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PERIODIC PERTURBATIONS OF SCALAR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. We prove the existence of periodic solutions for perturbations of some autonomous second order nonlinear differential equations by the use of the Poincaré-Birkhoff fixed point theorem.

1. Introduction. In this paper we deal with equations of the type

$$x'' + g(t, x, \varepsilon) = 0, \tag{1_ε}$$

where $\varepsilon > 0$ is a small parameter, g is T−periodic in t and $g(t, x, 0) = \bar{g}(x)$ is independent of t. We look for periodic solutions of (1_ε) whose periods are integer multiples of T.

This is a classical problem which has been studied by many authors. In [8], Loud proved the existence of periodic solutions of an equation like

$$x'' + cx' + g(x) = \varepsilon f(t)$$

by the use of a modified version of the implicit function theorem. This method has been developed by Lazer in [7] in order to deal with weakly uncoupled systems, and by Willem [10] who proved a general theorem for Hamiltonian systems. General results were also obtained by Hale et al. (see [1,5] and the references therein). The methods used in those papers require the function g to be continuously differentiable.

Here we use a different approach. We prove the existence of periodic solutions by finding fixed points of the Poincaré map, through a modified version of the Poincaré-Birkhoff fixed point theorem due to W. Ding [3]. This method permits to remove the regularity assumption on g : we only need g to be continuous.

2. Main results. We consider equation (1_ε), or equivalently the system

$$\begin{cases} x' = y \\ y' = -g(t, x, \varepsilon) . \end{cases} \tag{2_ε}$$

Here, $g : \mathbb{R} \times (\alpha, \beta) \times [0, \bar{\varepsilon}] \to \mathbb{R}$ is a continuous function, with $-\infty \leq \alpha < 0 < \beta \leq +\infty$, and $\bar{\varepsilon} > 0$. Moreover, we assume that, for every $(t, x, \varepsilon) \in \mathbb{R} \times (\alpha, \beta) \times [0, \bar{\varepsilon}]$, the following three conditions hold:

(a) $$g(t + T, x, \varepsilon) = g(t, x, \varepsilon),$$

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with $T > 0$,

(b) $$g(t,x,0) = \bar{g}(x),$$

and, if $x \neq 0$,

(c) $$\bar{g}(x)x > 0.$$

Hence, for $\varepsilon = 0$, the autonomous system

$$\begin{cases} x' = y \\ y' = -\bar{g}(x) \end{cases} \tag{2_0}$$

has the origin $(0,0)$ as a center.

Let $G(x) = \int_0^x \bar{g}(s)\,ds$. For convenience, we restrict our attention to a subinterval (α_1, β_1) of (α, β), with $-\infty \leq \alpha_1 < 0 < \beta_1 \leq +\infty$, for which

$$\sup_{(\alpha_1,0]} G = \sup_{[0,\beta_1)} G.$$

In this way, for any $r \in (0, \beta_1)$, the set

$$\Gamma_r = \{(x,y) \in (\alpha_1, \beta_1) \times \mathbb{R} : \frac{1}{2}y^2 + G(x) = G(r)\},$$

is a closed curve representing the orbit of a solution of (2_0), which delimitates a star-shaped set with respect to the origin. Clearly, if $r_1 \neq r_2$, then Γ_{r_1} and Γ_{r_2} are disjoint.

We consider the so-called time-map: for any $r \in (0, \beta_1)$,

$$\tau(r) = \sqrt{2} \int_s^r \frac{dx}{\sqrt{G(r) - G(x)}}\,,$$

where $s \in (\alpha_1, 0)$ is such that $G(s) = G(r)$. One can show that $\tau(r)$ is the minimal period of the solutions of (2_0) whose orbits lie on Γ_r. In the following, we denote by $\mathcal{J}$ the set of all such periods:

$$\mathcal{J} = \{\tau(r) : r \in (0, \beta_1)\}\,.$$

We say that a solution (x,y) of (2_ε) makes *at least* N rotations around the origin in the time τ if $(x(t), y(t)) \neq (0,0)$, for all $t \in [0, \tau]$, and, considering polar coordinates

$$x(t) = \rho(t)\cos\theta(t)\,, \quad y(t) = \rho(t)\sin\theta(t),$$

we have that $\theta(0) - \theta(\tau) \geq 2N\pi$. We say it makes *more than* N rotations if $\theta(0) - \theta(\tau) > 2N\pi$. In the same spirit, we speak of solutions making *at most, less than,* or *exactly* N rotations around the origin.

We now state our main results, which will be proved in the next section. First, we show that, provided that the interval $\mathcal{J}$ has a nonempty interior, there are periodic solutions when ε is small enough. More precisely, we have the following.

Theorem 1. *Let conditions (a), (b), (c) be satisfied. Then, given two positive integers k and N such that kT/N belongs to the interior of $\mathcal{J}$, there exists an $\varepsilon_{k,N} \in (0, \bar{\varepsilon}]$ such that, for any $\varepsilon \in [0, \varepsilon_{k,N}]$, equation (1_ε) has a kT-periodic solution which makes exactly N rotations around the origin.*

Next, we show that, when the time map is strictly monotone, the periodic solutions found can be located near to the solutions of the autonomous equation. Precisely, we will prove the following.

Theorem 2. *Assume conditions (a), (b), (c) to hold. Let k and N be two positive integers, and choose $(\bar{x}(t), \bar{y}(t))$, a (kT/N)-periodic solution of (2_0) with orbit Γ_r. If* *the time-map τ is strictly increasing or strictly decreasing at r, then for every $\gamma > 0$ there exists an $\varepsilon_{k,N} \in (0, \bar{\varepsilon}]$ such that, for any $\varepsilon \in [0, \varepsilon_{k,N}]$, (2_ε) has a kT-periodic solution $(x_\varepsilon(t), y_\varepsilon(t))$ which makes exactly N rotations around the origin and is such that, for some $\delta \in [0, kT/N)$,*

$$|x_\varepsilon(t) - \bar{x}(t+\delta)| + |y_\varepsilon(t) - \bar{y}(t+\delta)| \le \gamma,$$

for every $t \in [0, kT]$.

Remark that the kT-periodic solutions found in the above theorems have minimal period kT if, for example, one has $g(t,x,\varepsilon) = \bar{g}(x) + \varepsilon h(t)$, with h having minimal period T and, moreover, k is prime with N (the number of rotations).

3. Proof of the main results. We begin by stating a preliminary lemma which can be proved by standard arguments in the theory of ordinary differential equations (see e.g. Theorem 3.2 in [6]).

Lemma. *Let k be a positive integer and fix $r > 0$. Then for every $\gamma > 0$ there exists an $\varepsilon_\gamma \in (0, \bar{\varepsilon}]$ with the following property: for every $\varepsilon \in [0, \varepsilon_\gamma]$, and every $(x_0, y_0) \in \Gamma_r$, any solution $(x_\varepsilon(t), y_\varepsilon(t))$ of the Cauchy problem*

$$\begin{cases} x' = y \\ y' = -g(t,x,\varepsilon) \\ x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases} \qquad (P_\varepsilon)$$

is defined on $[t_0, t_0 + kT]$ and is such that

$$|x_\varepsilon(t) - x_0(t)| + |y_\varepsilon(t) - y_0(t)| \le \gamma,$$

for every $t \in [t_0, t_0 + kT]$ (here $(x_0(t), y_0(t))$ is a solution of (P_0)).

Now we give the proofs of our two main theorems.

Proof of Theorem 1. Choose T_1 and T_2 in $\mathcal{J}$ such that $T_1 < \frac{kT}{N} < T_2$. Let Γ^1 and Γ^2 be two orbits corresponding to two solutions of (2_0) having periods T_1 and T_2, respectively. Let $\mathcal{A}$ be the annulus determined by Γ^1 and Γ^2. As a consequence of the above lemma, there exists a $\bar{\bar{\varepsilon}} \in (0, \bar{\varepsilon}]$ such that every solution of (2_ε) with $\varepsilon \in [0, \bar{\bar{\varepsilon}}]$ starting at time $t_0 = 0$ from a point (x_0, y_0) in $\mathcal{A}$ is defined for all $t \in [0, kT]$.

Let us first assume the uniqueness for the Cauchy problems (P_ε) with $t_0 = 0$ and $(x_0, y_0) \in \mathcal{A}$. In this way, we can consider the Poincaré map which associates to each $(x_0, y_0) \in \mathcal{A}$ the point reached at time $t = kT$ by the solution starting at time $t = 0$ from (x_0, y_0). If $\varepsilon = 0$, a solution starting at time $t = 0$ from Γ^1 [resp. Γ^2] makes more than [resp. less than] N rotations around the origin in the time kT. By the above lemma, choosing γ small enough, there is a $\varepsilon_\gamma \in (0, \bar{\bar{\varepsilon}}]$ such that this property is maintained for the solutions of (2_ε), for every $\varepsilon \in [0, \varepsilon_\gamma]$. The generalized version of Poincaré-Birkhoff fixed point theorem due to W. Ding [3] yields, in this case, the existence of a kT-periodic solution of (2_ε) which makes exactly N rotations around the origin in the time kT.

If there is no uniqueness for the Cauchy problems (P_ε), consider a sequence $(g_n)_n$ such that $g_n(t,x,\varepsilon) \to g(t,x,\varepsilon)$ uniformly on compact sets, for which the Cauchy problems

$$\begin{cases} x' = y \\ y' = -g_n(t,x,\varepsilon) \\ x(0) = x_0 \\ y(0) = y_0 \end{cases} \qquad (P_{\varepsilon,n})$$

have the uniqueness property. At this point, we proceed like in [4] to prove that, if n is large enough, any solution of $(P_{\varepsilon,n})$ with (x_0, y_0) in Γ^1 [resp. Γ^2] makes more than [resp. less than] N rotations around the origin in the time kT. Hence, Ding's theorem yields the existence of a kT-periodic solution $(x_n(t), y_n(t))$ of

$$\begin{cases} x' = y \\ y' = -g_n(t,x,\varepsilon) \,, \end{cases}$$

starting at the time $t = 0$ from the annulus $\mathcal{A}$, which makes exactly N rotations around the origin in the time kT. By a standard compactness argument, a subsequence $(x_{n_k}(t), y_{n_k}(t))$ converges to a kT-periodic solution of (2_ε) which makes exactly N rotations around the origin in the time kT. □

Proof of Theorem 2. We will assume for definiteness that τ is strictly increasing at r. Since $\tau(r) = \frac{kT}{N}$, we can choose r_1 and r_2 such that $r_1 < r < r_2$ and in such a way that $\tau(s) < \tau(r)$ when $s \in [r_1, r)$ and $\tau(s) > \tau(r)$ when $s \in (r, r_2]$. Let $\mathcal{A}$ be the annulus determined by the orbits Γ_{r_1} and Γ_{r_2}. As shown in the proof of Theorem 1, for ε small enough there is a kT-periodic solution $(x_\varepsilon(t), y_\varepsilon(t))$ of (2_ε) starting at time $t = 0$ from a point of $\mathcal{A}$ which makes exactly N rotations around the origin in the time kT. Taking a sequence $\varepsilon_n \to 0$, a standard compactness argument yields the existence of a subsequence of $(x_{\varepsilon_n}(t), y_{\varepsilon_n}(t))$ converging towards a solution $(\tilde{x}(t), \tilde{y}(t))$ of (2_0) whose orbit is contained in $\mathcal{A}$, and which makes exactly N rotations around the origin in the time kT. Hence, such a solution has minimal period kT/N, and by the choice of $\mathcal{A}$ its orbit must be Γ_r. Therefore, for some $\delta \in [0, \frac{kT}{N})$, one has $(\tilde{x}(t), \tilde{y}(t)) = (\bar{x}(t+\delta), \bar{y}(t+\delta))$ for every t. The proof follows from the above arguments, by contradiction. □

In order to provide a simple situation when Theorem 2 can be applied, we give a condition which implies that the time-map is strictly increasing. In the following proposition, we consider the function $g(x)$ to be of "softening characteristic" (see [8,9]).

Proposition. *Assume (c) and the following condition:*

$$\text{(d)} \qquad \frac{\bar{g}(x)}{x} \ \textit{is} \ \begin{cases} \textit{strictly increasing for } x < 0, \\ \textit{strictly decreasing for } x > 0. \end{cases}$$

Then the time-map τ is strictly increasing on its domain.

Proof. Let, for $r \in (0, \beta_1)$,

$$\tau_1(r) = \sqrt{2} \int_0^r \frac{dx}{\sqrt{G(r) - G(x)}} \,.$$

Let us prove that τ_1 is strictly increasing on $(0, \beta_1)$. Consider r, r' such that $0 < r < r' < \beta_1$, and set $p = r'/r$. Define, for $x \in (0, \beta_1/p)$, the function

$$h(x) = \frac{\bar{g}(px)}{p}.$$

By our assumption, for every $x \in (0, \beta_1/p)$, one has

$$\bar{g}(x) = \frac{\bar{g}(x)}{x}x > \frac{\bar{g}(px)}{px}x = h(x),$$

and hence, setting $H(x) = \int_0^x h(u)\,du$, one has $G(r) - G(x) > H(r) - H(x)$. Then,

$$\tau_1(r) < \sqrt{2}\int_0^r \frac{dx}{\sqrt{H(r) - H(x)}} = \sqrt{2}\int_0^r \frac{p\,dx}{\sqrt{G(r') - G(px)}}$$
$$= \sqrt{2}\int_0^{r'} \frac{dx}{\sqrt{G(r') - G(x)}} = \tau_1(r'),$$

showing that τ_1 is strictly increasing on $(0, \beta_1)$. In a similar way, one proves that

$$\tau_2(s) = \sqrt{2}\int_s^0 \frac{dx}{\sqrt{G(s) - G(x)}}$$

is strictly decreasing on $(\alpha_1, 0)$. For any $r \in (0, \beta_1)$,

$$\tau(r) = \tau_1(r) + \tau_2(s),$$

where $s \in (\alpha_1, 0)$ is such that $G(s) = G(r)$. Being $G(x)$ strictly increasing for $x > 0$ and strictly decreasing for $x < 0$, one can easily conclude the proof. □

Corollary. *Assume (c) and the following condition:*

(e) $$\bar{g}(x) \text{ is } \begin{cases} \text{strictly convex for } x < 0, \\ \text{strictly concave for } x > 0. \end{cases}$$

Then the time-map τ is strictly increasing on its domain.

The situation described above can be found in mechanical systems where the restoring force is pendulum-like or Sitnikov-like (cf. [2]), even if we do not need it to be differentiable, but only continuous.

Similarly, one can provide a situation when the time-map is strictly decreasing, by assuming $g(x)$ to be of "hardening characteristic".

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