A logarithmic spiral in the complex plane interpolating between the exponential and the circular functions

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Abstract. We propose a unified construction of the real exponential function and the trigonometric functions. To this aim, we prove the existence of a unique curve in the complex plane starting at time 0 from 1 and arriving after some time τ at a point ζ, lying in the first quadrant, with the homomorphism property

$$f(x_1 + x_2) = f(x_1)f(x_2).$$

1 Introduction and statement of the result

The aim of this paper is to provide a unified construction of the exponential and the trigonometric functions, by the use of rather elementary arguments in the complex plane. In doing this, we are faced with a result which seems to be rather new in literature. Here it is.

Theorem 1 *Let ζ be a nonzero complex number, with $\Re\zeta \geq 0$ and $\Im\zeta \geq 0$, and let τ be a positive real number. There exists a unique continuous function $f : \mathbb{R} \to \mathbb{C} \setminus \{0\}$ with the following properties:*

(a) $f(0) = 1$, $f(\tau) = \zeta$;

(b) $f(x_1 + x_2) = f(x_1)f(x_2)$, for every $x_1, x_2 \in \mathbb{R}$;

(c) $\Re f(x) > 0$ and $\Im f(x) \geq 0$, for every $x \in]0, \tau[$.

It is worth to emphasize two special cases. When ζ is a positive real number, say $\zeta = a > 0$, and $\tau = 1$, then f will be the *real exponential function*

$$f(x) = a^x .$$

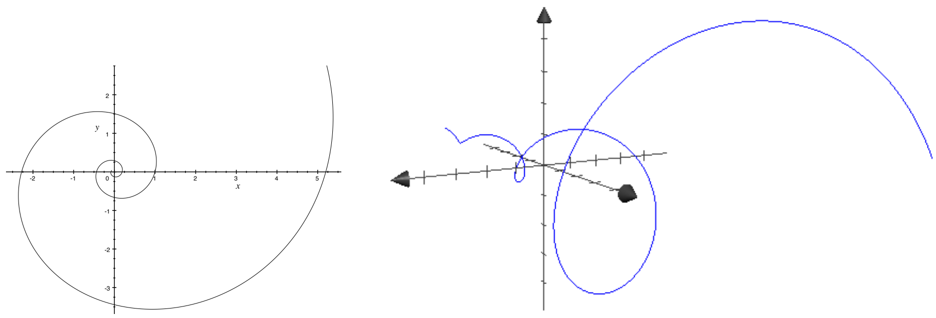

Figure 1: *The image and the graph of the function f*

On the other hand, when ζ is not real and $|\zeta| = 1$, we will obtain a *circular function* with minimal period

$$T = 2\pi \, \frac{\tau}{\mathrm{Arg}(\zeta)} \,.$$

For example, if $\zeta = i$ and $\tau = \pi/2$, we will get

$$f(x) = \cos x + i \sin x \,.$$

Our attention here, if not on the theorem itself, is focused on the construction of the function f, which permits to define the exponential and the trigonometric functions with a unified approach. In literature one may find several different ways for doing this (see the Appendix for a brief review), and we do not claim to have found the best possible one. However, this approach has the advantage to be rather elementary, and it has a somewhat geometrical insight.

The proof of Theorem 1 will be carried out in Section 2. It is mainly inspired by the construction of the circular functions provided by Giovanni Prodi in [14], who reported in his book some previous notes by Giorgio Letta [13]. In Sections 3 and 4 we will recover the main properties of the exponential and the trigonometric functions. Some final remarks will be made in Section 5. Let us remark that the paper contains ten figures: they have been inserted to help the reader's intuition, but they play no role, neither in the mathematical constructions, nor in the proof of the results.

2 The construction of the function

It is not restrictive to assume $\tau = 1$. Indeed, if we denote by $f_\tau : \mathbb{R} \to \mathbb{C} \setminus \{0\}$ the function we are looking for, once we have found $f_1 : \mathbb{R} \to \mathbb{C} \setminus \{0\}$, it is sufficient to define

$$f_\tau(x) = f_1\left(\frac{x}{\tau}\right),$$

so that all the requirements are satisfied.

We will denote the first quadrant of the complex plane by

$$\mathcal{Q}_1 = \{z \in \mathbb{C} : \Re(x) \geq 0,\, \Im(z) \geq 0\}.$$

Moreover, for every complex number z, we will denote by z^* its complex conjugate.

The proof is divided in several steps.

2.1 Preliminaries

We provide here a rigorous definition of the *argument* of the complex number ζ, passing through some sequences which have a simple geometric interpretation. However, we will not care about the geometrical aspect (only some pictures are provided to help the reader in this task), concentrating mainly on the analytical construction.

Let $(\sigma_n)_n$ be the sequence of complex numbers

$$\sigma_n = x_n + iy_n \in \mathcal{Q}_1,$$

such that

$$\sigma_0 = \zeta, \quad \text{and} \quad \sigma_{n+1}^2 = \sigma_n, \quad \text{for every } n \in \mathbb{N}.$$

More explicitly,

$$x_{n+1} = \sqrt{\frac{x_n + \sqrt{x_n^2 + y_n^2}}{2}}, \qquad y_{n+1} = \frac{y_n}{\sqrt{2\,(x_n + \sqrt{x_n^2 + y_n^2}\,)}}.$$

Notice that $x_n > 0$ for every $n \geq 1$. It is easily seen, by induction, that

$$\sigma_n^{2^n} = \zeta, \quad \text{for every } n \in \mathbb{N}. \tag{1}$$

Let now $(\tilde{\sigma}_n)_n$ be defined as

$$\tilde{\sigma}_n = \frac{\sigma_n}{|\sigma_n|}.$$

It is easily seen that $|\tilde{\sigma}_n| = 1$,

$$\tilde{\sigma}_0 = \tilde{\zeta} := \frac{\zeta}{|\zeta|}, \quad \text{and} \quad \tilde{\sigma}_{n+1}^2 = \tilde{\sigma}_n, \quad \text{for every } n \in \mathbb{N}.$$

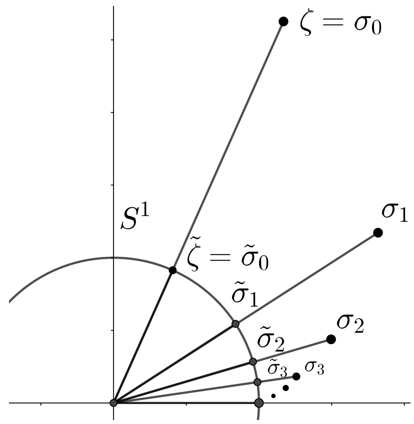

Figure 2: *The definition of* σ_n *and* $\tilde{\sigma}_n$

Moreover, let us define

$$\ell_n = |\tilde{\sigma}_n - 1|\,, \qquad L_n = \frac{2\ell_n}{\sqrt{4-\ell_n^2}}\,.$$

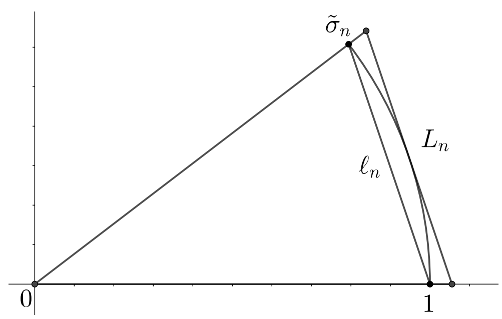

Figure 3: *The definition of* ℓ_n *and* L_n

We see that

$$|\tilde{\sigma}_n^{m+1} - \tilde{\sigma}_n^m| = |\tilde{\sigma}_n^m(\tilde{\sigma}_n - 1)| = |\tilde{\sigma}_n|^m\,|\tilde{\sigma}_n - 1| = |\tilde{\sigma}_n - 1| = \ell_n\,,$$

so that the points

$$1\,,\ \tilde{\sigma}_n\,,\ \tilde{\sigma}_n^2\,,\ \tilde{\sigma}_n^3\,,\ \tilde{\sigma}_n^4\,,\ \ldots$$

all lie on the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, and ℓ_n is the distance from each one of them to the next one.

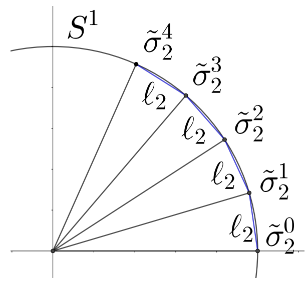

Figure 4: *The equidistant points*

Let us prove that

$$\ell_{n+1} = \sqrt{2 - \sqrt{4 - \ell_n^2}}\,. \tag{2}$$

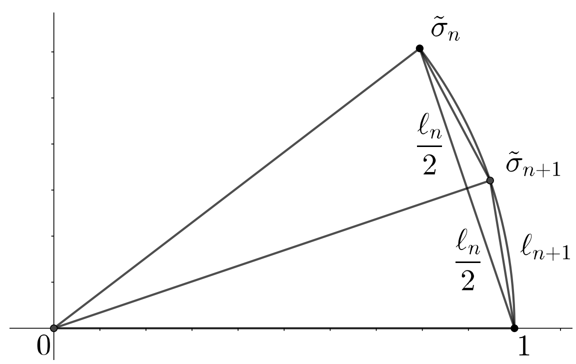

Figure 5: *A geometric view of ℓ_{n+1} in terms of ℓ_n*

Indeed, since $1 = |\tilde{\sigma}_n|^2 = \tilde{\sigma}_n \tilde{\sigma}_n^*$, for every $n \in \mathbb{N}$, we have that

$$\ell_n^2 = (\tilde{\sigma}_n - 1)(\tilde{\sigma}_n - 1)^* = (\tilde{\sigma}_n - 1)\left(\frac{1}{\tilde{\sigma}_n} - 1\right) = 2 - \tilde{\sigma}_n - \frac{1}{\tilde{\sigma}_n}\,,$$

hence

$$\begin{aligned}(2 - \ell_{n+1}^2)^2 &= \left(\tilde{\sigma}_{n+1} + \frac{1}{\tilde{\sigma}_{n+1}}\right)^2 = 2 + \tilde{\sigma}_{n+1}^2 + \frac{1}{\tilde{\sigma}_{n+1}^2} \\ &= 4 - \left(2 - \tilde{\sigma}_n - \frac{1}{\tilde{\sigma}_n}\right) = 4 - \ell_n^2\,,\end{aligned}$$

so that (2) is proved.

It can then be noticed that $\ell_0 \le \sqrt{2}$, and that $(\ell_n)_n$ is strictly decreasing. We now set

$$a_n = 2^n \ell_n\,, \qquad b_n = 2^n L_n\,.$$

Notice that, if ζ is a real number, then $\ell_n = 0$, hence $a_n = b_n = 0$, for every $n \in \mathbb{N}$. Let us now concentrate on the case $\Im\zeta > 0$. In this case, $\ell_n \in]0, 2[$ and

$$\frac{b_n}{a_n} = \frac{2}{\sqrt{4 - \ell_n^2}} > 1\,,$$

hence $a_n < b_n$, for every $n \in \mathbb{N}$. Let us see that the sequence $(a_n)_n$ is strictly increasing: by (2),

$$\frac{a_{n+1}}{a_n} = 2\frac{\ell_{n+1}}{\ell_n} = 2\frac{\sqrt{2 - \sqrt{4 - \ell_n^2}}}{\ell_n} = \frac{2}{\sqrt{2 + \sqrt{4 - \ell_n^2}}} > \frac{2}{\sqrt{2 + 2}} = 1\,.$$

Moreover, the sequence $(b_n)_n$ is strictly decreasing: by (2) again,

$$\begin{aligned}
\frac{b_n}{b_{n+1}} &= \frac{1}{2}\,\frac{\ell_n}{\sqrt{4 - \ell_n^2}}\,\frac{\sqrt{4 - \ell_{n+1}^2}}{\ell_{n+1}} \\
&= \frac{1}{2}\,\frac{\ell_n}{\sqrt{4 - \ell_n^2}}\,\frac{\sqrt{2 + \sqrt{4 - \ell_n^2}}}{\sqrt{2 - \sqrt{4 - \ell_n^2}}} \\
&= \frac{1}{2}\,\frac{2 + \sqrt{4 - \ell_n^2}}{\sqrt{4 - \ell_n^2}} \\
&= \frac{1}{2}\left(\frac{2}{\sqrt{4 - \ell_n^2}} + 1\right) \\
&> \frac{1}{2}(1 + 1) = 1\,.
\end{aligned}$$

Therefore, the sequences $(a_n)_n$ and $(b_n)_n$ both have a finite limit. Being then

$$\lim_n \ell_n = \lim_n \frac{a_n}{2^n} = 0\,,$$

we have that

$$\lim_n \frac{b_n}{a_n} = \lim_n \frac{2}{\sqrt{4 - \ell_n^2}} = 1\,,$$

so that we can conclude that the two sequences have indeed the same limit. We call such a number *argument of* ζ, and denote it by $\mathrm{Arg}(\zeta)$. In the case when ζ is a positive real number, we set $\mathrm{Arg}(\zeta) = 0$. Hence,

$$\mathrm{Arg}(\zeta) = \lim_n 2^n |\tilde{\sigma}_n - 1| \,. \tag{3}$$

In the following, we will need the inequality

$$|\tilde{\sigma}_n - 1| \leq \frac{1}{2^n} \mathrm{Arg}(\zeta)\,, \quad \text{for every } n \in \mathbb{N}\,, \tag{4}$$

which is a direct consequence of the fact that $(a_n)_n$ is increasing.

2.2 The definition on a dense set

We first define the function f on the set

$$E = \left\{ \frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\} ,$$

which is dense in $\mathbb{R}$. If we want the function $f : E \to \mathbb{C} \setminus \{0\}$ to satisfy the conditions (a), (b) and (c) of the statement, then $f(1) = \zeta = \sigma_0$ and, since

$$f(1) = f\left(\frac{1}{2} + \frac{1}{2}\right) = f\left(\frac{1}{2}\right) f\left(\frac{1}{2}\right) = \left[f\left(\frac{1}{2}\right)\right]^2 ,$$

and $\Re f\left(\frac{1}{2}\right) \geq 0$, $\Im f\left(\frac{1}{2}\right) \geq 0$, it has to be $f\left(\frac{1}{2}\right) = \sigma_1$. Iterating this process, we see that we must set

$$f\left(\frac{1}{2^n}\right) = \sigma_n\,, \quad \text{for every } n \in \mathbb{N}\,.$$

Moreover,

$$f\left(\frac{m}{2^n}\right) = f\left(m\,\frac{1}{2^n}\right) = \left[f\left(\frac{1}{2^n}\right)\right]^m .$$

This shows that, if (a), (b) and (c) hold, then the definition of f on the set E must be

$$f\left(\frac{m}{2^n}\right) = \sigma_n^m\,. \tag{5}$$

It is a good definition, since

$$\frac{m}{2^n} = \frac{m'}{2^{n'}} \quad \Rightarrow \quad \sigma_n^m = \sigma_{n'}^{m'}\,.$$

Indeed, if for instance $n' \geq n$, we see by (1) that $\sigma_n = \sigma_{n'}^{2^{n'-n}}$, hence

$$\sigma_n^m = (\sigma_{n'}^{2^{n'-n}})^m = \sigma_{n'}^{m2^{n'-n}} = \sigma_{n'}^{m'} .$$

Notice that $f(0) = 1$ and $f(1) = \zeta$, so that property (a) holds. Let us see that

$$f(x_1 + x_2) = f(x_1)f(x_2)\,, \quad \text{for every } x_1, x_2 \in E\,, \tag{6}$$

which is property (b) on the domain E. Taking $x_1 = \dfrac{k}{2^n}$ and $x_2 = \dfrac{m}{2^n}$ (we can now choose the same denominator), we have

$$f\left(\frac{k}{2^n} + \frac{m}{2^n}\right) = f\left(\frac{k+m}{2^n}\right) = \sigma_n^{k+m} = \sigma_n^k \sigma_n^m = f\left(\frac{k}{2^n}\right) f\left(\frac{m}{2^n}\right).$$

Finally, with the aim of veirfying property (c), we claim that

$$1, \tilde{\sigma}_n\,,\ \tilde{\sigma}_n^2\,,\ \ldots\ \tilde{\sigma}_n^{2^n} \ \text{ belong to } \mathcal{Q}_1\,, \quad \text{for every } n \in \mathbb{N}\,.$$

This is surely true if $n = 0$ or 1. If $n = 2$, we have that $1, \tilde{\sigma}_2$ and $\tilde{\sigma}_2^2 = \tilde{\sigma}_1$ surely belong to $\mathcal{Q}_1$, as well as $\tilde{\sigma}_2^4 = \tilde{\zeta}$. Concerning $\tilde{\sigma}_2^3$, we notice that

$$|\tilde{\sigma}_2^3| = 1 \qquad \text{and} \qquad |\tilde{\sigma}_2^3 - \tilde{\sigma}_2^2| = |\tilde{\sigma}_2^3 - \tilde{\sigma}_2^4| = \ell_2\,.$$

In principle, there are two points satisfying these properties, one in the first and one in the third quadrant. However, since $\ell_2 < \sqrt{2}$, it must be that $\tilde{\sigma}_2^3$ belongs to $\mathcal{Q}_1$. By induction, the same argument can be used to prove the claim, for every $n \in \mathbb{N}$.

So, we have constructed a function $f : E \to \mathbb{C} \setminus \{0\}$ which verifies the properties (a), (b) and (c) on its domain. And this is the only possible function with these properties.

2.3 The extension to the whole real line

We will now verify that the function $f : E \to \mathbb{C} \setminus \{0\}$ defined by (5) is uniformly continuous on any bounded subset of its domain. We fix a real number $R > 0$, and consider the restriction of f to $E \cap [-R, R]$.

We define the two functions $g : E \to]0, +\infty[\,$, $h : E \to S^1$ by

$$g(x) = |f(x)|\,, \qquad h(x) = \frac{f(x)}{|f(x)|}\,,$$

and we remark that

$$g(x_1 + x_2) = g(x_1)g(x_2)\,, \quad \text{for every } x_1, x_2 \in E\,,$$

and

$$h(x_1 + x_2) = h(x_1)h(x_2)\,, \quad \text{for every } x_1, x_2 \in E\,.$$

Let us first concentrate on the function g, and prove that it is uniformly continuous on $E \cap [-R, R]$. It is easily seen that, if $|\zeta| = 1$, then g is constant. Assume now that $|\zeta| > 1$. We want to show that, in this case, g is strictly increasing. Indeed, we first notice that

$$g(x) > 1\,, \quad \text{for every } x \in E \cap]0, +\infty[\,.$$

Consequently,

$$x_1 < x_2 \quad \Rightarrow \quad g(x_2) = g(x_1)g(x_2 - x_1) > g(x_1)\,,$$

proving that g is strictly increasing.

As a second step, let us now show that

$$\lim_{x \to 0^+} g(x) = 1\,.$$

Fix an $\epsilon > 0$. Let $\bar{n} \in \mathbb{N}$ be such that $\bar{n} \geq (|\zeta| - 1)/\epsilon$. Then, for every $n \geq \bar{n}$,

$$\left[g\left(\frac{1}{2^n} \right) \right]^{2^n} = |\zeta| \leq 1 + n\epsilon \leq 1 + 2^n \epsilon \leq (1 + \epsilon)^{2^n}\,,$$

so that

$$1 < g\left(\frac{1}{2^n} \right) \leq 1 + \epsilon\,.$$

Being g increasing, this proves the claim.

Let us now fix $\varepsilon > 0$. By the above, there exists $\delta > 0$ such that

$$0 < x < \delta \quad \Rightarrow \quad 1 < g(x) < 1 + \frac{\varepsilon}{g(R)}\,.$$

Then, taking $x_1, x_2 \in E \cap [-R, R]$ such that $x_1 < x_2$ and $x_2 - x_1 \leq \delta$, we have

$$0 < g(x_2) - g(x_1) = g(x_1)(g(x_2 - x_1) - 1) < g(R)\frac{\varepsilon}{g(R)} = \varepsilon\,.$$

This proves that g is uniformly continuous on $E \cap [-R, R]$, with values in $[g(-R), g(R)]$. If $|\zeta| < 1$, the proof is similar.

We now concentrate on the function h. Notice that

$$h\left(\frac{m}{2^n}\right) = \frac{\sigma_n^m}{|\sigma_n^m|} = \tilde{\sigma}_n^m \,.$$

Take $x_1 = \dfrac{k}{2^n}$ and $x_2 = \dfrac{m}{2^n}$, with $k < m$. Then,

$$\begin{aligned} h\left(\frac{m}{2^n}\right) - h\left(\frac{k}{2^n}\right) &= \tilde{\sigma}_n^m - \tilde{\sigma}_n^k = \tilde{\sigma}_n^k(\tilde{\sigma}_n^{m-k} - 1) \\ &= \tilde{\sigma}_n^k(\tilde{\sigma}_n - 1)(1 + \tilde{\sigma}_n + \tilde{\sigma}_n^2 + \ldots + \tilde{\sigma}_n^{m-k-1}) \,, \end{aligned}$$

and hence, by (4),

$$\left| h\left(\frac{m}{2^n}\right) - h\left(\frac{k}{2^n}\right) \right| \le |\tilde{\sigma}_n - 1|(m-k) \le \mathrm{Arg}(\zeta) \left| \frac{m}{2^n} - \frac{k}{2^n} \right| .$$

This proves that h is uniformly continuous on the whole domain E.

The restriction of the function f on $E \cap [-R, R]$ is uniformly continuous, since, when $x_1, x_2 \in E \cap [-R, R]$,

$$\begin{aligned} |f(x_2) - f(x_1)| &= |g(x_2)h(x_2) - g(x_1)h(x_1)| \\ &\le |g(x_2) - g(x_1)|\,|h(x_2)| + |h(x_2) - h(x_1)|\,|g(x_1)| \\ &\le |g(x_2) - g(x_1)| + |h(x_2) - h(x_1)|\, g(\pm R) \,, \end{aligned}$$

and both g and h are uniformly continuous. Its values are contained in the compact set

$$F = \{z \in \mathbb{C} : r_1 \le |z| \le r_2\} \,,$$

where $r_1 = g(-R)$ and $r_2 = g(R)$, or vice versa. Hence, it can be extended in a unique way to a continuous function on $[-R, R]$, with values in the same set F. In particular, $f(x) \ne 0$ for every $x \in [-R, R]$. Since this can be done for an arbitrary $R > 0$, we have thus defined a continuous extension of f on the whole real axis $\mathbb{R}$, with nonzero values. And this is the only possible continuous extension.

Recalling that $\tau = 1$, property (a) has already been verified above. Concerning property (b), take x_1, x_2 in $\mathbb{R}$ and let $(x_{1,n})_n$ and $(x_{2,n})_n$ be two sequences in E such that $x_{1,n} \to x_1$ and $x_{2,n} \to x_2$. Then, from (6), by continuity,

$$\begin{aligned} f(x_1 + x_2) &= f(\lim_n (x_{1,n} + x_{2,n})) = \lim_n f(x_{1,n} + x_{2,n}) = \lim_n f(x_{1,n}) f(x_{2,n}) \\ &= \lim_n f(x_{1,n}) \lim_n f(x_{2,n}) = f(x_1) f(x_2) \,. \end{aligned}$$

Similarly one can verify property (c), since $\mathcal{Q}_1$ is a closed set.

We have thus constructed a function $f : \mathbb{R} \to \mathbb{C} \setminus \{0\}$ which verifies the properties (a), (b) and (c). And this is the only possible one. Then, Theorem 1 is completely proved.

In the next two sections we will investigate more deeply the properties of our function f in two special cases: when ζ is a real number $a > 0$, and when $\zeta = i$.

3 The exponential function

Assume that ζ is a positive real number, say $\zeta = a > 0$. Then f coincides with $g : \mathbb{R} \to]0, +\infty[$. It is called *exponential with base* a, and one uses the notation

$$g(x) = \exp_a(x) \,, \ \text{ or } \ g(x) = a^x \,.$$

Let us see some of its properties. If $a = 1$, it is easy to see that the function g is constantly equal to 1. From the previous section, being E dense in $\mathbb{R}$, we deduce that, if $a > 1$, then g is strictly increasing. Moreover, since

$$g(m) = g\left(\frac{m}{2^0}\right) = \sigma_0^m = a^m \,,$$

for every $m \in \mathbb{Z}$, we deduce that

$$\lim_{x \to -\infty} g(x) = 0 \,, \qquad \lim_{x \to +\infty} g(x) = +\infty \,.$$

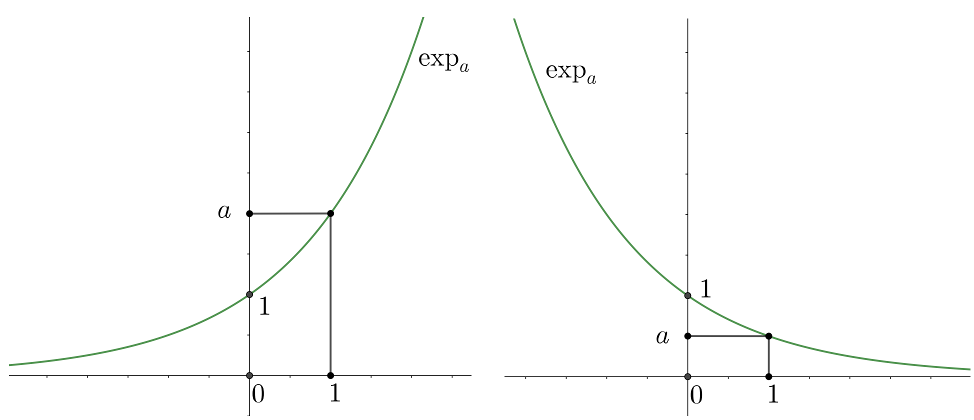

Figure 6: *The exponential function* $\exp_a$ *when* $a > 1$ *and* $a < 1$*, respectively*

Similarly, if $a < 1$, then g is strictly decreasing, and

$$\lim_{x \to -\infty} g(x) = +\infty\,, \qquad \lim_{x \to +\infty} g(x) = 0\,.$$

If $a \neq 1$, we can thus define the inverse function $\ell :]0, +\infty[\to \mathbb{R}$ by setting

$$\ell(y) = x \quad \Leftrightarrow \quad g(x) = y\,.$$

The function ℓ is called *logarithm with base* a, and we may use the notation

$$\ell(y) = \log_a(y)\,.$$

If $a > 1$, then ℓ is strictly increasing, and

$$\lim_{y \to 0^+} \ell(y) = -\infty\,, \qquad \lim_{y \to +\infty} \ell(y) = +\infty\,.$$

If $a < 1$, then ℓ is strictly decreasing, and

$$\lim_{y \to 0^+} \ell(y) = +\infty\,, \qquad \lim_{y \to +\infty} \ell(y) = -\infty\,.$$

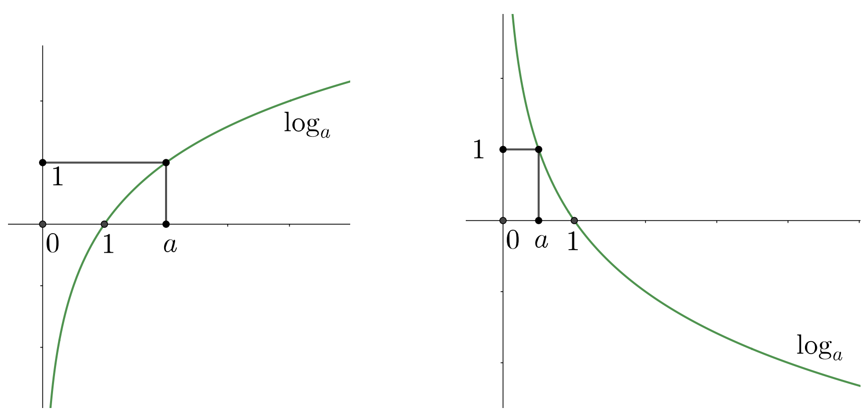

Figure 7: *The logarithmic function* $\log_a$ *when* $a > 1$ *and* $a < 1$*, respectively*

Let us prove the following properties of the exponential:

$$(ab)^x = a^x b^x\,, \quad \left(\frac{1}{a}\right)^x = \frac{1}{a^x} = a^{-x}\,, \quad (a^y)^x = a^{yx}\,.$$

The first one follows from the fact that the function $f(x) = a^x b^x$ verifies the properties (a), (b) and (c) in the statement of Theorem 1, with $\zeta = ab$, hence $f = \exp_{ab}$. The second is analogous, taking $f(x) = \frac{1}{a^x}$; for the third one, just take $f(x) = a^{yx}$.

We now present two useful properties of the logarithm:

$$\log_a(y^w) = w \log_a(y)\,, \qquad \log_b(y) = \frac{\log_a(y)}{\log_a(b)}\,.$$

Let us verify the first one: we set $u = \log_a(y^w)$ and $v = \log_a(y)$. Then, $a^u = y^w$ and $a^v = y$, so that $a^u = (a^v)^w = a^{vw}$. Hence, $u = vw$, which gives us the equality we wanted to prove. A similar procedure provides the proof of the second one, as well.

For the following theorem, we assume the reader to be familiar with the definition of the Euler constant e, and with the identity

$$e = \lim_{t\to\pm\infty} \left(1 + \frac{1}{t}\right)^t = 2.71828\dots\,.$$

Theorem 2 *We have the following limits:*

$$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a(e)\,, \qquad \lim_{x\to 0} \frac{\exp_a(x)-1}{x} = \frac{1}{\log_a(e)}\,.$$

<u>Proof</u> We have that

$$\lim_{x\to 0^\pm} \frac{\log_a(1+x)}{x} = \lim_{t\to\pm\infty} t \log_a\left(1+\frac{1}{t}\right) = \lim_{t\to\pm\infty} \log_a\left(1+\frac{1}{t}\right)^t = \log_a(e)\,,$$

and

$$\lim_{x\to 0} \frac{\exp_a(x)-1}{x} = \lim_{t\to 0} \frac{t}{\log_a(1+t)} = \frac{1}{\log_a(e)}\,,$$

by the well known properties of the limit. ■

If $a = e$, we will write $\exp(x)$ (or simply $\exp x$) instead of $\exp_e(x)$, and $\ln(x)$ (or simply $\ln x$) instead of $\log_e(x)$, and we have the fundamental limits

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1\,, \qquad \lim_{x\to 0} \frac{e^x-1}{x} = 1\,.$$

It is also useful to define the *hyperbolic functions*

$$\cosh(x) = \frac{e^x + e^{-x}}{2}\,, \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}\,.$$

The reader may verify the following identities:

(i) $\quad (\cosh(x))^2 - (\sinh(x))^2 = 1\,,$

(ii) $\quad \cosh(x_1 + x_2) = \cosh(x_1)\cosh(x_2) + \sinh(x_1)\sinh(x_2)\,,$

(iii) $\quad \sinh(x_1 + x_2) = \sinh(x_1)\cosh(x_2) + \cosh(x_1)\sinh(x_2)\,.$

4 The trigonometric functions

In this section, we take $\zeta = i$, and τ an arbitrary positive real number. Since $|\zeta| = 1$, the function g is constantly equal to 1, hence f coincides with the function $h : \mathbb{R} \to S^1$.

Now, since $|\zeta| = 1$, the points $\tilde{\sigma}_n$ and σ_n coincide. Moreover, we see that

$$1\,,\ \sigma_n\,,\ \sigma_n^2\,,\ \sigma_n^3\,,\ \ldots\ ,\sigma_n^{4\cdot 2^n - 1}$$

are the vertices of a regular polygon with 2^{n+2} sides inscribed in S^1, and $\ell_n = |\sigma_n - 1|$ is the length of its sides.

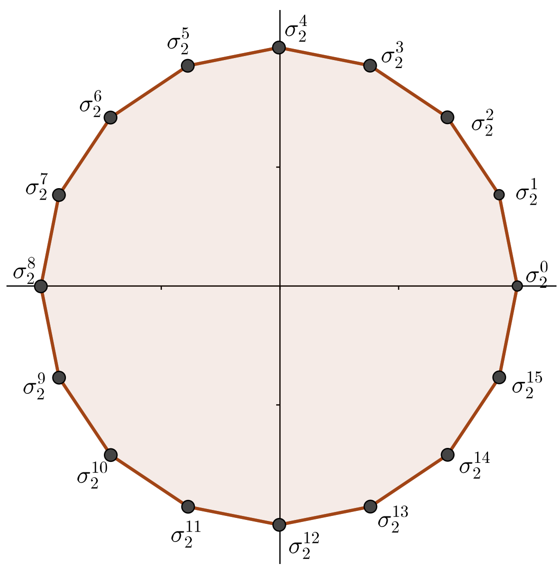

Figure 8: *The polygon with* 16 *vertices, when* $n = 2$

Hence, the semi-perimeter of such a polygon is equal to $2a_n$. On the other hand, $2b_n$ is the semi-perimeter of the corresponding circumscribed polygon. Recalling the definition (3), we now define the number π as follows:

$$\pi = 2\,\mathrm{Arg}(i) = 3.14159\ldots$$

Notice that, being $h(\tau) = i$,

$$\begin{aligned} h(2\tau) &= h(\tau + \tau) = h(\tau)^2 = i^2 = -1\,, \\ h(3\tau) &= h(2\tau + \tau) = h(2\tau)h(\tau) = -i\,, \\ h(4\tau) &= h(3\tau + \tau) = h(3\tau)h(\tau) = 1\,, \end{aligned}$$

and hence

$$h(x + 4\tau) = h(x)h(4\tau) = h(x)\,, \quad \text{for every } x \in \mathbb{R}\,,$$

showing that h is a periodic function, with period $T := 4\tau$. We would like to prove that T is indeed the *minimal* period.

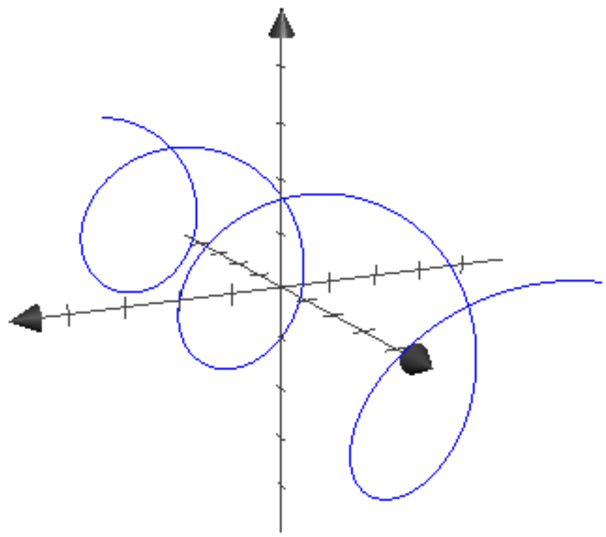

Figure 9: *The graph of the circular function h_T*

Since h is continuous and nonconstant, its minimal period is T/k, for some integer $k \geq 1$. Assume by contradiction that $k \geq 2$. Then,

$$1 = h\left(\frac{T}{k}\right) = \left[h\left(\frac{T}{2k}\right)\right]^2,$$

and since $h(T/2k) \in \mathcal{Q}_1$, it must be equal to 1. Then, it will be

$$1 = h\left(\frac{T}{2k}\right) = \left[h\left(\frac{T}{4k}\right)\right]^2,$$

and since $h(T/4k) \in \mathcal{Q}_1$, it must be equal to 1, too. And so on, proving that

$$h\left(\frac{T}{2^j k}\right) = 1\,, \quad \text{for every } j \in \mathbb{N}\,.$$

Hence, also

$$h\left(\frac{mT}{2^j k}\right) = 1\,, \quad \text{for every } j \in \mathbb{N} \text{ and } m \in \mathbb{Z}\,.$$

Since the set $\{mT/2^j k : j \in \mathbb{N}, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$ and h is continuous, this would imply that h is constantly equal to 1, a contradiction, since $h(\tau) = i$.

We have thus proved that

$$T = 4\tau \textit{ is the minimal period of } h\,,$$

and from now on we will write h_T instead of h.

Notice that, since

$$1 = h_T(0) = h_T(x - x) = h_T(x)h_T(-x)\,,$$

we have that $h_T(-x) = h_T(x)^{-1} = h_T(x)^*$, being $|h_T(x)| = 1$.

Theorem 3 *The function $\tilde{h}_T : [0, T[\, \to S^1$, restriction of h_T to the interval $[0, T[$, is bijective. Moreover,*

$$h_T\left([0, \tfrac{T}{4}]\right) = S^1 \cap \mathcal{Q}_1\,.$$

<u>Proof</u> Let us first prove the injectivity. Let $\alpha < \beta$ in $[0, T[$, and assume by contradiction that $h_T(\alpha) = h_T(\beta)$. Then,

$$h_T(\beta - \alpha) = h_T(\beta)h_T(-\alpha) = \frac{h_T(\beta)}{h_T(\alpha)} = 1\,,$$

so that

$$h_T(x + (\beta - \alpha)) = h_T(x)h_T(\beta - \alpha) = h_T(x)\,, \quad \text{for every } x \in \mathbb{R}\,.$$

Hence, $\beta - \alpha$ would be a period of h_T smaller than T, while we know that T is the minimal period.

Let us now prove that $\tilde{h}_T$ is surjective. Take a point $P = (X_1, X_2) \in S^1$. Clearly, $X_1 \in [-1, 1]$. The two cases $X_1 = -1$ or $X_1 = 1$ are treated immediately, since $h_T(\frac{T}{2}) = -1$ and $h_T(0) = 1$. Assume now $X_1 \in]-1, 1[$. We know that $\Re h_T(\frac{T}{2}) = -1$, $\Re h_T(0) = 1$ and that $\Re h_T : \mathbb{R} \to \mathbb{R}$ is continuous. By Bolzano Theorem, there exists $\bar{x} \in]0, \frac{T}{2}[$ such that $\Re h_T(\bar{x}) = X_1$. Then,

$$|\Im h_T(\bar{x})| = \sqrt{1 - (\Re h_T(\bar{x}))^2} = \sqrt{1 - X_1^2} = |X_2|\,.$$

We have two possibilities: either $\Im h_T(\bar{x}) = X_2$, so that $h_T(\bar{x}) = P$, or $\Im h_T(\bar{x}) = -X_2$, in which case

$$h_T(T - \bar{x}) = h_T(-\bar{x}) = h_T(\bar{x})^* = (X_1, X_2) = P\,.$$

Since $T - \bar{x} \in \left]\frac{T}{2}, T\right[$, this proves that $\tilde{h}_T$ is surjective.

Now we observe that $h_T\left(\left]0,\frac{T}{4}\right[\right)$ cannot contain any of the points 1, i, -1, $-i$. Hence, $\Re h_T\left(\left]0,\frac{T}{4}\right[\right)$ does not contain $-1,0,1$ and then, being connected, it must be a subset of $]-1,0[$ or of $]0,1[$, and the same is true for $\Im h_T\left(\left]0,\frac{T}{4}\right[\right)$. But both $\Re h_T\left(\left[0,\frac{T}{4}\right]\right)$ and $\Im h_T\left(\left[0,\frac{T}{4}\right]\right)$ contain 0 and 1, hence

$$\Re h_T\left(\left[0,\tfrac{T}{4}\right]\right)=[0,1]\,,\qquad \Im h_T\left(\left[0,\tfrac{T}{4}\right]\right)=[0,1]\,.$$

The conclusion follows. ■

We define the functions

$$\cos_T(x)=\Re h_T(x)\,,\qquad \sin_T(x)=\Im h_T(x)\,,$$

so that

$$h_T(x)=\cos_T(x)+i\sin_T(x)\,.$$

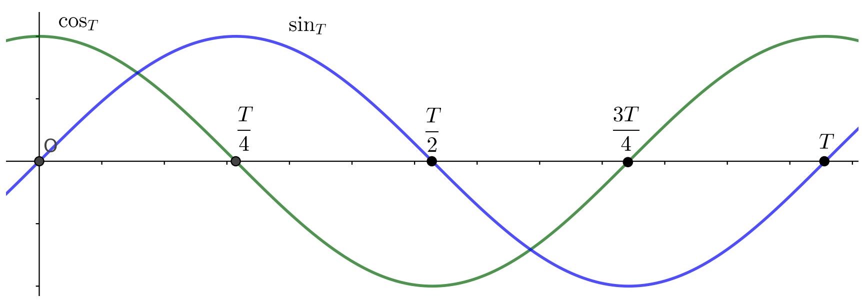

Figure 10: *The trigonometric functions* $\cos_T$ *and* $\sin_T$

These are T-periodic, and satisfy the following properties:

$$\begin{aligned}
&(i) && (\cos_T(x))^2+(\sin_T(x))^2=1\,,\\
&(ii) && \cos_T(x_1+x_2)=\cos_T(x_1)\cos_T(x_2)-\sin_T(x_1)\sin_T(x_2)\,,\\
&(iii) && \sin_T(x_1+x_2)=\sin_T(x_1)\cos_T(x_2)+\cos_T(x_1)\sin_T(x_2)\,,
\end{aligned}$$

those two last ones deriving from property (b). Moreover,

$$\begin{aligned}
&\cos_T(0)=1\,, && \sin_T(0)=0\,,\\
&\cos_T\left(\tfrac{T}{4}\right)=0\,, && \sin_T\left(\tfrac{T}{4}\right)=1\,,\\
&\cos_T\left(\tfrac{T}{2}\right)=-1\,, && \sin_T\left(\tfrac{T}{2}\right)=0\,,\\
&\cos_T\left(\tfrac{3T}{4}\right)=0\,, && \sin_T\left(\tfrac{3T}{4}\right)=-1\,.
\end{aligned}$$

Moreover, being $h_T(-x) = h_T(x)^*$, one has that

$$\cos_T(-x) = \cos_T(x)\,, \qquad \sin_T(-x) = -\sin_T(x)\,,$$

i.e., $\cos_T$ is an even function, while $\sin_T$ is odd.

Being

$$h_T\left(x + \tfrac{T}{2}\right) = h_T(x) h_T\left(\tfrac{T}{2}\right) = -h_T(x)\,,$$

we deduce that

$$\cos_T(x) \begin{cases} > 0 & \text{if } 0 < x < \frac{T}{4} \\ < 0 & \text{if } \frac{T}{4} < x < \frac{3T}{4} \\ > 0 & \text{if } \frac{3T}{4} < x < T \end{cases}, \qquad \sin_T(x) \begin{cases} > 0 & \text{if } 0 < x < \frac{T}{2} \\ < 0 & \text{if } \frac{T}{2} < x < T \end{cases}.$$

We can thus conclude that T is the *minimal* period of $\cos_T$ and $\sin_T$.

In the following theorem, the number π comes into the scene.

Theorem 4 *We have the following limit:*

$$\lim_{x \to 0^+} \frac{h_T(x) - 1}{x} = \frac{2\pi}{T}\, i\,.$$

Proof It will be equivalent proving that, when $\tau = 1$,

$$\lim_{x \to 0^+} \frac{h(x) - 1}{x} = \frac{\pi}{2}\, i\,.$$

First of all, we show that this is true when $x = \frac{1}{2^n}$, i.e., that

$$\lim_n 2^n(\sigma_n - 1) = \frac{\pi}{2}\, i\,. \tag{7}$$

We already know that

$$\lim_n |2^n(\sigma_n - 1)| = \mathrm{Arg}(i) = \frac{\pi}{2} = \left|\frac{\pi}{2}\, i\right|\,,$$

hence, being $\Im(2^n(\sigma_n - 1)) > 0$, it will be sufficient to show that $\lim_n 2^n \Re(\sigma_n - 1) = 0$. Being $\sigma_n^* = \sigma_n^{-1}$, we see that

$$\Re(\sigma_n - 1) = \frac{(\sigma_n - 1) + (\sigma_n - 1)^*}{2} = \frac{\sigma_n^2 + 1 - 2\sigma_n}{2\sigma_n} = \frac{(\sigma_n - 1)^2}{2\sigma_n}\,,$$

and recalling (4) we have that

$$|2^n \Re(\sigma_n - 1)| \le 2^n \frac{|\sigma_n - 1|^2}{|2\sigma_n|} \le \frac{\pi^2}{2^{n+1}}\,,$$

hence proving (7).

We now prove the stated limit when x varies in E, i.e., when $x = \frac{m}{2^n} > 0$. In such a case,

$$\begin{aligned}
&\left|\frac{h(x)-1}{x} - \frac{\pi}{2}\, i\right| = \left|\frac{\sigma_n^m - 1}{\frac{m}{2^n}} - \frac{\pi}{2}\, i\right| \\
&= \left|2^n(\sigma_n - 1)\frac{1+\sigma_n+\sigma_n^2+\ldots+\sigma_n^{m-1}}{m} - \frac{\pi}{2}\, i\right| \\
&\le 2^n|\sigma_n - 1|\left|\frac{1+\sigma_n+\sigma_n^2+\ldots+\sigma_n^{m-1}}{m} - 1\right| + \left|2^n(\sigma_n-1) - \frac{\pi}{2}\, i\right| \\
&\le 2^n|\sigma_n - 1|\,\frac{|\sigma_n - 1| + |\sigma_n^2 - 1| + \ldots + |\sigma_n^{m-1}-1|}{m} + \left|2^n(\sigma_n-1) - \frac{\pi}{2}\, i\right| \,.
\end{aligned}$$

Being $2^{n+1}|\sigma_n - 1| < \pi$, for $k = 1, 2, \ldots, m-1$ we have

$$|\sigma_n^k - 1| = \left|\sum_{j=1}^{k}(\sigma_n^j - \sigma_n^{j-1})\right| \le \sum_{j=1}^{k}\left|\sigma_n^j - \sigma_n^{j-1}\right| = k\,|\sigma_n - 1| \le k\,\frac{\pi}{2^{n+1}}\,.$$

Using the formula

$$1 + 2 + 3 + \ldots + (m-1) = \frac{(m-1)m}{2}\,,$$

we get

$$\begin{aligned}
&\frac{|\sigma_n - 1| + |\sigma_n^2 - 1| + \ldots + |\sigma_n^{m-1}-1|}{m} \le \\
&\qquad\qquad \le \frac{1}{m}\left[\frac{\pi}{2^{n+1}} + 2\,\frac{\pi}{2^{n+1}} + \ldots + (m-1)\frac{\pi}{2^{n+1}}\right] \\
&\qquad\qquad = \frac{1}{m}\,\frac{(m-1)m}{2}\,\frac{\pi}{2^{n+1}} < \frac{\pi}{4}\,\frac{m}{2^n}\,.
\end{aligned}$$

In conclusion, if $x = \frac{m}{2^n} > 0$, we have

$$\left|\frac{h(x)-1}{x} - \frac{\pi}{2}\, i\right| \le \frac{\pi}{2}\,\frac{\pi}{4}\,\frac{m}{2^n} + \left|2^n(\sigma_n - 1) - \frac{\pi}{2}\, i\right|\,.$$

As $x = \frac{m}{2^n}$ tends to 0, necessarily n tends to $+\infty$, and the result follows, by (7).

We finally look for the limit as $x \to 0^+$, without further restrictions on x, and assume by contradiction that either such a limit does not exist, or that it is not equal to $\frac{\pi}{2}i$. Then, there is $\varepsilon > 0$ and a sequence $(x_n)_n$, with $x_n \to 0^+$, such that, for every n,

$$\left| \frac{h(x_n) - 1}{x_n} - \frac{\pi}{2}\, i \right| > \varepsilon \, .$$

By the continuity of the function $\frac{h(x)-1}{x}$ and the density of E in $\mathbb{R}$, for every sufficiently large n one can find a positive number $x'_n \in E$ such that

$$|x_n - x'_n| \leq \frac{1}{n} \, , \qquad \text{and} \qquad \left| \frac{h(x'_n) - 1}{x'_n} - \frac{\pi}{2}\, i \right| > \varepsilon \, ,$$

contradicting the previous part of the proof. ■

Taking $T = 2\pi$, we will write $\cos(x)$, $\sin(x)$ (or simply $\cos x$, $\sin x$) instead of $\cos_T(x)$, $\sin_T(x)$. As a consequence of the above, we obtain the fundamental limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \, .$$

5 Final remarks

Given $\tau \in]0, +\infty[$ and $\zeta \in \mathcal{Q}_1 \setminus \{0\}$, we have constructed a continuous function $f : \mathbb{R} \to \mathbb{C} \setminus \{0\}$ with the properties (a), (b) and (c) stated in Theorem 1. The functions $g : \mathbb{R} \to]0, +\infty[$ and $h : \mathbb{R} \to S^1$ have been defined by

$$g(x) = |f(x)| \, , \qquad h(x) = \frac{f(x)}{|f(x)|} \, .$$

Having introduced the functions exp and ln, we can write

$$g(x) = \exp\left(\frac{\ln|\zeta|}{\tau}\, x \right) \, ;$$

moreover, in view of the definition of the functions cos and sin, we have

$$h(x) = \cos\left(\frac{\operatorname{Arg}(\zeta)}{\tau}\, x \right) + i \sin\left(\frac{\operatorname{Arg}(\zeta)}{\tau}\, x \right) \, .$$

Summing up,

$$f(x) = \exp\left(\frac{\ln|\zeta|}{\tau}\, x \right) \left[\cos\left(\frac{\operatorname{Arg}(\zeta)}{\tau}\, x \right) + i \sin\left(\frac{\operatorname{Arg}(\zeta)}{\tau}\, x \right) \right] \, .$$

We now remark that g is differentiable, being

$$g'(x) = \lim_{\omega\to 0} \frac{g(x+\omega)-g(x)}{\omega} = \lim_{\omega\to 0} \frac{g(\omega)-1}{\omega}\, g(x) = \frac{\ln|\zeta|}{\tau}\, g(x)\,,$$

for every $x \in \mathbb{R}$. We also have that h is differentiable, with

$$h'(x) = \lim_{\omega\to 0} \frac{h(x+\omega)-h(x)}{\omega} = \lim_{\omega\to 0} \frac{h(\omega)-1}{\omega}\, h(x) = i\,\frac{\mathrm{Arg}(\zeta)}{\tau}\, h(x)\,.$$

Since $f(x) = g(x)h(x)$, the function f is differentiable, as well, and

$$f'(x) = g'(x)h(x) + g(x)h'(x) = \left[\frac{\ln|\zeta|}{\tau}\, g(x)\right] h(x) + g(x)\left[i\,\frac{\mathrm{Arg}(\zeta)}{\tau}\, h(x)\right]\,,$$

i.e., f satisfies the differential equation

$$f'(x) = \frac{1}{\tau}\Big(\ln|\zeta| + i\,\mathrm{Arg}(\zeta)\Big) f(x)\,.$$

The continuity assumption in Theorem 1 cannot be deleted. Indeed, there surely exist several discontinuous functions f satisfying the properties (a), (b) and (c), and they can be constructed as follows.

As already observed by Cauchy [3], a function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$\psi(x_1 + x_2) = \psi(x_1) + \psi(x_2)\,, \quad \text{for every } x_1, x_2 \in \mathbb{R}\,.$$

is either continuous, or totally discontinuous. It has been proved by Hamel [8] that there are indeed discontinuous functions with this property and that their graph is dense in $\mathbb{R}^2$ (cf. [10]). Quoting from [7], it has been proved by Fréchet [6] that the measurability of such a function ψ implies its continuity. This theorem has been later proved again by Sierpinski [18], Banach [2], Kac [11], Alexiewicz and Orlicz [1], Kestleman [12], and later generalized in [19].

Let ψ be a discontinuous function as above, with $\psi(\tau) \neq 0$. Defining the function $f : \mathbb{R} \to \mathbb{C} \setminus \{0\}$ as

$$f(x) = \exp\left(\frac{\ln|\zeta|}{\psi(\tau)}\,\psi(x)\right)\left[\cos\left(\frac{\mathrm{Arg}(\zeta)}{\tau}\, x\right) + i\sin\left(\frac{\mathrm{Arg}(\zeta)}{\tau}\, x\right)\right]\,,$$

we see that f is discontinuous, and we can easily verify that (a), (b) and (c) hold.

Our continuity assumption in Theorem 1 could then be considerably weakened, in view of the results mentioned above. Most probably, measurability would be enough. However, we prefer not entering into these technical details.

6 Appendix

There are several different approaches to the definition of the exponential and trigonometric functions. In this appendix we briefly review some of them.

1. *The most often adopted definitions* in textbooks are frequently lacking mathematical rigor. The exponential function is usually first defined on the rationals as

$$\exp_a\left(\frac{p}{q}\right) = \sqrt[q]{a^p}\,,$$

and then "extended on all the real numbers by density", maintaining the continuity. This last part is usually left at an intuitive level. Concerning the trigonometric functions, the *geometrical method* is universally adopted in elementary textbooks. However, to be made rigorous, one would need a good definition of either the *length* of an arc, or the *area* of a circular sector. In this paper we have somewhat followed this approach, by approximating the arcs by regular polygons.

2. *The method of infinite series* provides the definitions

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \end{aligned}$$

All the known properties of these functions can then be derived from these definitions (see, e.g., [17]). One also obtains

$$\begin{aligned} \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \\ \sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \end{aligned}$$

A more direct approach (cf. [16]) would be to define the complex exponential function

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}\,,$$

and then set

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}\,, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}\,.$$

Equivalently,

$$e^{ix} = \cos x + i \sin x\,,$$

the well known Euler formula.

3. *The method of infinite products* permits to define

$$\cosh x = \left(1 + \frac{x^2}{\pi^2\left(1-\frac{1}{2}\right)^2}\right)\left(1 + \frac{x^2}{\pi^2\left(2-\frac{1}{2}\right)^2}\right)\left(1 + \frac{x^2}{\pi^2\left(3-\frac{1}{2}\right)^2}\right)\cdots$$
$$\sinh x = x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2\pi^2}\right)\left(1 + \frac{x^2}{3^2\pi^2}\right)\cdots$$
$$\cos x = \left(1 - \frac{x^2}{\pi^2\left(1-\frac{1}{2}\right)^2}\right)\left(1 - \frac{x^2}{\pi^2\left(2-\frac{1}{2}\right)^2}\right)\left(1 - \frac{x^2}{\pi^2\left(3-\frac{1}{2}\right)^2}\right)\cdots$$
$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots$$

For an explanation, see, e.g., [15]. Then, the exponential function is defined as

$$e^x = \cosh(x) + \sinh(x)\,.$$

As Hardy says in [9], "this method has many advantages, but naturally demands a knowledge of the theory of infinite products".

4. *The definition by integrals* of the logarithmic function,

$$\ln y = \int_1^y \frac{dt}{t}\,,$$

then provides the exponential function by inversion (see, e.g., [4]). Similarly, one can define as in [9] the function

$$\arctan y = \int_0^y \frac{dt}{1+t^2}\,,$$

which takes values in $]-\frac{\pi}{2}, \frac{\pi}{2}[$. Then, by inversion, define $\tan x$ as

$$y = \tan x \quad \Leftrightarrow \quad \arctan y = x\,,$$

for every $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ and $y \in \mathbb{R}$. Finally, if $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ set

$$\cos x = \frac{1}{\sqrt{1+(\tan x)^2}}\,, \qquad \sin x = \frac{\tan x}{\sqrt{1+(\tan x)^2}}\,,$$

and extend these functions to $\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$ by the formulas

$$\cos\left(x + \frac{\pi}{2}\right) = -\sin x\,, \qquad \sin\left(x + \frac{\pi}{2}\right) = \cos x\,,$$

and by continuity on the set $\{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$.

5. *The differential equations approach* provides $E(x) = \exp x$ as solution of the Cauchy problem

$$\begin{cases} E'(x) = E(x) \\ E(0) = 1\,. \end{cases}$$

Similarly, $F(x) = \cos x$ and $G(x) = \sin x$ can be defined as solutions of the Cauchy problems

$$\begin{cases} F''(x) = -F(x) \\ F(0) = 1\,,\ F'(0) = 0\,, \end{cases} \qquad \begin{cases} G''(x) = -G(x) \\ G(0) = 0\,,\ G'(0) = 1\,. \end{cases}$$

A concise exposition of this approach can be found in [5].

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