those in [17, 28], where a scalar second order differential equation was treated. We omit the proof of Theorem 13, for briefness, since it is a straightforward modification of the proof of Theorem 5.

6 Almost periodic solutions

Ortega and Tarallo [24] showed that, for an almost periodic differential equation, the existence of well-ordered almost periodic lower and upper solutions for the scalar equation (1) is not sufficient in general to guarantee the existence of an almost periodic solution. More precisely, they proved that, for any $\omega \in \mathbb{R} \setminus \mathbb{Q}$, there are a constant $c > 0$, a continuous function $G : \mathbb{R}^2 \to \mathbb{R}$, which is 2π-periodic in both variables, and constants $\alpha < \beta$ and δ such that

$$G(t, \omega t, \alpha) \le -\delta < 0 < \delta \le G(t, \omega t, \beta)\,, \quad \text{for every } t \in \mathbb{R}\,,$$

for which equation (1) with $g(t,x,y) = G(t, \omega t, x) - cy$ has no almost periodic solutions. This example, contradicting the result claimed in [29], shows how delicate the existence of almost periodic solutions can be. Nevertheless, Corduneanu [6] proved that if $g(t,x,y)$ is continuously differentiable and satisfies some strict monotonicity assumption with respect to x, then any bounded solution of (1) is almost periodic. We report the precise statement and proof of his theorem in the Appendix.

Applying Corduneanu's theorem, we easily get some corollaries of our results. Let us state a first one, where the Nagumo condition appears.

Corollary 14. *Let $g : \mathbb{R}^3 \to \mathbb{R}$ be continuous, and continuously differentiable with respect to (x,y). Assume the existence of an almost periodic lower solution α and an almost periodic upper solution β, with $\alpha(t) < \beta(t)$ for every $t \in \mathbb{R}$, and let $a = \inf \alpha$, $b = \sup \beta$. Moreover, let $\varphi : [0,+\infty[\to \mathbb{R}$ be a positive continuous function such that*

$$|g(t,x,y)| \le \varphi(|y|)\,, \qquad \text{for every } (t,x,y) \in \mathbb{R} \times [a,b] \times \mathbb{R}\,,$$

and

$$\int_0^{+\infty} \frac{s}{\varphi(s)}\,ds = +\infty\,.$$

If g is almost periodic in t uniformly with respect to $(x,y) \in [a,b] \times \mathbb{R}$, and there exists a constant m for which

$$\frac{\partial g}{\partial x}(t,x,y) \ge m > 0\,, \qquad \text{for every } (t,x,y) \in \mathbb{R} \times [a,b] \times \mathbb{R}\,, \tag{44}$$

then equation (1) *has an almost periodic solution satisfying*

$$\alpha(t) \le x(t) \le \beta(t)\,, \qquad \text{for every } t \in \mathbb{R}\,. \tag{45}$$

Proof. Just apply Corollary 7 and Corduneanu's Theorem. □

Concerning the mean curvature equation

$$\left(\frac{x'}{\sqrt{1+(x')^2}}\right)' = h(t,x,x')\,, \tag{46}$$

we have the following.

Corollary 15. *Let $h : \mathbb{R}^3 \to \mathbb{R}$ be continuous, and continuously differentiable with respect to (x,y). Assume the existence of an almost periodic lower solution α and an almost periodic upper solution β, with $\alpha(t) < \beta(t)$ for every $t \in \mathbb{R}$, and let $a = \inf \alpha$, $b = \sup \beta$. Assume moreover that there is a $c \ge 0$ such that*

$$|h(t,x,y)| \le c\,, \qquad \text{for every } (t,x,y) \in \mathbb{R} \times [a,b] \times \mathbb{R}\,,$$

with

$$c < \frac{1}{b-a} \min\left\{ \frac{1}{\sqrt{1+\|\alpha'\|_\infty^2}}\,,\ \frac{1}{\sqrt{1+\|\beta'\|_\infty^2}} \right\}.$$

If h is almost periodic in t uniformly with respect to $(x,y) \in [a,b] \times \mathbb{R}$, and there exists a constant m for which

$$\frac{\partial h}{\partial x}(t,x,y) \geq m > 0\,, \qquad \textit{for every } (t,x,y) \in \mathbb{R} \times [a,b] \times \mathbb{R}\,, \tag{47}$$

then equation (46) *has an almost periodic solution $x(t)$ satisfying* (45).

Proof. We can write equation (46) in the equivalent form

$$x'' = \big(1 + (x')^2\big)^{3/2} h(t,x,x')\,.$$

One quickly verifies that α and β are lower and upper solutions of (3) with $f(y) = y$ and $g(t,x,y) = (1+y^2)^{3/2} h(t,x,y)$, and that (44) holds. The result then follows from Proposition 8 and Corduneanu's Theorem. $\square$

Concerning the relativistic equation

$$\left(\frac{x'}{\sqrt{1-(x')^2}}\right)' = h(t,x,x')\,, \tag{48}$$

we have the following.

Corollary 16. *Let $h : \mathbb{R} \times \mathbb{R} \times \,]-1,1[\, \to \mathbb{R}$ be continuous, and continuously differentiable with respect to (x,y). Assume the existence of an almost periodic lower solution α and an almost periodic upper solution β, with $\alpha(t) < \beta(t)$ for every $t \in \mathbb{R}$, and let $a = \inf \alpha$, $b = \sup \beta$. Assume moreover that there is a $c \geq 0$ such that*

$$|h(t,x,y)| \leq c\,, \qquad \textit{for every } (t,x,y) \in \mathbb{R} \times [a,b] \times \mathbb{R}\,.$$

If h satisfies (47) *and is almost periodic in t uniformly with respect to $(x,y) \in [a,b] \times \,]-1,1[\,$, then equation* (48) *has an almost periodic solution $x(t)$ satisfying* (45).

Proof. We can write equation (48) in the equivalent form

$$x'' = \big(1 - (x')^2\big)^{3/2} h(t,x,x')\,.$$

This time we need to truncate and extend the function $g(t,x,y) = (1 - y^2)^{3/2} h(t,x,y)$, as in the proof of Proposition 10, but with some more care, so to preserve the continuous differentiability of the new function $g_\delta(t,x,y)$ in (x,y) and condition (44). To this aim, let $\mathcal{U}_\delta : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function such that

$$\mathcal{U}_\delta(s) = \begin{cases} -1 + \delta/2 & \text{if } s \leq -1\,, \\ s & \text{if } |s| \leq 1 - \delta\,, \\ 1 - \delta/2 & \text{if } s \geq 1\,, \end{cases}$$

and set $g_\delta(t,x,y) = g(t,x,\mathcal{U}_\delta(y))$. Once this is done, it is easy to see that (44) holds with m replaced by $(1 - (1 - \delta^2/4))^{3/2} m$. The result then follows from Corduneanu's Theorem, in view of Proposition 10.

As an example, consider the "relativistic pendulum" equation

$$\left(\frac{x'}{\sqrt{1-(x')^2}}\right)' + A\sin x = e(t)\,, \tag{49}$$

where $e:\mathbb{R}\to\mathbb{R}$ is continuous and almost periodic. If $|e(t)|\le A$, for every $t\in\mathbb{R}$, then there is an almost periodic solution such that

$$\frac{\pi}{2}\le x(t)\le\frac{3\pi}{2}\,, \qquad \text{for every } t\in\mathbb{R}\,.$$

7 Appendix: Corduneanu's Theorem

We will consider a scalar second order differential equation

$$x''=g(t,x,x')\,, \tag{50}$$

where $g:\mathbb{R}\times\mathcal{D}\to\mathbb{R}$ is continuous, with $\mathcal{D}$ an open convex subset of $\mathbb{R}^2$.

Let us start with the definition of almost periodic function.

Definition 17. *A function $\mathcal{F}:\mathbb{R}\to\mathbb{R}$ is said to be* almost periodic *if, for every $\varepsilon>0$, there is a $\ell(\varepsilon)>0$ with the following property: for every $\tau\in\mathbb{R}$ there is a $\xi\in[\tau,\tau+\ell(\varepsilon)]$ such that*

$$\sup\{|\mathcal{F}(t+\xi)-\mathcal{F}(t)|:t\in\mathbb{R}\}<\varepsilon\,. \tag{51}$$

Whenever we have a function $\mathcal{F}:\mathbb{R}\times\Lambda\to\mathbb{R}$, we say that $\mathcal{F}(t,\lambda)$ is almost periodic in t uniformly with respect to $\lambda\in\Lambda$ if the above holds with (51) *replaced by*

$$\sup\{|\mathcal{F}(t+\xi,\lambda)-\mathcal{F}(t,\lambda)|:t\in\mathbb{R},\lambda\in\Lambda\}<\varepsilon\,.$$

Let us now recall the statement of Corduneanu's Theorem, in a slightly improved version.

Theorem [Corduneanu, 1955]. *Assume that $g(t,x,y)$ is continuous, continuously differentiable with respect to (x,y), and that there exists a constant m for which*

$$\frac{\partial g}{\partial x}(t,x,y)\ge m>0\,, \qquad \text{for every } (t,x,y)\in\mathbb{R}\times\mathcal{D}\,. \tag{52}$$

If g is almost periodic in t, uniformly with respect to (x,y) in some set $\Lambda\subseteq\mathcal{D}$, then any bounded solution $x(t)$ of (50) *with $(x(t),x'(t))\in\Lambda$ is almost periodic.*

For the proof we will need the following estimate.

Lemma 18. *Let $a, b, c : \mathbb{R} \to \mathbb{R}$ be continuous functions, such that*

$$a(t) \geq m > 0\,, \qquad \textit{for every } t \in \mathbb{R}\,.$$

Then, any solution $u(t)$ of the two-point boundary value problem

$$\begin{cases} u'' = a(t)u + b(t)u' + c(t) \\ u(t_1) = A\,, \quad u(t_2) = B \end{cases}$$

satisfies

$$\max\{|u(t)| : t \in [t_1, t_2]\} \leq \frac{1}{m} \max\{|c(t)| : t \in [t_1, t_2]\} + \max\{|A|, |B|\}\,.$$

Proof. Let $t_0 \in [t_1, t_2]$ be such that $u(t_0) = \max\{u(t) : t \in [t_1, t_2]\}$. If $t_0 \in]t_1, t_2[$, then $u'(t_0) = 0$ and $u''(t_0) \leq 0$, so that $a(t_0)u(t_0) + c(t_0) \leq 0$, whence

$$u(t_0) \leq \frac{1}{a(t_0)} |c(t_0)| \leq \frac{1}{m} \max\{|c(t)| : t \in [t_1, t_2]\}\,.$$

On the other hand, if $t_0 \in \{t_1, t_2\}$, then $u(t_0) \leq \max\{|A|, |B|\}$. A similar argument for the minimum yields the conclusion. □

Proof of the Theorem. By the almost periodicity of g, given $\varepsilon > 0$ there is $\ell(\varepsilon) > 0$ such that in every interval with length $\ell(\varepsilon)$ there is a ξ for which

$$|g(t + \xi, x, y) - g(t, x, y)| \leq \frac{m\varepsilon}{3}\,, \qquad \text{for every } (t, x, y) \in \mathbb{R} \times \Lambda\,. \tag{53}$$

We want to prove that, for such a $\xi \neq 0$,

$$\sup\{|x(t + \xi) - x(t)| : t \in \mathbb{R}\} < \varepsilon\,. \tag{54}$$

Clearly, it cannot be that $\liminf_{t \to +\infty}(x(t+\xi) - x(t)) > 0$, since this would lead to a contradiction with the boundedness of x. Similarly, it cannot be that $\limsup_{t \to +\infty}(x(t + \xi) - x(t)) < 0$. Then, since

$$\liminf_{t \to +\infty}(x(t + \xi) - x(t)) \leq 0 \leq \limsup_{t \to +\infty}(x(t + \xi) - x(t))\,,$$

there is a strictly increasing sequence $(\tau_n)_n$ with $\tau_n \to +\infty$ such that

$$\lim_n \big(x(\tau_n + \xi) - x(\tau_n)\big) = 0\,. \tag{55}$$

Similarly, there is a strictly decreasing sequence $(\sigma_n)_n$ with $\sigma_n \to -\infty$ such that

$$\lim_n \big(x(\sigma_n + \xi) - x(\sigma_n)\big) = 0\,. \tag{56}$$

Since $x(t + \xi) - x(t)$ is bounded, there is a $t_0 \in \mathbb{R}$ such that

$$\sup\{|x(t + \xi) - x(t)| : t \in \mathbb{R}\} < |x(t_0 + \xi) - x(t_0)| + \frac{\varepsilon}{3}\,. \tag{57}$$

By (55) and (56), we can fix a positive integer p for which $\sigma_p < t_0 < \tau_p$ and

$$\big|x(\sigma_p + \xi) - x(\sigma_p)\big| < \frac{\varepsilon}{3}\,, \qquad \big|x(\tau_p + \xi) - x(\tau_p)\big| < \frac{\varepsilon}{3}\,. \tag{58}$$

We then have

$$\begin{aligned}
x''(t+\xi) - x''(t) &= g(t+\xi, x(t+\xi), x'(t+\xi)) - g(t, x(t), x'(t)) \\
&= g(t+\xi, x(t+\xi), x'(t+\xi)) - g(t+\xi, x(t), x'(t)) + \\
&\quad + g(t+\xi, x(t), x'(t)) - g(t, x(t), x'(t)) \\
&= \left(\int_0^1 \frac{\partial g}{\partial x}(\gamma(s,t))\, ds\right) \big(x(t+\xi) - x(t)\big) + \\
&\quad + \left(\int_0^1 \frac{\partial g}{\partial y}(\gamma(s,t))\, ds\right) \big(x'(t+\xi) - x'(t)\big) + \\
&\quad + g(t+\xi, x(t), x'(t)) - g(t, x(t), x'(t))\,,
\end{aligned}$$

where

$$\gamma(s,t) = (t+\xi, x(t) + s(x(t+\xi) - x(t)), x'(t) + s(x'(t+\xi) - x'(t)))\,.$$

Using Lemma 18, with

$$a(t) = \int_0^1 \frac{\partial g}{\partial x}(\gamma(s,t))\, ds\,, \qquad b(t) = \int_0^1 \frac{\partial g}{\partial y}(\gamma(s,t))\, ds\,,$$

$$c(t) = g(t+\xi, x(t), x'(t)) - g(t, x(t), x'(t))\,,$$

by (52), (53) and (58) we obtain

$$\sup\{|x(t+\xi) - x(t)| : t \in [\sigma_p, \tau_p]\} \le \frac{1}{m}\,\frac{m\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2}{3}\,\varepsilon\,.$$

Since $t_0 \in [\sigma_p, \tau_p]$, using (57), we conclude that (54) holds, proving that $x(t)$ is almost periodic. □

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Authors' addresses:

Alessandro Fonda
Dipartimento di Matematica e Geoscienze
Università di Trieste
P.le Europa 1
I-34127 Trieste
Italy
e-mail: a.fonda@units.it

Rodica Toader
Dipartimento di Scienze Matematiche, Informatiche e Fisiche
Università di Udine
Via delle Scienze 206
I-33100 Udine
Italy
e-mail: toader@uniud.it

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