A generalization of the parallelogram law to higher dimensions

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Abstract. We propose a generalization of the parallelogram identity in any dimension \( N \geq 2 \), establishing the ratio of the quadratic mean of the diagonals to the quadratic mean of the faces of a parallelotope.

1 Introduction and statement of the result

The well known parallelogram law states:

For any parallelogram, the sum of the squares of the lengths of its two diagonals is equal to the sum of the squares of the lengths of its four sides.

Equivalently: given two vectors \( \mathbf{a} \) and \( \mathbf{b} \), one has

\[
\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2) .
\]

This identity holds in any inner product space, but, since the two vectors belong to the same plane, we can see it as being of a two-dimensional nature. The aim of this paper is to provide a generalization to higher dimensions.
The parallelogram law has a natural geometric interpretation, involving the areas of the squares constructed on the sides and on the diagonals of the parallelogram. In particular, when $\|a + b\| = \|a - b\|$, it reduces to the Pythagorean theorem. In this paper, however, we will look at the parallelogram law from a rather unusual point of view: writing it in the form

$$\frac{\|a + b\|^2 + \|a - b\|^2}{2} = 2 \frac{\|a\|^2 + \|b\|^2 + \|a\|^2 + \|b\|^2}{4},$$

and taking the square roots, we can state it in the following equivalent form.

For any parallelogram, the ratio of the quadratic mean of the lengths of its diagonals to the quadratic mean of the lengths of its sides is equal to $\sqrt{2}$.

Now, instead of a parallelogram, we will consider an $N$-dimensional parallelotope, with an arbitrary $N \geq 2$, and our goal will be to prove that the same type of proposition holds in this general case. Indeed, our result can be stated as follows.

**Theorem** For any $N$-dimensional parallelotope, the ratio of the quadratic mean of the $(N - 1)$-dimensional measures of its diagonals to the quadratic mean of the $(N - 1)$-dimensional measures of its faces is equal to $\sqrt{2}$.

For $N = 2$, the 1-dimensional measure is the length, and we recover the parallelogram law. In the general case, we first need to specify what a diagonal should be, and indeed this will be clarified in Section 2. For example, if $N = 3$, the diagonals of a parallelepiped are precisely the parallelograms obtained joining the opposite edges of the parallelepiped (see Figure 2 below), so that the 2-dimensional measures of the diagonals are the areas of these parallelograms.

2 Proof of the theorem

Let $P$ be the parallelotope generated by the vectors $a_1, \ldots, a_N$, i.e.,

$$P = \left\{ \sum_{k=1}^{N} c_k a_k : c_k \in [0, 1], \text{ for } k = 1, \ldots, N \right\}.$$
Its $2N$ faces are defined by

$$F^-_n = \left\{ \sum_{k=1}^{N} c_k a_k \in \mathcal{P} : c_n = 0 \right\}, \quad F^+_n = \left\{ \sum_{k=1}^{N} c_k a_k \in \mathcal{P} : c_n = 1 \right\},$$

with $n = 1, \ldots, N$. Each $F^-_n$ is generated by the vectors $a_1, \ldots, \hat{a}_n, \ldots, a_N$, where, as usual, $\hat{a}_n$ means that $a_n$ is missing, while $F^+_n$ is a translation of $F^-_n$, for every $k = 1, \ldots, N$.

Concerning the diagonals, they are defined as

$$D^1_{i,j} = \left\{ \sum_{k=1}^{N} c_k a_k \in \mathcal{P} : c_i = c_j \right\}, \quad D^2_{i,j} = \left\{ \sum_{k=1}^{N} c_k a_k \in \mathcal{P} : c_i + c_j = 1 \right\},$$

with indices $i < j$ varying from 1 to $N$. There are $N(N-1)$ of them. (In Figure 2 we illustrate a diagonal in the case $N = 3$.)

Figure 2: One of the six diagonals of a three-dimensional parallelepiped

Hence, we have that

$D^1_{i,j}$ is generated by $a_i + a_j$ and $a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_N$, while $D^2_{i,j}$ is a translation of the set generated by $a_i - a_j$ and $a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_N$.

**Remark** Notice that our definition of a diagonal is not the same as the one given in [1, 2], where a different generalization of the parallelogram law has been proposed: in the three-dimensional case, e.g., their diagonals are triangles. We believe that our definition is somewhat more natural, since here the diagonals share the same geometrical shape of the faces.
In order to compute the \((N - 1)\)-dimensional measures of the faces and the diagonals of our parallelotope, we make use of the following proposition involving the exterior product of vectors in \(\mathbb{R}^N\). (See, e.g., [3] for the definition and the main properties of the exterior product.)

**Proposition** The \(M\)-dimensional measure of a parallelotope generated by \(M\) vectors \(v_1, \ldots, v_M\) in \(\mathbb{R}^N\), with \(1 \leq M \leq N\), is given by \(\|v_1 \wedge \ldots \wedge v_M\|\).

**Proof** If \(v_1, \ldots, v_M\) are linearly dependent, the \(M\)-dimensional measure of the parallelotope generated by \(v_1, \ldots, v_M\) is equal to zero, hence coincides with \(\|v_1 \wedge \ldots \wedge v_M\|\). Assume now that the vectors \(v_1, \ldots, v_M\) are linearly independent, and let \(V\) be the subspace generated by them. Choose an orthonormal basis \(e_1, \ldots, e_M\) of \(V\), and write
\[
v_1 = v_{11}e_1 + \ldots + v_{1M}e_M, \quad \vdots \quad v_M = v_{M1}e_1 + \ldots + v_{MM}e_M.
\]

Then,
\[
v_1 \wedge \ldots \wedge v_M = \det \begin{pmatrix}
    v_{11} & \cdots & v_{1M} \\
    \vdots & \ddots & \vdots \\
    v_{M1} & \cdots & v_{MM}
\end{pmatrix} e_1 \wedge \ldots \wedge e_M,
\]
so that
\[
\|v_1 \wedge \ldots \wedge v_M\| = \left| \det \begin{pmatrix}
    v_{11} & \cdots & v_{1M} \\
    \vdots & \ddots & \vdots \\
    v_{M1} & \cdots & v_{MM}
\end{pmatrix} \right|,
\]
which is indeed the \(M\)-dimensional measure of the parallelotope generated by \(v_1, \ldots, v_M\). 

Hence, the \((N - 1)\)-dimensional measures of the faces \(\mathcal{F}_n^\pm\) are given by
\[
\|a_1 \wedge \ldots \wedge \hat{a}_n \wedge \ldots \wedge a_N\|,
\]
while the \((N - 1)\)-dimensional measures of the diagonals \(\mathcal{D}_{i,j}^1\) are equal to
\[
\|(a_i + a_j) \wedge \bigwedge_{k \neq i,j} a_k\|,
\]
and those of the diagonals \(\mathcal{D}_{i,j}^2\) are equal to
\[
\|(a_i - a_j) \wedge \bigwedge_{k \neq i,j} a_k\|.
\]
Choosing any couple $i < j$, by the parallelogram law we have that
\[
\|(a_i + a_j) \wedge \bigwedge_{k \neq i,j} a_k\|^2 + \|(a_i - a_j) \wedge \bigwedge_{k \neq i,j} a_k\|^2 = \\
2\left(\|a_1 \wedge \ldots \wedge \hat{a}_j \wedge \ldots \wedge a_N\|^2 + \|a_1 \wedge \ldots \wedge \hat{a}_i \wedge \ldots \wedge a_N\|^2\right).
\]
We now want to take the sum of all these equalities, with $i < j$ varying form 1 to $N$. We claim that, for any $n = 1, \ldots, N$, when performing such a sum, in the right hand side,

the term $2\|a_1 \wedge \ldots \wedge \hat{a}_n \wedge \ldots \wedge a_N\|^2$ will appear $N - 1$ times.

Indeed, this term may appear with $j = n$, while $i$ varies from 1 to $n - 1$, or with $i = n$, while $j$ varies from $n + 1$ to $N$, and there are exactly $N - 1$ of such possibilities. Hence, summing all the equalities, we have that
\[
\sum_{i<j} \left(\|(a_i + a_j) \wedge \bigwedge_{k \neq i,j} a_k\|^2 + \|(a_i - a_j) \wedge \bigwedge_{k \neq i,j} a_k\|^2\right) = \\
= (N - 1)\sum_{n=1}^{N-1} 2\|a_1 \wedge \ldots \wedge \hat{a}_n \wedge \ldots \wedge a_N\|^2.
\]
So, we have proved the following.

For any $N$-dimensional parallelootope, the sum of the squares of the $(N-1)$-dimensional measures of its $N(N-1)$ diagonals is equal to $N - 1$ times the sum of the squares of the $(N-1)$-dimensional measures of its $2N$ faces.

The proof of the theorem is now easily completed, dividing each of the two sums by the number of their addends and taking the square roots. 

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References