On the topological degree of planar maps avoiding normal cones

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Abstract. The classical Poincaré–Bohl theorem provides the existence of a zero for a function avoiding external rays. When the domain is convex, the same holds true when avoiding normal cones. We consider here the possibility of dealing with nonconvex sets having inward corners or cusps, in which cases the normal cone vanishes. This allows us to deal with situations where the topological degree may be strictly greater than 1.

1 Introduction

Let Ω be an open and bounded planar set, whose boundary $\partial\Omega$ is a Jordan curve, and let $f : \overline{\Omega} \to \mathbb{R}^2$ be a continuous function such that $0 \notin f(\partial\Omega)$. The aim of this paper is to provide some conditions on the behaviour of the function at the boundary which guarantee that the Brouwer topological degree $\deg(f, \Omega)$ is a positive number. It is well known that, in such a case, there will be some $x \in \Omega$ such that $f(x) = 0$ (sometimes called *equilibria*).

In the case when Ω is convex, the normal cone at a given point $\bar{x} \in \partial\Omega$ is defined as

$$\mathcal{N}_\Omega(\bar{x}) = \big\{ v \in \mathbb{R}^2 : \langle v, x - \bar{x} \rangle \leq 0\,,\ \text{for every } x \in \Omega \big\}\ .$$

Here, as usual, $\langle \cdot\,, \cdot \rangle$ denotes the euclidean scalar product in $\mathbb{R}^2$, with associated norm $\|\cdot\|$. Let us recall the following known result.

Theorem 1. *Assume Ω to be convex, and that*

$$f(x) \notin \mathcal{N}_\Omega(x)\,, \qquad \textit{for every } x \in \partial\Omega\,. \tag{1}$$

Then, $\deg(f, \Omega) = 1$.

We call (1) an *avoiding cones condition.* (For a proof of Theorem 1, see, e.g., [6, 8].) In this paper we would like to investigate what happens when Ω is *not convex*. In this case, we adopt the following definition of normal cone (see, e.g., [9]):

$$\mathcal{N}_\Omega(\bar{x}) = \left\{ v \in \mathbb{R}^N : \limsup_{\substack{x \to \bar{x} \\ x \in \Omega}} \frac{\langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}. \tag{2}$$

This is the polar of the Bouligand cone (also named contingent cone). It has been called *regular normal cone* in [9, def. 6.3]. Since it could well happen that $\mathcal{N}_\Omega(\bar{x}) = \{0\}$ for some $\bar{x} \in \partial\Omega$, the avoiding cones condition at those points $\bar{x}$ gives *no restriction* for $f(\bar{x})$. However, despite this apparent difficulty, we will show that, if the avoiding cones condition (1) holds, the topological degree remains a positive number, at least when assuming some regularity for $\partial\Omega$.

There are many other possible definitions of normal cone in the nonconvex case (see [9, page 232] for a clarifying survey), and several theorems on the existence of equilibria are available (see, e.g., the well-written review paper [8]). The main novelty of our paper is allowing the normal cones to vanish at certain points, still recovering the existence result. However, we are able to do this only in the planar case, and we do not know if and how our results could be extended to higher dimensions.

Let us explain our main results, first introducing some notation. Since $\partial\Omega$ is a Jordan curve, there is a continuous function $\gamma : [0, 1] \to \mathbb{R}^2$, whose restriction to $[0, 1[$ is injective, with $\gamma(0) = \gamma(1)$ and $\gamma([0, 1]) = \partial\Omega$. Let us start assuming that $\partial\Omega$ is a *piecewise regular* Jordan curve. By this we mean that there are

$$0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1 \,,$$

such that, for every $j = 1, 2, \dots, n$, if we look at the function $\gamma_j : [a_{j-1}, a_j] \to \mathbb{R}^2$, restriction of γ to the closed interval $[a_{j-1}, a_j]$, this function is of class $\mathcal{C}^1$, and $\gamma_j'(s) \neq 0$ for every $s \in [a_{j-1}, a_j]$. Then, writing

$$\gamma_-'(a_j) = \gamma_j'(a_j) \,, \qquad \gamma_+'(a_j) = \gamma_{j+1}'(a_j) \,,$$

it may be that $\gamma_-'(a_j) \neq \gamma_+'(a_j)$. Among these, there could be inward and outward corner points (see Section 2 for a precise definition). Let us denote by N_ι *the number of inward corner points (or cusps).*

We will first prove the following result.

Theorem 2. *Assume $\partial\Omega$ to be a piecewise regular Jordan curve, and that*

$$f(x) \notin \mathcal{N}_\Omega(x)\,, \qquad \textit{for every } x \in \partial\Omega\,. \tag{3}$$

Then, $1 \leq \deg(f, \Omega) \leq N_\iota + 1$.

As we already said, at certain points a_j it may happen that $\mathcal{N}_\Omega(a_j) = \{0\}$, in which case $f(a_j)$ has no cone to avoid. Let us illustrate this with an example. Using complex notation, we consider the function $f : \mathbb{C} \to \mathbb{C}$ defined as $f(z) = z^2$. As for the set Ω, if we took the disk centered at the origin with radius 1, condition (3) would be violated at the point $(1, 0)$. So, we modify the disk in a small neighborhood of that point, by creating an inner corner, as in Figure 1. Now condition (3) is satisfied, and Theorem 2 tells us that $1 \leq \deg(f, \Omega) \leq 2$ (of course, we all know that $\deg(f, \Omega) = 2$ in this case).

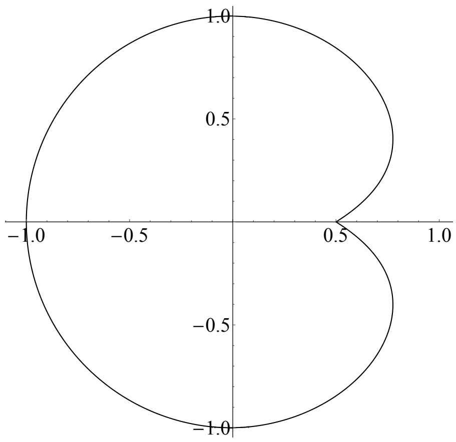

Figure 1: *Local deformation of the boundary*

The proof of Theorem 2 is provided in the next section. An important tool will be Hopf's Theorem (the so-called *Umlaufsatz*), adapted to our situation.

The extension of Theorem 2 to sets having an infinite number of corners is discussed in Section 3, where we focus our attention on sets whose boundary is piecewise the graph of a continuous function. This difficult task is not fully achieved here, since we eventually need to assume some additional regularity of the boundary. However, in view of some striking examples of sets whose boundary is locally the graph of nowhere differentiable functions (see, e.g., the one in [5]), we expect that further generalizations would require a much deeper insight in the theory of continuous functions. As expected, in this framework we loose the upper estimate on the degree, and finally only prove that $\deg(f, \Omega) \geq 1$.

Nevertheless, with the aim of extending Theorem 2, we will provide in Section 3.1 a generalization of Hopf's Theorem to some cases where the curve bounding the set Ω is not regular, and in Section 3.3 an extension of Darboux Theorem involving the Dini derivatives. These results could also have an independent interest.

The existence of equilibria of functions defined on sets in abstract spaces with very irregular boundaries has been investigated in [2, 3, 4], typically in situations when the associated topologically degree is equal to 1. Our results require a planar setting and stronger regularity assumptions on the boundary; we do not know whether they could be extended to higher dimensions.

Let us end this introduction by saying that Theorem 2 and its extension in Section 3 could be generalized assuming the vector field $f(x)$ to avoid some more general upper semicontinuous multivalued map having closed convex values. However, for briefness we prefer not entering into this subject, which will be treated elsewhere.

2 Proof of Theorem 2

Following the usual habit, we assume that $\gamma : [0,1] \to \mathbb{R}^2$ parametrizes $\partial\Omega$ in the *counter-clockwise* direction. Also, without loss of generality, we may ask that $\gamma(0) = \gamma(1)$ is a regular point, i.e., that $\gamma'_+(0) = \gamma'_-(1)$, and that $\gamma'_-(a_j) \neq \gamma'_+(a_j)$, for $j = 1, 2, \ldots, n-1$. Moreover, for simplicity we may also assume that γ is an arc-length parametrization.

2.1 The angular function

Denoting by $\mathcal{P}(\mathbb{R})$ the collection of all subsets of $\mathbb{R}$, we define a multivalued function $\omega : [0,1] \to \mathcal{P}(\mathbb{R})$, the so-called *angular function*, as follows.

In the open intervals $]a_{j-1}, a_j[$, the function will be single-valued, hence we can write

$$\gamma'(s) = e^{i\omega(s)}, \qquad \text{when} \;\; s \in]a_{j-1}, a_j[, \tag{4}$$

(recall that $\|\gamma'(s)\| = 1$) while at the points a_j, corresponding to corners or cusps, $\omega(a_j)$ will be a closed interval $[\alpha_j, \beta_j]$. Moreover, the multivalued function $\omega : [0,1] \to \mathcal{P}(\mathbb{R})$ will be upper semicontinuous (cf. [1, page 41]). We now enter into details.

Since we have assumed that $\gamma(0)$ is a regular point, we define $\omega(0)$ to be single-valued, such that $e^{i\omega(0)} = \gamma'_+(0)$ and $\omega(0) \in [0, 2\pi[$. Then, the function $\omega(s)$ is uniquely defined on $[0, a_1[$, by continuity, asking that (4) holds, and it is single-valued there.

Let us explain how $\omega(s)$ is defined on $[a_1, a_2[$. Since $\gamma'_-(a_1) \neq \gamma'_+(a_1)$, it is easily seen that we have the following alternative: either

(i) there is an $\varepsilon > 0$ such that $\gamma(a_1) + \lambda\gamma'_-(a_1) \notin \Omega$, for every $\lambda \in]0, \varepsilon[$,

in which case we say that $\gamma'_-(a_1)$ "points outward", so that $\gamma(a_1)$ is an "outward corner point", or

(ii) there is an $\varepsilon > 0$ such that $\gamma(a_1) + \lambda\gamma'_-(a_1) \in \Omega$, for every $\lambda \in]0, \varepsilon[$,

in which case we say that $\gamma'_-(a_1)$ "points inward", so that $\gamma(a_1)$ is an "inward corner point".

In case $\gamma'_-(a_1)$ points *outward*, let

$$\alpha_1 = \lim_{s \to a_1^-} \omega(s) . \tag{5}$$

Such a limit exists and is finite, since $\gamma(s) = \gamma_1(s)$ on $[0, a_1]$ and $\gamma_1 : [0, a_1] \to \mathbb{R}^2$ is of class $\mathcal{C}^1$, with $\|\gamma_1'(s)\| = 1$ for every $s \in [0, a_1]$. Moreover, $e^{i\alpha_1} = \gamma'_-(a_1)$. Let $\beta_1 \in]\alpha_1, \alpha_1 + \pi]$ be such that $e^{i\beta_1} = \gamma'_+(a_1)$, and define $\omega(a_1) = [\alpha_1, \beta_1]$. Now there is a unique way to define $\omega(s)$ on $]a_1, a_2[$, in such a way that (4) holds, preserving the upper semicontinuity of the multivalued function ω on the whole interval $[0, a_2[$. Notice that it has to be

$$\beta_1 = \lim_{s \to a_1^+} \omega(s) . \tag{6}$$

In case $\gamma'_-(a_1)$ points *inward*, let instead

$$\beta_1 = \lim_{s \to a_1^-} \omega(s) , \tag{7}$$

so that $e^{i\beta_1} = \gamma'_-(a_1)$, and let $\alpha_1 \in [\beta_1 - \pi, \beta_1[$ be such that $e^{i\alpha_1} = \gamma'_+(a_1)$. Define $\omega(a_1) = [\alpha_1, \beta_1]$, and extend $\omega(s)$ on $]a_1, a_2[$, in such a way that (4) holds, preserving the upper semicontinuity on the whole interval $[0, a_2[$. In this case, it has to be

$$\alpha_1 = \lim_{s \to a_1^+} \omega(s) . \tag{8}$$

The definition of $\omega(a_2)$ is analogous to that of $\omega(a_1)$, and we can continue recursively, thus defining $\omega(s)$ on $[a_{j-1}, a_j[$, for every $j = 1, 2, \ldots, n$. When we arrive at the last interval, we define $\omega(1)$ just by continuity: $\omega(1) = \lim_{s \to 1^-} \omega(s)$.

The following lemma will be crucial in the proof of Theorem 2.

Lemma 3. *One has that*

$$\omega(1) = \omega(0) + 2\pi .$$

Proof The function $\omega : [0,1] \to \mathcal{P}(\mathbb{R})$ defined above is upper semicontinuous and, since $\gamma(0) = \gamma(1)$ is a regular point, there must exist an integer N for which $\omega(1) = \omega(0) + 2\pi N$. If there are no singular points, i.e. if $n = 1$, we can apply Hopf's Theorem [7], stating that for any simple closed $\mathcal{C}^1$-curve γ in the plane it has to be $N = 1$.

Let us now assume $n \geq 2$. We will approximate the curve γ with a $\mathcal{C}^1$-curve $\tilde{\gamma} : [0,1] \to \mathbb{R}^2$, by *smoothing the angles*. We will thus correspondingly obtain an approximation of the multivalued function ω by a continuous single-valued function $\widetilde{\omega} : [0,1] \to \mathbb{R}$.

Let us explain how $\tilde{\gamma}$ is defined, assuming for simplicity $n = 2$, i.e., that a_1 is the only point of discontinuity of γ'. Recalling that ω is upper semicontinuous and $\omega(a_1) = [\alpha_1, \beta_1]$, for any $\varepsilon \in \left]0, \frac{\pi}{2}\right[$ there is a $\delta > 0$ such that

$$s \in [a_1 - \delta, a_1 + \delta] \quad \Rightarrow \quad \mathrm{dist}(\omega(s), [\alpha_1, \beta_1]) \leq \varepsilon\,.$$

(Here and in the following, $\mathrm{dist}(\mathcal{A}, \mathcal{B}) = \inf\{\|x - y\| : x \in \mathcal{A}, y \in \mathcal{B}\}$.) Take δ small enough, and consider the rectangle $I_1 = [a_1 - \delta, a_1 + \delta] \times [\alpha_1 - \varepsilon, \beta_1 + \varepsilon]$. We want the function $\widetilde{\omega}$ to coincide with ω on $[0, a_1 - \delta] \cup [a_1 + \delta, 1]$, while in the interval $[a_1 - \delta, a_1 + \delta]$ we will construct a $\mathcal{C}^1$-function whose graph is contained in I_1 and smoothly glues the endpoints $(a_1 - \delta, \omega(a_1 - \delta))$ and $(a_1 + \delta, \omega(a_1 + \delta))$.

Let $B(\gamma(a_1), r)$ be the open planar disk centered at $\gamma(a_1)$ with a small radius $r > 0$, so small that its boundary is crossed only twice by the curve γ. This choice is possible since there surely are $\bar{r} > 0$ and $\bar{\delta} > 0$ such that, if $r \in]0, \bar{r}]$ and $\mathrm{dist}(\gamma(s), \gamma(a_1)) = r$ for some $s \in]a_1 - \bar{\delta}, a_1 + \bar{\delta}[$, then $\gamma'(s)$ is transversal to $\partial B(\gamma(a_1), r)$. Moreover, there is a $\bar{\varepsilon} > 0$ such that, if $|s - a_1| \geq \bar{\delta}$, then $\mathrm{dist}(\gamma(s), \gamma(a_1)) \geq \bar{\varepsilon}$. It will then be sufficient to choose $r \leq \min\{\bar{r}, \bar{\varepsilon}\}$. With this choice of $r > 0$, there will be an "entrance point" $A = \gamma(a)$ and an "exit point" $B = \gamma(b)$. Notice that $a < a_1 < b$, and $b - a$ can be made arbitrarily small, by reducing the radius r.

Consider the segment AB joining A and B, and take the straight line $\mathcal{L}$, parallel to AB, at a small distance $\hat{\varepsilon} > 0$ from it, lying between the segment itself and the center of the ball $\gamma(a_1)$. Let A' and B' be the intersections of $\mathcal{L}$ with the lines

$$\mathcal{L}_A = \{\gamma(a) + t\gamma'(a) : t \in \mathbb{R}\} \quad \text{and} \quad \mathcal{L}_B = \{\gamma(b) + t\gamma'(b) : t \in \mathbb{R}\}\,,$$

respectively. Let A'' and B'' be the points on the segment $A'B'$ such that AA' and $A'A''$ have the same length, as well as for for BB' and $B'B''$. Taking $\hat{\varepsilon}$ small enough, the vector from A'' to B'' will have the same direction of the

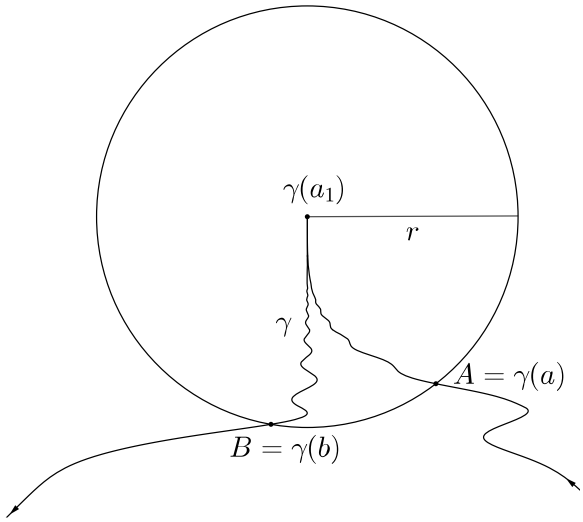

Figure 2: *The case of a cusp*

vector from A to B. Consider the circular arc $\mathcal{C}_{AA''}$, starting at A, arriving at A'', and tangent to both $\mathcal{L}$ and $\mathcal{L}_A$. Similarly, consider the circular arc $\mathcal{C}_{BB''}$, starting at B, arriving at B'', and tangent to both $\mathcal{L}$ and $\mathcal{L}_B$. The curve $\tilde{\gamma}$ will be defined as follows (see Figure 2): $\tilde{\gamma}(s)$ coincides with $\gamma(s)$ for $s < a$, i.e., until it reaches the point A; then, it follows the circular arc $\mathcal{C}_{AA''}$ until A''; at this point, it goes straight to B'', thus remaining on the line $\mathcal{L}$; then, it follows the circular arc $\mathcal{C}_{BB''}$ until B, where it rejoins the curve γ. (Notice that, since we must be careful to parametrize $\tilde{\gamma}$ in such a way that $\tilde{\gamma}(b) = B$, this curve will be regular but not necessarily parametrized by arc-length any more.) Finally, $\tilde{\gamma}(s)$ coincides with $\gamma(s)$ for $s > b$.

In the above construction, the constants r ε, δ and $\hat{\varepsilon}$ can be chosen to be arbitrarily small. Moreover, the angle function $\widetilde{\omega} : [0,1] \to \mathbb{R}$, defined by

$$\frac{\tilde{\gamma}'(s)}{\|\tilde{\gamma}'(s)\|} = e^{i\,\widetilde{\omega}(s)}\,, \tag{9}$$

with $\widetilde{\omega}(0) = \omega(0)$, is monotone as s varies in $[a,b]$, and continuous. These facts guarantee that

$$\mathrm{dist}(\widetilde{\omega}(s), \omega(s)) \leq \pi + 2\varepsilon < 2\pi\,, \qquad \text{for every } s \in [0,1]\,.$$

By Hopf's Theorem, $\widetilde{\omega}(1) = \widetilde{\omega}(0) + 2\pi$, hence also $\omega(1) = \omega(0) + 2\pi$, thus finishing the proof. $\square$

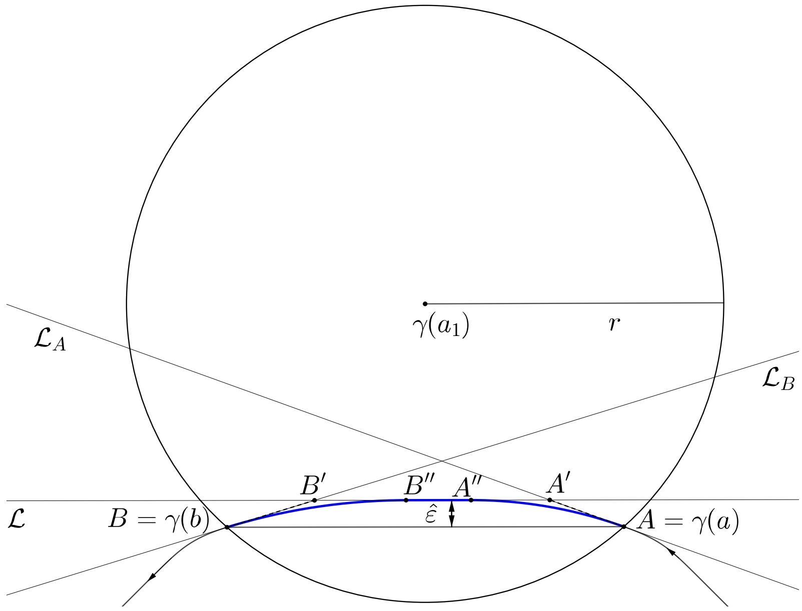

Figure 3: *An example of angle-smoothing*

2.2 The avoiding cones condition

We consider the restriction of our function $f : \overline{\Omega} \to \mathbb{R}^2$ to the boundary of Ω. More precisely, let us define the new function

$$g = f \circ \gamma : [0,1] \to \mathbb{R}^2 \setminus \{0\} .$$

Passing to polar coordinates, in complex notation, we can write

$$g(s) = \rho(s) e^{i\varphi(s)} ,$$

for some continuous functions $\rho : \mathbb{R} \to]0, +\infty[$ and $\varphi : \mathbb{R} \to \mathbb{R}$. Since $\gamma(0) = \gamma(1)$, the number $\varphi(1)$ differs from $\varphi(0)$ by an integer multiple of 2π, and

$$\deg(f, \Omega) = \frac{\varphi(1) - \varphi(0)}{2\pi} .$$

It will be useful to consider the multivalued function $\Theta : [0,1] \to \mathcal{P}(\mathbb{R})$ defined as

$$\Theta(s) = \begin{cases} \emptyset , & \text{if } s = a_j \text{ and } \gamma'_-(a_j) \text{ points inward} , \\ \omega(s) - \frac{1}{2}\pi + 2\pi\mathbb{Z} , & \text{otherwise} . \end{cases}$$

We can thus introduce an auxiliary cone $\mathcal{N}^*_\Omega(\gamma(s))$, made of the origin and the union of all the half-lines starting from the origin determined by the angles in $\Theta(s)$. Precisely,

$$\mathcal{N}^*_\Omega(\gamma(s)) = \begin{cases} \{0\}\,, & \text{if } s = a_j \text{ and } \gamma'_-(a_j) \text{ points inward,} \\ \{\alpha e^{i\theta} : \alpha \geq 0,\, \theta \in \Theta(s)\}\,, & \text{otherwise.} \end{cases} \tag{10}$$

In the sequel we will often use without further mention the elementary properties of this kind of cones, like e.g. being closed sets, translation invariant and rotation equivariant. Notice also that neither $\mathcal{N}_\Omega(\gamma(s))$ nor $\mathcal{N}^*_\Omega(\gamma(s))$ can be larger than a half-plane.

Lemma 4. *The cones $\mathcal{N}_\Omega(\gamma(s))$ and $\mathcal{N}^*_\Omega(\gamma(s))$ coincide. Therefore, the avoiding cones condition* (3) *is equivalent to*

$$\varphi(s) \notin \Theta(s)\,, \qquad \textit{for every } s \in [0,1]\,.$$

Proof We analyze several different situations.

If $s \neq a_j$ for every $j = 1, 2, \ldots, n-1$, the boundary of Ω is smooth at $\gamma(s)$, hence $\mathcal{N}_\Omega(\gamma(s))$ is just a single half-line, orthogonal to $\gamma'(s)$, with angle $\omega(s) - \frac{1}{2}\pi$. It thus coincides with $\mathcal{N}^*_\Omega(\gamma(s))$.

Assume that $s = a_j$ and that $\gamma'_-(a_j)$ points inward, so that $\Theta(a_j) = \emptyset$ and $\mathcal{N}^*_\Omega(\gamma(s)) = \{0\}$. We want to prove that $\mathcal{N}_\Omega(\gamma(a_j)) = \{0\}$, as well. Let us translate $\gamma(a_j)$ to the origin and rotate the reference system of axes in such a way that the two straight lines passing through it determined by $\gamma'_-(a_j)$ and $\gamma'_+(a_j)$ are symmetric with respect to the vertical axis and, roughly speaking, the set Ω locally stays below its boundary. More precisely, if these two lines coincide, in which case we have an inner cusp, they will be equal to $\{(x_1, x_2) : x_1 = 0\}$; otherwise, the first one will have a positive slope m, and the second one a negative slope $-m$. We may also assume, in both cases, that there are two constants $\bar{r} > 0$ and $\mu > 0$ such that

$$\{(x_1, x_2) : x_2 < \mu |x_1|\} \cap B(0, r) \subseteq \Omega\,, \qquad \text{for every } r \in\,]0, \bar{r}]\,.$$

Let $v = (v_1, v_2)$ be a vector with $\|v\| = 1$. We distinguish three cases.

Case 1: $v_2 \leq \mu |v_1|$. Then, choosing $x = \frac{r}{2} v$, we have that

$$\frac{\langle v, x \rangle}{\|x\|} = 1\,.$$

Case 2: $v_2 > \mu |v_1|$ and $v_1 \geq 0$. Here we choose $x = (\epsilon, \mu\epsilon)$, with $\epsilon > 0$ small enough, and we have that

$$\frac{\langle v, x \rangle}{\|x\|} \geq \frac{\mu}{\sqrt{1 + \mu^2}}\, v_2\,. \tag{11}$$

Case 3: $v_2 > \mu |v_1|$ and $v_1 < 0$. We then take $x = (-\epsilon, \mu\epsilon)$, with $\epsilon > 0$ small enough, and we have (11) again.

We have thus shown, in all the three cases, that $v \notin \mathcal{N}_\Omega(0)$. Since it cannot contain any unitary vector v, the cone $\mathcal{N}_\Omega(0)$ is reduced to $\{0\}$.

Assume now that $s = a_j$ and that $\gamma'_-(a_j)$ points outward. In this case, $\omega(a_j) = [\alpha_j, \beta_j]$, so that $\Theta(a_j) = [\alpha_j - \frac{1}{2}\pi, \beta_j - \frac{1}{2}\pi] + 2\pi\mathbb{Z}$. As above, we translate $\gamma(a_j)$ to the origin and take a reference system of axes so that the two straight lines passing through the origin determined by $\gamma'_-(a_j)$ and $\gamma'_+(a_j)$ are symmetric with respect to the vertical axis. If they coincide (in which case $\alpha_j = \pi/2$ and $\beta_j = 3\pi/2 \mod 2\pi$), we have an outer cusp, and they will be equal to $\{(x_1, x_2) : x_1 = 0\}$; otherwise, the first one will have a negative slope $-m$, and the second one a positive slope m (in this case, $\alpha_j = \pi - \arctan(m)$ and $\beta_j = \pi + \arctan(m) \mod 2\pi$). We want to prove that, in the first case, $\mathcal{N}_\Omega(0) = \{(x_1, x_2) : x_2 \geq 0\}$ while, in the second case, $\mathcal{N}_\Omega(0) = \{(x_1, x_2) : x_2 \geq \frac{1}{m}|x_1|\}$. This will imply that $\mathcal{N}_\Omega(0) = \mathcal{N}^*_\Omega(0)$.

Let us consider the case of an outer cusp. We first prove the inclusion $\{(x_1, x_2) : x_2 > 0\} \subseteq \mathcal{N}_\Omega(0)$. Let v be a vector in $\{(x_1, x_2) : x_2 > 0\}$, and let $m_v > 0$ be such that $v \in \{(x_1, x_2) : x_2 \geq m_v|x_1|\}$. There is a $\bar{r} > 0$ such that

$$\Omega \cap B(0, r) \subseteq \left\{(x_1, x_2) : x_2 < -\frac{2}{m_v}\,|x_1|\right\}, \quad \text{for every } r \in \,]0, \bar{r}]\,.$$

Therefore, for any $r \in \,]0, \bar{r}]$ and every $x \in \Omega \cap B(0, r) \setminus \{0\}$, one has that $\langle v, x\rangle < 0$, showing that $v \in \mathcal{N}_\Omega(0)$. Since $\mathcal{N}_\Omega(0)$ is closed (cf. [9, Proposition 6.5]), we conclude that $\{(x_1, x_2) : x_2 \geq 0\} \subseteq \mathcal{N}_\Omega(0)$.

To prove the opposite inclusion, let $v = (v_1, v_2)$ be such that $v_2 < 0$. There exist $c_v > 0$ and $\tilde{\mu}_v > 0$ such that, for every nonzero vector $x = (x_1, x_2)$ with $x_2 \leq -\tilde{\mu}_v|x_1|$, one has

$$\frac{\langle v, x\rangle}{\|x\|} \geq c_v\,. \tag{12}$$

Now, there is a $\bar{r}_v > 0$ such that

$$\Omega \cap B(0, r) \subseteq \{(x_1, x_2) : x_2 < -\tilde{\mu}_v\,|x_1|\}\,, \quad \text{for every } r \in \,]0, \bar{r}_v]\,.$$

Therefore, for any $r \in \,]0, \bar{r}_v]$ and every $x \in \Omega \cap B(0, r) \setminus \{0\}$, one has that (12) holds, showing that $v \notin \mathcal{N}_\Omega(0)$.

Assume now that $\gamma'_-(a_j)$ points outward, but is not a cusp. Let us first prove the inclusion $\{(x_1, x_2) : x_2 > \frac{1}{m}|x_1|\} \subseteq \mathcal{N}_\Omega(0)$. Let v be a vector in $\{(x_1, x_2) : x_2 > \frac{1}{m}|x_1|\}$, and let $m'_v \in \,]0, m[$ be such that $v \in \{(x_1, x_2) : x_2 \geq \frac{1}{m'_v}|x_1|\}$. There is a $\bar{r} > 0$ such that

$$\Omega \cap B(0, r) \subseteq \{(x_1, x_2) : x_2 < -m'_v\,|x_1|\}\,, \qquad \text{for every } r \in \,]0, \bar{r}]\,.$$

Therefore, for any $r \in]0, \bar{r}]$ and every $x \in \Omega \cap B(0,r) \setminus \{0\}$, one has that $\langle v, x \rangle < 0$, showing that $v \in \mathcal{N}_\Omega(0)$. Since $\mathcal{N}_\Omega(0)$ is a closed cone, we conclude that $\{(x_1,x_2) : x_2 \geq \frac{1}{m}|x_1|\} \subseteq \mathcal{N}_\Omega(0)$.

Let us now prove the opposite inclusion. Let $v = (v_1, v_2) \notin \{(x_1,x_2) : x_2 \geq \frac{1}{m}|x_1|\}$, and let $\mu_v > m$ be such that $v \notin \{(x_1,x_2) : x_2 \geq \frac{1}{\mu_v}|x_1|\}$. There is a $\bar{r} > 0$ such that

$$\{(x_1,x_2) : x_2 < -\mu_v\,|x_1|\} \cap B(0,r) \subseteq \Omega\,, \qquad \text{for every } r \in]0, \bar{r}]\,.$$

Assume $v_1 \geq 0$, and hence $v_2 < \frac{1}{\mu_v} v_1$. Then, taking $x = (\delta, -\mu_v \delta)$, for any sufficiently small $\delta > 0$ we have that $x \in \Omega$, and

$$\frac{\langle v, x \rangle}{\|x\|} = \frac{1}{\sqrt{1+\mu_v^2}}(v_1 - v_2\mu_v) > 0\,,$$

showing that $v \notin \mathcal{N}_\Omega(0)$. The case $v_1 \leq 0$ is analogous.

The proof of the lemma is thus completed. □

2.3 Conclusion of the proof

Recalling that $\gamma(0)$ is a regular point and that, by Lemma 4,

$$\varphi(0) \notin \omega(0) - \tfrac{1}{2}\pi + 2\pi\mathbb{Z}\,,$$

there is a $K \in \mathbb{Z}$ such that

$$\omega(0) - \tfrac{1}{2}\pi + 2\pi K < \varphi(0) < \omega(0) - \tfrac{1}{2}\pi + 2\pi(K+1)\,. \tag{13}$$

Then, by continuity and Lemma 4, it has to be that

$$\varphi(s) > \omega(s) - \tfrac{1}{2}\pi + 2\pi K\,, \qquad \text{for every } s \in [0, a_1[\,. \tag{14}$$

(Notice that $\omega(s)$ is single-valued in $[0, a_1[$, and in each interval $]a_{j-1}, a_j[$.) When we arrive at $s = a_1$, we have two possibilities: either $\gamma'_-(a_1)$ points outward, or it points inward. If it points outward, then

$$\varphi(a_1) \notin \Theta(a_1) = \omega(a_1) - \tfrac{1}{2}\pi + 2\pi\mathbb{Z} = [\alpha_1, \beta_1] - \tfrac{1}{2}\pi + 2\pi\mathbb{Z}\,. \tag{15}$$

By (14) and (5), we know that

$$\varphi(a_1) = \lim_{s \to a_1^-} \varphi(s) \geq \lim_{s \to a_1^-} \omega(s) - \tfrac{1}{2}\pi + 2\pi K = \alpha_1 - \tfrac{1}{2}\pi + 2\pi K\,,$$

hence, by (15) and (6), it has to be

$$\varphi(a_1) > \beta_1 - \tfrac{1}{2}\pi + 2\pi K = \lim_{s \to a_1^+} \omega(s) - \tfrac{1}{2}\pi + 2\pi K\,.$$

Consequently, if $s > a_1$ and s is sufficiently near a_1, then $\varphi(s) > \omega(s) - \frac{1}{2}\pi + 2\pi K$. This inequality will persist, by continuity and Lemma 4, for every $s \in]a_1, a_2[$.

On the other hand, if $\gamma'_-(a_1)$ points inward, there is no cone to avoid. However, by (14), (7) and (8),

$$\varphi(a_1) = \lim_{s\to a_1^-} \varphi(s) \geq \lim_{s\to a_1^-} \omega(s) - \tfrac{1}{2}\pi + 2\pi K = \beta_1 - \tfrac{1}{2}\pi + 2\pi K >$$
$$> \alpha_1 - \tfrac{1}{2}\pi + 2\pi K = \lim_{s\to a_1^+} \omega(s) - \tfrac{1}{2}\pi + 2\pi K\,.$$

Hence, by the same argument as above, we will have that $\varphi(s) > \omega(s) - \frac{1}{2}\pi + 2\pi K$, for every $s \in]a_1, a_2[$.

Iterating this process, we have that

$$\varphi(s) > \omega(s) - \tfrac{1}{2}\pi + 2\pi K\,, \qquad \text{for every } s \in \bigcup_{j=1}^{n}]a_{j-1}, a_j[\,,$$

and finally, by continuity, Lemma 3 and (13),

$$\varphi(1) \geq \omega(1) - \tfrac{1}{2}\pi + 2\pi K = \omega(0) - \tfrac{1}{2}\pi + 2\pi(K+1) > \varphi(0)\,.$$

Since $\varphi(1) - \varphi(0)$ is an integer multiple of 2π, we then deduce that

$$\varphi(1) - \varphi(0) \geq 2\pi\,,$$

i.e., that $\deg(f, \Omega) \geq 1$.

In order to show that $\deg(f, \Omega) \leq N_\iota + 1$, let us go back to $[0, a_1[$. Arguing as above, by (13) we have that

$$\varphi(s) < \omega(s) + \tfrac{3}{2}\pi + 2\pi K\,, \qquad \text{for every } s \in [0, a_1[\,. \tag{16}$$

If $\gamma'_-(a_1)$ points outward,

$$\varphi(a_1) < \alpha_1 + \tfrac{3}{2}\pi + 2\pi K\,, \tag{17}$$

and we see that, if $s > a_1$ and s is sufficiently near a_1, then $\varphi(s) < \omega(s) + \frac{3}{2}\pi + 2\pi K$, and this inequality will persist for every $s \in]a_1, a_2[$.

Now, if $\gamma'_-(a_j)$ points outward for every j, we would have

$$\varphi(s) < \omega(s) + \tfrac{3}{2}\pi + 2\pi K\,, \qquad \text{for every } s \in \bigcup_{j=1}^{n}]a_{j-1}, a_j[\,,$$

and, by Lemma 3 and (13),

$$\varphi(1) \leq \omega(1) + \tfrac{3}{2}\pi + 2\pi K = \omega(0) + \tfrac{3}{2}\pi + 2\pi(K+1) < \varphi(0) + 4\pi\,.$$

Then, $\varphi(1) - \varphi(0) \leq 2\pi$, so that $\deg(f, \Omega) \leq 1$.

On the other hand, if $\gamma'_-(a_1)$ points inward, there is no control like (17), and it could be as well that

$$\alpha_1 + \tfrac{3}{2}\pi + 2\pi K < \varphi(a_1) < \beta_1 + \tfrac{3}{2}\pi + 2\pi K \,,$$

giving an increase of 1 in the final computation of the degree. Clearly, the same could happen for any of the N_ι inward corner points.

The proof of Theorem 2 is thus completed. $\square$

3 An extension of Theorem 2

The aim of this section is to extend Theorem 2 to the case when $\partial\Omega$ is piecewise the graph of a continuous function. However, this difficult task will not be completely achieved, and we will eventually need to assume some additional regularity on that set. Moreover, as may be expected, in this framework we will loose the upper estimate on the degree, and finally only prove that $\deg(f, \Omega) \geq 1$.

Let us start by giving a precise definition of what we mean by "piecewise graph of a continuous function". As usual, $\partial\Omega$ is a Jordan curve parametrized by a continuous function $\gamma : [0,1] \to \mathbb{R}^2$, in counter-clockwise direction.

Definition 5. *We say that $\partial\Omega$ is* piecewise the graph of a continuous function *if there are*

$$0 = \hat{a}_0 < \hat{a}_1 < \cdots < \hat{a}_{m-1} < \hat{a}_m = 1 \,,$$

such that, writing $p_k = \gamma(\hat{a}_k)$,

the closed polygonal curve $\Gamma = p_0 p_1 \cdots p_m$ has no self-intersections;

moreover, denoting by ν_k the outer normal to the segment $\overline{p_{k-1}p_k}$ joining the two points p_{k-1} and p_k, for every $k = 1, 2, \ldots, m$ there are $h_k > 0$ and a continuous function $g_k : \overline{p_{k-1}p_k} \to [-h_k, h_k]$ such that, defining the rectangles

$$R_k = \overline{p_{k-1}p_k} + [-h_k, h_k]\nu_k \,,$$

we have that

$$\begin{aligned} \Omega \cap R_k &= \{p + y\nu_k : p \in \overline{p_{k-1}p_k}\,, y \in [-h_k, g_k(p)[\,\} \,, \\ \partial\Omega \cap R_k &= \{p + y\nu_k : p \in \overline{p_{k-1}p_k}\,, y = g_k(p)\} \,. \end{aligned}$$

Notice that the polygonal curve Γ, being a piecewise regular Jordan curve, can be parametrized by a piecewise regular function $\gamma_\Gamma : [0,1] \to \mathbb{R}^2$ such that $\gamma_\Gamma(\hat{a}_k) = \gamma(\hat{a}_k)$, for every $k = 1, 2, \dots, m$. Then, there is an associated *angular function* $\omega_\Gamma : [0,1] \to \mathcal{P}(\mathbb{R})$, defined precisely as in Section 2 (to simplify the exposition, we may assume that $\gamma_\Gamma(0)$ is a regular point for Γ, i.e., that $\gamma'_\Gamma(0) = \gamma'_\Gamma(1)$). Notice that there are no cusps for Γ, and that $\omega_\Gamma(1) = \omega_\Gamma(0) + 2\pi$, by Lemma 3.

Let us now introduce the concept of "vanishing set". Given a set $\mathcal{S}$, we denote by $\mathcal{S}'$ the *derived set* of $\mathcal{S}$, i.e., the set of cluster points of $\mathcal{S}$.

Definition 6. *Looking at the iterated derived sets*

$$\mathcal{S}^{(1)} = \mathcal{S}' \,, \qquad \mathcal{S}^{(n+1)} = [\mathcal{S}^{(n)}]' \,,$$

we call $\mathcal{S}$ *a* vanishing set *if, for some positive integer* N*, the iterated derived set* $S^{(N)}$ *is empty.*

We will prove the following extension of Theorem 2.

Theorem 7. *Assume* $\partial\Omega$ *to be a Jordan curve, piecewise graph of a continuous function. Let* $\gamma : [0,1] \to \mathbb{R}^2$ *be a continuous parametrization of* $\partial\Omega$*, with the property that there are a countable number of non-overlapping intervals* $[a_j, b_j]$*, contained in* $[0,1]$*, on the interior of which* γ *is of class* $\mathcal{C}^1$*, and* $\mathcal{S} = [0,1] \setminus \bigcup_j \,]a_j, b_j[$ *is a vanishing set. If the avoiding cones condition* (3) *holds, then* $\deg(f, \Omega) \geq 1$*.*

The proof will be carried out in the next four subsections. We will first need to extend Hopf's Theorem in this new setting, and to characterize the normal cones with the new angular function, similarly as in Lemma 4. We will then make a small detour to provide us with some useful properties of the Dini derivatives (which could also have some independent interest). The proof of Theorem 7 will then be given first assuming the number of intervals $[a_j, b_j]$ to be finite, and finally in its general form.

3.1 An extension of Hopf's Theorem

We need to define the *angular function* $\omega : [0,1] \to \mathcal{P}(\mathbb{R})$ in the case when $\partial\Omega$ is piecewise the graph of a continuous function. This will eventually lead us to an extension of Hopf's Theorem.

So, take some $x \in \partial\Omega$, and assume first that $x = \gamma(s)$ for some $s \in \,]\hat{a}_{k-1}, \hat{a}_k[\,$. After a roto-translation $\mathcal{S}_k$, in which the segment $\overline{p_{k-1}p_k}$ becomes horizontal, of the type $[c_k, d_k] \times \{0\}$, we have a corresponding continuous function $F_k : [c_k, d_k] \to \mathbb{R}$, whose graph is the transformation of the graph of g_k by $\mathcal{S}_k$, and $\mathcal{S}_k(\Omega)$ locally "stays below" this graph. More precisely, we can write $\mathcal{S}_k = \mathcal{T}_k \circ \mathcal{R}_k$, where $\mathcal{T}_k$ is a translation and $\mathcal{R}_k$ is the rotation around the origin with angle

$$\hat{\theta}_\Gamma^k = \pi - \omega_\Gamma \left(\frac{\hat{a}_{k-1} + \hat{a}_k}{2} \right) .$$

(Notice that ω_Γ is constant on $]\hat{a}_{k-1}, \hat{a}_k[$.) The interval $[c_k, d_k]$ has the same length as the segment $\overline{p_{k-1}p_k}$, and we will have that

$$\mathcal{S}_k(\gamma(s)) = (t(s), F_k(t(s))) ,$$

with $t(s) \in]c_k, d_k[$ continuously determined by $s \in]\hat{a}_{k-1}, \hat{a}_k[$ through the formula

$$t(s) = c_k + \frac{d_k - c_k}{\hat{a}_k - \hat{a}_{k-1}} (\hat{a}_k - s) .$$

Moreover, $t(\hat{a}_{k-1}) = d_k$, $t(\hat{a}_k) = c_k$, and

$$F_k(t(s)) = [\mathcal{S}_k \circ g_k \circ \mathcal{S}_k^{-1}](t(s), 0) .$$

To simplify the notation, we will now write F instead of F_k, and t instead of $t(s)$. We consider the four Dini derivatives

$$D_\pm^\ell F(t) = \liminf_{h \to 0^\pm} \frac{F(t+h) - F(t)}{h} , \quad D_\pm^u F(t) = \limsup_{h \to 0^\pm} \frac{F(t+h) - F(t)}{h} .$$

(In the above, the letter ℓ stands for "lower", while u means "upper".) Let

$$\begin{aligned} \mathcal{L}_-^\ell(t) &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le 0 \,,\, x_2 = D_-^\ell F(t)\, x_1\} , \\ \mathcal{L}_+^u(t) &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le 0 \,,\, x_2 = D_+^u F(t)\, x_1\} , \end{aligned}$$

where it is implicitly assumed that

$$\begin{aligned} D_-^\ell F(t) = -\infty \quad &\Rightarrow \quad \mathcal{L}_-^\ell(t) = \{0\} \times [0, +\infty[\,, \\ D_-^\ell F(t) = +\infty \quad &\Rightarrow \quad \mathcal{L}_-^\ell(t) = \{0\} \times]-\infty, 0] \,, \\ D_+^u F(t) = -\infty \quad &\Rightarrow \quad \mathcal{L}_+^u(t) = \{0\} \times [0, +\infty[\,, \\ D_+^u F(t) = +\infty \quad &\Rightarrow \quad \mathcal{L}_+^u(t) = \{0\} \times]-\infty, 0] \,. \end{aligned}$$

Let $\theta_-^\ell(t)$, $\theta_+^u(t)$ be the two real numbers in $[\frac{\pi}{2}, \frac{3\pi}{2}]$ such that, in complex notation,

$$\mathcal{L}_-^\ell(t) = \{\alpha e^{i\theta_-^\ell(t)} : \alpha \ge 0\} , \qquad \mathcal{L}_+^u(t) = \{\alpha e^{i\theta_+^u(t)} : \alpha \ge 0\} .$$

(Notice that, whenever the right and left derivatives exist and are finite, the case $\theta^\ell_-(t) < \theta^u_+(t)$ corresponds to an inward corner point, while the case $\theta^\ell_-(t) > \theta^u_+(t)$ corresponds to an outward corner point.) We thus define

$$\omega(s) = [\alpha(s), \beta(s)]\,,$$

where

$$\alpha(s) = \theta^u_+(t(s)) - \hat{\theta}^k_\Gamma\,, \qquad \beta(s) = \theta^\ell_-(t(s)) - \hat{\theta}^k_\Gamma\,, \tag{18}$$

with the convention that $[a, b] = [b, a]$ when $b < a$.

Now we look at the cases when $s = \hat{a}_k$, for some $k = 1, 2, \dots, m$. At these points, the limits from the left have to be made with one reference function, while those from the right concern a different one. For example, looking at $s = \hat{a}_k$, the angle $\theta^u_+(t(\hat{a}_k))$ must be defined through the function $F_k : [c_k, d_k] \to \mathbb{R}$, with $t(\hat{a}_k) = c_k$, while $\theta^\ell_-(t(\hat{a}_k))$ is defined using $F_{k+1} : [c_{k+1}, d_{k+1}] \to \mathbb{R}$, with $t(\hat{a}_k) = d_{k+1}$. Once this is done, the definition of $\omega(\hat{a}_k)$ is

$$\omega(\hat{a}_k) = [\alpha(\hat{a}_k), \beta(\hat{a}_k)]\,,$$

where

$$\alpha(\hat{a}_k) = \theta^u_+(t(\hat{a}_k)) - \hat{\theta}^k_\Gamma\,, \qquad \beta(\hat{a}_k) = \theta^\ell_-(t(\hat{a}_k)) - \hat{\theta}^{k+1}_\Gamma\,, \tag{19}$$

with the usual convention for $[a, b]$ when $b < a$.

Having defined the multivalued function $\omega : [0, 1] \to \mathcal{P}(\mathbb{R})$, we can now state an analogue of Hopf's Theorem.

Theorem 8. *Assume that $\partial\Omega$ is piecewise the graph of a continuous function. Then,*

$$\omega(1) = \omega(0) + 2\pi\,.$$

Proof We know that $\omega_\Gamma(1) = \omega_\Gamma(0) + 2\pi$ and, for every $s \in [0, 1]$,

$$\alpha'\,, \beta' \in \omega_\Gamma(s) \quad \Rightarrow \quad |\alpha' - \beta'| < \pi\,.$$

Moreover, recalling the assumption that $\partial\Omega$ is piecewise the graph of a continuous function,

$$s \in \,]\hat{a}_{k-1}, \hat{a}_k[\quad \Rightarrow \quad \mathrm{dist}(\omega_\Gamma(s), \omega(s)) \le \frac{\pi}{2}\,.$$

The conclusion easily follows. $\square$

3.2 A characterization of normal cones

We now give a characterization of normal cones, similarly as in Section 2, when $\partial\Omega$ is piecewise the graph of a continuous function. It will be useful to consider the following multivalued function $\Theta : [0,1] \to \mathcal{P}(\mathbb{R})$. Recalling how we have defined $\omega(s) = [\alpha(s), \beta(s)]$, we set

$$\Theta(s) = \begin{cases} \emptyset\,, & \text{if } \alpha(s) > \beta(s)\,, \\ \omega(s) - \frac{1}{2}\pi + 2\pi\mathbb{Z}\,, & \text{if } \alpha(s) \leq \beta(s)\,. \end{cases}$$

We can thus introduce an auxiliary cone $\mathcal{N}^\star_\Omega(\gamma(s))$, made of the origin and the union of all the half-lines starting from the origin determined by the angles in $\Theta(s)$, as in (10).

Lemma 9. *The cones $\mathcal{N}_\Omega(\gamma(s))$ and $\mathcal{N}^\star_\Omega(\gamma(s))$ coincide. Therefore, the avoiding cones condition* (3) *is equivalent to*

$$\varphi(s) \notin \Theta(s)\,, \qquad \textit{for every } s \in [0,1]\,.$$

Proof We fix $s \in [0,1]$ and assume first that $s \in \,]\hat{a}_{k-1}, \hat{a}_k[\,$, for some k. After operating the roto-translation $\mathcal{S}_k$, we can assume that the segment $\overline{p_{k-1}p_k}$ coincides with $[c_k, d_k] \times \{0\}$. Moreover, without loss of generality, we can assume that $c_k < 0 < d_k$ and that $\mathcal{S}_k(\gamma(s))$ coincides with the origin.

Let $\alpha(s) > \beta(s)$, so that $\Theta(s) = \emptyset$ and $\mathcal{N}^\star_\Omega(\gamma(s)) = \{0\}$. We want to prove that $\mathcal{N}_\Omega(\gamma(s)) = \{0\}$, as well. In this case, there are two real constants $\bar\mu > \bar\nu$ such that, for every $\mu \leq \bar\mu$ and every $\nu \geq \bar\nu$, the half-lines

$$\ell^+_\mu = \{(x_1, x_2) : x_1 \geq 0,\ x_2 = \mu x_1\}\,, \qquad \ell^-_\nu = \{(x_1, x_2) : x_1 \leq 0,\ x_2 = \nu x_1\}$$

intersect the set Ω infinitely many times in every small neighborhood of the origin. Hence, for every $v \in \mathbb{R}^2 \setminus \{0\}$, it is possible to find a vector x with $\|x\| = 1$ on one of such half-lines for which $\langle v, x\rangle = \delta > 0$. Hence, there is a sequence of points $(x_n)_n$ of $\Omega \setminus \{0\}$ on this half-line such that $x_n \to 0$ and $\langle v, x_n\rangle = \delta\|x_n\|$. Therefore, if $v \neq 0$, then $v \notin \mathcal{N}_\Omega(\gamma(s))$.

Assume now that $\alpha(s) = \beta(s)$, so that $\omega(s)$ is single-valued and $\mathcal{N}^\star_\Omega(\gamma(s))$ is a half-line. For every $\varepsilon > 0$ there are two sectors $S^\varepsilon_1 \subseteq S^\varepsilon_2$, with the following properties. First of all, both sectors are symmetrical with respect to $\mathcal{N}^\star_\Omega(\gamma(s))$. The sector S^ε_2 has angular amplitude equal to $\pi + 2\varepsilon$, and there is a $\bar r > 0$ such that $S^\varepsilon_2 \cap B(0,r)$ contains $\Omega \cap B(0,r)$, for every $r \in \,]0, \bar r[\,$. The sector S^ε_1 has angular amplitude equal to $\pi - 2\varepsilon$, and every half-line of this sector intersects the set Ω infinitely many times in every small neighborhood of the origin.

Let $v \neq 0$ be a vector not belonging to the half-line $\mathcal{N}_\Omega^\star(\gamma(s))$. Then, taking $\varepsilon > 0$ small enough, it is possible to find a half-line in S_1^ε and a point x on it, with with $\|x\| = 1$, for which $\langle v, x\rangle = \delta > 0$. Then, there is a sequence of points $(x_n)_n$ of $\Omega \setminus \{0\}$ on this half-line such that $x_n \to 0$ and $\langle v, x_n\rangle = \delta\|x_n\|$, showing that $v \notin \mathcal{N}_\Omega(\gamma(s))$. We have thus proved that $\mathcal{N}_\Omega(\gamma(s)) \subseteq \mathcal{N}_\Omega^\star(\gamma(s))$.

On the other hand, let $v \in \mathcal{N}_\Omega^\star(\gamma(s))$ be a vector with norm $\|v\| = 1$. For every $\varepsilon > 0$, there is a $\bar{r} > 0$ such that, for every $x \in \Omega \cap B(0, \bar{r})$, being $x \in S_2^\varepsilon$, one has

$$\frac{\langle v, x\rangle}{\|x\|} \leq \cos\left(\frac{\pi}{2} - \varepsilon\right) . \tag{20}$$

Since ε is arbitrary, this shows that $v \in \mathcal{N}_\Omega(\gamma(s))$, and since $\mathcal{N}_\Omega(\gamma(s))$ is a cone, we have proved that $\mathcal{N}_\Omega^\star(\gamma(s)) \subseteq \mathcal{N}_\Omega(\gamma(s))$.

Finally, let $\alpha(s) < \beta(s)$. In this case, $\Theta(s) = [\alpha(s) - \frac{1}{2}\pi, \beta(s) - \frac{1}{2}\pi] + 2\pi\mathbb{Z}$, and $\mathcal{N}_\Omega^\star(\gamma(s))$ is a cone whose angular amplitude is $\iota(s) = \beta(s) - \alpha(s)$. We distinguish two subcases.

Case 1: $\iota(s) < \pi$. For every $\varepsilon \in \left]0, \frac{1}{2}(\pi - \iota(s))\right[$ there are two sectors $S_1^\varepsilon \subseteq S_2^\varepsilon$, symmetrical with respect to $\mathcal{N}_\Omega^\star(\gamma(s))$. The sector S_2^ε has angular amplitude equal to $\pi - \iota(s) + 2\varepsilon$, and there is a $\bar{r} > 0$ such that $S_2^\varepsilon \cap B(0, r)$ contains $\Omega \cap B(0, r)$, for every $r \in \left]0, \bar{r}\right[$. The sector S_1^ε has angular amplitude equal to $\pi - \iota(s) - 2\varepsilon$, and every half-line of this sector intersects the set Ω infinitely many times in every small neighborhood of the origin. The proof now is the same as the one seen above in the case $\alpha(s) = \beta(s)$.

Case 2: $\iota(s) = \pi$. In this case, $\mathcal{N}_\Omega^\star(\gamma(s))$ is the half-plane $\{(x_1, x_2) : x_2 \geq 0\}$. For every $\varepsilon > 0$ there is a sector S^ε, symmetrical with respect to the vertical axis, having angular amplitude equal to 2ε, and there is a $\bar{r} > 0$ such that $S^\varepsilon \cap B(0, r)$ contains $\Omega \cap B(0, r)$, for every $r \in \left]0, \bar{r}\right[$. Let $v = (v_1, v_2)$ be a vector with $\|v\| = 1$ and $v_2 > 0$. Then, for every sufficiently small $\varepsilon > 0$, taking $r \in \left]0, \bar{r}\right[$, we see that, for every $x \in \Omega \cap B(0, r)$, being $x \in S^\varepsilon$, the inequality (20) holds true. Since ε is arbitrary, this shows that $v \in \mathcal{N}_\Omega(\gamma(s))$. We have thus proved that $\mathcal{N}_\Omega(\gamma(s))$ contains the open set $\{(x_1, x_2) : x_2 > 0\}$. Being a closed cone, it contains $\{(x_1, x_2) : x_2 \geq 0\}$, hence $\mathcal{N}_\Omega^\star(\gamma(s)) \subseteq \mathcal{N}_\Omega(\gamma(s))$. Then, equality must hold, since $\mathcal{N}_\Omega(\gamma(s))$ cannot be larger than a half-plane.

In the case when $s = \hat{a}_k$ for some $k \in \{0, 1, \ldots, m\}$, the proof is essentially the same, in view of (19), taking care of distinguishing the behaviour to the left from the one to the right. We avoid the details, for briefness. $\square$

3.3 A generalized version of Darboux's Theorem

In the following theorem and related corollary, we provide some important properties of the Dini derivatives, in the spirit of Darboux's Theorem.

Theorem 10. *Let $F : [a,b] \to \mathbb{R}$ be a continuous function such that, for some $\mu \in \mathbb{R}$,*

$$D^u_+ F(a) > \mu > D^\ell_- F(b)\,. \tag{21}$$

Then, there is a $\xi \in]a,b[$ such that

$$D^\ell_- F(\xi) \geq \mu \geq D^u_+ F(\xi)\,.$$

Proof By Weierstrass Theorem, the function $\widetilde{F}(t) = F(t) - \mu t$ has a maximum in $[a,b]$. By (21), a maximum point ξ must be in $]a,b[$. Then, $D^\ell_- \widetilde{F}(\xi) \geq 0 \geq D^u_+ \widetilde{F}(\xi)$, and since

$$D^\ell_- \widetilde{F}(\xi) = D^\ell_- F(\xi) - \mu\,, \qquad D^u_+ \widetilde{F}(\xi) = D^u_+ F(\xi) - \mu\,,$$

the result follows. □

The following corollary will play an important role in the proof of Theorem 7.

Corollary 11. *Let $F : I \to \mathbb{R}$ be a continuous function, defined on some interval I, and let τ_0 be a point of I. Consider the set*

$$E = \{\tau \in I : D^u_+ F(\tau) \leq D^\ell_- F(\tau)\}\,.$$

If τ_0 is a cluster point for E from the left, then

$$D^\ell_- F(\tau_0) \geq \liminf_{\substack{\tau \to \tau_0^- \\ \tau \in E}} D^u_+ F(\tau)\,. \tag{22}$$

Similarly, if τ_0 is a cluster point for E from the right, then

$$D^u_+ F(\tau_0) \leq \limsup_{\substack{\tau \to \tau_0^+ \\ \tau \in E}} D^\ell_- F(\tau)\,. \tag{23}$$

Proof Let us prove (22). Assume by contradiction that the opposite inequality holds. Then, we can find a $\delta > 0$ and a real number μ such that $[\tau_0 - \delta, \tau_0] \subseteq I$ and

$$D^u_+ F(\tau) > \mu > D^\ell_- F(\tau_0)\,, \quad \text{for every } \tau \in [\tau_0 - \delta, \tau_0[\cap E\,. \tag{24}$$

Fix $\bar{\tau} \in [\tau_0 - \delta, \tau_0[\cap E$. By Theorem 10, there is a $\xi \in]\bar{\tau}, \tau_0[$ such that

$$D^\ell_- F(\xi) \geq \mu \geq D^u_+ F(\xi)\,.$$

Then, we see that $\xi \in E$ and, by (24), it should be $D^u_+ F(\xi) > \mu$, a contradiction. The proof of (23) is analogous. □

3.4 The proof of Theorem 7

The proof will be divided in three steps.

Step 1. First, we assume that the number of intervals $[a_j, b_j]$ is finite. Hence, besides assuming that $\partial\Omega$ is piecewise the graph of a continuous function, we also ask that there are

$$0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1\,,$$

such that, for every $j = 1, 2, \ldots, n$, the restriction of γ to the *open* interval $]a_{j-1}, a_j[$ is of class $\mathcal{C}^1$, and $\gamma_j'(s) \neq 0$ for every $s \in]a_{j-1}, a_j[$. Notice that, in this setting, the limits $\lim_{s\to a_j^\pm} \gamma'(s)$ do not have to exist.

In the following, for simplicity, we will ask that $\gamma(0) = \gamma(1)$ is a regular point, i.e., that $\gamma'_+(0) = \gamma'_-(1)$. Let us start by assuming that each point a_j is contained in the interior of some $]\hat{a}_{k-1}, \hat{a}_k[$.

We consider the function $g = f \circ \gamma : [0,1] \to \mathbb{R}^2 \setminus \{0\}$ and, extending it by 1-periodicity, we write

$$g(s) = \rho(s)e^{i\varphi(s)}\,,$$

for some continuous functions $\rho : \mathbb{R} \to]0, +\infty[$ and $\varphi : \mathbb{R} \to \mathbb{R}$.

Being $\varphi(0) \notin \Theta(0)$, let $K \in \mathbb{Z}$ be such that

$$\beta(0) + 2\pi K < \varphi(0) + \tfrac{1}{2}\pi < \alpha(0) + 2\pi(K+1)\,. \tag{25}$$

(Here, since $\omega(0)$ is single-valued, $\alpha(0) = \beta(0)$.) By continuity and Lemma 9, it has to be that

$$\varphi(s) + \tfrac{1}{2}\pi > \beta(s) + 2\pi K\,, \qquad \text{for every } s \in [0, a_1[\,.$$

We know that $a_1 \in]\hat{a}_{k-1}, \hat{a}_k[$, for some $k \in \{1, 2, \ldots, m\}$. We consider the corresponding function $t :]\hat{a}_{k-1}, \hat{a}_k[\to]c_k, d_k[$, and set $\tau_0 = t(a_1)$. Then, recalling (18), there exists some $\delta > 0$ for which

$$\varphi(t^{-1}(\tau)) + \tfrac{1}{2}\pi > \theta_-^\ell(\tau) - \hat{\theta}_\Gamma^k + 2\pi K\,, \qquad \text{for every } \tau \in]\tau_0, \tau_0 + \delta[\,.$$

Then, by (23), recalling (18) again,

$$\begin{aligned}
\varphi(a_1) + \tfrac{1}{2}\pi &= \limsup_{\tau\to\tau_0^+} \varphi(t^{-1}(\tau)) + \tfrac{1}{2}\pi \\
&\geq \limsup_{\tau\to\tau_0^+} \theta_-^\ell(\tau) - \hat{\theta}_\Gamma^k + 2\pi K \\
&\geq \theta_+^u(\tau_0) - \hat{\theta}_\Gamma^k + 2\pi K = \alpha(a_1) + 2\pi K\,.
\end{aligned} \tag{26}$$

(Here the set E plays no role.) We have two possibilities.

Case 1: $D^\ell_- F_k(\tau_0) \geq D^u_+ F_k(\tau_0)$. Then, by Lemma 9 and (26), it has to be that

$$\varphi(a_1) + \tfrac{1}{2}\pi > \beta(a_1) + 2\pi K\,. \tag{27}$$

Case 2: $D^\ell_- F_k(\tau_0) < D^u_+ F_k(\tau_0)$. Then, $\alpha(a_1) > \beta(a_1)$, and from (26) we get (27) again.

On the other hand, by (18) and (22),

$$\beta(a_1) = \theta^\ell_-(\tau_0) - \hat{\theta}^k_\Gamma \geq \liminf_{\tau\to\tau_0^-} \theta^u_+(\tau) - \hat{\theta}^k_\Gamma\,.$$

(Even here the set E plays no role.) So, by (27), there are a sufficiently small $\varepsilon > 0$ and a strictly increasing sequence $(\tau_n)_n$ such that $\lim_n \tau_n = \tau_0$ and, setting $s_n = t^{-1}(\tau_n)$, by (18),

$$\varphi(a_1) + \tfrac{1}{2}\pi - \varepsilon > \theta^u_+(\tau_n) - \hat{\theta}^k_\Gamma + 2\pi K = \alpha(s_n) + 2\pi K = \beta(s_n) + 2\pi K\,.$$

(Here $\alpha(s_n) = \beta(s_n)$, being γ of class $\mathcal{C}^1$ on $]a_1, a_2[$.) Since $s_n \to a_1$, by continuity, for n large enough,

$$\varphi(s_n) + \tfrac{1}{2}\pi > \beta(s_n) + 2\pi K\,.$$

Hence, by Lemma 9 and the continuity of φ and β on $]a_1, a_2[$,

$$\varphi(s) + \tfrac{1}{2}\pi > \beta(s) + 2\pi K\,, \qquad \text{for every } s \in\,]a_1, a_2[\,.$$

Iterating this procedure on each interval $]a_{j-1}, a_j[$, we thus prove that

$$\varphi(s) + \tfrac{1}{2}\pi > \beta(s) + 2\pi K\,, \qquad \text{for every } s \in\,]a_{j-1}, a_j[\,.$$

By continuity and Theorem 8, recalling that $\omega(1)$ is single-valued and using (25),

$$\varphi(1) + \tfrac{1}{2}\pi \geq \beta(1) + 2\pi K = \alpha(1) + 2\pi K = \alpha(0) + 2\pi(K+1) > \varphi(0) + \tfrac{1}{2}\pi\,.$$

Since $\varphi(1) - \varphi(0)$ is an integer multiple of 2π, we then deduce that

$$\varphi(1) - \varphi(0) \geq 2\pi\,,$$

and the proof is completed. In the case when some a_j coincides with some $\hat{a}_k$ the proof is easily adapted, in view of the definition given in (19), taking care of the different functions involved when approaching a_j from the left and from the right.

Step 2. As a second step, we now assume that there are a countable number of non-overlapping intervals $[a_j, b_j]$, contained in $[0,1]$, on the interior of which γ is of class $\mathcal{C}^1$, and that the *singular set*

$$\mathcal{S} = [0,1] \setminus \bigcup_{j=0}^{\infty}]a_j, b_j[$$

has a finite number of cluster points $a'_1 < a'_2 < \cdots < a'_N$.

Keeping the same notations as above, for simplicity we ask that $\gamma(0) = \gamma(1)$ be a regular point, and we first assume that each point $a'_1, a'_2, \ldots, a'_N$ is contained in the interior of some $]\hat{a}_{k-1}, \hat{a}_k[$. Let $K \in \mathbb{Z}$ be such that (25) holds. Then, by induction, using the result proved in Step 1, we see that

$$\varphi(s) + \tfrac{1}{2}\pi > \beta(s) + 2\pi K\,, \qquad \text{for every } s \in [0, a'_1[\,.$$

We know that $a'_1 \in]\hat{a}_{k-1}, \hat{a}_k[$, for some $k \in \{1, 2, \ldots, m\}$. We consider the corresponding function $t :]\hat{a}_{k-1}, \hat{a}_k[\to]c_k, d_k[$, and set $\tau'_0 = t(a'_1)$. Using (23), we see like in (26) that $\varphi(a'_1) + \frac{1}{2}\pi \geq \alpha(a'_1) + 2\pi K$ (here the set E plays no role). We now have two possibilities.

Case 1: $D^\ell_- F_k(\tau'_0) \geq D^u_+ F_k(\tau'_0)$. Then, by Lemma 9, it has to be that

$$\varphi(a'_1) + \tfrac{1}{2}\pi > \beta(a'_1) + 2\pi K\,. \tag{28}$$

Case 2: $D^\ell_- F_k(\tau'_0) < D^u_+ F_k(\tau'_0)$. Then, $\alpha(a'_1) > \beta(a'_1)$, and we get (28) again.

Now, using (22), there is a strictly decreasing sequence $(s_n)_n$ such that

$$\lim_n s_n = a'_1\,, \quad \alpha(s_n) \leq \beta(s_n)\,, \quad \lim_n \alpha(s_n) \leq \beta(a'_1)\,.$$

(In this case, the set E plays a crucial role.) By (28), taking $\varepsilon > 0$ small enough,

$$\varphi(a'_1) + \tfrac{1}{2}\pi - \varepsilon > \alpha(s_n) + 2\pi K\,,$$

for every sufficiently large n. Being $\Theta(s_n) = [\alpha(s_n), \beta(s_n)] - \frac{1}{2}\pi + 2\pi\mathbb{Z}$, with $\alpha(s_n) \leq \beta(s_n)$, by Lemma 9 we have that

$$\varphi(a'_1) + \tfrac{1}{2}\pi - \varepsilon > \beta(s_n) + 2\pi K\,,$$

so that, by continuity, for n large enough,

$$\varphi(s_n) + \tfrac{1}{2}\pi > \beta(s_n) + 2\pi K\,.$$

We can now use the argument at the end of Step 1 to show that

$$\varphi(s) + \tfrac{1}{2}\pi > \beta(s) + 2\pi K\,, \qquad \text{for every } s \in]a'_1, a'_2[\,.$$

Iterating this procedure, we easily conclude the proof. The case when some a'_j coincides with some $\hat{a}_k$ is treated similarly, as already observed above.

Step 3. We have thus shown that the topological degree is a positive number if $\mathcal{S}'$, the derived set of $\mathcal{S}$, is finite. We can now repeat the argument in Step 2 assuming that $\mathcal{S}'$ is an infinite set, with a finite number of cluster points. And this procedure can be carried on an arbitrary finite number of times. Since we have assumed that $\mathcal{S}$ is a vanishing set, we will eventually reach an iterated derived set having only a finite number of points. The proof is then completed using once again the argument in Step 2. $\square$

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